APPROXIMATING SUBOPTIMAL DYNAMIC EQUILIBRIA: AN EULER EQUATION APPROACH

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Approximating Suboptimal Dynamic Equilibria:

An Euler Equation Approach

by

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ABSTRACT

This paper develops a new method for approximating dynamic competitive equilibria in economies in which competitive equilibrium is not necessarily Pareto optimal. The method involves finding approximate equilibrium policy functions by iterating on the stochastic Euler equations which characterize the economy's equilibrium. Two applications are presented: the stochastic growth model of Brock and Mirman (1971) modified to allow distortionary taxation, and a model of inflation and capital accumulation based on Stockman (1981). The computational speed and accuracy of this approach suggests that it may be a feasible method for studying suboptimal economies with large state spaces.

1. Introduction

Many central research questions necessarily involve the study of suboptimal dynamic equilibria. For example, one might be interested in studying the effect of a change in the income tax laws on the joint time series behavior of investment, production, and asset returns under the assumption of rational expectations. But because the wedge between private and social returns induced by the tax policy means that the resulting dynamic equilibrium is suboptimal, this problem cannot generally be studied with methods which require the Pareto optimality of competitive equilibrium.

This paper develops a new method for approximating dynamic competitive equilibria which can be applied to a wide variety of economic environments.[1] Within the model economy, individual agents are assumed to make their decisions in a privately rational manner. The result of this maximization is a set of first-order necessary conditions or "stochastic Euler equations" for the individual's problem; these conditions restrict the dynamic evolution of the individual's choice variables. When combined with aggregate consistency conditions, the stochastic Euler equations restrict the dynamic behavior of the entire economic system. A dynamic competitive equilibrium, then, is a set of functions that satisfy the stochastic Euler equations. The properties of equilibrium can be explored by finding approximations to these equilibrium functions. This paper presents a method for approximating the equilibrium functions that solve the stochastic Euler equations. A virtue of this method is that it is applicable to economies in which competitive equilibrium is not necessarily Pareto optimal.

Approximate equilibria are computed by an algorithm which involves discretization of the state space as in Bertsekas (1976) and Sargent (1980), combined with iteration on the stochastic Euler equations. This paper provides a detailed discussion of this method, and demonstrates its use by application to two examples. The first example is based on the stochastic one-sector neoclassical growth model of Brock and Mirman (1982) modified to allow distortionary taxation. The second example is a model of a monetary economy. Stockman (1981) takes the deterministic one-sector growth model, imposes a cash-in-advance constraint on purchases of consumption goods and investment goods, and characterizes the steady state levels of capital and inflation under a constant money growth rule. The present paper approximates equilibrium capital accumulation rules within a stochastic version of the cash-in-advance model, where the uncertainty stems from random monetary growth.

While this paper is organized around two problems of capital accumulation in the presence of distortions, the basic computational approach is applicable to a wide variety of problems in which competitive equilibrium can be characterized as a system of Euler equations. For example, this method can be used to study overlapping generations (OLG) economies with long-lived agents, as discussed in Baxter (1987). Because equilibria in OLG economies are generally suboptimal, these equilibria cannot be studied using methods that rely on the optimality of competitive equilibrium. Other types of monetary economies that can be studied include those in which money is introduced via the utility function or via an explicit transactions technology. This methodology is also potentially applicable to the study of economies in which suboptimality of competitive equilibrium is due to the existence of monopoly power at the firm level, as in Blanchard and Kiyotaki (1987); due to productive externalities of the types studied by Romer (1986), Lucas (1988) and Baxter and King (1988); or due to incompleteness in asset markets, as in Persson and Svensson (1987) and Svensson (1988).

The paper is organized as follows. Section 2 describes the method of obtaining approximate equilibrium policy functions via iteration on Euler equations. The presentation is organized around the stochastic one-sector growth model with distortionary taxation, and highlights the conceptual similarity between this method and the more familiar method of value function iteration. The section concludes with the presentation of policy functions computed for several example economies and evaluates the computational accuracy of the approach. This is done by comparing approximate policy functions to exact policy functions in the context of an example economy possessing a closed-form solution for the policy function. Section 3 presents the results of applying this computational method to a stochastic version of Stockman's (1981) model of capital accumulation in a cash-in-advance model. Section 4 contains concluding remarks and discusses directions for future research.

2. The equilibrium Euler equation approach

The equilibrium Euler equation approach is illustrated within the basic dynamic framework of the neoclassical model of capital accumulation under uncertainty. In this model, individuals maximize expected utility:

$$E \sum_{t=0}^{\infty} \beta^t u(c_t) | A_0, k_0 \tag{1}$$

where β is a discount factor between zero and one, where the utility function u(.) is assumed to be twice continuously differentiable, and where the expectation taken at time zero is conditioned on the initial capital stock k_0 and the initial value of the technology shock, A_0. Agents face a sequence of resource constraints of the form

$$A_t f(k_t) + (1-\delta)k_t \leq c_t + k_{t+1} \tag{2}$$

where A_t is a technology shock; k_t is the capital stock, predetermined as of the beginning of period t; f(.) is the production function, assumed to be twice continuously differentiable; and δ is the rate of depreciation of capital. The technology shock, A_t, follows a discrete Markov process with state transition matrix Π. Agents in this model are viewed as owning the capital stock and directly operating the technology. In period t, they receive output from production, $A_t f(k_t)$, and there is undepreciated capital left over after production in the amount $(1-\delta)k_t$. They allocate this gross output between current consumption, c_t, and capital to be used in production in the subsequent period, k_{t+1}. Thus the period t+1 capital stock is determined at the end of period t, and cannot be adjusted after the period technology shock A_{t+1} is realized at the beginning of period t+1. We shall assume that there is a maximum sustainable capital stock so that stationary distribution of capital is bounded.

The first-order necessary conditions for the consumer's problem are:

$$Du(c_t)=\beta E \{[A_{t+1}Df(k_{t+1})+(1-\delta)]Du(c_{t+1})\} | A_t, k_{t+1}. \tag{3a}$$

$$E \{\lim_{t\to\infty} \beta^t Du(c_t)k_{t+1}\} | A_0, k_0 = 0 \tag{3b}$$

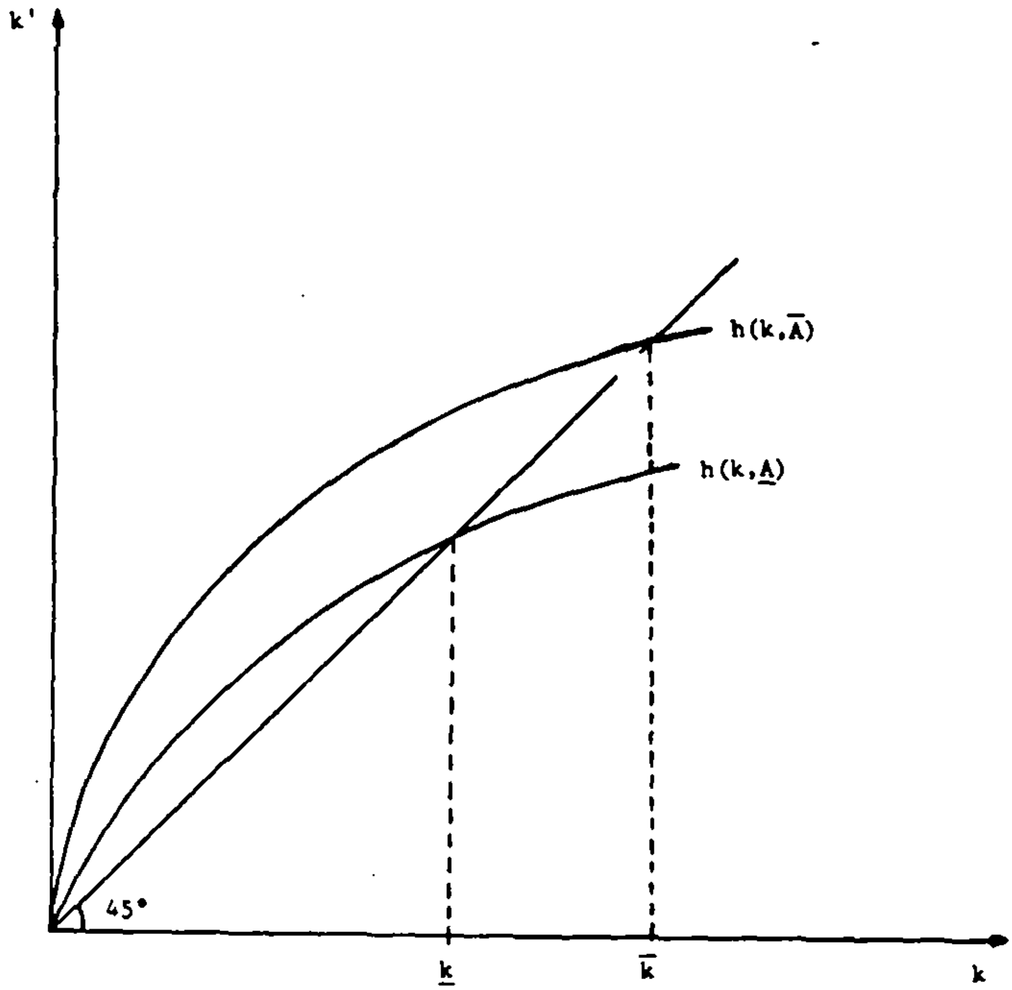
and the resource constraints, (2). Since this problem is recursive (i.e., does not involve time in an essential way) we let unprimed variables denote period t, single primes denote period t+1, and double primes denote period t+2. Making these substitutions, and using the resource constraint to substitute for c, equation (3a) becomes:

$$Du(Af(k)+(1-\delta)k-k')=\beta E\ \{[A'Df(k')+(1-\delta)]Du(A'f(k')+(1-\delta)k'-k'')\}|A,k'. \quad (4)$$

Under the assumptions imposed on this problem, there is unique function relating the optimal choice of k' to the current level of k and the current technology shock A; call this function h:

$$k' = h(k,A). \quad (5)$$

To take a specific example, suppose that there are only two possible realizations of the technology shock, $A_t \in \{\bar{A}, \underline{A}\}$, and that A_t follows a Markov process with transition function F. Graphed below are the functions relating k' to k and the technology shock. One steady state (or fixed point) with a constant level of k is at $k=\bar{k}$; this is the level of capital that would obtain if the economy turned out always to have the high realization of the technology shock, $A_t=\bar{A}$ for all t (even though, each period, there is positive probability that $A=\underline{A}$ in some future period.) There is a second steady state with a constant level of k, at $k=\underline{k}$, which is the level of capital that would obtain if the economy always had the low realization of the technology shock, $A_t=\underline{A}$ for all t. In addition, all the points in the interval $(\bar{k}, \underline{k})$ generally have positive mass in the stationary distribution of k.



Except under very special conditions on preferences and technologies, it is not possible to solve (4) to obtain a closed-form solution for the function h(k,A). We turn now to a discussion of two approaches to computing approximations to the equilibrium policy function. The first is the approach of stochastic dynamic programming and value function iteration. This method relies on the equivalence between competitive equilibrium and Pareto optimum in the economy under consideration. (Two examples of papers which use this computational approach are Sargent (1979) and Greenwood, Hercowitz and Huffman (1988).) The second approach is new, and involves iteration on a stochastic Euler equation. This approach does not rely on the Pareto optimality of competitive equilibrium.

2.1 Stochastic Dynamic Programming and Value Function Iteration

Since the problem described above has a recursive structure, it can be studied using the methods of stochastic dynamic programming. Thus, the problem can be rewritten as:

$$v(k_t,A_t) = \max_{c_t,k_{t+1}} \quad u(c_t) + \beta E \,\{v(k_{t+1},A_{t+1})\}\,|k_{t+1},A_t \tag{6}$$

subject to the constraint (2). The function v is commonly referred to as the value function; it gives the value, in utility terms, of entering a period with capital equal to k_t and encountering the technology shock A_t, assuming that the agent makes individually optimal decisions. Equation (6) is a functional equation in the unknown function v. Using (2) to substitute for c_t in equation (4), we obtain:

$$v(k_t,A_t) = \max_{k_{t+1}} \quad u(A_t f(k_t)+(1-\delta)k_t-k_{t+1}) + \beta E_t\{v(k_{t+1},A_{t+1})\}\,|k_{t+1},A_t. \tag{7}$$

Define the operator T by

$$Tv \;=\; \max_{k_{t+1}} \quad u(A_t f(k_t)+(1-\delta)k_t-k_{t+1}) + \beta E_t\{v(k_{t+1},A_{t+1})\}\,|k_{t+1},A_t. \tag{8}$$

Since the form of (8) does not depend on the time period, t, time subscripts can be suppressed and (8) can be written:

$$Tv = \max_{k'} \quad \{u(Af(k)+(1-\delta)k-k') + \beta Ev(k',A')\}\,|k',A \tag{9}$$

where, as above, variables without superscripts refer to the current period (t) and primed variables refer to the subsequent period (t+1).

Solving for the unknown function v involves finding a fixed point (in the space of continuous functions) of the mapping T, i.e., finding the function v for which Tv=v. Because the mapping T defined by equation (8) is a contraction mapping, iteration on the mapping converges to the function v which is the unique fixed point of the mapping. This indicates that iteration on the mapping can be used as a computational approach to finding an approximation to the optimal value function. The approximate nature of the solution is due to the computational necessity of "discretizing the state space", i.e., choosing a discrete grid for k and A over which the value function will be defined. Having done this, the computational problem involves finding the value function defined on the (k,A) grid that solves the equation Tv=v.

The iterative procedure begins by choosing an initial v function (defined on the (k,A) grid) from the domain of T; call this function v_0. Given v_0, apply the operator T yields a new v function; call this new function v_1:

$$v_1 = Tv_0 = \max_{k'} \{u(Af(k)+(1-\delta)k-k') + \beta E v_0(k',A') | k',A\}$$

where the maximization is over values of k in the chosen grid, and is conditional on the current value of A. Subsequent iterations proceed in the same way, generating a sequence of v functions, $\{v_j\}$. Because the operator T defined in (8) is a contraction, this sequence of functions converges to the true value function: $\lim_{j\to\infty} v_j = v$.

Often the value function chosen as the starting point for the iterative procedure is the zero function, $v_0=0$. This choice of v_0 means that the sequence of functions produced by application of the operator T has an economic interpretation as the sequence of value functions for finite economies. Thus, v_1 is the value function for an economy with one period left to go, v_2 is the value function for an economy with two periods left to go, and so forth. In this problem, the limit of the value functions for the finite horizon economies is the value function for the infinite horizon economy. We will return to this interpretation when discussing iteration on Euler equations below, except that there we will be generating a sequence of policy functions instead of a sequence of value functions.

If the economy under study satisfies the conditions of the second welfare theorem, the optimal solution obtained by value function iteration may be interpreted as a competitive equilibrium. In cases where competitive equilibrium is not optimal, the approach outlined above is generally invalid. It can be used only if there is a way to rewrite the competitive problem as an optimum problem which properly reflects the constraints of the competitive problem. The class of problems for which this is possible, however, is not very large. Studying suboptimal equilibria generally requires a direct attack on the first-order necessary conditions of the individual's problem; it is to this approach that we now turn.

2.2 The Equilibrium Euler Equation Approach

Unlike value function iteration, the method described here does not rely on the second welfare theorem. For illustrative purposes, we consider the the neoclassical model of capital accumulation described above, modified to allow distortionary taxation in the form of an income tax with lump-sum rebates of the proceeds of the tax. In this example, taxes can be functions of the Markovian technology shock and the level of the aggregate capital stock. Per capita aggregate capital will be denoted by K, and the agent's choice of capital will be denoted by k. The tax function is denoted $\tau(K,A)$.

The problem facing the representative agent in this economy is:

$$\max_{\{c_t,\ k_{t+1}\}} E\ \{ \sum_{t=0}^{\infty} \beta^t u(c_t)\} | A_0, k_0 \tag{10}$$

subject to:

$$c_t + k_{t+1} \le (1-\tau(K_t,A_t))A_t f(k_t) + (1-\delta)k_t + [\tau(K_t,A_t)A_t f(K_t)] \tag{11}$$

$$K_{t+1} = H(K_t,\ A_t) \tag{12}$$

where all variables are as defined earlier. Equation (11) is the individual's resource constraint; the first two terms on the right hand side are after tax gross output, and the last term is the lump sum rebate of the government's tax revenues. Equation (12) is agents' perceived law of motion for aggregate capital, K_t. As before, it is convenient to suppress time subscripts, and the arguments of the tax function are suppressed as well: τ should be read as $\tau(K,A)$. The first-order necessary condition for maximization with respect to choice of capital is:

$$Du(c) = \beta E\{[(1-\tau')A'Df(k')+(1-\delta)]Du(c')\} | A,k' \ .$$

Using (11) to substitute for c_t yields:

$$Du((1-\tau)Af(k)+(1-\delta)k+\tau(A,K)f(K)-k') = \tag{13}$$

$$\beta E\{[(1-\tau')A'Df(k')+(1-\delta)]Du((1-\tau')A'f(k')+(1-\delta)k'+\tau'A'f(K')-k'')\} | A,k'.$$

Individual maximization yields equilibrium decision rules of the form

$$k' = h(k,A;K,H). \tag{14}$$

In equilibrium, the capital agents choose to carry out of the period is a function of capital brought into the period, k, and the current technology shock, A. Individuals take as given the current level of the aggregate capital stock, K, and condition on their beliefs about the law of motion for aggregate capital as summarized by the function H.

A rational expectations equilibrium requires, in the case of a single representative agent, that the law of motion for k_{t+1} coincides the the perceived law of motion for K_{t+1}:

$$h(k,A;K,H) = H(K,A). \tag{15}$$

This condition is sometimes referred to as a "consistency condition", meaning that individual's beliefs are consistent with the outcomes of the economy's equilibrium: in equilibrium, k (capital chosen by the representative agent) must equal aggregate capital, K. Imposing this consistency condition on the first order condition yields:

$$Du(Af(k)+(1-\delta)k-k') = \beta E\{(1-\tau')A'Df(k')+(1-\delta)]Du(A'f(k')+(1-\delta)k'-k'')\}. \tag{16}$$

Finding the competitive equilbrium means finding the function h of the form given by (15) which solves (16) and for which the implied function H is such that $h(k,A;K,H) = H(K,A)$. Below, we use the notation $h(k,A)$ when referring to equilibrium policy functions, functions for which $h(k,A;K,H) = H(K,A)$.

2.3 An Iterative Approach to Approximating Stochastic Euler Equations

This subsection provides a detailed description of the computational procedure for computing approximate equilibrium policy rules.[2] This procedure is similar in spirit to the method of iterating on the value function described earlier. As before, the first step is to discretize the state space by choosing a grid for k and A. And as with value function iteration, the method of iterating on Euler equations can be viewed as generating a sequence of optimal policy rules for finite economies with the horizon lengthening one period at each iteration. This perspective will be used in the following discussion of the computational algorithm. Under this perspective, we view ourselves as working backward from the end of the economy, in a manner similar to stochastic dynamic programming.

Therefore, consider an economy that will terminate at the end of period N. In an N-period economy agents will plan to consume all of their capital by the end of period N, setting $k_{N+1}=0$ regardless of the levels of k_N and A_N. Thus, the equilibrium policy function relating k' to (k,A) for an economy with zero periods to go is the zero function: $k_{N+1}=h_0(k_N,A_N)=0$. Now, step back one period and consider the problem of the optimal choice of capital in period N-1. This involves solving the period N-1 version of equation (16), using the fact that $k_{N+1}=h_0(k_N,A_N)=0$. Thus, the period N-1 version of (16) is:

$$Du[A_{N-1}f(k_{N-1})+(1-\delta)k_{N-1}-k_N] = \beta E\{[(1-\tau_N)A_N Df(k_N)+(1-\delta)]Du(A_N f(k_N)+(1-\delta)k_N)\}|k_N,A_{N-1}. \tag{17}$$

This is a first-order stochastic difference equation in k. Solving the equation means finding, for each (k_{N-1},A_{N-1}) pair, the equilibrium amount of capital to take out of the period, k_N. That is, the solution is a function: $k_N=h_1(k_{N-1},A_{N-1})$. Thus we will use the initial policy function, h_0, together with the stochastic Euler equation to generate a new policy function, h_1.

A two step computational procedure is used to trace out the new policy function, h_1. The first step is to compute the "marginal value" of k'--the right hand side of (16)--for each (k_N,A_{N-1}) pair in the grid. This generates a "marginal value matrix", call it $MV(k_N,A_{N-1})$. The second step is to find, for each (k_{N-1},A_{N-1}) pair, the value of k in the matrix which comes closest to solving (17); i.e., the value of k_N which comes nearest to solving

$Du(Af(k_{N-1})+(1-\delta)k_{N-1}-k_N)=MV(k_N,A_{N-1})$. This yields a function giving k_N as a function of A_{N-1} and k_{N-1}, call this function h_1: $k_N = h_1(k_{N-1},A_{N-1})$. We now have the equilibrium policy function yielding k' as a function of (k,A) for an economy with one period left.

Now, step back again, and consider the version of equation (16) that applies to the economy in period N-2. This equation is:

$$Du(A_{N-2}f(k_{N-2})+(1-\delta)k_{N-2}-k_{N-1}) = \tag{18}$$
$$\beta E\{[(1-\tau_{N-1})A_{N-1}Df(k_{N-1})+(1-\delta)]Du(A_{N-1}f(k_{N-1})+(1-\delta)k_{N-1}-k_N)\}|k_{N-1},A_{N-2}.$$

This is a second order difference equation in k. But in the previous paragraph, we discussed how to find the function $k_N=h_1(k_{N-1},A_{N-1})$. We can therefore use this function to substitute for k_N in equation (18), obtaining

$$Du(A_{N-2}f(k_{N-2})+(1-\delta)k_{N-2}-k_{N-1}) = \beta E\{[(1-\tau_{N-1})A_{N-1}Df(k_{N-1})+(1-\delta)]$$
$$Du(A_{N-1}f(k_{N-1})+(1-\delta)k_{N-1}-h_1(k_{N-1},A_{N-1}))\}|k_{N-1},A_{N-2}\ . \tag{19}$$

Now we have only a first-order difference equation in k. Proceeding as in the first iteration, compute the right-hand-side of (19) to yield $MV(A_{N-2},k_{N-1})$ for every (A_{N-2},k_{N-1}) pair. Now, for each pair (A_{N-2},k_{N-2}) find the value of k in the grid which, when substituted for k_{N-1}, comes closest to solving (19). Call this function h_2. $(k_{N-1}=h_2(k_{N-2},A_{N-2}))$.

The way to proceed in the third and subsequent steps should, by now, be clear. At step j, the policy function h_{j-1} is used to replace k" on the right- hand side of equation (16), and the resulting equation is solved to obtain a new policy function h_j as described above.

Iteration continues until the sequence of functions, $\{h_j\}$ converges, i.e., when h_j changes only a small amount between iterations. Thus, one might choose a tolerance level d and stop when

$$\max_{A,k} [h_j(k,A) - h_{j-1}(k,A)] \leq d.$$

In the applications presented in this paper, however, the iterative procedure was run a fixed number of times (usually 100) and then terminated.[3]

To summarize, the computational strategy begins by choosing an initial policy function, h_0. Using h_0 to evaluate k'' as a function of k' and A', equation (16) becomes a first-order stochastic difference equation determining k' as a function of k and A; call this function $h_1(k,A)$. However, this function is not the equilibrium function h. The iterative process involves replacing the initial function h_0 with the function h_1 computed as described above; in the second iteration, evaluate k'' using $k''=h_1(k',A')$. Now, equation (16) is a new difference equation determining k' as a function of k and A; call this new function $h_2(k,A)$. Step j of the iterative process involves replacing h_{j-2} by h_{j-1} as the function determining k'', and solving (16) for a new function h_j yielding k' as a function of k and A. The iterative process stops when the sequence of functions $\{h_j\}$ converges.

In the course of implementing this algorithm, the initial function $h_0=0$ has been found to work well in the sense that convergence is quite rapid. With $h_0=0$, the sequence of functions $\{h_j\}$ generated by the iterative procedure has a natural economic interpretation in much the same way as in value function iteration discussed earlier. The sequence $\{h_j\}$ can be viewed as an approximation to the sequence of equilibrium policy functions for finite economies with j periods left to go. But computation is more rapid still if the initial function h_0 is a positive, nondecreasing function of k and A. The interpretation of a positive h_0 is that the economy is required to end with a positive level of the capital stock. The reason is that iterative scheme essentially involves exploiting the "turnpike" property of the finite horizon economy. Thus, loosely speaking, the nearer you are to the turnpike when you start, the sooner you arrive on the turnpike (i.e., the sooner the policy function converges to that for the infinite horizon problem).

There do not appear to be any general theoretical results available which give the conditions under which this procedure will converge to the infinite horizon solution. The following subsection presents approximate policy functions for a several examples of the distorted stochastic growth model, including one example with a nonmonotonic, discontinuous tax function. The algorithm is well behaved and converges even in this case. Based on experimentation with a variety of economies, it is conjectured that the iterative scheme described above converges for any economy which possesses a turnpike property, in the sense that the limit of the sequence of equilibrium policy functions for finite horizon economies is the equilibrium policy function for an infinite horizon economy. However, this conjecture has not yet been formally established.

2.4 Some examples

This section presents the results of applying the equilibrium Euler equation approach to several specific examples of capital accumulation problems in distorted economies. It begins by examining a special case of an undistorted economy for which a closed form solution exists. The approximate policy rules are compared to those computed from the closed form.

Subsequently, an example of an economy with distortionary taxation is presented.

A closed-form example

As an initial appplication of the approximation methodology, we study an economy possessing a closed form solution for the policy function $k' = h(k,A)$. This closed form is used to check the accuracy of the approximation methodology. In this economy, individuals maximize an objective function of the following form:

$$E \sum_{t=0}^{\infty} \beta^t \ln(c_t) | A_0, k_0. \tag{20}$$

The production function is Cobb-Douglas and is subject to Markovian technology shocks, A:

$$Af(k) = Ak^{\alpha}. \tag{21}$$

and there is 100% depreciation of capital in each period: $\delta=1$. Thus the resource constraint is given by:

$$Ak^{\alpha} \leq c + k' \quad . \tag{22}$$

In this economy, the solution for the equilibrium path of capital is given by:

$$k' = (\alpha\beta)Ak^{\alpha}. \tag{23}$$

Figure 1 plots the exact equilibrium policy functions for this economy with a grid containing 500 points for k and two points for A, with the following parameter values: $\beta=.95$, $\alpha=.4$, $\underline{A}=1.0$, $\bar{A}=1.2$. On the graph there is one policy function for each value of the technology shock, and one fixed point or steady state corresponding to each value of the technology shock. Figure 2 exhibits the approximate equilibrium policy functions for this economy, computed with the same capital grid of 500 points, and for 100 iterations. Figure 3 exhibits the approximate functions together with the exact functions. As seen from Figure 3, the approximate function is essentially indistinguishable from the exact function. To get a closer look at the approximation error, Figure 4 graphs the approximation error (approximate minus exact) against the capital stock for each of the two policy functions. In the range of k containing the stationary distribution for k (roughly .30 to .65) the average approximation error is less than 1%. Convergence of the policy functions is quite rapid, and after only 20 iterations the approximate policy rule is visually indistinguishable from the exact policy rule, in the sense that these functions are indistinguishable in Figure 3. The initial function h_0 was a constant function close to zero.

Other examples

Figure 5 plots equilibrium policy functions for the economy described above with the modification that depreciation is a more realistic 10% per year. A notable feature of the policy functions is that they appear approximately linear in the capital stock. There are two steady states, one for each value of the technology shock; the upper one is at a level of capital of 6.75, and the lower one is at a level of 4.98. The stationary distribution of capital is contained in the interval (4.98,6.75). Figure 6 plots equilibrium policy functions for the economy with 10% depreciation, and in which there is also a 25% tax on output (not including the undepreciated component of the capital stock). The tax rate is not state-dependent. In this economy, the steady states are at 4.669 and 2.645. The upper steady state in the taxed economy is 31% below that of the untaxed economy, and the lower steady state is 47% below the corresponding steady state in the untaxed economy. Thus a tax rate of 25% on output leads to a greater than proportional decline in the stationary distribution of the capital stock in this economy. Computation of the stationary distribution of capital is straightforward, and would be preliminary to answering questions about relative welfare in the taxed and untaxed economies.

Finally, Figure 7 graphs the equilibrium response of the economy with 10% depreciation a particularly strange tax function: the tax is zero for capital stocks between zero and 3.2 and between 4.4 and infinity, and is equal to 30% for capital stocks between 3.2 and 4.4. This example is presented to demonstrate that convergence does not depend on smooth or monotonic tax functions. As one would guess, the tax depresses capital accumulation in the range over which it operates. The fact that the algorithm converges easily even in the presence of such a strange tax function suggests that it may be possible to develop a theoretical proof of the conjecture that convergence requires only fairly weak conditions on the structure of the economy; i.e., the turnpike condition discussed earlier.

3. Anticipated inflation and the capital stock in a cash-in-advance economy

In a paper with the above title, Alan Stockman (1981) presented a model in which higher expected inflation leads to a fall in the capital stock. The adverse effect of inflation stems from the fact that the model's economic environment requires that money be accumulated in advance of purchases of investment goods so that inflation acts as a tax on investment. In his paper, Stockman characterizes the steady state of the economy under a constant rate of monetary growth and no uncertainty elsewhere in the model. Abel (1985) studies a version of Stockman's economy which has been linearized about the steady state. He develops expressions relating the near-steady-state speed of adjustment in the economy to preference parameters and the monetary growth rate.

Using the computational methods developed in this paper, we can assess the quantitative effects of money growth on capital by looking directly at numerical approximations to the equilibrium policy rules, obviating the need for linear approximations of the sort used by Abel. In particular, we are not constrained to studying deterministic money growth, and the productive environment can easily be generalized to allow technology stocks and distortionary taxation of output as in the economy of Section 2 above. Thus, we may ask the questions: (i) given reasonable parameters for preferences, technology, and the stochastic process for money growth, what is the quantitative effect of money growth and anticipated inflation on the steady state level of the capital stock? (ii) quantitatively, what is the welfare loss associated with inflation? (iii) what is the marginal welfare loss from inflation (i.e., is 7% inflation much worse than 6% inflation)?, and (iv) what is the joint stationary distribution of money and capital? After describing the model, we shall present a first attempt at answering some of these questions.

3.1 The model

This model is identical to that in Stockman (1981) except that money growth is permitted to follow a Markov process. The representative individual is assumed to maximize

$$E\{\sum_{t=0}^{\infty} \beta^t u(c_t)\}|k_0,\omega_0 \tag{24}$$

subject to two constraints--a resource constraint and the cash-in-advance constraint:

$$f(k_t) + (m_{t-1}(1+\omega_t))/p_t - c_t - k_{t+1} + (1-\delta)k_t - (m_t^d/p_t) = 0 \tag{25}$$

$$(m_{t-1}(1+\omega_t))/p_t \geq c_t - k_{t+1} + (1-\delta)k_t \tag{26}$$

where ω_t denotes the random lump-sum monetary transfer paid out at the beginning of period t, $m_t \equiv m_{t-1}(1+\omega_t)$ is the post-transfer nominal money holdings at the beginning of period t, and m_t^d is "money demand", the amount of money held at the end of period t. The random monetary growth rate ω_t is assumed to follow a discrete Markov process with state transition matrix Π. The force of the cash-in-advance constraint (26) is that both consumption and investment must be paid for in cash out of money carried over from the previous period plus current period monetary transfers. Let λ and μ be the Kuhn-Tucker multipliers for the constraints (25) and (26). The first-order conditions for this problem are:

$$Du(c_t) = \lambda_t + \mu_t \tag{27}$$

$$\beta\lambda_{t+1}[Df(k_{t+1}) + (1-\delta)] + \beta\mu_{t+1}(1-\delta) = \lambda_t + \mu_t \tag{28}$$

$$\beta\lambda_{t+1}(1/p_{t+1}) + \beta\mu_{t+1}(1/p_{t+1}) = \lambda_t(1/p_t) \tag{29}$$

$$f(k_t) + (m_{t-1}(1+\omega_t))/p_t - c_t - k_{t+1} + (1-\delta)k_t - (m_t^d/p_t) = 0 \tag{30}$$

$$(m_{t-1}(1+\omega_t))/p_t \geq [c_t - k_{t+1} + (1-\delta)k_t]\mu_t = 0, \quad [.]\geq 0, \quad \mu_t \geq 0. \tag{31}$$

The multiplier λ may be interpretated as the marginal utility of wealth, and the multiplier μ may be interpreted as the marginal utility of real cash balances. Under conditions guaranteeing that the nominal rate of interest is always positive, which we shall impose, the cash-in-advance constraint (26) always holds with equality, so that the multiplier μ is always positive. Under these assumptions, (29)-(31) can be solved to yield the fundamental dynamic equation of the system:

$$Du(c) = E\ \{\beta^2(p'/p'')Df(k')Du(c'') + \beta(1-\delta)Du(c')\}\,|\omega,k' \tag{32}$$

where unprimed variables denote the current period (period t), single primes denote next period (period t+1), double primes denote period t+2, and triple primes (which will occur below) denote period t+3. Because the cash-in-advance is assumed always to bind, consumption is equal to:

$$c = f(k) + (1-\delta)k - k', \tag{33}$$

and the price level is equal to:

$$p = m/(c + k' - (1-\delta)k), \tag{34}$$

so that (32) becomes

$$\begin{aligned} &Du(f(k)+(1-\delta)k-k') = \\ &\quad E\ \{\beta^2 Df(k')(1/(1+\omega''))(f(k'')/f(k'))Du(f(k'')+(1-\delta)k''-k''') + \\ &\qquad \beta(1-\delta)Du(f(k')+(1-\delta)k'-k'')\}\,|\omega,k' \end{aligned} \tag{35}$$

Thus, the basic equation of the system is a a third-order nonlinear stochastic difference equation in k. As before, we seek a solution in the form of an equilibrium policy function $k'=h(k,\omega)$ yielding next period's capital stock, k', as a function of the current capital stock, k and the current realization of the monetary transfer, ω, given the state transition matrix Π.

3.2 Computation of Equilibrium

Despite the fact that this equation is one order higher than the one studied in the last section, it can be attacked in exactly the same way. In the model of the last section, an initial function, h_0, was chosen giving k_{N+1} as a function of k_N and A_N. This function was used to evaluate terms in k" in equation (16), that economy's analogue to equation (28). In this environment, we must specify a pair of initial functions, h_0 and h_{-1} that will be used to evaluate k" and k''' in equation (28). Thus, in the first iteration, let

$$k'' = h_0\ (k', \omega')$$
$$k''' = h_{-1}(k'', \omega'').$$

In the case where $h_0 = h_{-1} = \theta$, where θ is a small positive number, we retain our earlier economic interpretation of the iterative process as finding equilibrium policy functions for a sequence of finite horizon economies (with the terminal condition that the economy end with a level of capital equal to θ.)[4] As before, the first iteration involves finding, for each (k, ω) pair, the value of k' that solves (35), conditional on the functions h_0 and h_{-1}. This gives a new value of the equilibrium policy function, call it h_1: $k' = h_1(k, \omega)$. For the second iteration, we replace the function h_{-1} with the function h_0, and we replace the function h_0 with the newly calculated function, h_1. Thus, we view ourselves as stepping back one period in time, and evaluating k" and k''' by:

$$k'' = h_1(k', \omega')$$
$$k''' = h_0(k'', \omega'').$$

Iteration continues in this way until the policy function h converges.

The increased order of the difference equation does not mean that computation of equilibrium policy functions is significantly more time consuming. Recall that the computational strategy involves fixing k' and ω, and computing the right-hand-side of (35), which we can think of as the marginal value of k' conditional on ω. The second step is to find a value of k that makes the right-hand-side of the equation equal to the left-hand-side. The only effects of having a high order system are (i) to increase slightly the amount of arithmetic involved in computing the marginal value of k', and (ii) to carry along an additional h_j function at each step.

3.3 The Quantitative Effects of Inflation on Capital Accumulation

The time interval is taken to be a year, and parameter values for the economy were chosen as follows: α=.40, a number roughly consistent with estimates of capital's share in GNP; β=.95, implying a steady state real interest rate of about 5%; σ=1 which is logarithmic utility; and, unless stated otherwise, δ=.10 implying a 10% annual rate of depreciation of capital.

As a check on the computational accuracy of the program, we computed the equilibrium policy function for a closed form example which is essentially identical to that of Section 2.4. With 100% depreciation (δ=1) and a constant money growth rate ω, the equilibrium policy function has the form:

$$k' = \left[\frac{\alpha\beta^2}{(1+\omega)}\right] k^{\alpha}. \tag{36}$$

Figure 8 plots exact and approximate policy functions for this special case, with the constant money growth rate equal to -.05. This rate was chosen by setting $(1/(1+\omega))=\beta$; i.e., the money supply contracts at the rate of time preference. This is the optimal rate of monetary growth, and the result of choosing this monetary growth rate is that the equilibrium of the cash-in-advance economy is identical to the equilibrium of the undistorted economy studied in Section 2. (Notice that when $(1/(1+\omega))=\beta$, the equilibrium policy rule (36) becomes $k'=\alpha\beta k^{\alpha}$, exactly the equilibrium policy rule (23) in Section 2.4.) Thus another check on the cash-in-advance computer program is provided by setting $(1/(1+\omega))=\beta$ and comparing the results to those computed with the programs for Section 2. This check can be used even for economies with less than 100% depreciation of capital. Finally, in economies with deterministic money growth, Stockman provides the following equation implicitly determining the steady state level of capital:

$$Df(k) = \beta^{-2}(1+\omega)(1-(1-\delta)\beta).$$

This equation can be used to check that the algorithm delivers the correct steady state with deterministic money growth. All of these checks were carried out for a variety of parameter values, and the approximation error in each case was found to be very small.

Figure 9 graphs the equilibrium policy function for the economy with 10% depreciation and two possible values for the monetary growth rate: $\omega_1=.03$ and $\omega_2=.07$. The average steady state inflation rate is 5%, a figure roughly consistent with recent U.S. experience. The states are serially independent; every element of the state transition matrix is equal to .5. (In U.S. data, however, inflation is more persistent; we consider an example of persistent inflation below.) As seen from Figure 9, there is a single equilibrium policy function for this economy despite the fact that inflation is stochastic. The reason is that the states are independently distributed over time, and examination of equation (35) shows that only conditional expected inflation is important for capital accumulation. Since the states are i.i.d., conditional expected inflation is invariant to the current state. For comparison, the upper line in Figure 9 is the equilibrium policy function for the same economy except that capital and consumption goods may be obtained by barter; the cash-in-advance constraint is removed from the problem. The steady state capital stock in that economy is 4.98; in the cash-in-advance economy the steady state capital stock is 4.19, a level of capital which is 16% below that of the barter economy. Steady state utility is 11% lower in the cash-in-advance economy relative to the barter economy.

The average rate of monetary growth in this economy is 5%. If the monetary growth rate was a constant 5% rate, the steady state capital stock would be 4.25. Thus there is a sense in which there is an additional loss because of the stochastic nature of money growth and inflation.

Figure 10 plots the equilibrium policy functions for an economy with the same parameters as above, except that money growth is either .02 or .11. The probability of staying in the same monetary growth state is .95 (i.e. the probability of 2% monetary growth next year conditional on having 2% monetary growth this year is .95). These parameter values were chosen to simulate an economy which experiences either high inflation or low inflation, and in which the inflation rate is very persistent. In this example, the expected duration of the current inflation rate is 20 years (expected duration is $(1-\rho)^{-1}$ where ρ is the probability of staying in the same state next year.) The upper steady state, corresponding to the policy function for 2% current monetary growth, is at a capital level of 4.26. The lower steady state, corresponding to the policy function for 11% current monetary growth, is at a capital level of 3.95. In this model, the stationary distribution of capital is not a point, as it was with i.i.d. money supply growth rates. Instead, the stationary distribution of capital is contained in the interval (3.96,4.20). As in the examples of Section 2, the equilibrium decision rules are very close to linear. This suggests that for some applications the linear approximations used, for example, by King, Plosser anmd Rebelo (1988a,b) may be good approximations to the exact decision rules, and they have a definite advantage in terms of speed of computation.

4. Conclusions

This paper has developed a new method for obtaining equilibrium policy functions by means of iteration on stochastic Euler equations. The chief advantage of this method is that it can be used to study economies in which competitive equilibrium is not Pareto optimal. Previously, such economies could only be studied if the problem could be recast as a fictitious planner's problem, rendering the problem amenable to study by means of value function iteration. This new method is computationally fast and accurate, as demonstrated in Sections 2 and 3.[5]

With this new technology in hand, we can quantitatively evaluate a much wider range of theoretical economies. Our hunch is that the class includes any model whose equilibrium are characterized by a set stochastic Euler equations and which possesses a kind of turnpike property. Thus, the stochastic neoclassical growth model with distortionary taxation which was the focal point of this paper could be generalized to allow variable labor, other kinds of tax policies, productive externalities, and additional sources of randomness such as preference shocks or labor augmenting technical change.

But the applications are not limited to neoclassical capital theory. For example, one could study economies whose equilibria are suboptimal because of (i) monopolistic market structure; (ii) absence of complete markets, perhaps due to private information, or (iii) money introduced via money in the utility function or via an explicit transactions technology.

Another application of these methods is to the study of stochastic overlapping generations models with agents who live for realistic lengths of time. Because the state space in such a model is very large (the state variables include the beginning-of-period wealth positions of everyone alive in the economy) it is essential that the computational algorithm is one which runs rapidly for economies with small state vectors. The results presented here are encouraging. Thus, while the economic structure of the OLG economy maps neatly into the framework developed here (see Baxter (1987)), the next step in this line of research is to determine whether applications of these methods to this problem is computationally feasible.

Another use of this method is as a check on the computational accuracy of the sort of linear approximation methods currently used in studying equilibrium business cycle models. The fact that many of the decision rules computed in this paper are so nearly linear suggests that, for many applications, linear approximations will work well (and run many times faster!).

Finally, it is important to address the question of how to evaluate the "fit" of a model constructed and simulated along the lines developed in this paper. One method, popular in the study of real business cycle models,[6] begins by choosing key parameters from microeconomic studies and the national accounts, together with parameters for the stochastic processes of the exogenous shocks. Then, moments of the simulated time series are compared to a subset of the moments of actual time series. The model is said to fit well if the moments match up in a sense chosen by the researcher. But many researchers prefer an evaluation procedure grounded in classical statistical theory and are consequently uncomfortable with this informal approach. In a recent paper, Singleton (1988) discusses econometric methods for evaluation of real business cycle models. The methods he discusses are also applicable to the class of models for which the methods in this paper were developed. An important component of future work in this area is statistical evaluation of the empirical adequacy of these models along the lines suggested by Singleton.

Footnotes

[1]This paper develops in more detail the computational strategy outlined in Baxter (1987). That paper presented a Markovian representation of equilibrium in overlapping generations models with long-lived agents, together with an algorithm for generating numerical approximations to equilibrium decision rules in that economy. To keep the size of the state space small, the computational algorithm was developed and discussed in the context of the stochastic one-sector growth model.

[2]The computer code to execute the algorithm described in this paper was written in Fortran. The programs were run using Microsoft Fortran Version 3.31 on IBM-compatibe personal computers. On an IBM Personal System 2/ Model 80, the computation time for 100 iterations with a capital grid of 500 points and two values of the technology shock was about six and a half minutes.

[3]By a variety of reasonable convergence criteria, the functions presented in this paper have generally converged after 40 iterations.

[4]The cash-in-advance constraint guarantees that money is valued in a finite horizon economy since without holding money, one literally cannot eat. In other monetary economies--the OLG economy, for example--this is not the case. In order to ensure that the limit of the finite horizon economies converges to the infinite horizon equilibrium with valued money, it would be necessary to impose a terminal condition requiring positive money holdings.

[5]Informal "horseraces" suggest that this method is significantly faster than value function iteration when studying economies for which value function iteration is a valid computational approach.

[6]See for example, Kydland and Prescott (1982, Hansen (1985), and Prescott (1986).

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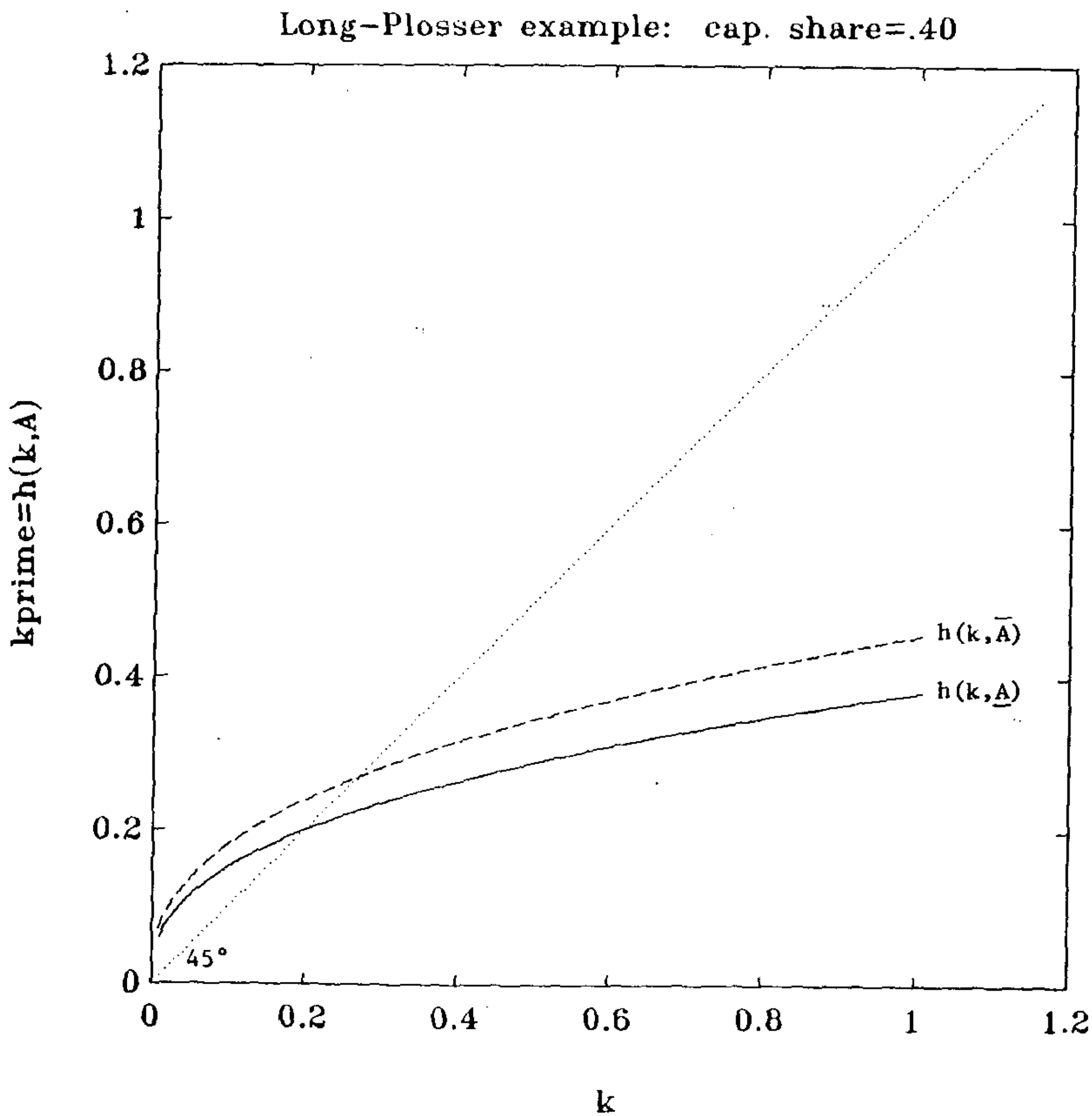
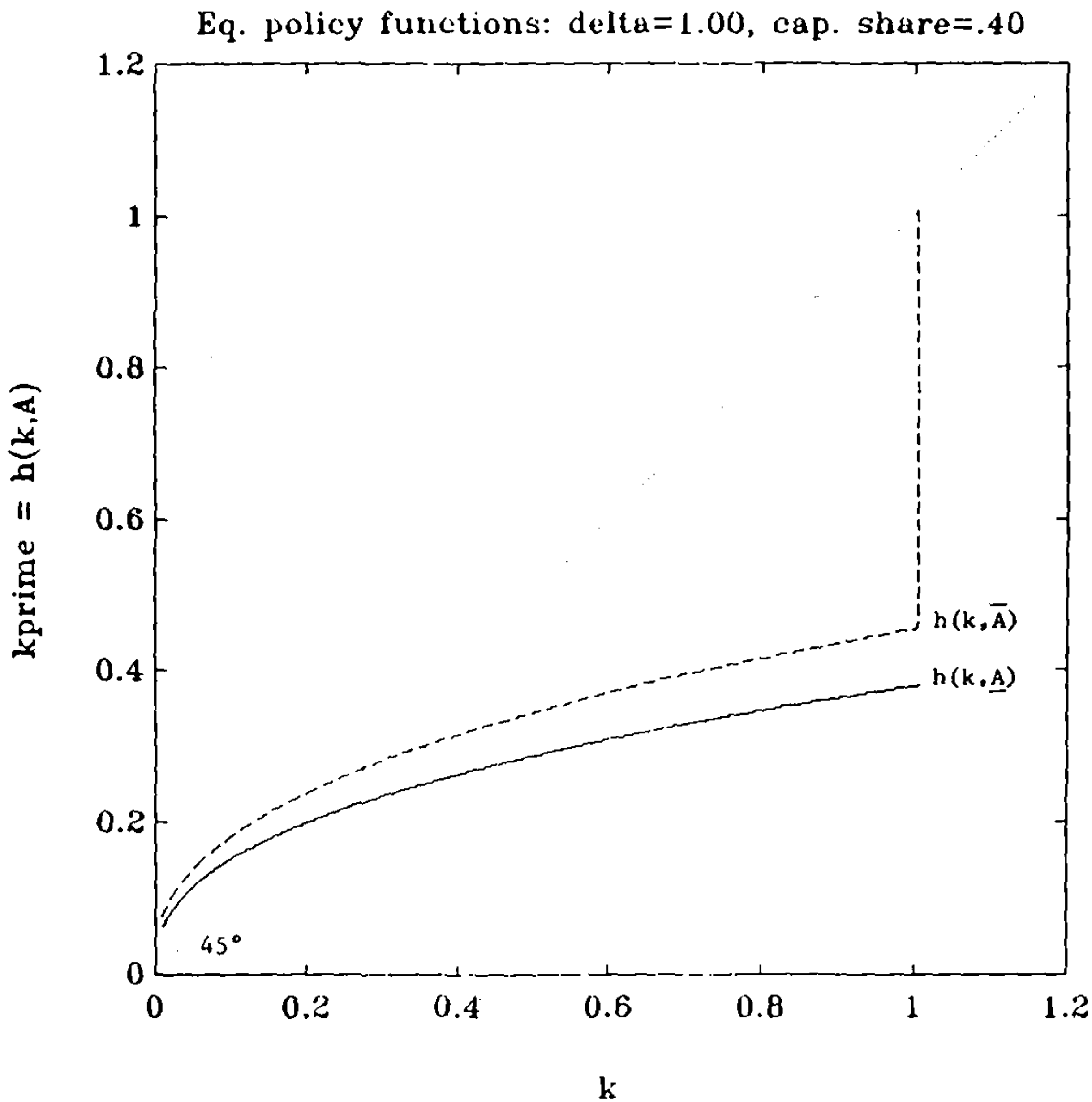
Figure 1



Long-Plosser example: cap. share=.40
kprime=h(k,A)
k
h(k,$\bar{A}$)
h(k,$\underline{A}$)
45°
0
0.2
0.4
0.6
0.8
1
1.2

Figure 2



Eq. policy functions: delta=1.00, cap. share=.40
kprime = h(k,A)
k
0
0.2
0.4
0.6
0.8
1
1.2
h(k,Ā)
h(k,A)
45°

Figure 3

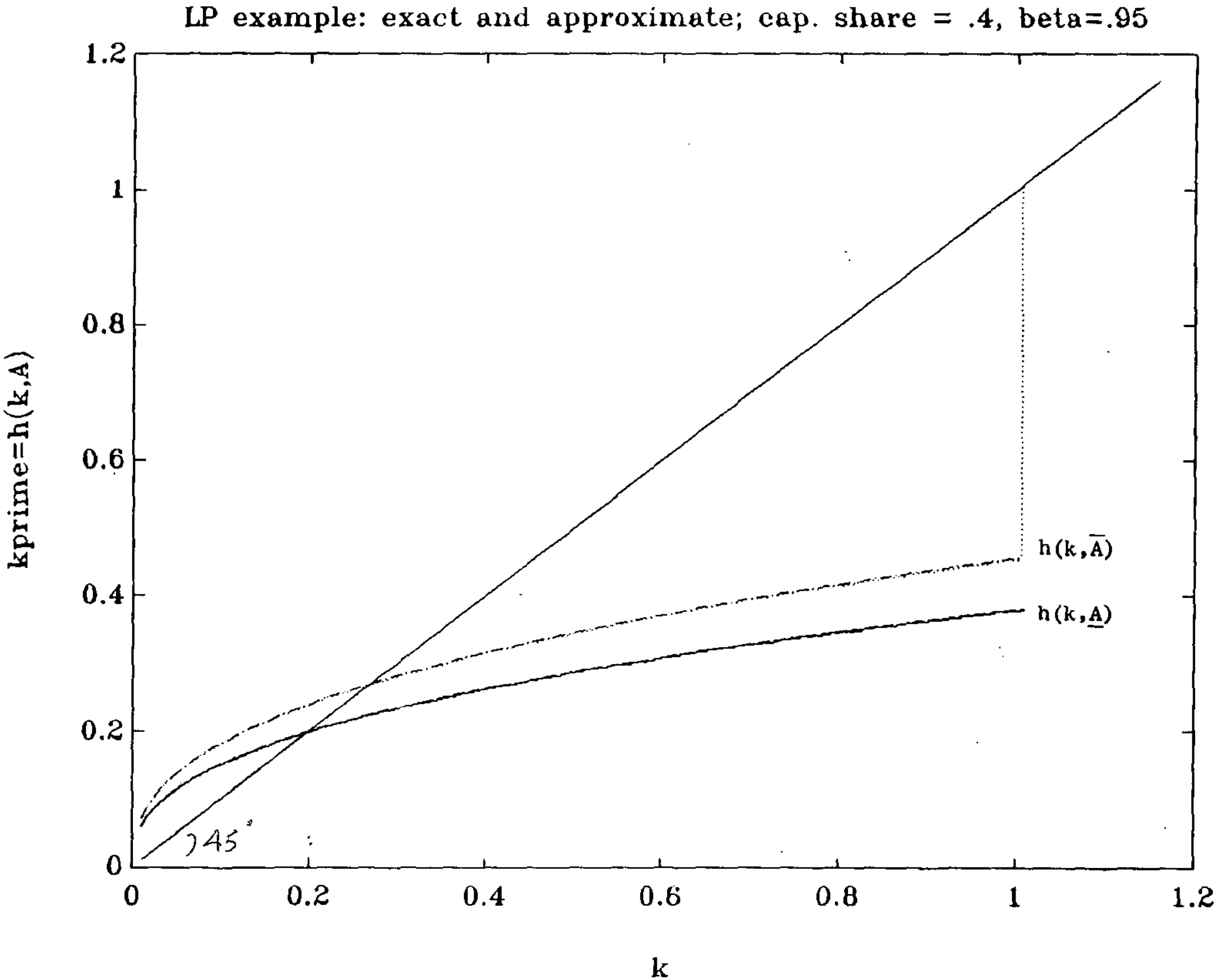
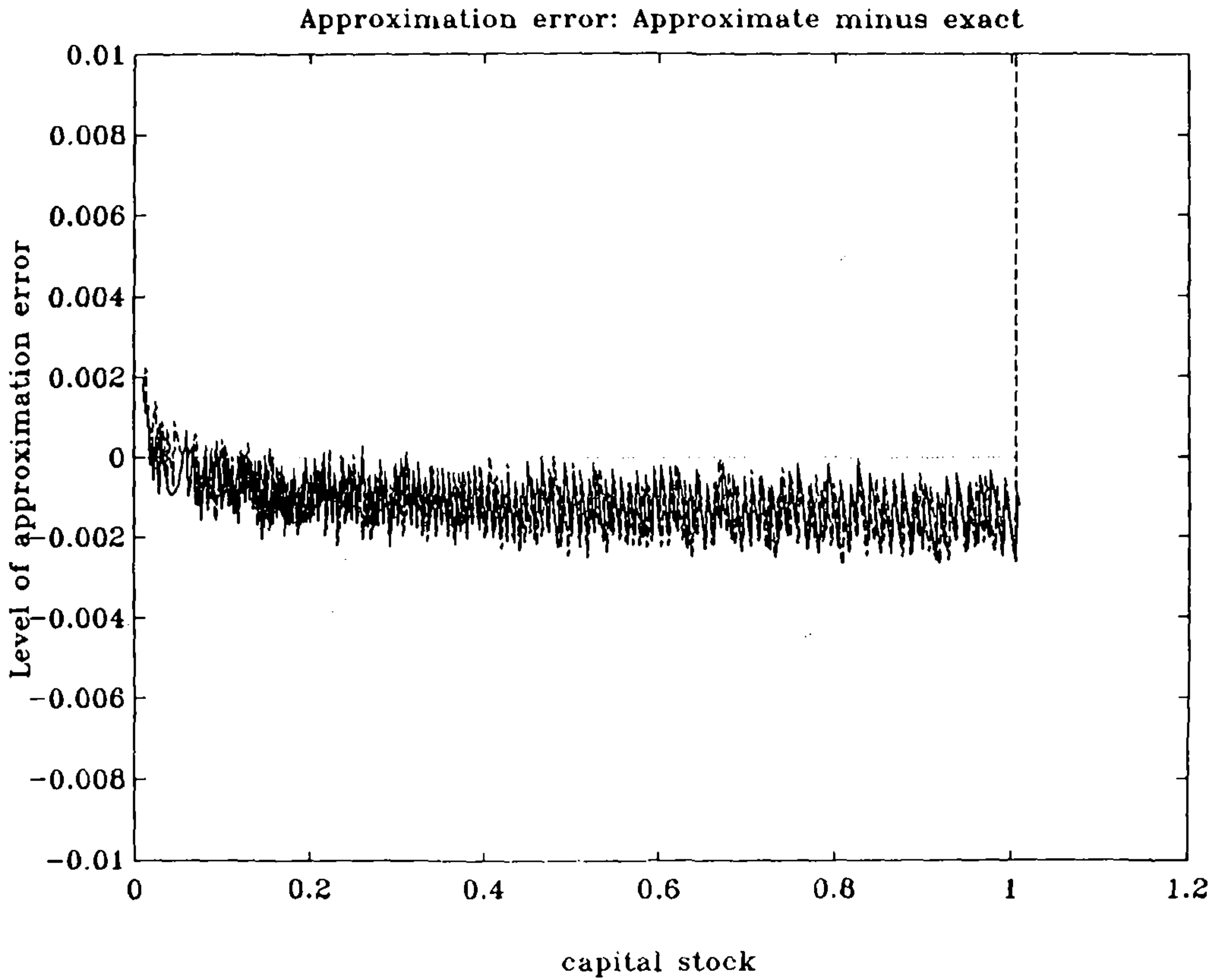

LP example: exact and approximate; cap. share = .4, beta=.95
kprime=h(k,A)
k
h(k,$\bar{A}$)
h(k,$\underline{A}$)
45°
0
0.2
0.4
0.6
0.8
1
1.2

Figure 4



Approximation error: Approximate minus exact
Level of approximation error
0.01
0.008
0.006
0.004
0.002
0
−0.002
−0.004
−0.006
−0.008
−0.01
0
0.2
0.4
0.6
0.8
1
1.2
capital stock

Figure 5

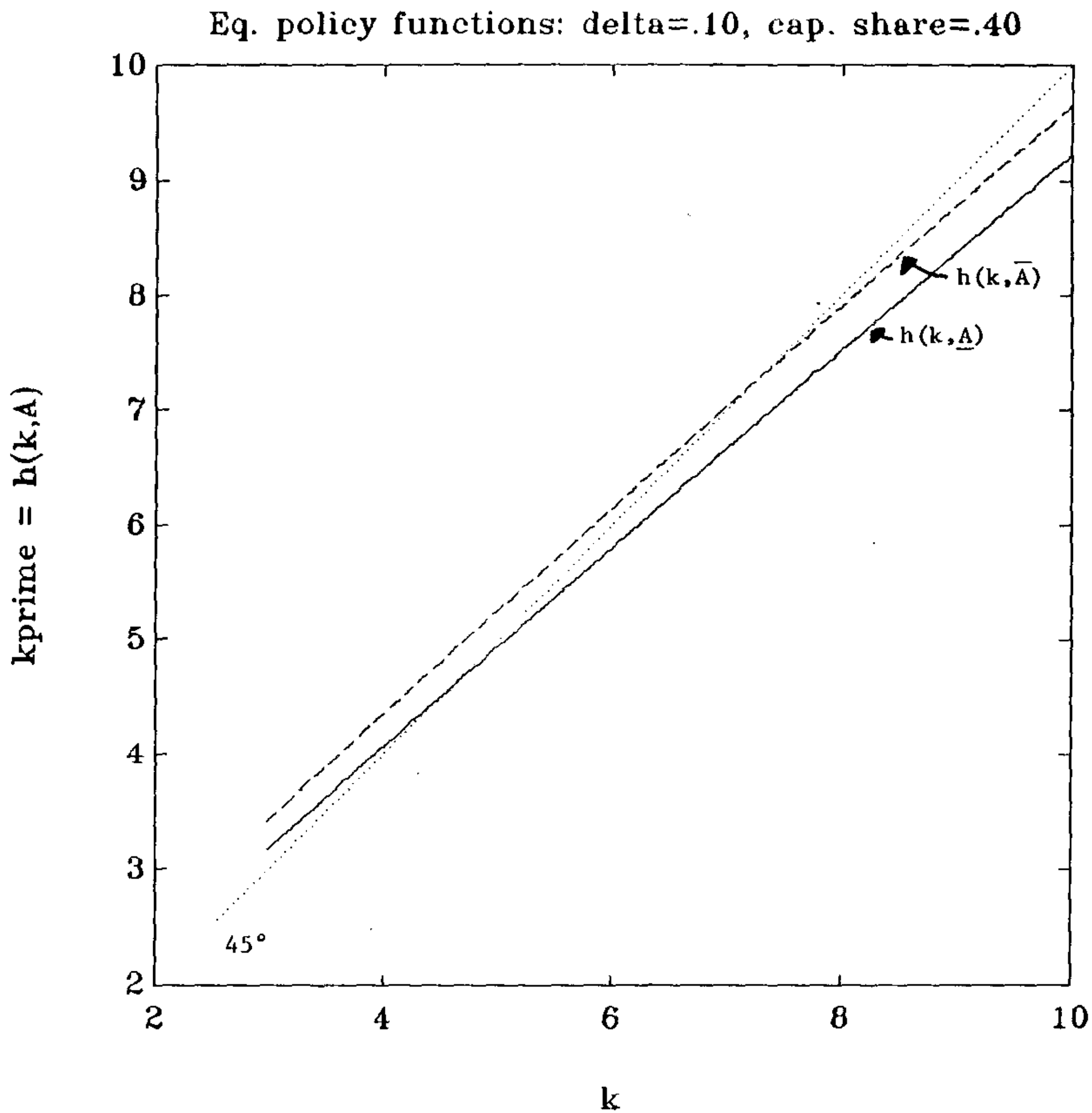
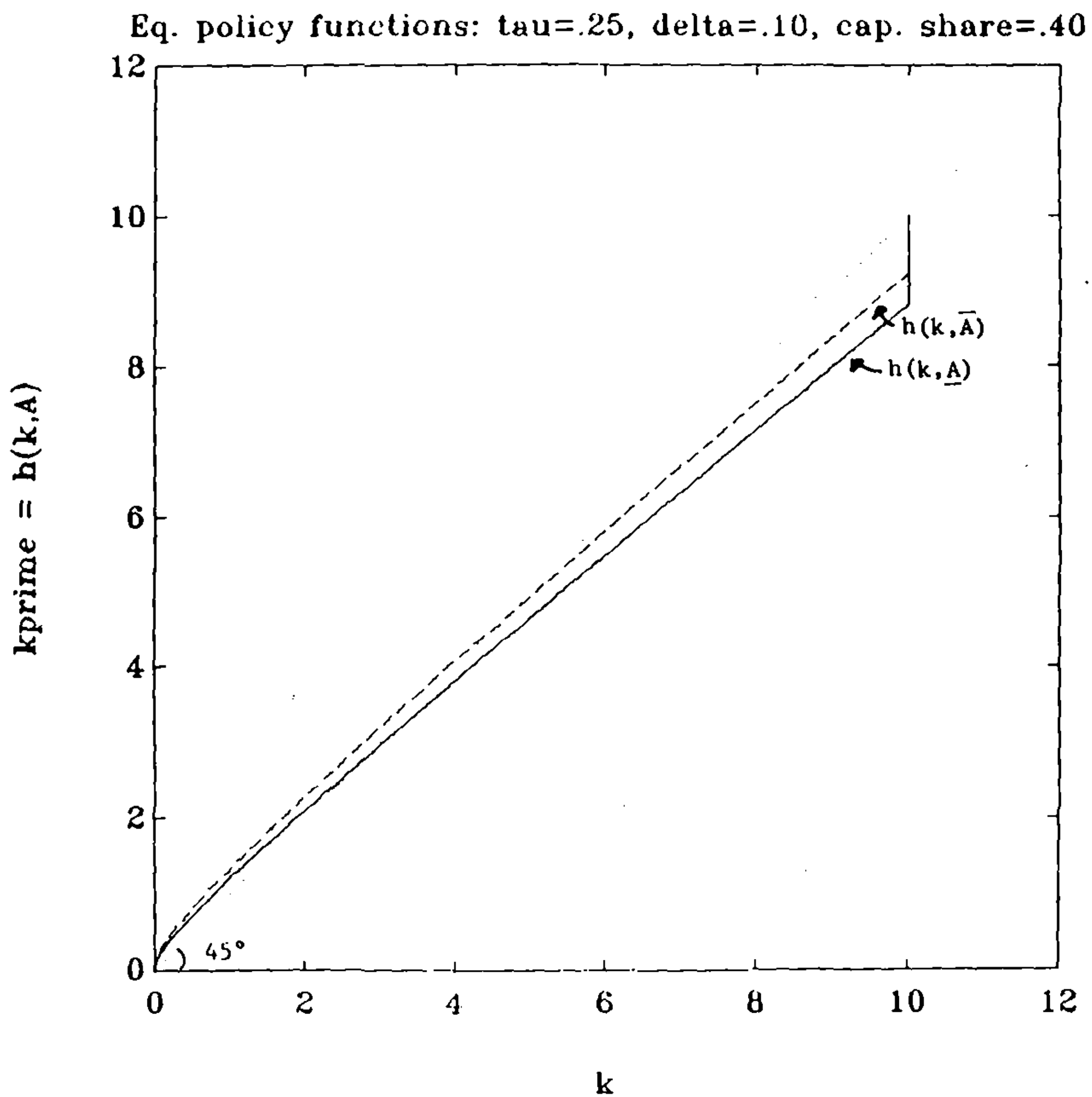
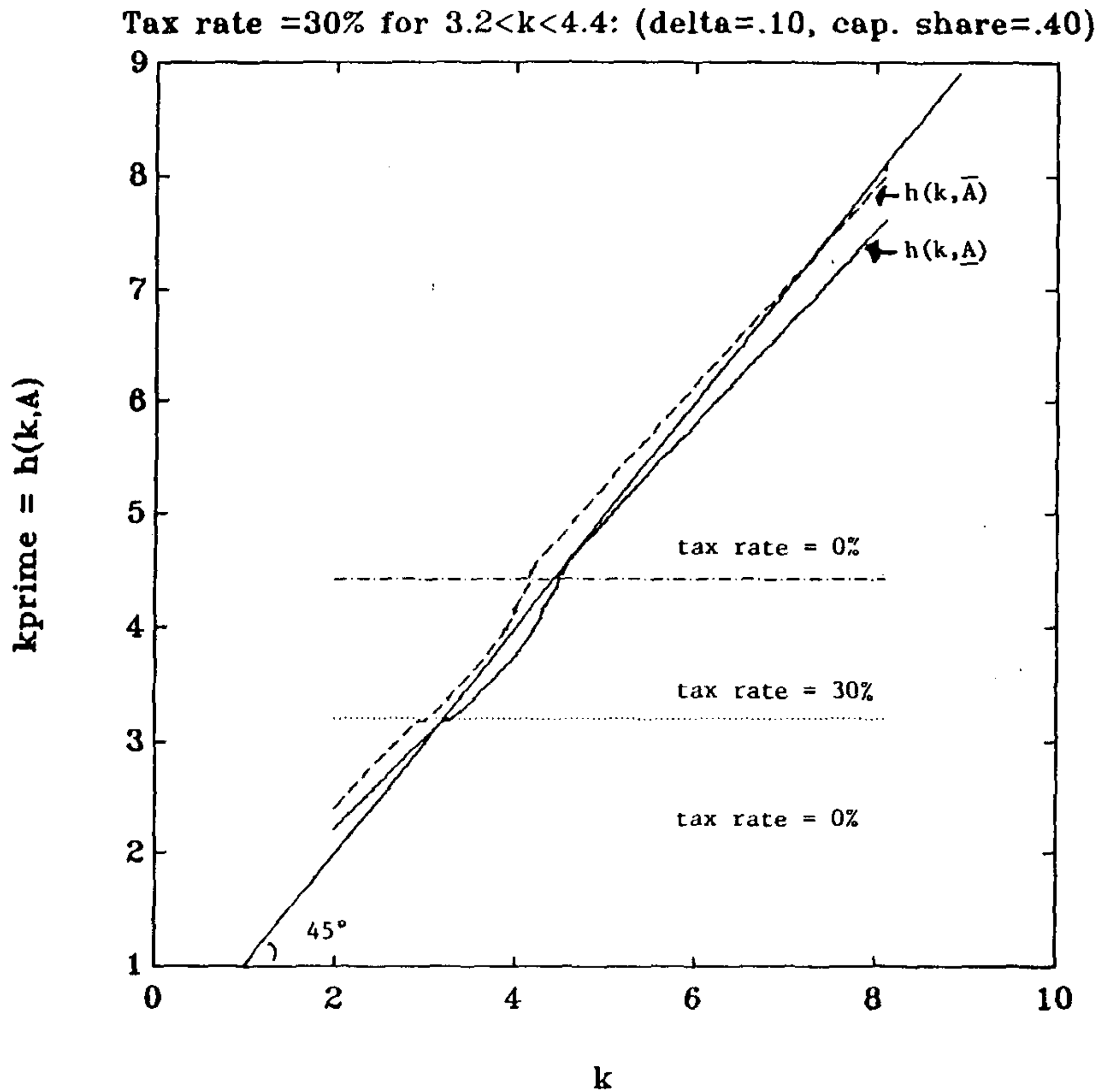

Eq. policy functions: delta=.10, cap. share=.40
kprime = h(k,A)
k
h(k,Ā)
h(k,A̲)
45°
10
9
8
7
6
5
4
3
2
2
4
6
8
10

Figure 6



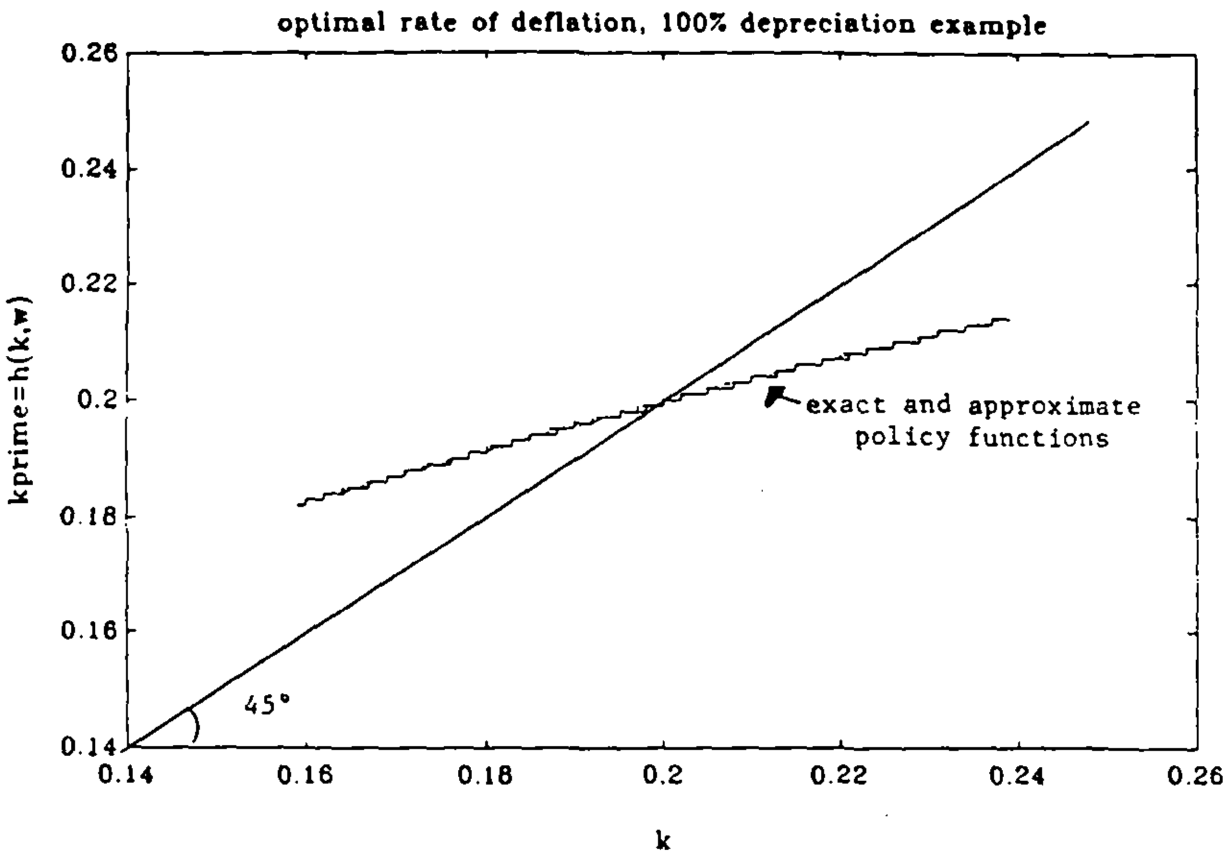
Eq. policy functions: tau=.25, delta=.10, cap. share=.40
kprime = h(k,A)
k
0
2
4
6
8
10
12
45°
h(k,Ā)
h(k,A̲)

Figure 7



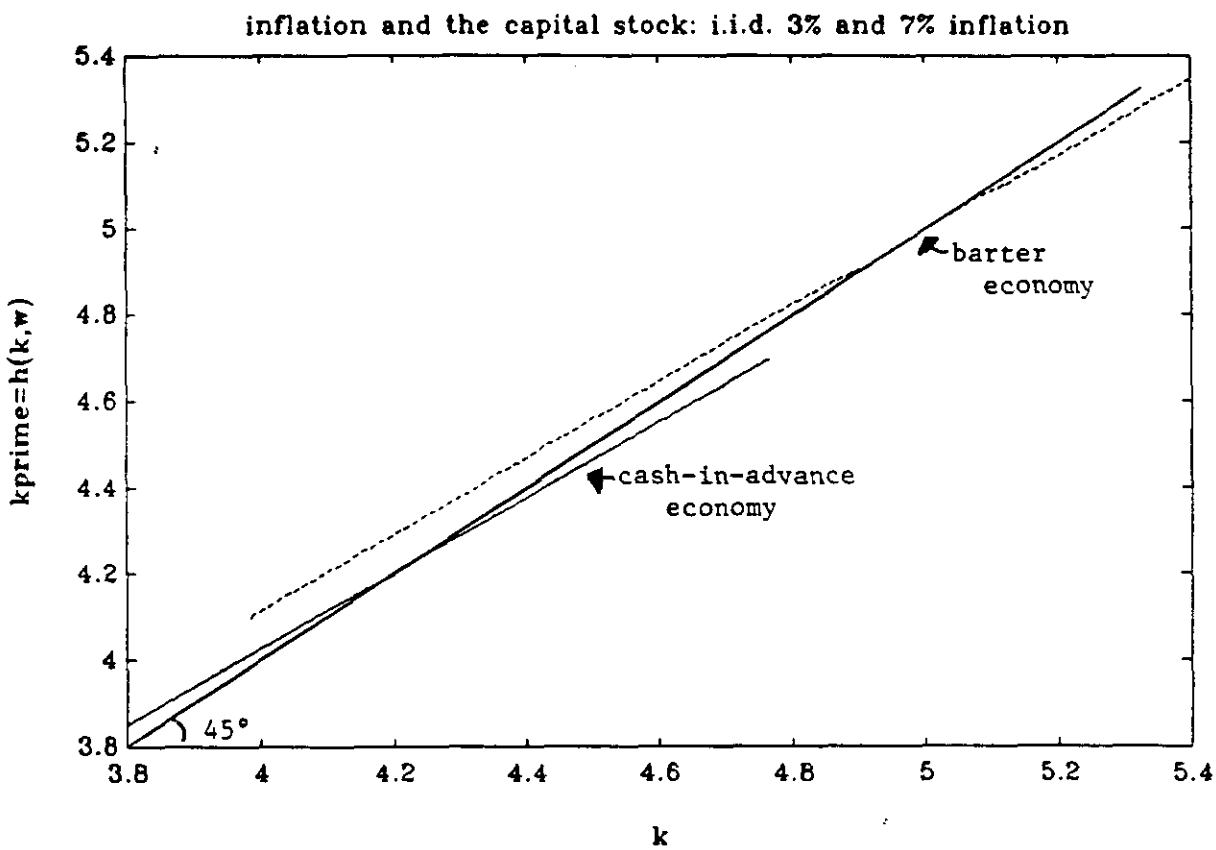
Tax rate =30% for 3.2<k<4.4: (delta=.10, cap. share=.40)
kprime = h(k,A)
k
h(k,$\bar{A}$)
h(k,$\underline{A}$)
tax rate = 0%
tax rate = 30%
tax rate = 0%
45°
0
2
4
6
8
10
1
2
3
4
5
6
7
8
9

Figure 8



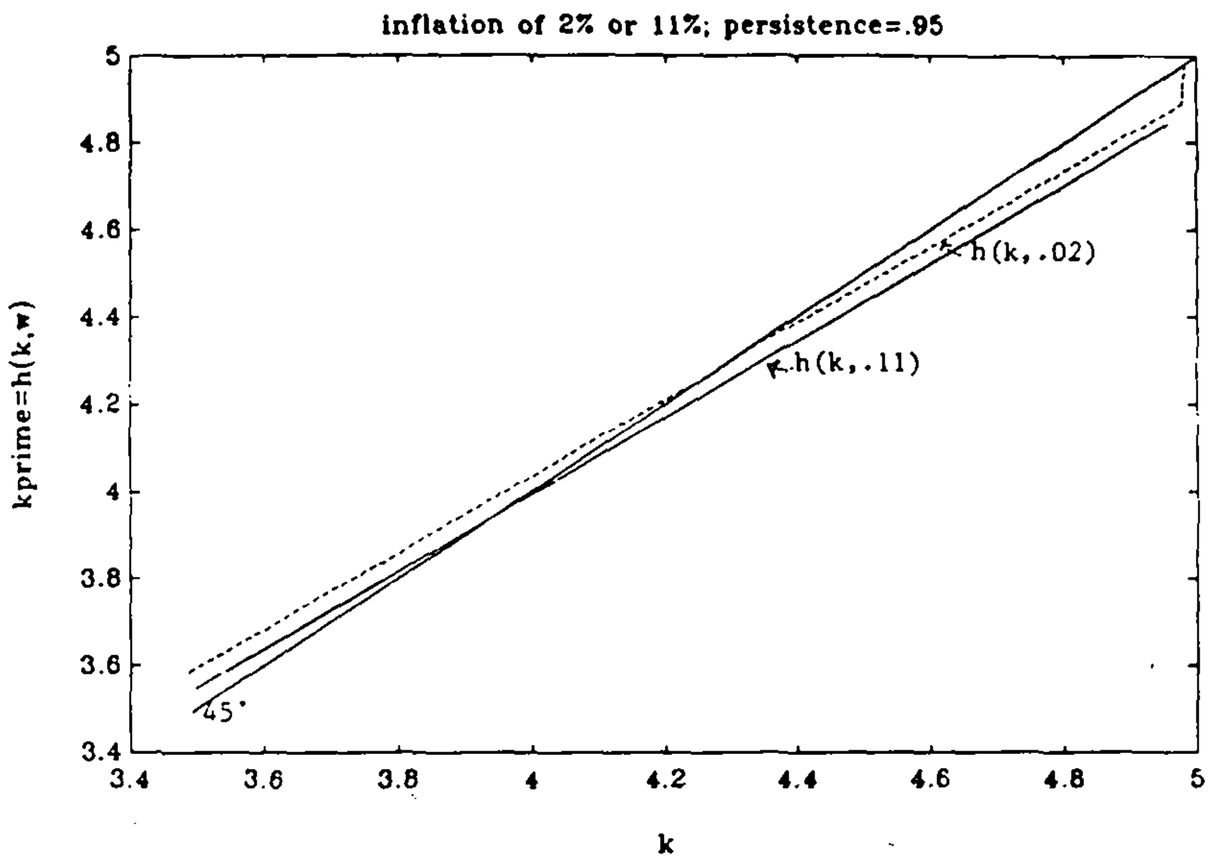
optimal rate of deflation, 100% depreciation example
kprime=h(k,w)
k
0.14
0.16
0.18
0.2
0.22
0.24
0.26
45°
exact and approximate policy functions

Figure 9



inflation and the capital stock: i.i.d. 3% and 7% inflation
kprime=h(k,w)
k
5.4
5.2
5
4.8
4.6
4.4
4.2
4
3.8
3.8
4
4.2
4.4
4.6
4.8
5
5.2
5.4
barter economy
cash-in-advance economy
45°

Figure 10



inflation of 2% or 11%; persistence=.95
5
4.8
4.6
4.4
4.2
4
3.8
3.6
3.4
3.4
3.6
3.8
4
4.2
4.4
4.6
4.8
5
kprime=h(k,w)
k
h(k,.02)
h(k,.11)
45°