Legal Restrictions and Welfare in a Simple Model of Money

Will Roberds, 12/14/90

I. Introduction

A recent paper by Kiyotaki and Wright (1989) presents an elegantly simple model of the transactions role of money. In the model of that paper, money serves as universally acceptable good, given the transaction technologies available to agents in the models. Kiyotaki and Wright show that intrinsically worthless fiat money can improve welfare by enhancing the possibility of a "double coincidence of wants."

The originality of the Kiyotaki–Wright approach hardly needs further comment. A particularly interesting aspect of their paper is its complete avoidance of "legal restrictions" or "cash–in–advance" constraints. In a footnote to their paper, Kiyotaki and Wright take some pains to distinguish their analysis from the standard cash–in–advance approach. They assert that cash–in–advance models "...have no hope of explaining endogenously either the nature of money or the development of monetary exchange."

The analysis that follows attempts to qualify the last statement. Using a simple model of exchange that incorporates many elements of the Kiyotaki–Wright setup, equilibria are derived for a number of assumptions regarding transactions technologies. Initially it is assumed that all transactions are barter transactions. Fiat money is then introduced into the model, first without and then with a cash–in–advance constraint. Without a cash–in–advance constraint, equilibrium with fiat money can be either welfare increasing or decreasing relative to equilibrium under barter. However, the introduction of fiat money with a cash–in–advance constraint is consistent with an equilibrium which welfare dominates equilibria under either barter or under fiat money without a cash–in–advance constraint. Finally, a model is considered where agents do not trade goods for money, but instead are required to trade goods for a simple form of debt that resembles the an early form of trade credit known as "bills of exchange." The resulting equilibrium is shown to dominate barter in welfare terms, but to be weakly dominated by the "best" cash–in–advance equilibrium.

These results attempt to provide an indirect justification for cash–in–advance constraints by describing model economies in which adoption of such a constraint (or social convention) can be beneficial to everyone. This is done with the larger goal of convincing the reader that there may be still be some role for legal restrictions to play, even in models where transactions are carefully modelled and nontrivial barriers to transactions exist.

II. Model Setup Under Barter

In the present setup, there are two locations (L1,L2) and two goods ($\gamma 1,\gamma 2$). There are two types of nonatomic people (T1,T2) native to each location in sufficiently large number so as to eliminate aggregate uncertainty. A person of type i can receives utility $u>0$ from consumption of a discrete unit of good γi, and otherwise receives zero utility. All people of a given type have the same preferences, and the constant u is common to both types. Consumption by a type Ti person can only take place at location Li. Time is discrete and unbounded, and people seek to maximize the discounted sum of expected future utility over the infinite horizon. The time invariant discount factor $\beta \in (0,1)$ is common to both types of people.

While at location i, each person of type i can produce one discrete unit of good $j \neq i$ at zero cost. Each person can also hold a maximum of one unit of either good in inventory, also at zero cost, with no depreciation.

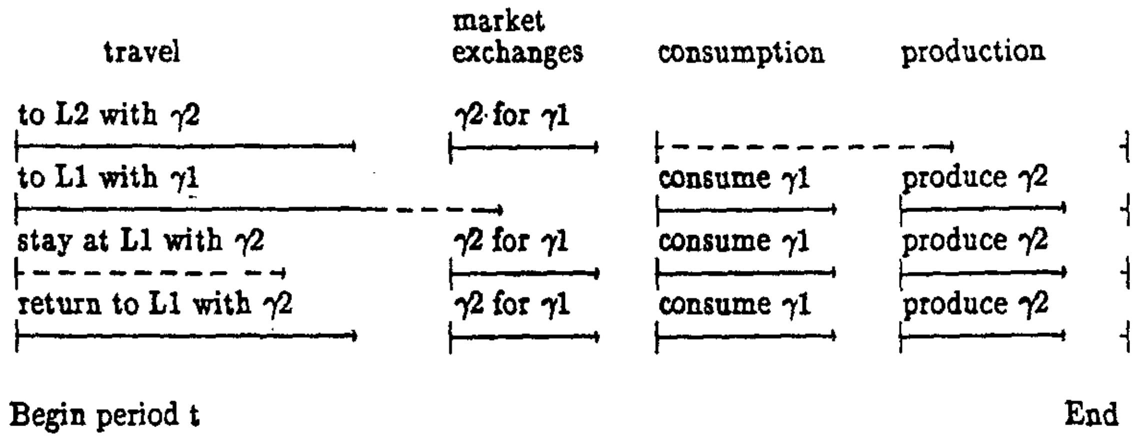
Transactions must occur at a particular location. That is, there is no centralized marketplace, but two distinct markets at locations L1 and L2. At each market, barter transactions take place purely at random, with the probability of a match determined by the relative numbers of each type at the given location, in a sense to be made precise below.

There are two impediments to trade in this model. First, there is an explicit transportation cost $c>0$ that must be paid by anyone transporting one unit of any good between locations. Second, transportation of goods takes time. In particular, it is assumed that a complete "trading mission" will take up more than one unit of time. A person cannot travel to a foreign market, trade his production good for his consumption good, travel back home, consume his consumption good, and produce another unit of his production good all in one period. To capture this notion precisely, it is assumed that after successfully completing a trade in a foreign market, a person must wait until the next time period to return home, consume the acquired consumption good, and produce another unit of the production good. One effect of the two disincentives to trade will be to preclude trade unless the expected return to a trading mission is nonnegative. The expected return to a trading mission will be bounded above by $-c+\beta(u-c)$, which is the return to a trading mission when the probability of exchange in the foreign market is equal to one. Therefore, a necessary condition for existence of trade in this model will be

$$c/u \leq \beta/(1+\beta) \tag{1}$$

The beginning-of-period state vector for a person of type α is $S_\alpha = (i,J)$ indicating that a person of type α is at location i with good J. Since we consider only symmetric equilibria, in what follows it is assumed without loss of generality that $\alpha=1$. The partitioning of a given time period for a T1 person is depicted in Figure 1, together with various possible sequences of events that could occur during the time period.

Figure 1: Partitioning of Time Period t



Given the setup described above, the only way a given person can influence their own welfare is by choice of location in which they will attempt to trade. Accordingly, let the strategy x(S) denote the probability of locating at the home location, given the value of the state vector S.

A stationary <u>equilibrium</u> under barter can now be defined as a strategy function for type T1 people $x^*(S)$ and a symmetric strategy function for T2 people, such that the following conditions are satisfied.

First, the stationary distribution across states π, that is induced by x^*, must satisfy the standard markov equation given transition probabilities p. Under the setup described above, feasible states for a T1 person are (1,2),(2,1), and (2,2) [henceforth labeled states 1,2,3]. The following transitions are possible:

(1,2) ⟶ (1,2), (2,1), (2,2)

{stay home no trade; or trade, consume & produce}, {travel & trade}, or {travel & no trade}

(2,1) ⟶ (1,2) with probability one

{go home & consume after trade}

$(2,2) \longrightarrow (1,2), (2,1), (2,2)$

{return & no trade; or return,trade, consume & produce},

{stay & trade}, or {stay & no trade}

and the stationary distribution π must satisfy

$$[\pi_1\ \pi_2\ \pi_3] = [\pi_1\ \pi_2\ \pi_3] \begin{bmatrix} p_1 & p_2 & p_3 \\ 1 & 0 & 0 \\ q_1 & q_2 & q_3 \end{bmatrix} \tag{2}$$

where the p's and q's are defined as follows. Let $Q_1 \equiv$ the proportion of type i at location i who are in the market during a given period. Let $Q_2 \equiv$ the proportion of type j at location i (or vice–versa) who are in the market. Since π_2 of a given type are busy lugging home their consumption goods during any given period, it must hold that

$$Q_1+Q_2 = \pi_1+\pi_3 \tag{3}$$

From (2) it must hold that

$$Q_1 = p_1\pi_1+q_1\pi_3 \tag{4}$$

$$Q_2 = (p_2+p_3)\pi_1 + (q_2+q_3)\pi_3 \tag{5}$$

where in equilibrium

$$p_1 = x^*(1,2) \tag{6}$$

$$p_2+p_3 = 1-x^*(1,2) \tag{7}$$

$$q_1 = x^*(2,2) \tag{8}$$

$$q_2+q_3 = 1-x^*(2,2) \tag{9}$$

and the probability of exchange, given participation in the home market, is

$$M_1 \equiv \min\{Q_2/Q_1,1\}$$

and the probability of exchange, given participation in the foreign market is

$$M_2 \equiv \min\{1,Q_1/Q_2\}$$

so that in equilibrium the transition probabilities p and q are given by

$$[1-x^*(1,2)]M_2 = p_2 \quad (10)$$

$$[1-x^*(1,2)](1-M_2) = p_3 \quad (11)$$

$$[1-x^*(2,2)]M_2 = q_2 \quad (12)$$

$$[1-x^*(2,2)](1-M_2) = q_3 \quad (13)$$

In addition to equations (2)–(13), equilibrium strategies x^* must be optimal given that all other T1 people are playing x^*. This completes the definition of equilibrium.

The solution of the barter model is relatively straightforward. To begin, consider the following results, which are helpful in characterizing stationary equilibrium under barter.

Lemma 1. In equilibrium, $M_2=1$ and $\pi_3=0$. Proof: Clearly either $M_1=1$ or $M_2=1$. Since travel is costly, in equilibrium this cost must be offset by a higher probability of market participation hence $M_2=1$. By equation (11) it follows that $p_3=0$. Hence, state (2,2) is never visited once a T1 person has left it, implying that $\pi_3=0$.□

Lemma 2. There is no equilibrium with trade in pure strategies. Proof: Assuming equation (1) holds, type T1's strategy for state (2,1) is pure, i.e., to return to L1 and consume $\gamma 1$. Also, by Lemma 1, state (2,2) does not occur, so choice of strategy for this state is irrelevant. Now consider the strategy for the state (1,2). If there is to be trade it cannot be the case that $x^*(1,2)= 1$ or 0, otherwise there will not be positive proportions of both types of people at both locations, which is a precondition for trade. Hence $x^*(1,2)\in(0,1]$.□

In view of Lemma 1 and Lemma 2, one can write Bellman's equations for this model in a fairly simple form. Using Lemma 2, we can set $M_2=1$ and need not consider state (2,2). Also, we simplify notation by setting $x(1,2)=x$ and $M_1=M$, obtaining

$$v(1,2) = x[Mu+\beta v(1,2)] + (1-x)[-c+\beta v(2,1)] \quad (14)$$

$$v(2,1) = u-c+\beta v(1,2) \quad (15)$$

Equation (14) says the for a T1 person at L1 with their production good $\gamma 2$, the value of being in this state is equal to a convex combination of the expected value of remaining at

L1 and the expected value of traveling to L2. Equation (15) says that the value of holding one's consumption good at the foreign location at the beginning of the period is equal to the value of consuming the consumption good, minus the cost of transporting the consumption good home, plus the discounted value of producting one's production good and attempting to return to the marketplace in the following period.

In addition to equations (14) and (15), the following condition is necessary for x to be in (0,1):

$$Mu+\beta v(1,2) = -c+\beta v(2,1) \tag{16}$$

which states that the value of staying put for some T1 person in state (1,2) is equal to the value of traveling. If T1 persons were not indifferent between moving and not moving, given that they are in state (1,2), a mixed strategy would not be optimal for state S=(1,2), and no equilibrium with trade would exist by Lemma 2.

III. Solution of the Barter Model.

Solving for the symmetric, stationary equilibrium of the model under barter is a matter of straightforward, if rather tedious solution of equations (2)–(16) for the unknowns M, v(1,2), v(2,1), π_1, π_2, and x. Applying Lemma 2 to equations (2)–(13) yields solutions for M, π_1, and π_2 in terms of x, i.e.,

$$M = (1-x)/x \tag{17}$$

$$\pi_1 = (2-x)^{-1} \tag{18}$$

$$\pi_2 = (1-x)/(2-x) \tag{19}$$

Substituting (17) into (14) and solving (14) and (15) for v(1,2) and v(2,1) as a function of x yields

$$v(1,2) = \frac{(1+\beta)(1-x)(u-c)}{(1-\beta)(1+\beta-x\beta)} \tag{20}$$

$$v(2,1) = \frac{(1-2x\beta+\beta)(u-c)}{(1-\beta)(1+\beta-x\beta)} \tag{21}$$

To obtain the equilibrium value of x, we use condition (16) to obtain the following condition

$$(1+\beta)Mu = \beta u-(1+\beta)c \tag{22}$$

which states that in equilibrium the expected payoff from staying home two periods in succession [LHS (22)] must equal the certain payoff from going on a two-period trading mission [RHS (22)]. Eliminating M via (17) and solving for x yields

$$x^* = [1-(c/u)+\beta/(1+\beta)]^{-1} \tag{23}$$

which is decreasing in β and increasing in c/u. If we bound β to be in (0,1) and c/u so that $0<c/u<\beta/(1+\beta)$, then x^* must be in the interval (2/3,1).

These results are summarized in the following theorem.

Theorem 1 (Description of Barter Equilibrium). If trade occurs in equilibrium, then the equilibrium strategy $x^*(1,2)$ is given in equation (23), $x^*(2,1)=1$, and state(2,2) occurs with probability zero. Equilibrium values of π_1 and π_2 are given by substitution of $x^*(1,2)$ in equations (18) and (19). Transition probabilities p_1 and p_2 are given by $x^*(1,2)$ and $1-x^*(1,2)$ respectively. Proof: see discussion above.□

Before discussing welfare in the barter model, it is useful to to derive an expression for $V\equiv E[v(\cdot)]$ as a function of x, i.e., the probability of potential traders staying at home. Solving equations (18)–(21) for V yields

$$V(x) = \frac{2(1-x)(u-c)}{(2-x)(1-\beta)} \tag{24}$$

Following Kiyotaki and Wright, we take the welfare criterion for this model to be the average level of steady state utility or $W\equiv(1-\beta)V$. The measure W can also be parametrized by x in the obvious way, i.e.,

$$W(x) \equiv (1-\beta)V(x) \tag{25}$$

Clearly, W(x) is decreasing in x, so that the best obtainable symmetric equilibrium (if x could be specified exogenously) is when x=1/2 and $W = W_I \equiv (2/3)(u-c)$. But a strategy of $x=1/2$ is not an equilibrium strategy under barter for any values of c/u or β, Individuals will find it in their self interest to remain at home with probability of at least 2/3. Hence the barter economy does not attain this benchmark level of welfare, but is strictly bounded above by $W_B^u \equiv (1/2)(u-c)$ when $c>0$ or $\beta<1$.

To calculate the actual value of W attained in barter equilibrium, let x take on its equilibrium value x^* as given in equation (23). Doing so yields $W_B \equiv W(x^*)$, which after simplification reduces to

$$W_B = \frac{2\,[\,\beta-(\,1+\beta\,)\,(\,c\,/\,u\,)\,]}{1+3\,\beta-2\,(\,c\,/\,u\,)\,(\,1+\beta\,)}\,(u-c) \tag{26}$$

For various values of $\beta\epsilon[0,1]$ and $c/u\epsilon[0,\beta/(1+\beta)]$, the welfare measure associated with the barter economy, W_B is plotted in Figure 2, assuming $u=1$. Figure 2 reveals (as can be shown with a little calculus) that W_B decreases with c/u and increases with β. These results are intuitive, as they suggest that welfare is decreasing as transportation and hence trade becomes more costly, and increases as people become more patient and therefore willing to wait for the potential benefits of trade. If β is sufficiently small or c/u sufficiently large, there will be no trade.

IV. Model with Kiyotaki–Wright Fiat Money

Now suppose that the model setup is same as before, except that we introduce fiat money along the lines of Kiyotaki and Wright. Fiat money is a good ($\gamma 0$) with no intrinsic value, but possible value in exchange for other goods. By assumption fiat money, unlike goods $\gamma 1$ and $\gamma 2$, can be transported costlessly between L1 and L2. It is also assumed that transport of fiat money to a person's home location does not entail the loss of time that is incurred with the import of the person's consumption good. However, fiat money takes up "space" in the sense that holding an indivisible unit of fiat money precludes holding an inventory of any other good.

The state space for an agent of type α must be expanded in the model with fiat money, to include a state $S_\alpha=(*,0)$ which indicates that a person of type α is either location at the beginning of the period with one unit of money. The beginning–of–period location of the person is not important when the person holds money, since the transport cost of money is zero by assumption.

Initially, it is assumed that at each location, money holders and goods holders of both types are thrown into the same market. The probability of exchange is determined by the relative proportions of each type. As is the case with barter economy, in equilibrium there will be at least as many individuals of a given type in their home market as there are individuals of a given type in the foreign market. So the probability of market participation is $M_2=1$ in the foreign market and $M_1\leq 1$ in the home market. All individuals of a given type Ti, holding either the good γj or good $\gamma 0$ (money), attempt to participate in market exchanges with people of type Tj who are holding good γi or money . Probability of participation for T1, given location in market i, is M_i which is defined as before. But the number of T1 people at location 1 is now given by $Q_1=C_1+G_1$, where C_1 is the number of T1 people holding money, and G_1 is the number of T1 people holding good $\gamma 2$. The number of T2 people at location 1 (or T1 people at location 2) is similarly given by $Q_2=C_2+G_2$. The probability of a T1 person making an exchange for goods or money, given that he is able to participate in the market, is determined by the relative numbers of goods and money holders of type T2 at the given location, and does not depend on whether a person holds goods or cash. To keep this straight the following table can be helpful:

Table 1: Transactions in Economy with K–W Money (Market at L1)

	Located at L1	Active in Market	End up with Good 1	End up with Cash
Type 1				
with $\gamma 2$	G_1	$\frac{Q_2}{Q_1} G_1$	$\frac{Q_2}{Q_1} G_1 \frac{G_2}{Q_2}$	$\frac{Q_2}{Q_1} G_1 \frac{C_2}{Q_2}$
with $\gamma 0$	C_1	$\frac{Q_2}{Q_1} C_1$	$\frac{Q_2}{Q_1} C_1 \frac{G_2}{Q_2}$	$\frac{Q_2}{Q_1} C_1 \frac{C_2}{Q_2}$
Type 2			(with Good 2)	
with $\gamma 1$	G_2	G_2	$G_2 \frac{G_1}{Q_1}$	$G_2 \frac{C_1}{Q_1}$
with $\gamma 0$	C_2	C_2	$C_2 \frac{G_1}{Q_1}$	$C_2 \frac{C_1}{Q_1}$

Given the transactions technology described aboved, it is now possible to define a symmetric, stationary equilibrium for the economy with fiat money. An <u>equilibrium</u> in the economy with fiat money will again consist of a strategy function for type T1 people, $x^*(S)$, and a symmetric strategy function for type T2 people, such that the following conditions will be satisfied. First, denoting state (*,0) as state 0, the steady state distribution π induced by x^* across states 0,1, and 2 must satisfy

$$[\pi_0\ \pi_1\ \pi_2] = [\pi_0\ \pi_1\ \pi_2] \begin{bmatrix} r_0 & r_1 & r_2 \\ p_0 & p_1 & p_2 \\ 0 & 1 & 0 \end{bmatrix} \tag{27}$$

where in equilibrium the r's and p's are defined by

$$Q_1 = G_1 + C_1 \tag{28}$$

$$Q_2 = G_2 + C_2 \tag{29}$$

$$G_1 = x^*(1,2)\pi_1 \tag{30}$$

$$G_2 = [1 - x^*(1,2)]\pi_1 \tag{31}$$

$$C_1 = x^*(*,0)\pi_0 \tag{32}$$

$$C_2 = [1-x^*(*,0)]\pi_0 \tag{33}$$

$$\pi_0 = \{\text{money supply per capita}\} \tag{34}$$

$$p_1 = x^*(1,2)(1-M_1\frac{C_2}{Q_2}) = x^*(1,2)(1-\frac{C_2}{Q_1}) \tag{35}$$

$$p_2 = [1-x^*(1,2)](1-\frac{C_1}{Q_1}) \tag{36}$$

$$p_0 = x^*(1,2)(\frac{C_2}{Q_1}) + [1-x^*(1,2)](\frac{C_1}{Q_1}) \tag{37}$$

$$r_1 = x^*(*,0)(\frac{G_2}{Q_1}) \tag{38}$$

$$r_2 = [1-x^*(*,0)](\frac{G_1}{Q_1}) \tag{39}$$

$$r_0 = x^*(*,0)(1-\frac{G_2}{Q_1}) + [1-x^*(*,0)](1-\frac{G_1}{Q_1}) \tag{40}$$

Second, it must be true that equilibrium strategies x^* are optimal, given that other T1 people are playing x^*.

The following result characterizes fiat money equilibrium.

Theorem 2 (Description of equilibria with fiat money). There are two possible types of equilibrium with fiat money. In the first type of equilibrium, per capita money supply π_0=\$1/3, $x^*(1,2)=1$, and $x^*(*,0)=0$. Equilibrium values[1] of π_0, π_1, and π_2 will be 1/3, and average utility will be $W^1_{KW}=(u-c)/3$. Trade will occur as long as $\beta(1-\beta)/[1+\beta(1-\beta)] \leq c/u<1$. In the second type of equilibrium with money, π_0 will be "small" (<1/3), $x^*(1,2)$ will be in (0,1) and $x^*(*,0)=0$. Trade will occur as long as $c/u \leq \beta/(1+\beta)$, and average utility will be given by

$$W^2_{KW} = W_B + \pi_0 \frac{(1-\beta)[1+2(1+\beta)(c/u)]}{1+3\beta-2(c/u)(1+\beta)}(u-c) \tag{41}$$

which dominates barter but is strictly bounded above by $(1/2)(u-c)$ for $c>0$ or $\beta<1$.

Proof: See Appendix A.

Intuitively, the first type of equilibrium fiat money sets up the following deterministic pattern of trade for all the people in the model. A person who has just produced their production good remains at their home location with probability one. They then are able sell their production good to people of the other type for fiat money, again with certainty. Holders of fiat money then travel to the foreign location to buy their consumption good. Finally, the buyers of the consumption good return with probability one to their home location, consume, and produce another unit of the production good. To maintain this pattern of trade, transport costs must be sufficiently high so that people are not tempted to transport their production goods to the foreign market.

There are three sets of circumstances where welfare under fiat money will be greater than under barter. The first case is when the transport cost c is relatively high and/or β is relatively small, so that $c/u \geq \beta/(1+\beta)$, in which case there will be no barter equilibrium with trade. In this case, disincentives to trade are so high that no one wants to carry goods abroad. In the case of the pure strategies equilibrium with fiat money, however, the pattern of trade is such that goods are only transported by people who intend to immediately consume them, effectively overcoming the these disincentives.

The second case in which welfare under fiat money will be greater than under barter is where $\beta(1-\beta)/[1+\beta(1-\beta)] < c/u < \beta/(1+\beta)$, so that both the barter and the pure strategies fiat money equilibria exist, but where the probability of produces staying home in the barter equilibrium is greater than 80 percent, i.e., $x^* > 4/5$ in the barter model. The last inequality implies that

$$(u-c)/3 > 2(1-x^*)(u-c)/(2-x^*), \tag{42}$$

i.e., that the welfare measure for the monetary economy W^1_{KW} will exceed the corresponding measure for the barter economy, W_B.

The third case in which welfare is greater under fiat money will be when the second type of equilibrium described in Theorem 2 occurs. From equation (41), it is clear that this equilibrium welfare dominates barter, but in appendix A, it is shown that this equilibrium is in turn dominated by (1/2)(u–c).

In all of the cases described above, fiat money results in an improvement over barter because of the relatively low incidence of trade under barter (in the first two cases, less than one–eighth of the population is trading in any given period under barter). Fiat money overcomes the barriers to trade posed by the available transportation technology, but does so at a cost. This cost is incurred when people holding their production good wait one period at their home location for money holders to purchase their production good.

V. Model with Kiyotaki–Wright Fiat Money (with legal restrictions)

In this section, we assume the same setup as before, except that the following restriction holds: goods cannot be traded directly for goods but must be traded for money. At first glance the impact of this restriction might to be relatively minor, since the equilibrium described in Theorem 2 does not violate this restriction. However, it is shown below that the introduction of a cash–in–advance constraint allows for the existence of at least one other monetary equilibrium in the model, in which welfare will be higher than is the case in the equilibrium described in Theorem 2.

Note that the cash–in–advance requirement in effect creates the potential for two markets at each location, one for each good. To describe the operation of these markets, let $x=x^*(1,2)$ and $y=x^*(*,0)$. At location L1, there will be $y\pi_0$ people of type T1 seeking to buy $\gamma 1$ for cash, and $x\pi_1$ people of type T1 seeking to sell γ_2 for cash. Also, there will be $(1-x)\pi_1$ people of type T2 at location L1, each seeking to sell good γ_1, and $(1-y)\pi_0$ people of type T2 will be at L1 seeking to buy γ_2 for cash. Symmetrical remarks apply to the markets at location L2. If there is a positive number of participants on both sides of both markets, the market participation probabilities $M_{i,j}$ can be defined as the probability of a person in beginning of period state i (=0,1 as defined in section IV) being a market participant in the appropriate market at location j:

$$M_{0,1} \equiv \min\left\{\frac{\pi_1(1-x)}{\pi_0 y}, 1\right\} \tag{43}$$

$$M_{0,2} \equiv \min\left\{\frac{\pi_1 x}{\pi_0(1-y)}, 1\right\} \tag{44}$$

$$M_{1,1} \equiv \min\left\{\frac{\pi_0(1-y)}{\pi_1 x}, 1\right\} \tag{45}$$

$$M_{1,2} \equiv \min\left\{\frac{\pi_0 y}{\pi_1(1-x)}, 1\right\} \tag{46}$$

However, in contrast to the previous models, it can happen that some markets have no participants even when trade occurs, in which case some of the expressions in (43)–(46) are undefined. To complete the definitions of the $M_{i,j}$'s, we require that $M_{i,j}=0$ in the case that the first expression inside the parentheses in (43)–(46) is undefined.

The key to analyzing the model with legal restrictions is to recognize that under the assumption of symmetric equilibrium, the markets operate in pairs. The following lemmas establish the mechanics of the paired markets.

Lemma 3. $M_{0,1}=0$ if and only if $M_{1,2}=0$. Also $M_{0,2}=0$ if and only if $M_{1,1}=0$.

Lemma 4. If $M_{0,1}$ and $M_{1,2}$ are not both zero, then they are both positive with $M_{0,1}=1$ and/or $M_{1,2}=1$. Also, either $M_{0,2}$ and $M_{1,1}$ are both positive with $M_{0,2}=1$ and/or $M_{1,1}=1$, or they are both equal to zero.

The proofs of Lemmas 3 and 4 follow immediately from the definition of $M_{i,j}$. Intutively, Lemma 3 says that the domestic market for the home type person's consumption good will be open if and only if the foreign market for their production good is open. Also the home market for the home type's production good will be open if and only if there is a foreign market for their consumption good. Lemma 4 simply says that either buyers or sellers must be in a majority (or in equal numbers) in each market, provided that the market exists. Using Lemmas 3 and 4, we can prove the following result.

Lemma 5. In equilibrium, $M_{1,2}=0$ or 1, and $M_{0,2}=0$ or 1. Proof: Suppose that $0<M_{1,2}<1$, which implies by Lemmas 3 and 4 that $M_{0,1}=1$. In effect this means, in the context of the L1 market for γ_1, that sellers (type T2 with their production good γ_1) outnumber buyers (type T1 with cash). This cannot be an equilibrium, for it would clearly pay some T1 money holders who are journeying to L2 to buy $\gamma 1$, to stay at location L1 to buy $\gamma 1$. By staying at home they would avoid the explicit cost c of transporting $\gamma 1$ back home, as well as the time cost of this transportation. Hence $M_{1,2} = 0$ or 1.

Now suppose that $0<M_{0,2}<1$. By Lemmas 3 and 4 this implies that $M_{1,1}=1$, which means that people can sell their production good at home with probability one. This, in turn implies that people would have no incentive to take their production good abroad for sale, since doing so would incur the transport cost c, and the odds of selling in the foreign market would be no greater. Hence $M_{0,2} = 0$ or 1.□

To formally define an equilibrium in the cash-in-advance model, we require that the equilibrium optimal strategies for type T1, $x^*(\cdot)$, together with the symmetric strategies for type T2, induce a stationary distribution π which satisfies the the following markov relation:

$$[\pi_0\ \pi_1\ \pi_2] = [\pi_0\ \pi_1\ \pi_2] *$$

$$\begin{bmatrix} (1-M_{0,1})\,y+(1-M_{0,2})(1-y) & M_{0,1}y & (1-y)M_{0,2} \\ M_{1,1}x+M_{1,2}(1-x) & (1-M_{1,1})x & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{47}$$

Note that the term $(1-M_{1,2})(1-x)$, i.e., the probability of transition from state (1,2) to (2,2) for a T1 person, is omitted from the second row of the transition matrix in (47). This can be done because Lemma 5 says that either there is no foreign market for T1's production good ($M_{1,2}=0$) or T1 people attempting to sell their production good in the foreign market will be successful with probability one ($M_{1,2}=1$). In neither case is it possible that people will end up holding their production good at a foreign location at the end of the period. Hence no transition to (2,2) is possible.

It is now possible to characterize the model's first–best equilibrium under the cash–in–advance constraint.

Theorem 3. When $c/u \leq \beta$, one equilibrium under cash–in–advance which attains the highest possible level of welfare (for a cash–in–advance economy) occurs when equilibrium strategies for a type T1 person are x=0 and y=1. This equilibrium dominates all other equilibria under cash–in–advance. Proof: First, it can be verified by direct substitution that when x=0 and y=1, the transition equation (47) reduces to

$$[\pi_0\ \pi_1\ \pi_2] = [\pi_0\ \pi_1\ \pi_2]\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (48)$$

which is satisfied only when $\pi_0=\pi_1=.5$, and $\pi_2=0$. Intutitively, if x=0 and y=1, then the home market for both types' production good and the foreign market for their consumption good are both shut down. Thus, under the cash–in–advance constraint, people wishing to take place in market exchanges have no choice but to sell their production good abroad and to buy their consumption good at home. Since there are no choices other than the pattern of trade implied by equation (48) or autarky, people will sell abroad and buy at home as long as $c/u \leq \beta$. Hence, the strategies x=0 and y=1 constitute an equilibrium.

In this equilibrium, the expected value of utility in steady state is $W_{LR} \equiv (1/2)(u-c)$, since in equilibrium each person is consuming half the time with payoff u and transporting his production good half the time with payoff –c. With the cash–in–advance constraint, no more than half of the population can consume at any given time period, since each person's transactions for consumption and production cannot take place in the same time period. Also, for consumption to take place, someone must transport the good that is being consumed. Hence this equilibrium yields the highest possible level of welfare under the cash–in–advance constraint.

To show that no other equilibrium attains this level of welfare, first note that the only other equilibrium in pure strategies is that described in Theorem 2, which is dominated by the equilibrium described above as long as $c/u \leq \beta$. Now consider the case of mixed equilibria. If a mixed equilibrium exists, then both x and y are in (0,1) by Lemma 3. It also follows from Lemma 3 that all the $M_{i,j}$ are positive. By Lemma 5, it must then hold that $M_{1,2}=M_{0,2}=1$. The equilibrium level of welfare can then be evaluated as the fraction of people able to buy their consumption good, $\pi_0[yM_{0,1}+(1-y)]$, where $\pi_0 \leq .5$, times the net return to consumption (u–c). Hence, in the case of a mixed equilibrium, welfare can only attain (1/2)(u–c) when $yM_{0,1}+(1-y)=1$. This in turn, can only happen when y=0 or $M_{0,1}=1$. If y=0, the equilibrium cannot be mixed. Similarly, if $M_{0,1}=1$, then there is no incentive for buyers of consumption goods to go abroad, implying y=0 and that the equilibrium is not mixed. Thus, a mixed equilibrium cannot attain level of welfare that results from the equilibrium described above.□

Note that without the cash–in–advance constraint, the equilibrium described in Theorem 3 cannot hold. This is because people holding their production goods at their home location will not have an incentive to transport this production good and pay the transport costs, since they can always stay home and barter their production good for their consumption good. The equilibrium described in Theorem 3 results in a welfare gain because it forces producers to sell their good abroad and not to wait for buyers to seek them out. In return for bearing the costs of transporting their production good, producers are guaranteed a market for their product. This pattern of trade is more efficient than the pattern implied by the pure strategies equilibrium described in Theorem 2, and corresponds at a very rough level to what is seen in the real world, i.e., producers bearing the cost of transporting their goods to markets where such goods are in demand.

The equilibrium described in Theorem 3 attains the benchmark level W_B^u of welfare for the barter economy. This level of welfare corresponds to the level of welfare that would be attained in the barter economy if it were the case that $x^*=2/3$ in the barter equilibrium. However, this level of welfare is never actually attained in the barter economy except in the limiting case as $\beta \uparrow 1$ and $c \downarrow 0$, i.e., as the disincentives to trade vanish. As in the model of Kiyotaki and Wright, the welfare gain under a system with fiat money obtains paradoxically when half of the population of the model holds only intrinsically worthless pieces of paper in inventory. The distinguishing feature of the present model is that the cash–in–advance constraint is necessary (but not sufficient) to generate an equilibrium that attains the upper bound on barter equilibrium W_B^u.

VI. Model with "Bills of Exchange"

Theorem 3 establishes that the introduction of fiat money and a cash–in–advance constraint <u>can</u> lead to a welfare improvement over barter, but this need not be the case. For example, the equilibrium described in Theorem 2 is also an equilibrium under a cash–in–advance constraint, but is inferior to barter for regions of the parameter space where the discount factor β is close to one and the transport cost c is close to zero.

The nonuniqueness of equilibrium in the present model under cash–in–advance raises the question of whether adoption of fiat money would necessarily constitute a social good. While a complete answer to this question has not been forthcoming, the analysis below offers some additional motivation as to why only equilibria such as that described in Theorem 3 would likely to obtain. Specifically, it is shown that the first–best fiat money equilibrium of Theorem 3 not only welfare dominates barter, but also a payment system based on a simple but historically important form of trade credit known as a "bill of exchange." The equilibrium obtained under a bill of exchange payment system, in turn, is shown to dominate that obtained under barter.

Standard accounts (e.g. Clough and Cole 1941) of the early development of European capitalism stress the role of the bills of exchange as a mechanism for trade, rather than commodity or other types of money. Credit had certain advantages over commodity money for purposes of intercity trade, credit being less subject to theft, high transport costs, and legal restrictions on its export. The simplest kind of bill of exchange is aptly described by Cole (19441 p.77): "G, a merchant of Genoa, buys goods in Genoa from

V, a merchant of Venice. Instead of paying him in cash, he agrees to pay him in Venetian money in Venice before a certain date. The document by which he agrees is the bill of exchange."

In the context of the model of this paper, this kind of contract is interpreted in the following fashion. At time t, at location L1, a person of type T2 sells good $\gamma 1$ to a person of type T1. Instead of receiving payment in kind or in money, however, initially assume that the payment system is structured so that all transactions for goods must be paid for with bills of exchange; that is, in return for time t delivery of $\gamma 1$ at L1, T1 promises to deliver good $\gamma 2$ to the T2 person at L2, at time t+1. At time t+1, the T1 person writing the bill of exchange delivers the promised unit of $\gamma 2$ to L2, and returns to location L1 empty–handed. Since the T1 person is not involved in market exchanges or consumption, it is assumed that this person then has time to produce another unit of the T1 production good $\gamma 2$.

To accommodate this kind of contract, it is necessary to add an additional component to a person of Type T1's state vector, so that current beginning of period state vector for T1 is S(i,J,k), where i denotes the person's location, J the type of commodity held, and $k \in \{-1,0,1\}$ denotes the person's outstanding debt in terms of bills of exchange. To keep the setup tractable, an arbitrary limit of one bill of exchange is allowed each individual. It is also assumed that the bill of exchange contract is enforcable at zero cost. Under these assumptions, the following transitions are possible, assuming that the number of people traveling to participate in a market will not exceed the number staying at home:

(1,2,0) ⟶ (1,2,1), (1,2,0), (1,2,–1)

{stay home issue BOE, and consume},

{stay at home and don't trade},

or {travel & receive BOE in exchange for $\gamma 2$, assuming $(c/u) \leq \beta$}

(1,2,1) ⟶ (1,2,0) with probability one

{travel, pay off BOE and return}

(1,2,–1) ⟶ (1,2,0) with probability one

{stay at home and receive γ1 in return for cancellation of BOE}

As before, let M represent the probability of participation by Type T1 in his home market. Let $z \equiv x(1,2,0)$ represent the probability of remaining in the home market for a person with state (1,2,0). Then the stationary distribution π across the three states described above must satisfy the transition equation

$$[\pi_0\ \pi_1\ \pi_{-1}] = [\pi_0\ \pi_1\ \pi_{-1}] \begin{bmatrix} p_0 & p_1 & p_{-1} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{49}$$

where the subscripts indicate the person's net indebtedness in the given state. In equilibrium, the p's are defined by

$$p_0 = z^*(1-M) \tag{50}$$

$$p_1 = z^* M \tag{51}$$

$$p_{-1} = 1-z^* \tag{52}$$

$$M = (1-z^*)/z^* \tag{53}$$

Writing out the first equation of (49) and substituting from (50)–(53) yields

$$\pi_0 = 1/(3-2z^*) \tag{54}$$

$$\pi_1 = \pi_{-1} = (1-z^*)/(3-2z^*) \tag{55}$$

The value of expected utiltity in stationary equilibrium can be derived by noting that in steady state, unconditional probability of consumption is $z^* M\pi_0 + \pi_{-1}$, and unconditional probability of paying transport costs is $(1-z^*)\pi_0+\pi_1$ (which is equal to the same thing, since somebody has to transport a good before it can be consumed). Hence in equilibrium, average utility will be

$$W_{BOE}(z^*) = [(1-z^*)\pi_0+\pi_1](u-c) = \frac{2\,(1-z^*)\,(u-c)}{3-2\,z^*} \tag{56}$$

To carry out welfare comparisons with other payment mechanisms, it is necessary to calculate the equilibrium value of z^*. Mimicking the approach of Section II yields Bellman's equations for the BOE model:

$$v(1,2,0) = z[Mu+M\beta v(1,2,1)+(1-M)\beta v(1,2,0)] + (1-z)[-c+\beta v(1,2,-1)] \tag{57}$$

$$v(1,2,1) = -c+\beta v(1,2,0) \tag{58}$$

$$v(1,2,-1) = u+\beta v(1,2,0) \tag{59}$$

to which can be appended the condition necessary for z to be in (0,1), i.e.,

$$Mu+M\beta v(1,2,1)+(1-M)\beta v(1,2,0) = -c+\beta v(1,2,-1) \tag{60}$$

Substituting (57) and (58) into (59) and solving yields

$$v(1,2,0) = \frac{(1+\beta)(1-z)(u-c)}{(1-\beta)(1+2\beta-2z\beta)} \tag{61}$$

Substituting (61) into (58)–(60) and solving for z^* yields

$$z^* = \frac{\beta - (2\beta-1)(c/u)}{2\beta-(\beta^2+2\beta-1)(c/u)} \tag{62}$$

which necessarily lies in [1/2,1) for β in (0,1] and c in [0,1).

Substituting (62) into (56) yields an expression for expected utility in the bill of exchange economy

$$W_{BOE} = \frac{u - c}{2+(1-\beta)^2(c/u)} \tag{63}$$

The preceding discussion can now be stated as a theorem.

Theorem 4 (Description Bill of Exchange Equilibrium with Legal Restriction). In equilibrium, T1's strategy for state (1,2,0), i.e., z^* is given by equation (62). The equilibrium probability distribution π for T1's states is given by equations (54)–(55), and the expected value of utility in equilibrium is given by equation (63). Trade will occur so long as $(c/u) \leq \beta$. Proof: See discussion above.□

To characterize welfare under bills of exchange, the following result is useful.

Theorem 5. Suppose that $(c/u) \leq \beta/(1+\beta)$, so that there is trade under barter. Then

$W_{BOE} \geq W_B$, i.e., the bills–of–exchange economy welfare dominates the barter economy, with strict inequality as long as $c>0$ or $\beta<1$. Proof: See Appendix B.□

The intuitive basis for Theorem 5 is the following observation. Due to the technological restriction that people cannot return from the foreign market, consume, and produce during the same period, there are basically two patterns of trade in the models that we consider. The low cost pattern of trade, which is embodied in the Theorem 3 equilibrium, occurs when all transportation is done by producers. The high cost pattern of trade occurs when all transportation is done by consumers, as occurs in the equilibrium described in Theorem 2. While equilibrium under barter in effect involves a mixture of these two patterns, the bills–of–exchange regime excludes the inefficient pattern of trade by requiring that debts be incurred at home. On the other hand, the bills–of–exchange equilibrium does not work as well as the best cash–in–advance equilibrium, since clearly $W_{BOE} \leq W_{LR}$, with strict inequality for $\beta<1$ and $c>0$. Note that this result obtains even though both equilibria result in exactly the same pattern of trade in equilibrium: i.e., transportation costs are borne only by producers, and all trading for consumption goods is done at home. The fundamental difference between these two equilibria, from the standpoint of individuals' decision problems, is that individuals can create bills of exchange but cannot create fiat money. Rather than being forced to carry their production good to the foreign market, people have the option of staying at home and trying to find a person of the opposite type who will accept their debt. Since not everyone who chooses to remain at home can be successfully matched, there will be $z^*(1-M)\pi_0 = (2z^*-1)/(3-2z^*)$ people of each type who fail to trade each period. Under the cash–in–advance equilibrium of Theorem 3, everyone trades with certainty in every time period.

VII. Conclusion

In the model presented above, fiat money serves to overcome disincentives to trade imposed by the available means of transportation. Fiat money can work considerably better in this capacity if a cash–in–advance requirement is imposed on the model. When disincentives to trade are sufficiently small, there exists a monetary equilibrium under the cash–in–advance constraint that dominates equilibrium under barter, monetary equilibria without the cash–in–advance constraint, and equilibrium where payments consist of "bills of exchange."

This dominance result provides a rationale for a societal preference for the use of fiat money, together with a cash–in–advance constraint, as a mechanism for exchange. While monetary equilibria don't always dominate barter, equilibrium with bills of exchange always dominates barter. The bills–of–exchange equilibrium, in turn, is always dominated by the best monetary equilibrium, i.e., the equilibrium described in Theorem 3. These results appear at least broadly consistent with the historical evolution of European economies from barter to trade credit to fiat money. That is, given the model results, it is not difficult to imagine the payment system of an economy evolving along these lines. Moving from equilibrium under barter to the bills–of–exchange equilibrium eliminates the high cost pattern of trade that occurs under barter. Moving from bills–of–exchange equilibrium to the best fiat money equilibrium, in turn, improves welfare by eliminating uncertainty over who will give and who will receive credit. The deterministic nature of the best monetary equilibrium also masks the potential uncertainty that underlies the search problem faced by people in the model. That is, in the best monetary equilibrium, the model economy behaves almost as if it were a nonmonetary economy, in which both types of people would meet at a central location to exchange their production goods for consumption goods. The importance of money in overcoming barriers to trade would not be obvious to an outside observer, who would always see a perfect "double coincidence of wants" at both locations.

As is always the case with such mathematical parables, there is the temptation to read too much into the model results. However, the model does provide an argument for taking cash–in–advance constraints seriously, even as economists are modeling the transactions role of money with ever greater sophistication. After all is said and done, the best rationale for money may be that there is always a ready market for it.

Appendix A. Proof of Theorem 2

To begin, consider Bellman's equations for the model with money. To simplify notation, denote $x^*(1,2)$ as x, M_1 as M, and $x^*(*,0)$ as y. Bellman's equation for a T1 person's optimization problem can then be written

$$v(1,2) = \max_x\{x[(\frac{G_2}{Q_1})u+\beta(\frac{G_2}{Q_1})v(1,2)+(1-M_1)\beta v(1,2)+(\frac{C_2}{Q_1})\beta v(*,0)] + (1-x)[-c+\beta(\frac{G_1}{Q_1})v(2,1)+\beta(\frac{C_1}{Q_1})v(0.*)]\} \tag{a1}$$

$$v(2,1) = u-c+\beta v(1,2) \tag{a2}$$

$$v(*,0) = \max_y\{y[(\frac{G_2}{Q_1})u+(\frac{G_2}{Q_1})\beta v(1,2)+(1-M_1)\beta v(*,0)+(\frac{C_2}{Q_1})\beta v(*,0)] + (1-y)[\beta(\frac{G_1}{Q_1})v(2,1)+\beta(\frac{C_1}{Q_1})v(0.*)]\} \tag{a3}$$

Using an induction argument, it is easy to show that equations (a1) and (a3) can be rewritten in the form

$$v(1,2) = \max_x\{x[1-(1-M_1)\beta]^{-1}[(\frac{G_2}{Q_1})u+\beta(\frac{G_2}{Q_1})v(1,2)+(\frac{C_2}{Q_1})v(*,0)] + (1-x)[-c+\beta(\frac{G_1}{Q_1})v(2,1)+\beta(\frac{C_1}{Q_1})v(0.*)]\} \tag{a4}$$

$$v(*,0) = \max_y\{y[1-(1-M_1)\beta]^{-1}[(\frac{G_2}{Q_1})u+(\frac{G_2}{Q_1})\beta v(1,2)+(\frac{C_2}{Q_1})\beta v(*,0)] + (1-y)[\beta(\frac{G_1}{Q_1})v(2,1)+\beta(\frac{C_1}{Q_1})v(0.*)]\} \tag{a5}$$

From equations (a4) and (a5), it follows that $v(*,0)\geq v(1,2)$.

Continuing to assume that people prefer to return home with their consumption good above other alternatives, there are nine possibilities for pairs of equilibrium strategies followed by T1 people in state (1,2) (shorthand notation x) and state(*,0) (shorthand notation y. We consider these below:

(a) x=1,y=1: T1 stays home in both states. Cannot be (symmetric) equilibrium strategies for an equilibrium with trade since no one ever leaves home to trade.

(b) x=1,y=0: Possible equilibrium strategies, since the value of T1 money holders of locating at L2 is higher than that of T1 holders of $\gamma 2$, due to lack of explicit transport costs for money. See discussion below.

(c) x=1,yϵ(0,1): Seems possible since mixed strategies of T1 money holders implies equal value of locating at either L1 or L2. But since the value of locating at L1 is the same for either money or $\gamma 2$ holders, then staying at home makes sense for $\gamma 2$ holders, who would have to pay transport costs to locate at L2. But if $\gamma 1$ is not offered at L1, it makes no sense for T1 money holders to stay at home when they can travel (unless v=0 for all alternatives). Hence no nontrivial equilibrium possible.

(d) x=0,y=0: Everyone leaves home. Cannot be (symmetric) equilibrium strategy since there will be no one to trade with at L2.

(e) x=0,y=1: Cannot be equilibrium strategy since the value of locating at L2 for T1 money holders dominates that of locating at L2 for $\gamma 2$ holders.

(f) x=0,yϵ(0,1): Cannot be equilibrium for same reason as in (e).

(g) xϵ(0,1),y=1: Cannot be equilibrium because mixed strategies for T1 holders of $\gamma 2$ implies that the value of locating at L1 and L2 are equal for $\gamma 2$ holders. But then the value of locating at L2 should be higher than the value of L1 for money holders, since the value of locating at L2 is higher for money holders than for $\gamma 2$ holders.

(h) xϵ(0,1),y=0: Possible equilibrium since value of locating at L2 is greater for money holders.

(i) xϵ(0,1),yϵ(0,1): Impossible for same reason as (g).

To summarize, even under the assumption that T1 holders of $\gamma 1$ at L2 always return to L1 with their consumption good, there are two possible equilibrium patterns of trade: (1) an equilibrium in pure strategies whereby goods are always sold in the home market for money and money holders always move to the foreign market; (2) an equilibrium where goods holders play mixed strategies and money holders always move to the foreign market.

First consider equilibria in pure strategies. The only possible equilibrium in pure strategies is where x=1, y=0. This pattern of trade requires a transition matrix of the form:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

which in turn requires a steady state distribution of $\pi_i=1/3$ for all i. Given that π_0 is determined by the money supply per capita, such an equilibrium is possible only when the money supply per capita = \$1/3. Bellman's equations (a1)–(a3) reduce to

$$v(1,2) = \beta v(*,0) \tag{a6}$$

$$v(2,1) = u-c+\beta v(1,2) \tag{a7}$$

$$v(*,0) = \beta v(2,1) \tag{a8}$$

subject to the additional side condition that $\beta^2(u-c)\leq -c+\beta(u-c)$, i.e., that the transportation cost are sufficiently high, or that the discount factor β is sufficiently low, so that T1 holders of $\gamma 2$ do not find it advantageous to transport $\gamma 2$ to L2. Solving (a6)–(a8) yields

$$v(2,1) = (u-c)/(1-\beta^3) \tag{a9}$$

$$v(1,2) = \beta^2(u-c)/(1-\beta^3) \tag{a10}$$

$$v(*,0) = \beta(u-c)/(1-\beta^3) \tag{a11}$$

implying an expected value of steady state utility

$$W^1_{KW} = (u-c)/3 \tag{a12}$$

Now consider possible monetary equilibria with mixed strategies. Again there is only one possibility: T1 people holding money locate at L2 and T1 people holding $\gamma 2$ play mixed strategies, which requires that $(c/u)\leq\beta/(1+\beta)$. We begin by solving for v(1,2), v(2,1) and v(*,0). This can be done by solving equations (a1)–(a3) with x=0 and y=0 (it is legitimate to solve for v in this fashion even though $x^* \neq 0$ since if $x^* \in (0,1)$, then both expressions inside square brackets on the RHS of (a1) must be equal). Doing this yields

$$v(1,2) = [\beta u-(1+\beta)c]/(1-\beta^2) \tag{a13}$$

$$v(2,1) = [u-(1+\beta)c]/(1-\beta^2) \tag{a14}$$

$$v(*,0) = \beta v(2,1) \tag{a15}$$

We can also use equations (35)–(40) to derive expressions for π_1 and π_2 for a given per capita money supply π_0 and a given probability x of T1 staying at L1 with $\gamma 2$. These are:

$$\pi_1 = (1-2\pi_0)/(2-x) \tag{a16}$$

$$\pi_2 = (1-x+x\pi_0)/(2-x) \tag{a17}$$

which reduce to (18) and (19) if $\pi_0=0$.

To derive the equilibrium value of x, we require both expessions in square brackets on the RHS of (a1) to be equal. This equality requires that the value of staying home be equal to the value of going abroad for an individual holding their production good. Taking π_0 parametrically and using equations (a13)–(a17) yields the solution for x^*:

$$x^* = \frac{\{1-2\pi_0[1-\beta(c/u)]\}\quad(1+\beta)}{1+2\beta-(c/u)+\pi_0[(1+\beta)(2+\beta)(c/u)-(1+2\beta)]} \tag{a18}$$

The expected value of utility in equilibrium can be calculated as

$$W^2_{KW} = \{\text{probability of trading for the cons. good}\}\,(u-c)$$

$$= [\pi_0+\frac{2(1-x^*)}{2-x^*}(1-2\pi_0)](u-c) \tag{a19}$$

$$= W_B + \pi_0\frac{(1-\beta)[1+2(1+\beta)(c/u)]}{1+3\beta-2(c/u)(1+\beta)}(u-c) \tag{a20}$$

Equation (a20) establishes that for small values of the per capita money supply π_0, $W^2_{KW} > W_B$ and W^2_{KW} is increasing in π_0. That is, small infusions of fiat money into a barter economy are welfare increasing. However, the extent to which such infusions can improve welfare is limited. To see this, first note that π_0 cannot exceed \$1/3 in the mixed strategies equilibrium, since otherwise the probability of exchange in the foreign market would fall below one. It follows that

$$W_{KW}^2 \leq W_B + (1/3)\frac{(1-\beta)[1+2(1+\beta)(c/u)]}{1+3\beta-(c/u)(1+\beta)}(u-c)$$

$$< \quad W_B + \frac{(1/2)(1-\beta)+(1+\beta)(c/u)}{1+3\beta-(c/u)(1+\beta)}(u-c)$$

$$= \quad W_B + [(1/2)(u-c)-W_B]$$

$$= \quad (1/2)(u-c). \qquad \text{(a21)}$$

Hence W_{KW}^2 does not attain W_B^u.□

Appendix B. Proof of Theorem 5.

We need to show $W_{BOE} > W_B$. From equations (26) and (63), this is equivalent to

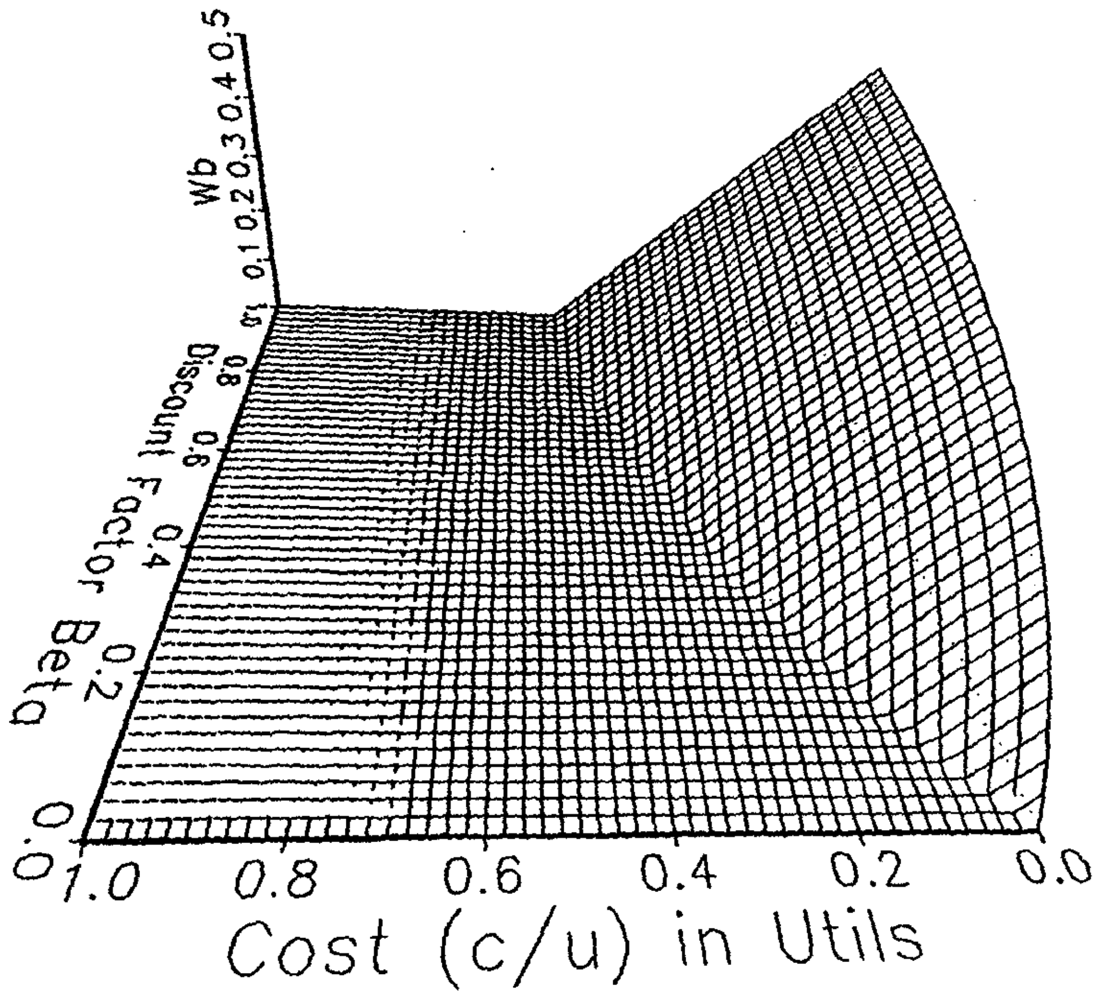
$$1+3\beta-2(1+\beta)(c/u) > 2[\beta-(1+\beta)(c/u)][2+(1-\beta)^2(c/u)] \qquad \text{(b1)}$$

Multiplying out the RHS of (b1) and collecting terms yields the equivalent expression,

$$4\beta - 2\{(1+\beta)+[(1-\beta)^2(1+\beta)(c/u)]+(1+2\beta^2-\beta^3)\}(c/u) \qquad \text{(b2)}$$

Comparing the LHS of (b1) to (b2), it follows that $W_{BOE} > W_B$.□

Figure 2


Wb
0.5
0.4
0.3
0.2
0.1
0
Discount Factor Beta
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
Cost (c/u) in Utils

References

Clough, Shepard Bancroft and Charles Woolsey Cole. 1941. Economic History of Europe. Boston: D.C. Heath and Company.

Kiyotaki, Nobuhiro and Randall Wright. 1989. "On Money as a Medium of Exchange." Journal of Political Economy 97(4): 927–954.

Notes

[1]The requirement in the first part of Theorem 2 that the per capita money supply equal \$1/3 may seem unduly restrictive. This requirement can be relaxed, however, if the monetary "unit" is redefined in the obvious way. In other words, fiat money equilibrium is compatible with any positive nominal amount of money \$k per capita, as long as the rate of exchange between money and goods is 3k to 1.