VERY PRELIMINARY

SOLVING HETEROGENEOUS AGENT MODELS: AN APPLICATION TO ASSET PRICING WITH INCOMPLETE MARKETS

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ABSTRACT

This paper is part of a project to model the interaction between heterogeneous agents in intertemporal stochastic models and to develop numerical algorithms to solve these kind of models. It is well-known that solving dynamic heterogeneous agent models is a challenging problem, since in these models the distribution of wealth and other characteristics evolve endogenously over time. Existing dynamic models in the literature contain therefore just two agents or other simplifying assumptions to limit the heterogeneity.

In this paper we study short-term interest rates in a heterogeneous agent economy with incomplete markets. We first look at examples in which agents are ex-ante identical but different realizations of the stochastic income process causes the agents to be different ex-post. Consequently their accumulation of wealth and consumption stream will be different. We analyze the importance of borrowing constraints, the supply of government bonds, the number of agents and the persistence of the stochastic shocks. We also look at examples in which agents are different ex-ante. Examples are economies in which agents differ because they have different levels of risk aversion, face a different stochastic income processes or use different information sets.

We argue that incomplete markets by itself can not generate substantial premiums in asset markets. It is also shown that the result found in the literature that borrowing constraints are effective in generating premiums disappears if there is a positive supply of government bonds. A more positive result of this paper is that substantial premiums are possible in models in which only a small fraction of the agents face a (very) high variability in income.

1. INTRODUCTION

For over a decade the representative agent model has played an important role in the macro and finance literature. One of the reasons is that it delivered a convenient framework in which we could handle both dynamics and uncertainty. By now many analytical tools have been developed to investigate important issues like the existence and uniqueness of equilibrium[1]. Although the economic environment in this type of model is usually very simple, the introduction of dynamics, uncertainty and rational expectations makes it hard to solve these models. But recently a lot of progress has been made in developing numerical algorithms and we can find many applications of these algorithms in the literature[2]. Because of the empirical failure of representative agent models, however, the literature has shifted attention to heterogeneous agent models. This creates a problem since very little progress has been made in numerically solving heterogeneous agent models. In this paper we develop an algorithm to solve dynamic general equilibrium asset pricing models with heterogeneous agents. It is clear that such an algorithm can be used for a wide variety of other topics in which uncertainty, intertemporal aspects and the interaction between agents is important. Examples are questions involving economic growth and international trade, monetary policy, inflationary uncertainty and the utility loss of near-rational decision rules. We discuss these topics for future research in the last section.

In Section 1.A we motivate the use of heterogeneous agent models and in Section 1.B we discuss the difficulties that arise in numerically solving heterogeneous agent models. In the second section we give a general description of the model and in the Section 3 we describe the algorithm. The model we use is an equilibrium version of the model in Imrohoroglu (1989) with heterogeneous agents and incomplete markets. In Imrohoroglu (1989) the interest rate is a constant but in our version the interest rate adjusts at each point in time to equilibrate the demand and supply of financial assets.

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1 See Stockey and Lucas (1989) for a detailed discussion.

2 See Taylor and Uhlig (1990) and Marcet (1991) for an overview.

We use the method of parameterized expectations discussed in Marcet (1989) and Den Haan and Marcet (1990) to solve this problem. It is important to note that we can very easily run the algorithm on a pc[3]. In Section 4 the example economies are discussed in more detail together with the simulation results. We conclude this paper by a a description of other problems that could be handled with this algorithm.

1.A Criticism against the representative agent model.

i. Rejections by the data. The representative agent has not been very successfull empirically. Several empirical studies have rejected the representative agent models using powerful techniques like Generalized Method of Moments[4]. More harmful for the representative agent models has been the fact that the current representative agent models cannot explain some important stylized facts. The most famous example is the equity premium puzzle discussed in Mehra and Prescott (1985).

ii. Unsensible assumptions. The second form of criticism focuses on the assumptions under which there is a single agent that "represents" the behavior of all the individuals in the economy. These assumptions involve restrictions on the utility functions[5] and the market structure. Concerning the market structure, the representative agent paradigm (almost always) assumes that markets are complete so individuals can diversify all idiosyncratic risk. Suppose for a moment that these assumptions are valid. It is clear that a lot of trade has to take place in an economy before it will be in the "representative agent" equilibrium. But trade takes time so there is an important question about how the economy will behave out of equilibrium. This is a tricky question since it involves thinking about trade while the idea of the representative agent framework is that we do not have to worry about trade anymore.

It is especially the complete market assumption that has received a lot of attention in the literature for the following reasons. Because of moral

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3 It is kind of a fast PC (486/66).

4 See Singleton (1989) for an overview.

5 See Kirman (1992) for a critical survey and Lewbel (1989).

hazard some important types of individual-specific risk like labor market risk cannot be insured. For instance, I don't know any bank who is willing to sell a claim contingent on the rejection of an NSF proposal. It is also believed that a lot of agents face restrictions in their financial trading because of transaction costs or borrowing constraints, which means that in reality much less trade takes place than is assumed in the representative agent economy. Several empirical studies document this lack of risk sharing. Mankiw & Zeldes (1991), for instance, show that the standard deviation of consumption is about 50% larger for stockholders than it is for a group of non-stockholders. Cochrane (1991) also rejects the perfect insurance hypothesis using PSID data. Wincoop (1992) discusses the lack of diversification within countries and among countries. This type of heterogeneity that is caused by incomplete markets is the one usually addressed in the literature[6,7].

iii. Limited applications. For many interesting economic questions the differences and interactions between economic agents are crucial. Examples are coordination problems, asymmetric information, the effects of government policies on income distributions, the effects of trade restrictions or the rising wage gap between skilled and unskilled laborers. The representative agent model is clearly inadequate to answer these questions.

Some researchers try to get rid of the shortcomings of the representative agent models by using non-conventional utility functions[8] or by using different driving processes[9]. But a lot of researchers think that the explicit introduction of heterogeneity is a fruitful line of research. Some progress has been made in the theoretical literature on the existence

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6 Examples are Aiyagari and Gertler (1992), Wincoop (1992), Marcet and Singleton (1990), Heaton and Lucas (1992), Telmer (1991), Weil (1992), Danthine, Donaldson and Mehra (1992).

7 It has to be pointed out that complete markets are not necessary for the existence of a representative agent. Constantinides and Duffie (1991) and Zin (1992) give some counter-examples. They make very special assumptions on the driving processes, however.

8 See Constantinides (1990) and Epstein and Zin (1989).

9 See Reitz (1988).

and multiplicity of equilibria[10]. However, very little progress has been made in solving heterogeneous equilibrium models.

1.B Difficulties in solving heterogeneous agent models.

As we mentioned above we recently have made quite a bit of progress in solving representative agent models and, actually, there are already a few applied examples in the literature that solve heterogeneous agent models. But these examples deal with heterogeneity in a very limited way. To understand the difficulties in solving heterogeneous agent economies consider a very simple dynamic endowment economy in which agents receive different income streams and can smooth consumption only by trading one-period bonds with each other. Since not all idiosyncratic risk can be diversified away the price of this bond will depend on the whole distribution of wealth; We will have at least as many state variables as we have different agents. Compare this with the representative agent economy in which a simple Markov assumption on the driving process would reduce the number of state variables to one. In a static or two-period problem one can usually deal with this dimension problem by choosing the initial distribution in a convenient way. But in a dynamic model the distribution of wealth will change endogenously and each period you would have to keep track of this distribution. The literature has taken care of this problem by simplifying the problem. The easiest way of course is to limit the heterogeneity by working with just two types of agents[11]. Another approach is to have a large number of agents but not to have any interaction between agents[12]. In this case you just have a collection of representative agent economies, where the collection is not an equilibrium system. There are some papers in the literature that deal with this problem in an ingenious way giving up some other aspect of standard equilibrium models. Aiyagari and Gertler (1991) and Aiyagari (1992) work

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10 See Duffie (1990) and Geanakoplos (1990).

11 This approach is taken in Marcet and Singleton (1990), Heaton and Lucas (1992), Telmer (1991), Weil (1992), Danthine, Donaldson and Mehra (1992) and Wincoop (1992).

12 Examples are Imrohoroglu (1989), Barro and Sala-i-Martin (1992) Den Haan (1992), Mankiw and Romer and Weil (1992).

with an infinite number of agents, but have no aggregate uncertainty. In this case the distribution of wealth does not change over time. The disadvantage of this approach of course is that returns also do not change over time. In Díaz-Giménez and Prescott (1992) and Díaz-Giménez, Prescott, Fitzgerald and Alvarez (1992) the distribution of wealth does not enter the bond price or the agent's decision rules because the government sets bond prices in such a way that they only depend on a limited set of state variables. Only in Díaz-Giménez and Rios-Rull (1992) a first attempt is made to numerically solve the "real" heterogeneous equilibrium model. They approximate the distribution of wealth by its mean and standard deviation. For wealth distributions that resemble the one we observe in the US this would be a bad approximation since we observe considerable positive skewness (median < mean). In models with a bimodal distribution you also would not want to approximate the distribution with just the mean and the variance. Klenov (1991) shows you get a bimodal distribution in a neoclassical model with a local externality in the production function.

2. THE EXAMPLE ECONOMIES

The economies that we look at consist of N infinitely lived agents who maximize

$$E_0 \ \Sigma_{t=0}^{\infty} \ln(c_t^j),$$

where E_t is the expectation conditional on period t information and c_t^j is the amount of consumption of agent j. To simplify the discussion somewhat we assume at this point that the agents are different only in their asset holdings and employment opportunities. So for instance the information set and the utility function are the same for all agents. The same is true for the stochastic processes that determine their employment opportunities. Employment opportunities will differ, however, because agents receive a different draw from this driving process.

We start in Section 2A by using an equilibrium version of the model in Imrohoroglu (1989). In this economy there are only two realizations for the individual income process. In Section 2B we discuss an example with a continuum of realizations.

2. A Imrohoroglu (1989).

Agent j is either employed ($i^j=e$) in which case he earns y units or he is unemployed ($i^j=u$) in which case he earns θy units through household production ($\theta < 1$). The agent can smooth consumption only by trading in a one-period bond. Given the rich stochastic structure this means that markets are clearly incomplete. Let q_t be the price of a bond at period t that delivers one unit of consumption in the next period and let a_{t+1}^j be the amount of those bonds bought at period t. We assume that agents can go short in bonds only for a certain amount. Thus $a_{t+1} \geq \bar{a}$, where $\bar{a}$ is a negative number. The budget constraint for agent j is given by

$$c_t^j + q_t a_{t+1}^j = y + a_t^j \quad \text{if} \quad i^j = e, \quad \text{or}$$

$$c_t^j + q_t a_{t+1}^j = \theta y + a_t^j \quad \text{if} \quad i^j = u$$

The employment state is assumed to follow a first-order Markov chain, but the transition probabilities depend on the national economy. The national economy can be in a good ($n=g$) and a bad state ($n=b$) and is also assumed to follow a first-order Markov chain. In the good state agents are more likely to leave the unemployment state and less likely to leave the employment state. The transition matrix for i is $\mathbf{P}^g$ in good times and $\mathbf{P}^u$ in bad times. Let

$$\mathbf{P}^g = \begin{bmatrix} p^g_{u|u} & p^g_{e|u} \\ p^g_{u|e} & p^g_{e|e} \end{bmatrix} \quad \text{and} \quad \mathbf{P}^u = \begin{bmatrix} p^u_{u|u} & p^u_{e|u} \\ p^u_{u|e} & p^u_{e|e} \end{bmatrix}$$

The following differences in the transition probabilities characterize the differences between good and bad times.

$$p^g_{e|e} > p^b_{e|e}, \quad p^g_{e|u} > p^b_{e|u}, \quad p^g_{u|u} < p^b_{u|u}, \quad p^g_{u|e} < p^b_{e|u}.$$

The parameters are chosen in such a way that the average duration of good or bad times is 16 periods. With a model period equal to 6 weeks, the average duration of one whole business cycle is 4 years. The average duration of being unemployed is 10 (14) weeks in good (bad) times. And the average unemployment rate is 4% in good times and 12% in bad times. The first-order conditions of this problem are given by the following equations:

$$(2.1) \qquad (a_{t+1} - \bar{a}\ )\ (q_t \ \frac{\partial U(c_t^j)}{\partial c_t^j} - \mathrm{E}_t \ \frac{\partial U(c_{t+1}^j)}{\partial c_{t+1}^j}\ ) \ = \ 0, \ j = 1,\dots,\mathrm{N},$$

$$(2.1)^* \qquad (q_t \ \frac{\partial U(c_t^j)}{\partial c_t^j} \geq \mathrm{E}_t \ \frac{\partial U(c_{t+1}^j)}{\partial c_{t+1}^j}\ ) \qquad , \ j = 1,\dots,\mathrm{N},$$

$$(2.2) \qquad c_t^j \ + \ q_t \ a_{t+1}^j \ = \ y \ + a_t^j \quad \text{if} \quad i^j = e, \quad \text{or}$$

$$c_t^j \ + \ q_t \ a_{t+1}^j \ = \ \theta y \ + a_t^j \quad \text{if} \quad i^j = u, \quad j = 1,\dots,\mathrm{N},$$

If agents are not at their borrowing constraint $\bar{a}$, then Equation $(2.1)^*$ holds with equality. The intuition behind this equation is that at the margin the disutility of investing q_t dollars in bonds this period should be equal to the expected discounted utility of next period's pay-off. If the borrowing constraint is (strictly) binding, i.e. the agent would want to borrow more, then Equation 2.1 tells us that the marginal utility of receiving q_t dollars is more than the expected discounted disutility of the repayment of the debt. It is clear from 2.1 that if agents are constrained then there is less downward pressure on q_t. Thus in the case with frequently binding constraints we can expect higher bond prices, i.e. lower interest rates.

In the first few examples bonds are in zero net supply. This gives us the following equilibrium condition

$$(2.3) \qquad \Sigma_{j=0}^{N} \ a_t^j \ = \ 0,$$

but in Section 4 we will allow the government to issue bonds in which case the aggregate demand for bonds should be equal to the supply of government bonds. By Walras' law we know that equilibrium on the bonds market implies equilibrium on the commodity market.

The state variables for the individual are his own employment state, i, the state of the national economy, n, his wealth level a_t^j and everything that influences current and future bond prices. This means that the whole distribution of wealth and income is part of the list of state variables. With N agents we therefore have 2N state variables[13]. Moreover the distribution of wealth will evolve endogenously over time in response to shocks to the system. Note that in this specification all agents will have the same decision rules. Let W and Y be respectively the N-dimensional vector of wealth levels and employment status and let W^{-j} (Y^{-j}) be the N-1 vector excluding the wealth level (employment status) of individual j. A solution to this economy then consists of decision rules $a(n,i^j,w^j,W^{-j},Y^{-j})$ and $c(n,i^j,w^j,W^{-j},Y^{-j})$ and a bond price $q(n,i^j,w^j,W^{-j},Y^{-j})$ which are consistent with utility maximization and market clearing.

Imrohoroglu (1989) solves several versions of this model by assuming that the bond price is a fixed constant. In that case the individual's state variables are just n, i and the individual's own wealth level. Note that in this case you do not have a real heterogeneous agent model since there is no interaction between the agents at all; you just have N separate economies each consisting of a single agent. It is clear that for a lot of purposes having constant asset returns is a very undesirable property. For one thing it is very realistic. In the next section we show how to solve this model without making these type of simplifying assumptions.

2.B Real-valued driving processes

We also look at examples in which both the aggregate state of the economy and the individual shock have continuous support. More precise, agent j's income is given by

$$y_t^j = ea_t \, e_t^j,$$

13 You do not need the whole (N×1) vector of bond holdings since in equilibrium we know that they sum up to 0.

where ea_t is the aggregate shock and e_t^j is the idiosyncratic shock at period t. The e_t^j's are assumed to be independent across agents. The law of motion for these process are given by

$$\log(ea_t) = \rho_a \log(ea_{t-1}) + \sigma_a \varepsilon_{a,t} \quad \text{and}$$

$$\log(e_t^j) = \rho \log(e_{t-1}^j) + \sigma \varepsilon_t^j.$$

The error terms $\varepsilon_{a,t}$ and ε_t^j are white noise and distributed $N(0,1)$.

3. The Algorithm

In this section we will describe how the algorithm works. To give some idea about the difficulty of this problem I will include some quotes from the literature.

From Díaz-Giménez and Prescott (1992):

> When aggregate uncertainty is considered, the distribution of agents as indexed by their current asset holdings and idiosyncratic shocks is no longer invariant over time and it becomes a part of the state of the economy. The resulting high-dimensional state precludes the use of standard recursive computational methods.
> Both the price of bills and the inflation rate are restricted to being a function of the exogenous component of the economy-wide state only. This restriction was dictated by computational considerations.

From Aiyagari and Gertler (1992):

> We have abstracted from aggregate uncertainty because the general computational problem is quite formidable if, for example, dividends are stochastic.

From Weil (1992):

> To keep the analysis as transparent as possible and eliminate the need to track the wealth distribution over time and across states of nature, I will assume that the economy lasts for two periods only with all consumers identical ex-ante.

A solution to the Imrohoroglu model satisfies the following equations:

$$(3.1) \qquad (a_{t+1} - \bar{a})\ (q_t \frac{\partial U(c_t^j)}{\partial c_t^j} - E_t \frac{\partial U(c_{t+1}^j)}{\partial c_{t+1}^j}) = 0, \quad j = 1,\ldots,N,$$

$$(3.1)^* \qquad (q_t \frac{\partial U(c_t^j)}{\partial c_t^j} \geq E_t \frac{\partial U(c_{t+1}^j)}{\partial c_{t+1}^j}) \qquad , \quad j = 1,\ldots,N,$$

$$(3.2) \qquad c_t^j + q_t\, a_{t+1}^j = y + a_t^j \quad \text{if} \quad i^j = e, \quad \text{or}$$

$$c_t^j + q_t\, a_{t+1}^j = \theta y + a_t^j \quad \text{if} \quad i^j = u, \quad j = 1,\ldots,N,$$

$$(3.3) \qquad \Sigma_{j=0}^{N}\ a_t^j = 0,$$

$$(3.4) \qquad a_{t+1} \geq \bar{a}$$

Basicly we have a system of 2N+1 equations with which we have to solve for the policy functions of consumption of the N agents, their financial investments and the interest rate. In the first examples agents only differ in their wealth level and employment status. This means that the policy rules for each agent will be the same. We use parameterized expectations to solve for these functions[14]. First we will discuss the general idea behind parameterized expectations; Then we will discuss the problem that arises because of the large number of state variables. The problem in solving this system is that we do not know the functional form of the conditional expectation in equation (3.1). If we would know the policy rule for consumption we should be able to (at least numerically) calculate this expectation. But the policy rule for consumption is exactly what we are trying to find. So without knowing the policy functions we cannot evaluate the expectation and without an expression for the expectation we cannot solve for the policy functions. Now we do know that the conditional expectation is a function of the state variables. In the method of parameterized expectations we replace this conditional expectation by a function of the state variables of the agent, $\psi(n, i^j, w^j, W^{-j}, Y^{-j}; \delta)$. We will choose $\psi(\cdot)$, the

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[14] See Marcet (1989) and Den Haan and Marcet (1990) for a description of the method. Marcet and Singleton (1990) use parameterized expectations to solve a heterogeneous agent model with two agents.

functional form and δ, the vector of parameters, to make $\psi(\cdot;\delta)$ as close as possible to the conditional expectation. Let $P_n(x)$ denote a polynomial of degree n on the vector x. We choose ψ to be of the form $\exp(P_n(n,i^j,w^j,W^{-j},Y^{-j}))$. Since it can be shown that functions of the form $\exp(P_n(x))$ can approximate any function mapping $R^{2N+2}\to R^+$ arbitrarily well we can approximate the equilibrium arbitrarily well. We will start with a first-order polynomial and see whether the results change if we increase the order of the polynomial. It has to be pointed out that for a given function ψ and parameters δ the system is still non-linear, but a very simple iteration algorithm can solve for the equilibrium bond price at each point in time very fast. The remaining problem is of course to find the parameters of the polynomial. To find these parameters the following iteration scheme is used. Each iteration consists of two steps.

Step 1. In the first step we solve the model with the parameters from the preceding iteration and at each point in time solve for each agents consumption and investment decision and for the equilibrium bond price. With these decision and next period's shocks we can calculate next period's distribution of wealth. Note that the random shocks are generated only once and are kept the same in every iteration.

Step 2. In the second step we estimate the parameters of the power functions with the data generated in step 1. Since all the agents use the same forecasting rules we have to do the estimation for only one agent. In order to estimate the first-order power function, for instance, the following equation has to be estimated with nonlinear least squares.

$$g_t^j = \exp(P_1(n_t, i_t^j, w_t^j, W_t^{-j}, Y_t^{-j}) + \nu_t^j ,$$

where $g_t^j = \partial U(c_{t+1}^j)/\partial c_{t+1}^j$ and ν_t^j is an error term.

We conclude that the iteration scheme has converged if the parameters that are used for the simulation of other series are 'close' to the estimated parameters in the second step of the iteration. If we interpret every iteration as a mapping from the space of power functions to itself, then we can say that the solution of the model is a fixed point of this mapping. After we have solved the model using $P_n(x)$ we solve the model using $P_{n+1}(x)$ and check whether the results change.

The algorithm is relatively simple since ex ante all agents are the same. That is, if at period t two agents have exactly the same wealth level and the same employment status, then they will make the same decisions. In general, however, agents do not have the same individual state variables since there are individual specific shocks. In the next section we will also look at economies in which agents are different ex ante, for instance because they have a different level of risk-aversion. It is relatively straightforward to extend the algorithm to deal with this case. If there are two types of agents, say with a low and a high level of risk aversion, then we have two sets of policy functions. In this case you just do the regression described in step 2 for each type of agent. Since the regressions are not computer intensive this does not slow down the algorithm very much[15].

The curse of dimensionality.

For numerical algorithms that discretize the state space a large number of state variables creates a serious problem since the number of grid points, i.e. the number of points at which you have to calculate the decision rules grows exponentially with the number of state variables. A big advantage of parameterized expectations is that adding another state variables just means that we have to include another variable in the list of potential regressors.[16]. However not all state variables have to be included in the regression and this shows another, maybe even bigger advantage of using parameterized expectations for large problems. Note that we only have to

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15 Although each iteration does not take much longer it is still possible that it takes longer to solve the model, since it takes more iterations to get to the fixed point. This did happen sometimes.

16 As an example we mention that with parameterized expectations we have been able to solve the time-to-build model from Kydland and Prescott (1982) which has 7 state variables.

include the additional state variable in $\psi(\cdot)$ if it helps predicting what is inside the conditional expectation, i.e. next period's marginal utility of agent j's consumption in this example. It turns out to be the case that in some of the examples the distribution of wealth is of little importance in predicting next period's marginal utility, while it is very important to determine today's interest rate. That is, in some cases we can approximate the decision rules of the agents with a small set of the model's state variables, but if we combine the policy rules of all these agents we get an equilibrium interest rate that does depend on the distributional characteristics. With algorithms that discretize the state space all the endogenous variables, i.e. choice variables and prices have to be on the same grid.

In the examples below we will characterize the distribution by a discretized distribution function. By increasing the number of grid points we can approximate the exact distribution. We also approximated the distribution by several moments and obtained similar results.

4. SIMULATION RESULTS

Compared to the incomplete markets framework, there is something very convenient about the representative agent framework: it imposes much more discipline on the researcher. In the representative agent version of an endowment economy the researcher can play around with the utility function and the driving process. If we give up the representative agent framework we are immediately forced to make a lot of choices. But since computational constraints keep the model very abstract it is often hard to justify these choices by some insights into the "real world" economy. In our model agents can only smooth consumption by investing in one-period bonds, a clearly unrealistic assumption. Other choices involve the number of agents in the economy, the distribution of shocks, the level of borrowing constraints and other characteristics of the agent. The examples below are therefore mainly meant as a learning tool. We want to see which features of the model have potential in explaining the bad performance of representative agent economies. To do this we will solve and simulate many versions of the model. The following table gives an overview of the models that we look at:

A. **HETEROGENEOUS BUT EX-ANTE IDENTICAL AGENTS**

i. different levels of borrowing constraints.
ii. number of agents participating in the financial markets.
iii. persistence of the driving processes.
iv. the existence of government bonds / money.

B. **EX-ANTE DIFFERENT AGENTS**

i. some agents do not read the newspapers.
ii. being unemployed is worse for some agents than it is for others.
iii. some agents are more risk averse.

For all cases we will use the Imrohoroglu driving processes except when we look at the persistence of the data.

4.A Heterogeneous but ex-ante identical agents

In the first set of models that we look at all agents are ex-ante identical. That is they have the same utility function, they face the same borrowing constraints and their endowment, or employment status, is generated by an identical stochastic process. Ex-post the agents are different, however, since the realizations of the endowment process will be different. Thus at each point in time we will have a variety of agents who differ in their asset holdings and employment status. We start this section by analyzing the impact of borrowing constraints in the Imrohoroglu version of the model.

i. The impact of borrowing constraints.

We look at two levels of the borrowing constraint. In the first example $\bar{a}$ = −150 which means that the maximum debt an agent can have is 5 times his income level when he is unemployed and 1.5 times his income when working. With this level the constraint is binding very infrequently. In the alternative case $\bar{a} = -10$. In this case the agent is basicly constrained in the amount he wants to borrow whenever he is unemployed. In Table 4.1 we give some summary statistics for these two cases. The message of Table 4.1 is very clear. The presence of borrowing constraints is very effective in generating a very low real interest rate[17]. The intuition for this result is straight forward. Each individual faces the possibility of getting into a "crash state" in which it receives a very low income, but he cannot borrow very much. He is therefore very eager to buy bonds when he is employed, but the net-supply of bonds is zero in this economy. This means that the equilibrium bond price is high and the interest rate is low. It is clear that we cannot take this result very serious because in most economies agents can build up net-savings by investing in capital, government bonds or money. We will allow for investment in government bonds below.

An interesting aspect of this model is that the interest rate does not change very much if we increase the number of agents. Of course the variance of per capita consumption (= per capita income) drops if we increase the number of agents. For the case with 100 agents it is still higher than what we observe in the data[18]. But since the interest rate does not seem to be sensitive to the number of agents we should be able to get the same low interest rate with a realistic value of the standard deviation of per capita consumption by increasing the number of agents in this economy. We will now discuss some other characteristics of the time series of these economies.

The interest rate and the aggregate state

In Figure 4.1 we plot the indicator for the aggregate state of the economy and the interest rate with the very restrictive and less restrictive borrowing constraint. Recall that a change in the aggregate state of the

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17 Telmer (1991) and Heaton and Lucas (1992) find similar results for models with only two agents.

18 On a quarterly basis the standard deviation of per capita consumption is around 0.005 for the post-war period. Note that we would want a smaller number for this model which is based on one model period being 1.5 months.

economy does not have a direct effect on the available budget of the agents. Being in a bad state only means that the probability of becoming unemployed increases. So if the economy gets into the bad state agents would like to save more to insure themselves against the higher probability of becoming unemployed. This increase in the demand for bonds lowers the interest rate as we clearly see in the graph. When the bad state continues more and more agents get unemployed. These agents do not want to save anymore but they would like to borrow. If borrowing (selling bonds) is unconstrained then this puts upward pressure on interest rates. In the graph we see that this is exactly what happens in the case with the less restrictive borrowing constraint. In the case with the tight borrowing constraints, however, this supply of bonds never enter the market and there is no upward pressure on interest rates.

Close to the representative agent case?

If we would have complete markets, we can represent this economy by a representative agent economy and the consumption of each agent would just be equal to per capita consumption which is equal to per capita income. We see that in none of the examples this is the case. In all cases the standard deviation of an individual agent's consumption is much higher than the standard deviation of per capita consumption. We also see that in these and in none of the other tables in this paper, there is a clear relation between the real interest rate and the variability of aggregate consumption. In the representative agent economy there is such a relation.

ii. The number of agents.

In the smaller economies we can expect more variation in the percentage of people to be unemployed and consequently less potential for risk sharing. If people do not face borrowing constraints than indeed we see (in Table 4.1) that the standard deviation of individual consumption is going down and the average interest is going up if we increase the number of agents. In the case with borrowing constraints this additional potential for risk sharing is less likely to be exploited. We see indeed that with tight borrowing constraints the number of agents is not very important.

iii. Persistence of the driving process

We will use the continuous process to look at the importance of the persistence of individual risk. We let the autocorrelation coefficient of the idiosyncratic risk component be respectively 0.95 and 0.10. We adjust the standard deviation of the random process to get the same unconditional standard deviation of the change in income in both cases. The maximum amount of short sales is equal to 50, a level that will restrict the agent's borrowing behavior quite often. In Table 4.2 we see that the results are quite different for the persistent process. The difference in the average interest rate between the two cases is a little over 1% and the standard deviation of each agent's consumption is much higher with the persistent process. Note that the standard deviation of each agent's change in income is the same in both cases. These results are very intuitive. If you are hit by a very persistent shock it is going to be very hard to borrow since it is going to take very long before you can repay your debt. This is very similar to the results of Constantinides and Duffie (1992) who show that if all individual specific risk is permanent there will be no trade.

iv. The existence of government bonds / money.

Above we mentioned that the real problem is not so much that agents can not borrow when they face a very low income, but that the private sector as a whole can not save. It is clear that with assets like capital or money this is no longer the case. To get some idea about the importance of this we add a fixed positive supply of government bonds to the model. In these first experiments we assume that the government has some production process available to pay for the interest payments[19]. We haven't tried yet the obvious alternative in which the government has to raise taxes to pay for the interest payments. We let the per capita amount of government bonds be equal to 100 and 200 which is equal to respectively once and twice the income when employed. In Table 4.3 we see that with an average bond supply equal to 100,

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19 Alternatively, we could think of this example as an open economy where the foreign sector supplies a fixed amount of bonds every period.

the interest rate is equal to 0.4% which is around 4% more than it was in the case of a zero net supply of bonds. If we increase the average bond supply to 200 the interest rate increases even more to 0.76%. Corresponding with this increase in the interest rate is a substantial decrease in the standard deviation of individual consumption. This clearly indicates that the result found in the literature with two heterogeneous agents is not robust to the existence of positive net savings.

4.B EX ANTE DIFFERENT AGENTS

We will now discuss some examples in which agents are not only ex post but also ex ante different. There are several reasons to motivate and there are numerous ways to implement this. We want to study ex-ante different agents for two reasons. First we want to look at the disutility of non-rational decision rules. Smith (1992) points out that the disutility of non-rational decision rules is very small in representative agent economies because the standard deviation of consumption in these models is very small. In heterogeneous agent models this is not necessarily the case. We discuss this issue in more detail in the last section. Here we just try one very simple example. To be precise we let 10 of the 100 agents only use individual state variables to predict next period's consumption. Apparently these guys don't read their newspapers to find out what is happening with the rest of the economy. Of course these agents do observe the interest rate in making their consumption decision. The other 90 agents are fully informed and do use all the state variables. The second reason that we want to look at ex-ante heterogeneous agents is that we do not think that the income generating stochastic process is very realistic for a lot of people. We therefore change the parameters in such a way that only a fraction of the agents has a big drop in income when they become unemployed.

i. Some agents do not read the newspapers.

In this section we look at an economy in which 10 of the 100 agents only use individual state variables to predict next period's marginal utility of consumption. Those state variables are the employment status and the bond holdings. In this economy we set the borrowing constraint equal to 150 a level that is usually not binding. We find the results of the reported statistics to be very similar to the results reported above. However, there are some noticeable differences between the policy rules of the two agents. If the aggregate economy reaches it's bad state, then the rational agent realizes that there is a higher probability of becoming unemployed. So he wants to consume less and save more which pushes down the interest rate. The agent who ignores the state of the aggregate economy will actually save less then he otherwise would have since the interest rate is going down. We clearly see this difference in behavior in Figure 2A and 2B. We also calculated the utility loss of using the limited information set and found this to be basicly equal to zero[20]. The reason is also clear from Figure 2A. The main swings in consumption are caused by changes in his employment status. In this graph the agent is unemployed in period 122. Relative to these unavoidable big swings, the reactions to the aggregate state of the economy are of minor importance.

ii. being unemployed is not the same for everybody

In this section we look at the case where for the majority of people it is not as bad anymore to be unemployed. That is, for 90 agents the level of their income when unemployed is now 70% of their regular income; the remaining 10 agents still get 30% of their regular income. The restrictive level of the borrowing constraint is used. We see in Table 4.4 that the average interest rate is equal to -0.24%. Recall that in the case where everybody got 30% the average interest rate was equal to -4.3%. If we change the unemployment income of those 10 agents to 10% of their regular income then the interest rates drops somewhat from -0.24% to -0.47%.

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20 We calculated this utility loss in the following way. We ran this economy 250 times always starting at the same initial conditions. For each data set we calculated the discounted utility for agent 1. Taking the average over the 250 simulations gives the expected utility for agent 1. For each economy we also used the discounted utility for agent 1 if he would have used the policy rule of the agents with the limited information set. So we use the same income stream, the same interest rate and the same economy as before to calculate the discounted utility. We calculated the permanent increase in consumption that corresponds with this change.

iii. some agents are more risk averse

We continue the example mentioned in the last section in which 90 agents receive 70% of their regular income when unemployed and 10 agents receive 10% of their regular income when unemployed. But now we let the risk aversion of those 10 agents be 2 and 2.5 instead of 1. We see in Table 4.4 that this has an enormous impact on the average interest rate. If we let the risk aversion parameter increase from 1 to 2 we see that the standard deviation of the risk averse agents decreases substantially and that of the less risk averse agents increases. If the risk aversion increases from 2 to 2.5 this process continues but the changes are not as big. There still is a big decrease in the average interest rate.

4.C SUMMARY OF THE RESULTS

One of the main puzzles in the literature has been the observed large difference between the average return on equity and the average return on short-term bonds (the equity premium). Note that we can always get the correct average return on one investment right by an appropriate choice of the rate of time preference. By looking at the difference between the (fixed) rate of time preference and the average interest rate we can get an idea how successfull the models in this paper are in generating a substantial equity premium.

The first important conclusion is that just having incomplete markets is not enough for generating huge premiums. This was illustrated by the case in which agents do not face restrictive borrowing constraints, agents have a highly volatile individual income and can smooth consumption only by trading in one-period bonds. Although individual consumption is still much more volatile than per capita consumption (the representative agent solution) the average interest rate is not much lower then it is in the representative agent case.

Adding borrowing constraints to the model clearly decreased the amount of risk sharing and did generate substantial premiums. This result was already established in the literature for heterogeneous agent models with two types of agents. In this paper we showed that this result is not robust to the introduction of a positive supply of government bonds.

Not all results are negative, however. We saw that we can still get large premiums if only a small fraction of the population faces a highly volatile income stream. We will focus on this finding in the future. Note that we would not have been able to find this result without the algorithm developed in this paper.

5. FUTURE RESEARCH

In this section we describe the other parts of this research project. In the first place we want to spend more time to look at the properties of the algorithm. Future research will focus on the following:

i. Comparison with other methods. For the economies with a small number of agents we would like to compare our solution with Coleman's method of policy function iteration. This method discretizes the state space.

ii. Accuracy of the algorithm. We will see whether the solutions are robust to using higher-order terms in the parameterization. It also has to be noted that with parameterized expectations as with other solution algorithms there are several different ways to solve the model. Comparison of these solutions will be done. Also we will use the accuracy test from Den Haan and Marcet (1989) to check for accuracy.

iii. More complicated models. It should be clear that the models presented in this paper are just very simple examples, although typical for the models used in the literature. The next step will be to include other assets like physical capital and to increase the heterogeneity.

Given the enormous interest in the literature for heterogeneous agent models but the lack of a powerful numerical algorithm it won't be hard to start a whole series of interesting research projects. Below we discuss some projects. The first project is related to the work in Den Haan (1992) using representative agents and the second to Den Haan (1990) again using a representative agent framework.

i. Convergence of GNP and international trade. Recently the empirical question whether different economies converge or diverge has received a lot of attention. An important contribution was given by Barro and Sala-i-Martin (1992) and Mankiw, Romer and Weil (1992) who show that the U.S. states provide clear evidence of convergence in the sense that poor economies tend to grow faster than rich ones in per capita terms. Barro and Sala-i-Martin conclude that their empirical findings can be reconciled quantitatively with the non-stochastic version of the neo-classical growth model, but the share of capital has to be quite high[21]. In Barro and Sala-i-Martin (1992) the states differ in their initial level of capital and there is no interaction between the separate states. In Den Haan (1992) I show that the conclusion can be quite different if we add stochastic technology shocks to the model. Moreover, Den Haan (1992) also takes care of the problem in Barro and Sala-i-Martin that the perfect-foresight version has unsensible implications for the comovements of GNP growth of the separate economies. By letting the different economies have a common and an idiosyncratic technology shock it was possible to match some key statistics about the comovement of US states' GNP in a parsimonious way. In Den Haan we just choose the parameters of the model and check how likely the empirical results of Barro and Sala-i-Martin are using the stochastic growth model with the chosen parameters. Given that the numerical solution of this model is very easy, it should be feasible to estimate the parameters and test the model using Estimation by Simulation. This estimation technique has been developed recently by Ingram and Lee (1991) and Duffie and Singleton (1988) to estimate and test models in which the moment conditions obtain unobservables. In most intertemporal models the unobservables appear in a non-linear fashion which means that they do not drop out of the moment condition.

As another part of this research project I want to include several forms of trade to the model. To solve this model we would need an algorithm for heterogeneous agent models. All growth models mentioned above are examples of the embarrassing nature of heterogeneous agent literature, since there is no trade at all between the states. In this research project I want to look

[21] Mankiw, Romer and Weil (1992) reach the same conclusion.

at the speed of convergence when we allow certain types of trade between the states. My conjecture is that the introduction of even a limited amount of trade will increase the speed of convergence substantially. This will make it much harder to reconcile the neo-classical model with the slow speed of convergence observed empirically. The next step would be to go to more interesting models in which countries use different technologies as in Krussel (1991) and Lucas (1988) or in which local non-convexities may force economies to operate at different stochastic steady states as in Klenov (1991).

ii. Inflationary uncertainty and the term structure. In Den Haan (1991) I discuss the difficulties of real and monetary models to explain some stylized facts about the term structure. I look at the random walk character of the interest rate and the term premiums. I show that the random walk character of interest rates can be easily explained using a representative agent economy with production but the positive term premiums that we usually observe are hard to explain both in real and monetary representative agent models. First some intuition for the real model. In a real model an n-period bond delivers one unit of consumption n periods in the future. But the further in the future you go the more uncertainty there is. This makes this certain unit of consumption more attractive and will push down the average return on long-term bonds. In a monetary model the pay-off of a bond will be uncertain as well due to inflationary uncertainty. But in a representative agent model there is little reason for inflationary uncertainty to be priced. The inflationary risk of nominal contracts between individual agents should not be priced since it can be diversified away completely. And the aggregate risk of inflationary uncertainty is very small for the following reason. An unexpectedly high inflation means that the pay-off of government assets is unexpectedly low but this is offset by an unexpected decrease in the tax burden since in real terms the obligations of the government went down. The main effect of inflationary uncertainty works through the effect that real money balances have on transaction services and this channel is not very important. There is no important effect on the agent's budget if there is an unexpected inflationary shock. In a model with heterogeneous agents in which inflationary risk can not be diversified away this is not the case.

The plan is to solve several heterogeneous agent models. It has been pointed out in the literature that less wealthy agents mainly use short-term nominal assets to insure against unforeseen shocks. It would be interesting to analyze the importance of inflationary uncertainty when a fraction of the agents of the economy can only invest in nominal assets while the rest of the economy also has access to investment in equity or capital. An interesting exercise would be to see what happens with the role of inflationary uncertainty if the fraction of constrained investors increases.

iii. Utility losses from near-rationality. Smith (1992) shows that in standard neo-classical models using the representative agent framework there are many very different non-rational decision rules, that are nevertheless close to the rational expectations solution in terms of utility. The main restriction on the alternative policy rules is that the mean is the same as in the rational expectations solution. So a policy in which the capital stock is always held constant, for instance, is in terms of utility almost the same as using the rational expectations decision rule. The problem with these small utility losses is that the outcome of a model is not very robust to small changes. By introducing tiny adjustment cost you can get any variance of investment that you want. This "trick" of introducing adjustment costs is used in Christiano and Eichenbaum (1992) and Mendoza (1991). The explanation for Smith's findings is of course the lack of variation in aggregate consumption. In a heterogeneous agent model in which the link between aggregate and individual variation is broken these results are likely to be different in general.

iv. Monetary policy

Recently following Rotemberg (1984) and Grossman and Weiss (1983) a lot of attention has been given to the idea that monetary policy has real effects because the increase in liquidity is not spread evenly over the different groups of the economy and within the period the separate members of the economy are limited in trading with each other. This idea was pursued for instance by Christiano and Eichenbaum (1990). They use the following idea from Lucas (1990) and Fuerst (1992) to keep the heterogeneity tractable. At the beginning of the period the members of a representative household split up as different members of the economy like a shopper, a banker, a producer. During the period the members act as separate individuals but at the end of the period they come back to the same household. At the end of the period all households are thus the same again and thus have the same wealth level. There are several other papers that point at the importance of heterogeneity for understanding the impact of monetary policy. Gertler and Gilchrist (1991) distinguish between small and large firms. Taylor (1980), Lucas (1986) and Levin (1990) use staggered contracts to distinguish people in the economy. With the algorithm developed in this paper you could keep the groups separate over time and take wealth effects into account.

FIGURE 1: INTEREST RATES AND THE AGGREGATE ECONOMY

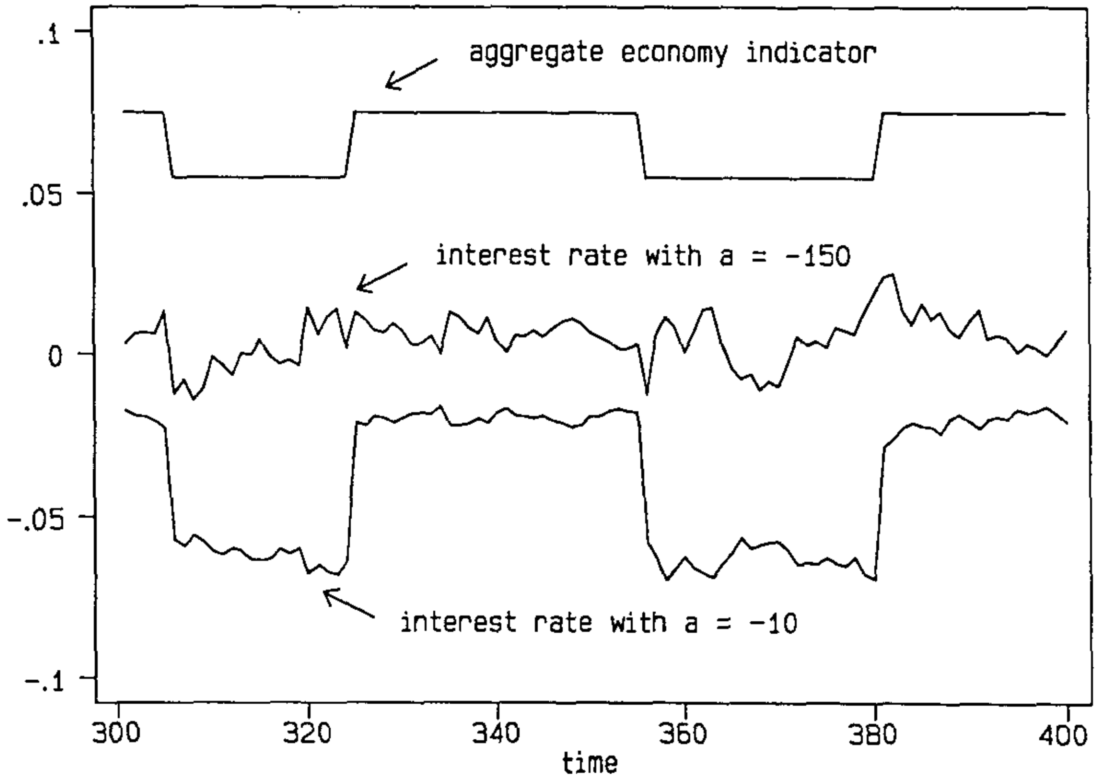

FIGURE 2A: CONSUMPTION AND THE AGGREGATE ECONOMY

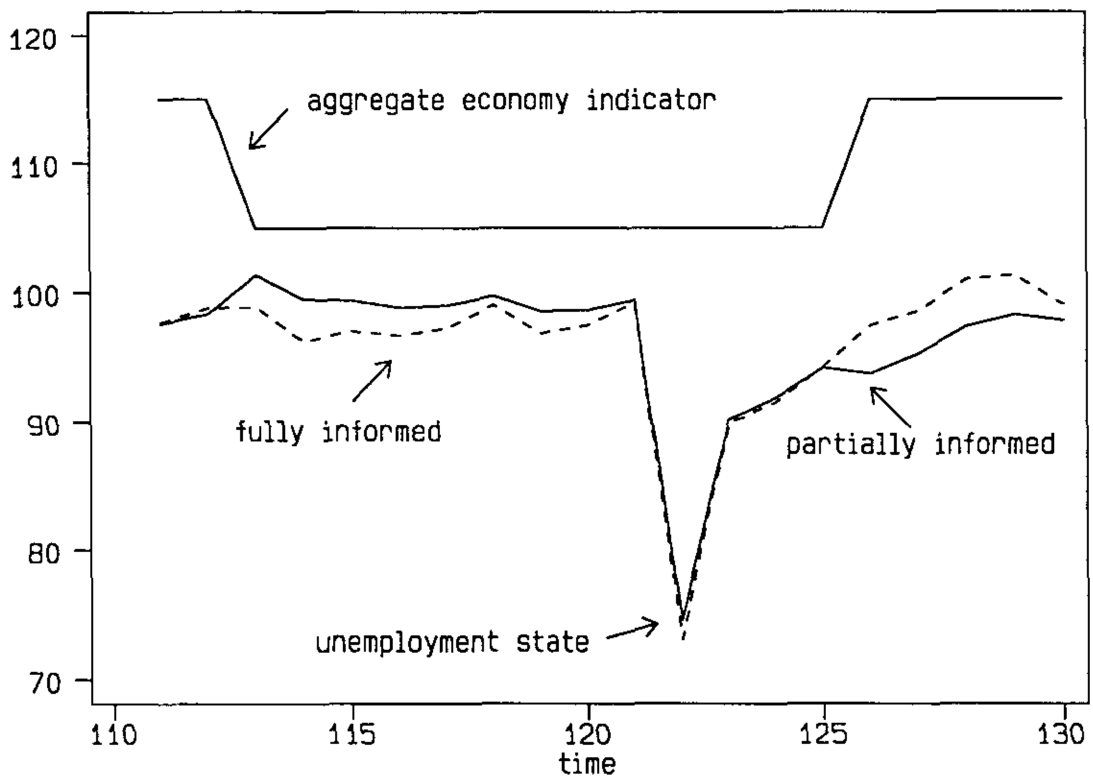

FIGURE 2B: SAVINGS AND THE AGGREGATE ECONOMY

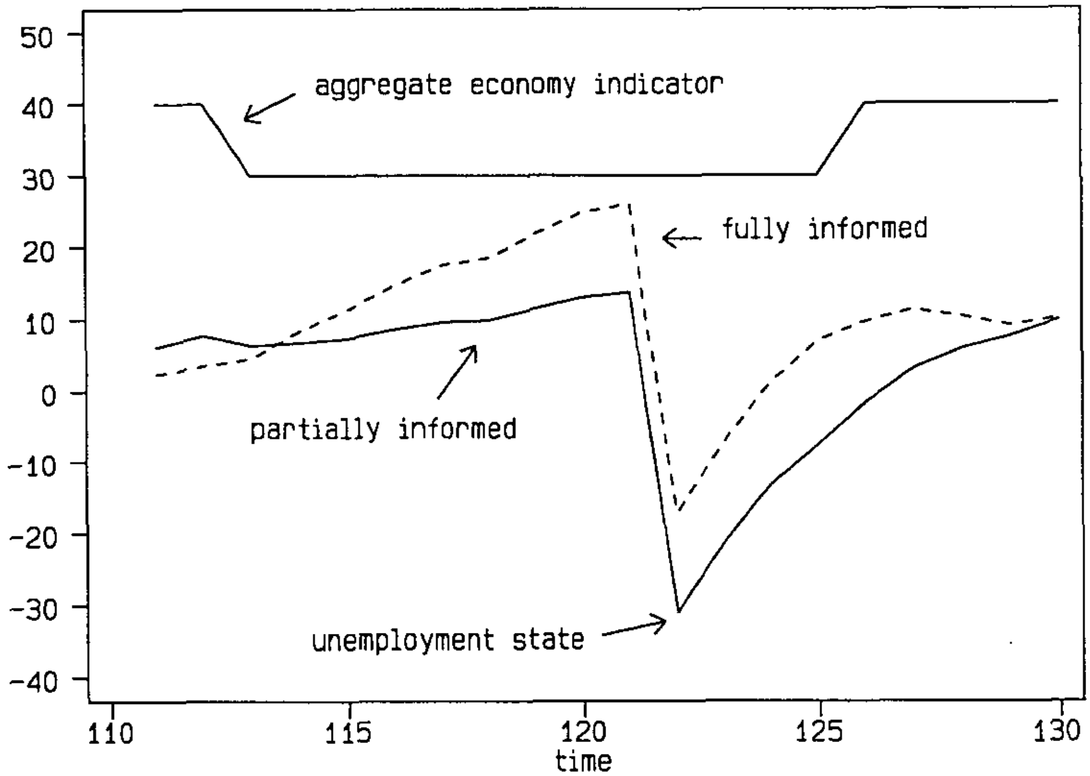

TABLE 4.1 THE IMPORTANCE OF BORROWING CONSTRAINTS

BORROWING CONSTRAINT	a = -10		a = -150	
# AGENTS	10	100	10	100
stan. dev. $\Delta\Sigma c_j/N$	0.0686	0.0227	0.0686	0.0227
stan. dev. Δy_1	0.3236	0.3236	0.3236	0.3236
stan. dev. Δc_1	0.2634	0.2610	0.1160	0.0918
mean interest rate	-4.26%	-4.30%	0.47%	0.58%

The transition probabilities are as in Imrohoroglu (1989), the rate of time preference is equal to 1%. All variables are in natural logs.

TABLE 4.2 THE IMPORTANCE OF PERSISTENCE

PERSISTENCE(ρ)	.10	.95
stan. dev. $\Delta\Sigma c_j/N$	0.0280	0.0317
stan. dev. Δy_1	0.2064	0.2049
stan. dev. Δc_1	0.0479	0.1849
mean interest rate	0.60%	-0.62%

ρ is the AR(1) coefficient of the idiosyncratic shock, σ is chosen to give a similar amount of individual income variation. This means that σ is equal to 0.15 and 0.20 for respectively ρ equal to .10 and .95. The other parameters are ρ_a = 0.95, σ_a = 0.02, β = 0.99, $\bar{a}$= -50 and the number of agents is equal to 100.

TABLE 4.3 POSITIVE SUPPLY OF GOVERNMENT BONDS

PER CAPITA GOVERNMENT BONDS	$1 \times y$	$2 \times y$
stan. dev. $\Delta\Sigma c_j/N$	0.0227	0.0227
stan. dev. Δy_1	0.3236	0.3236
stan. dev. Δc_1	0.1075	0.0675
mean interest rate	0.38%	0.76%

The transition probabilities are as in Imrohoroglu (1989), the rate of time preference is equal to 1%. All variables are in natural logs.

TABLE 4.4 EX-ANTE DIFFERENT AGENTS

CASE	A	B	C	D
stan. dev. $\Delta\Sigma c_j / N$	0.0112	0.0124	0.0124	0.0124
stan. dev. Δy_1	0.0959	0.0959	0.0959	0.0959
stan. dev. Δc_1	0.0809	0.0886	0.0950	0.0944
stan. dev. Δy_{100}	0.3326	0.6361	0.6361	0.6361
stan. dev. Δc_{100}	0.1517	0.2345	0.2068	0.2009
mean interest rate	-0.24%	-0.47%	-7.56%	-13.16%

The transition probabilities are as in Imrohoroglu (1989), the rate of time preference is equal to 1%. All variables are in natural logs.

Case A: For the first 90 agents we have θ = 0.7 and for the last 10 agents we have θ = 0.3. Coefficient of relative risk aversion for all agents is equal to 1.

Case B: For the first 90 agents we have θ = 0.7 and for the last 10 agents we have θ = 0.1. Coefficient of relative risk aversion for all agents is equal to 1.

Case C: For the first 90 agents we have θ = 0.7 and for the last 10 agents we have θ = 0.1. Coefficient of relative risk aversion is equal to 1 for the first 90 agents and equal to 2 for the last 10 agents.

Case D: For the first 90 agents we have θ = 0.7 and for the last 10 agents we have θ = 0.1. Coefficient of relative risk aversion is equal to 1 for the first 90 agents and equal to 2.5 for the last 10 agents.