MONEY AND INVENTORIES IN AN ECONOMY WITH UNCERTAIN AND SEQUENTIAL

TRADE

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Abstract
We propose a model in which an unanticipated reduction in the money
supply leads to a contemporaneous increase in inventories followed by
periods with lower output. This persistent real effect does not
require price-rigidity or real shocks and confusion. It is obtained
in a model in which markets are cleared and agents are price-takers.

NTRODUCTION

Uncertain and sequential trading (UST) models are based on ideas in Prescott (1975) and Butters (1977). Prescott considers an environment in which sellers set prices before they know how many buyers will eventually appear. He assumes that less expensive goods will be sold before more expensive ones. Free entry implies that in equilibrium there will be a distribution of prices rather than a single price even though the product is homogeneous. Butters stresses the trade-off between the level of prices and the probability that a sale will be realized. In both models sellers commit to prices before the realization of demand. In the UST approach taken by Eden (1990) an equilibrium distribution of prices is obtained even though sellers are allowed to change their prices during trade.

Recently Eden (1994) and Lucas and Woodford (1994) study money non-neutrality in models which utilize the above idea. Eden (1994) considers a UST overlapping generations model, in which buyers (the old) receive monetary transfers in a sequential manner. They spend the money immediately and therefore from the point of view of the sellers (the young) purchasing power arrives sequentially. Sellers make contingent plans which specify the quantity that they will sell to each batch of dollars that may arrive. They choose not to sell everything they produced to the first batch that arrived because they speculate on the event that more batches of dollars will arrive and will buy at a higher price. Accordingly, when the realized money supply is low, some goods are not sold (and are lost).

Lucas and Woodford (1994) study an infinite-horizon, cash-in-advance model. In their framework, trade is sequential but the

transfer of money is not: trade starts only after buyers know the total amount of money transferred. Sellers who do not know that amount, choose to put different price tags on various units so that goods offered at low prices are rationed if the money supply is high. As they point out the source of non-neutrality is different from the source in Lucas (1972). In particular, in Lucas (1972) in addition to the monetary shocks there are real shocks, so that in contrast to Lucas and Woodford (1994), producers cannot assess correctly the real rate of return on money.

In the above models there are no real shocks and no confusion. Money is non-neutral because prices at the beginning of the trading process cannot depend on information that becomes public only at the end of the process. Unlike in fixed price models, here sellers have no incentive to change prices during trade.

The persistence of the real effect of monetary shocks is absent in Lucas (1972). This motivated Blinder and Fischer (1981) to consider inventories as a propagation mechanism. Building on Lucas' confusion hypothesis, an unanticipated increase in the money supply is interpreted in part as a real demand shock. Firms increase sales and to restore inventories to their target-level, output increases as well. However, to rationalize the existence of inventories, Blinder and Fischer require that firms have monopoly power.

This paper combines the money non-neutrality analysis in Eden (1994) with the Bental and Eden (1993) UST analysis of inventories, to get persistent real effects of unanticipated monetary shocks.

Following Lucas (1980), we employ a cash-in-advance economy populated by infinitely lived households which consist of two people: a seller (producer) and a buyer. At the beginning of each period the household has money and inventories. The seller takes the inventories

and goes to work. The seller produces some additional output and tries to sell some or all of the accumulated stock. The buyer takes the money and goes shopping. On the way to the shopping location, the buyer may receive a monetary transfer. Once the buyer arrives at the marketplace he spends part or all of the money he has and returns home.[1] The household consumes whatever the buyer managed to buy. The only uncertainty in the model is about the number of buyers that will receive the transfer payment.

The seller stays in one location. He knows that a certain minimal amount of money will arrive. We say that this minimal amount buys in the first market. With some probability, more buyers will get a transfer and more money will arrive. The additional money, if it arrives opens the second market and so on. The seller, after having produced, allocates the available supply (output + beginning of period inventories) among all potential markets. If a particular market opens the seller sells the supply allocated to that market for cash. If that market does not open, the supply is carried over to the following period as inventories. Inventories may also be held for purely speculative reasons.

A low realization of the money supply at time t, will cause an increase in time t + 1 inventories and a reduction in output. As in Bental and Eden (1993), the reduction of output is smaller in absolute value than the increase in inventories. Therefore, total supply rises and inventories at t + 2 will be higher on average. Thus, a monetary shock may have long lasting effects, and a causality test will reveal that money "causes" output. However, this

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[1] Like Eden (1990) and Bental and Eden (1993) and unlike Eden (1994) we do not allow here more than one trip per period to the market.

"causation" cannot be used by a policy maker, since agents use the correct probabilities that markets will open.

2. THE MODEL

There are N households. Each household consists of two people: a seller and buyer. The typical household is similar to the one described in Lucas and Stokey (1983). It engages in production and shopping for consumption goods. Labor (L_t) is the only input and output equals labor input.

The amount of money available to household h at the beginning of period t (M_t^h) equals the proceeds of period t-1 sales and any amount carried over from that period. At the beginning of period t the buyer takes the M_t^h dollars and goes shopping. On the way to the market the buyer may receive a transfer of T_t dollars. In general, not all buyers will receive a transfer payment.

Buyers arrive in the market sequentially in an order that is randomly determined. At the beginning of the period, a buyer does not know the order at which he will arrive at the market-place. The order of arrival is identically and independently distributed across periods. The amount of money that a buyer can spend is M_t^h if he did not get a transfer and $M_t^h + T_t$ if he did. Upon arrival, each buyer sees all the available selling offers. He chooses whether to spend, on the basis of the lowest price offer. Cheaper goods are bought first and therefore buyers that arrive late may face a higher price.

We assume that utility is linear in consumption and therefore the decision whether or not to buy does not depend on the household's wealth. Furthermore, in equilibrium, if a buyer chooses to spend he spends the entire amount available.

The number of buyers that will get a transfer is $N_{\tilde{s}t}$ where the index $\tilde{s}_t$ is an identically and independently distributed (i.i.d.) random variable that can take the realizations: $1, \ldots, S$. It is assumed that $0 \leq N_1 < N_2 < \ldots < N_S = N$. The probability that $\tilde{s}_t = s$ is denoted by Π_s and the probability that $\tilde{s}_t \geq s$ is denoted by q_s.

The total amount of dollars at the beginning of period t (before the beginning of the transfer process) is: $M_t = \sum_{h=1}^{N} M_t^h$. We assume that the transfer payment that a buyer may get is proportional to the current money supply: $T_t = \lambda M_t$. The total amount available for spending depends on the realization of $\tilde{s}_t$. It is:

$M_t + N_s T_t = (1 + \lambda N_s) M_t$ if N_s buyers got the transfer. Information about the realization of $\tilde{s}_t$ becomes public in a sequential manner. Everyone knows that at least $\Delta_{1t} = M_t + N_1 T_t$ dollars will be available for spending. So the first real news come when additional $\Delta_{2t} = (N_2 - N_1) T_t$ dollars arrive (with probability q_2). At this stage everyone knows that $\tilde{s}_t \geq 2$. Then, if additional $\Delta_{3t} = (N_3 - N_2) T_t$ dollars arrive everyone learns that $\tilde{s}_t \geq 3$ and so on.

From the sellers' point of view purchasing power arrives in batches. The first batch of $\Delta_{1t} = M_t + N_1 T_t$ dollars arrives with certainty. Buyers who own the dollars in the first batch spend $D_{1t} \leq \Delta_{1t}$ and return home. Since there is no wealth effect, the choice of D_{1t} does not depend on the identity of the owners of the first batch of dollars.

The second batch of purchasing power consists of $\Delta_{2t} = (N_2 - N_1) T_t$ dollars and will arrive with probability q_2. The buyers who own the dollars in the second batch spend $D_{2t} \leq \Delta_{2t}$ and go home. In general, for $s > 1$, the purchasing power of

$\Delta_{st} = (N_s - N_{s-1})T_t$, will arrive with probability q_s and the owners of dollars in this batch will spend $D_{st} \leq \Delta_{st}$ dollars out of it and then go home.

It is assumed that each buyer can make only one trip to the market and does not hang around: He finishes shopping and goes home.

The probability that the dollars of buyer h are in batch j is equal to the fraction of dollars in batch j out of the post-transfer money supply. The probability that a dollar is in batch 1, given that exactly 1 batch of dollars arrives is: $\upsilon_1^1 = 1$. The probability that a dollar is in batch j, given that exactly $s > 1$ batches of dollars arrive $(j \leq s)$ is: $\upsilon_j^s = \Delta_{jt}/(M_t + N_s T_t) =$

$= (N_j - N_{j-1})\lambda M_t/((1 + \lambda N_s)M_t) = (N_j - N_{j-1})\lambda/(1 + \lambda N_s)$. Note that given our assumption about the transfer payment these probabilities do not depend on the beginning of period money supply.

Given that s batches arrive, the probability that buyer h will get a transfer is: $\phi_s = N_s/N$. This probability does not depend on the buyer's place in the line (the order at which the buyer arrives). In every batch of dollars that arrives there are typically buyers who have received a transfer and others who have not.

A buyer in batch j spends:

$$(1) \qquad \chi_{jt}^{hi} \leq M_t^h + iT_t,$$

dollars, where $i = 1$ if the buyer got a transfer and zero otherwise. The amount of consumption and nominal balances that the buyer brings home in this case are:

$$(2) \qquad c_{jt}^{hi} = \chi_{jt}^{hi}/P_{jt} \; ; \; \gamma_{jt}^{hi} = M_t^h + iT_t - \chi_{jt}^{hi}.$$

Given that s batches arrive, the expected consumption for a buyer in batch j is thus: $[\phi_s c_{jt}^{h1} + (1-\phi_s)c_{jt}^{h0}]$.

At the beginning of the period the seller (producer) chooses the amount of labor which given the beginning of period inventories (I_t) determines total supply (k_t):

$$(3) \qquad\qquad k_t = L_t + I_t.$$

The seller makes a contingent plan on how to sell the available supply, which specifies the amount that the seller will sell to batch j if it arrives.

We assume that the amount of dollars that arrive at the market place is observed by everyone, and say that the arrival of a batch of dollars opens a new market. We describe the contingent plan of the seller by the quantities that the seller chooses to sell in each market that opens. If batch j of dollars arrives but is not used for spending, we say that market j is open but not active. We later show that this does not occur: In equilibrium all markets which open are active.

The seller allocates the available supply to the S markets. In addition some of the supply may be carried to the next period for purely speculative reasons. We say that this amount is allocated to market S+1 which opens, this period, with probability zero. Thus, the seller chooses the supply to market i (k_{it}) subject to:

$$(4) \qquad\qquad \sum_{i=1}^{S+1} k_{it} = k_t.$$

It is assumed that inventories depreciate at the rate

$0 \leq \delta \leq 1$. If j markets open, the next-period inventories are:

$$(5) \qquad I_{t+1}^{j} = (1 - \delta)\sum_{i=j+1}^{S+1} k_{it}.$$

In addition, the seller will contribute to the beginning of next period's money balances the sum of:

$$(6) \qquad M_{t+1}^{j} = \sum_{i=1}^{j} P_{it}k_{it}.$$

It is assumed that the household is risk neutral and its single period utility is $c_t - v(L_t)$ where $v(\ )$ measures the disutility of work in terms of current consumption. We assume that the marginal cost schedule starts at zero and is strictly increasing: $v'(\ ) > 0$, $v'(0) = 0$, $v''(\ ) > 0$. Furthermore, we assume that $\sup[v''(\ )]$ is finite.

The household has a discount factor of β. Taking prices, the conditional probabilities of buying in market j, and the probability of a transfer payment as given, the household chooses $0 \leq L_t \leq 1$, k_{it} and χ_{jt}^{hi} to maximize:

$$(7) \qquad E\sum_t \beta^t (c_t - v(L_t)), \quad \text{s.t. } (1) - (6).$$

We assume that given the information available at time t, the household can form point estimates of the prices in all markets at any period $t + \tau$ as a function of the sequence of realizations of $\tilde{s}_t, \tilde{s}_{t+1}, \ldots, \tilde{s}_{t+\tau}$. In particular, its point estimate of the prices next period given that s markets were opened this period is:

$(P_{1t+1}^{s}, \ldots, P_{St+1}^{s}).$

Let $V(I_t, M_t; P_{1t}, \ldots, P_{St})$ denote the expected utility of a household that starts period t with I_t units of inventories and M_t dollars given that prices in the S markets are $(P_{1t}, \ldots, P_{St})$. The Bellman equation which defines V( ) is:

(8) $\qquad V(I_t, M_t^h; P_{1t}, \ldots, P_{St}) =$

$$\max \sum_{s=1}^{S} \Pi_s \sum_{j=1}^{s} \upsilon_{jt}^{s} [\phi_s c_{jt}^{h1} + (1-\phi_s) c_{jt}^{h0}] - v(L_t)$$

$$+ \beta \sum_{s=1}^{S} \Pi_s \sum_{j=1}^{s} \upsilon_{j}^{s} \{\phi_s V(I_{t+1}^{s}, M_{t+1}^{s} + \gamma_{jt}^{h1}; P_{1t+1}^{s}, \ldots, P_{St+1}^{s})$$

$$+ (1 - \phi_s) V(I_{t+1}^{s}, M_{t+1}^{s} + \gamma_{jt}^{h0}; P_{1t+1}^{s}, \ldots, P_{St+1}^{s})\};$$

$$\text{s.t. } (1) - (6).$$

We now turn to the first order conditions associated with (8), under the assumption that the seller be ready to sell in each of the S markets and produce a strictly positive amount.[1]

<u>Arbitrage conditions:</u> It is assumed that the household can form a point estimate about the maximum expected amount of consumption, discounted to t+1, that a dollar held by the buyer at t + 1 in market j' will buy, if exactly s markets are opened at t. These expectations are denoted by: $(R_{1t+1}^{s}, \ldots, R_{St+1}^{s})$.

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[1] Thus we derive the first order condition for a solution $k_s > 0$ for all $s \leq S$, but allow $k_s = 0$ in case there is no demand. If sellers were allowed to set prices, like in Lucas and Woodford (1994), this restriction on pricing strategies means that a seller must be willing to sell a strictly positive amount at each price he advertises. In equilibrium, the seller's supply to any particular market is perfectly elastic.

Given these expectations the maximum expected present value of consumption (expected discounted consumption) that a dollar can buy if its owner is currently in market j is R_{jt} which satisfies:

$$(9) \qquad R_{jt} = \max\{1/P_{jt} \ , \ \beta\sum_{s=j}^{S}(\Pi_s/q_j)\sum_{s'=1}^{S}\Pi_s \cdot \sum_{j'=1}^{s'}v_{j'}^{s'} \cdot R_{j' \ t+1}^{s}\}$$

$$= \max\{1/P_{jt} \ , \ \beta Z_{jt}\} .$$

To understand the second term in the max expression, note that the probability that the buyer will find himself in market j' at time t+1 depends on the number of markets opened at t+1 (s'). Therefore we take expectations over j' and s' to get the maximum expected consumption, discounted to t+1, given that s markets are opened at t: $X^s = \beta\sum_{s'=1}^{S}\Pi_s \cdot \sum_{j'=1}^{s'}v_{j'}^{s'}R_{j' \ t+1}^{s}$. Since Π_s/q_j is the probability that exactly $s \geq j$ markets open at t given that market j has already opened, the expected discounted consumption that a dollar will buy, if it is not spent at time t in market j, is: $\beta Z_{jt} = \sum_{s=j}^{S}(\Pi_s/q_j)X^s$. If the dollar is spent at time t in market j it yields $1/P_{jt}$ units of consumption. And therefore R_{jt} is the maximum discounted consumption that a dollar can buy if at time t the buyer is in market j.

From the seller's point of view, the expected purchasing power of a dollar earned at time t, given that market j opened at time t is:

$$(10) \qquad Z_{jt} = \sum_{s=j}^{S}(\Pi_s/q_j)\sum_{s'=1}^{S}\Pi_s \cdot \sum_{j'=1}^{s'}v_{j'}^{s'}R_{j' \ t+1}^{s} .$$

Note that Z_{jt} was implicitly defined in (9) as the expected purchasing power of a dollar held by the buyer in market j at time t, if the buyer chooses not to spend it.

The expected consumption from a unit of output supplied to market 1 is: $P_{1t}Z_{1t}$. In equilibrium marginal cost must equal the present value of the ensuing expected future consumption:

$$(11) \qquad v'(L_t) = \beta P_{1t}Z_{1t}.$$

Let π_s denote the probability that market s will open, given that market s-1 opens. The expected pay-off from not selling one unit when market s-1 opens is: $\pi_s P_{st}Z_{st} + (1-\pi_s)(1-\delta)v'(L_{t+1}^{s-1})$. The first element represents the present value of expected purchasing power which results from a sale in market s if it opens. The second element reflects the substitution between inventories and next period production: An addition of a unit to inventories can be used to cut production by $(1-\delta)$ units. The second element is therefore the present value of inventories if market s does not open. The arbitrage condition is:

$$(12) \quad P_{s-1t}Z_{s-1t} = \pi_s P_{st}Z_{st} + (1-\pi_s)(1-\delta)v'(L_{t+1}^{s-1}) \; ; \text{ for all } s > 1,$$

where L_{t+1}^{s-1} is the amount of labor that the household plans to supply at time t+1, if exactly s-1 markets open this period.

An alternative to selling one unit in market S when it opens is to hold it as inventories and use it next period to reduce production. The value of inventories is $(1-\delta)v'(L_{t+1}^{S})$ and the arbitrage condition is:

(13) $P_{st}Z_{st} \geq (1 - \delta)v'(L_{t+1}^{S})$ with strict equality if $k_{s+1} > 0$.

The buyer chooses to spend the entire amount in market s if the purchasing power of a dollar is higher than the expected purchasing power of this dollar in the next period. To simplify, we also assume that he spends the entire amount in case of indifference. Thus,

(14) $D_{st} = 0$, if $\beta Z_{st} > 1/P_{st}$ and $D_{st} = \Delta_{st}$ if $\beta Z_{st} \leq 1/P_{st}$.

The market clearing conditions are:

(15) $D_{st}/P_{st} = k_{st}$; for all s.

Equilibrium requires (11) - (15).

We will show that there exists an equilibrium in which the beginning of period inventories and money supply are sufficient statistics for information available at time t. Within this class, we restrict our attention to prices that can be multiplicatively decomposed into two elements: a normalized price $p_s(I_t)$ and the beginning of period money M_t:

(16) $P_{st} = p_s(I_t)M_t.$

It is useful to think of the whole money supply (M_t) as a normalized dollar. The price in terms of a normalized dollar in market s is $p_s(I_t) = P_{st}/M_t.$ We use $\theta_j = 1 + (N_j/N)\lambda$ to denote the gross rate of change in the money supply. Thus, $M_{t+1} = \theta_j M_t,$ when N_j buyers got a transfer at time t. For notational convenience we set

$\theta_0 = 0.$

We define the expected purchasing power of a normalized dollar (held by a buyer) at time t in market j, $r_j(I_t)$, and the expected purchasing power of a normalized dollar earned (by a seller) in market j at time t, $z_j(I_t)$, by:

(17)
$$r_j(I_t) =$$
$$= \max\{1/p_j(I_t) \ , \ \beta\sum_{s=j}^{S}(\Pi_s/q_j)\sum_{s'=1}^{S}\Pi_{s'}\cdot\sum_{j'=1}^{S'}\upsilon_{j'}^{s'}\cdot r_{j'}\cdot(I_{t+1}^s)/\theta_s\}$$

$$= \max\{1/p_j(I_t) \ , \ \beta z_j(I_t)\}.$$

Note that $r_j(I_t)/M_t = R_{jt}$ and $z_j(I_t)/M_t = Z_{jt}$. This can be shown by dividing (17) by M_t and comparing with (9).

Expressing conditions (11) to (14) in terms of normalized dollars leads to:

(11')
$$v'(L(I_t)) = \beta p_1(I_t)z_1(I_t);$$

(12') $$p_{s-1}(I_t)z_{s-1}(I_t) = \pi_s p_s(I_t)z_s(I_t) + (1-\pi_s)(1-\delta)v'(L(I_{t+1}^{s-1}));$$

(13')
$$p_S(I_t)z_S(I_t) \geq (1-\delta)v'(L(I_{t+1}^S))$$

with strict equality if $k_{S+1} > 0$.

(14')
$$d_{st} = 0 \text{ if } \beta z_s(I_t) > 1/p_s(I_t) \text{ and}$$

$$d_{st} = D_{st}/M_t = (\theta_s - \theta_{s-1}) \text{ if } \beta z_s(I_t) \leq 1/p_s(I_t).$$

We now turn to define a temporary equilibrium that takes expectations about future magnitudes as given, in a particular form.

We assume that next period's labor supply can be specified as a strictly decreasing function of the beginning of next period's inventories, $L(I)$, and define next period's marginal cost by: $\omega(I) = v'(L(I))$, where $\omega(I)$ is strictly decreasing in I. Likewise, we assume that the expected purchasing power of a normalized dollar (held by a buyer) next period in market j is a function of the beginning of next period's inventories, $\alpha_j(I)$. We assume that the $\alpha_j(I)$ functions are strictly increasing and define a temporary equilibrium as follows.

The vector

$(p_1, \ldots, p_S, L, k_1, \ldots, k_S, k_{S+1}, d_1, \ldots, d_S, r_1, \ldots, r_S, z_1, \ldots, z_S)$ is a temporary equilibrium for given I_t, strictly increasing $\alpha_1, \ldots, \alpha_S$, and strictly decreasing ω if it satisfies identities (3)-(5); the arbitrage conditions (11')-(14'); the market-clearing conditions,

(18)
$$d_s/p_s = k_s \, ,$$

And

(17')
$$r_j =$$
$$= \max\{1/p_j \, , \, \beta\Sigma_{s=j}^{S}(\Pi_s/q_j)\Sigma_{s'=1}^{S}\Pi_s \cdot \Sigma_{j'=1}^{S'} v_j^{s'} \cdot \alpha_j \cdot (I_{t+1}^s)/\theta_s\}$$

$$= \max\{1/p_j \, , \, \beta z_j\}.$$

We show later that a temporary equilibrium exists when the functions $\alpha_j(.)$ and ω belong to a set that will be specified soon.

Lemma 1: At a temporary equilibrium we must have $p_1 \leq p_2 \leq \cdots \leq p_S$;

$z_1 \geq z_2 \geq \dots \geq z_S$; $p_1 z_1 \leq p_2 z_2 \leq \dots \leq p_S z_S$, and $r_1 \geq r_2 \geq \dots \geq r_S$.

Proof: Omitting time index we see that (5) implies
$I^1 \geq I^2 \geq \dots \geq I^S$. Since ω is strictly decreasing, it follows that
$\omega(I^1) \leq \omega(I^2) \leq \dots \leq \omega(I^S)$ and condition (12') implies:

$$(19) \qquad p_{S-1} z_{S-1} \leq \pi_S p_S z_S + (1-\pi_S)(1 - \delta)\omega(I^S);$$

Using (13'), (19) leads to:

$$(20) \qquad p_{S-1} z_{S-1} \leq p_S z_S.$$

Since (12') implies that $p_{S-1} z_{S-1}$ is a weighted average of $p_S z_S$ and
$(1 - \delta)\omega(I^{S-1})$, condition (20) implies

$$(21) \qquad p_{S-1} z_{S-1} \geq (1 - \delta)\omega(I^{S-1}).$$

We can now proceed by induction, replacing S by s and showing that if
(21) holds for s, then

$$(22) \qquad p_{s-1} z_{s-1} \leq p_s z_s,$$

and (21) holds for $s - 1$.

Let $x^s = \beta \sum_{s'=1}^{S} \Pi_s \cdot \sum_{j'=1}^{s'} v_{j'}^{s'} \cdot \alpha_j \cdot (I_{t+1}^s)/\theta_s$. Since $I^1 \geq I^2 \geq \dots \geq I^S$
it follows that $x^1 \geq x^2 \geq \dots \geq x^S$. Since $z_j = \sum_{s=j}^{S}(\Pi_s/q_j)x^s$, it
follows that $z_1 \geq z_2 \dots, \geq z_S$. This and (22) implies $p_{s-1} \leq p_s$.
Finally since $p_{s-1} \leq p_s$ and $x^{s-1} \geq x^s$ it follows that $r_{s-1} \geq r_s$. $\square$

We now show that if $\omega \leq 1$, then in a temporary equilibrium all markets which open are active. We later show that, once expectations are endogenous, there exists a stationary equilibrium with the property $\omega \leq 1$. The intuition is that in a stationary equilibrium, increasing output by one unit will lead to an increase in consumption by at most one unit in the future. Therefore, the present value of the addition to consumption is less than a unit. Since the marginal cost is equal to the expected present value of the additional consumption, it must be less than unity.

Lemma 2: If $\omega \leq 1$, then in a temporary equilibrium $1/p_j \geq \beta z_j$ for all $j \leq S$ and all markets which open are active.

Proof: Suppose first that market s-1 is not active. Then it must be the case that market s is not active. To show this claim, note that by Lemma 1, $p_s z_s$ increases with s and since the buyer will spend only if $\beta p_s z_s \leq 1$, it follows that if buyers do not spend in market s-1, they do not spend in market s.

Next, we show that if markets $j > s$ are not active, then $p_j z_j = (1 - \delta)\omega(I^s)$ is a constant for all $j \geq s$. To show this claim, note that if markets $j > s$ are not active, then $I^s = I^{s+1} = \ldots = I^S$. Condition (12') can therefore be written as:
$[p_{j-1}z_{j-1} - (1-\pi_j)(1 - \delta)\omega(I^s)]/\pi_j = p_j z_j$. A solution to these equations is:

(23) $\qquad p_{j-1}z_{j-1} = (1 - \delta)\omega(I^s), \quad$ for all $j > s$.

Without limiting generality, we assume that there exists an index $s \leq S$ such that all markets $j \leq s$ are active if they open

($d_j > 0$) and all markets $j > s$ are not active even when they open. For all $j \leq s$, (14') implies that $1/p_j \geq \beta z_j$.

For $j > s$, (23) and $\omega \leq 1$ imply

$1/p_j = z_j/(1 - \delta)\omega(I^s) > z_j$. Thus $1/p_j \geq \beta z_j$, and condition (14') implies that $d_j > 0$, for all $1 \leq j \leq S$. $\quad\square$

The implication of Lemma 2 is that if the household wants to save it uses purely speculative inventories (supply to market S+1) and not money. Thus, inventories weakly dominate money.

We now endogenize the functions α_s and ω. We look for a vector of functions ($p_1(I)$, ..., $p_S(I)$, $L(I)$, $k_1(I)$, ..., $k_S(I)$, $k_{S+1}(I)$, $d_1(I)$, ..., $d_S(I)$, $\alpha_1(I)$, ..., $\alpha_S(I)$, $\omega(I)$, $z_1(I)$, ..., $z_S(I)$) that satisfies identities (3)-(5), arbitrage conditions (11')-(14'), market-clearing conditions:

(15') $\qquad d_s(I_t)/p_s(I_t) = k_s(I_t)$ for all $s \leq S$;

And for all I_t,

(24) $\qquad \alpha_j(I_t) =$
$= \max\{1/p_j(I_t) , \beta\sum_{s=j}^{S}(\Pi_s/q_j)\sum_{s'=1}^{S}\Pi_{s'}\sum_{j'=1}^{s'}\upsilon_{j'}^{s'}\alpha_{j'}(I_{t+1}^s)/\theta_s\}$;

(25) $\qquad v'(L(I_t)) = \omega(I_t)$;

And

(26) $\qquad z_j(I_t) = \sum_{s=j}^{S}(\Pi_s/q_j)\sum_{s'=1}^{S}\Pi_{s'}\sum_{j'=1}^{s'}\upsilon_{j'}^{s'}\alpha_{j'}(I_{t+1}^s)/\theta_s$.

We call such a vector a _stationary equilibrium_. Note that

$$\alpha_j(I_t) = \max\{1/p_j(I_t) , \beta z_j(I_t)\}.$$

We use the following notation:

p_{min} = the lowest possible price in the first market;

z_{max} = highest expected purchasing power of a dollar earned in the first market;

I_{max} = the maximum amount of inventories;

p' = a cutoff price: if the price in the first market is below p' the entire new output is sold in the first market.

Since $M_t/M_{t+1} = 1/\theta$, the money supply this period will constitute on average $E(1/\theta)$ of the next period money supply, a normalized dollar earned in the first market will become on average $E(1/\theta)$ normalized dollars next period.

If the lowest possible price in the first market is p_{min} the expected purchasing power of $E(1/\theta)$ normalized dollars must be smaller than: $z_{max} = E(1/\theta)/p_{min}$, because the probability that the dollars will be spent in the first market is less than unity and prices in other markets are higher. Suppose that inventories cannot exceed I_{max}, then we can define $\bar{p}_{min}$, and $\bar{z}_{max}$ as the solution to:

(27) $v'(y)/\beta z_{max} = p_{min}$; $\theta_1/p_{min} = Y + I_{max}$; $z_{max} = E(1/\theta)/p_{min}$.

Note that θ_1/p is the lowest total demand, $v'(y)/\beta z_{max}$ is the lowest marginal cost schedule. Given z_{max} and I_{max}, a solution to the first two equations in (27) is a lower bound on the price in the

first market. This is because not all the supply goes to the first market.

We now define the cutoff price $\bar{p}'$ by the solution to:

(28) $\qquad\qquad v'(x)/\beta z_{max} = p'; \quad (b) \quad \theta_1/p' = x.$

Note that when the price in the first market is $p \leq \bar{p}'$ the amount produced is always sold and therefore inventories do not grow.

We define the maximum amount of inventories by:

(29) $\qquad\qquad\qquad \theta_S/p' = I_{max}.$

We can now define: $\bar{p}_{min}$, $\bar{p}'$, $\bar{I}_{max}$, $\bar{z}_{max}$ as a solution to (27) - (29). These definitions are illustrated by Figure 1.
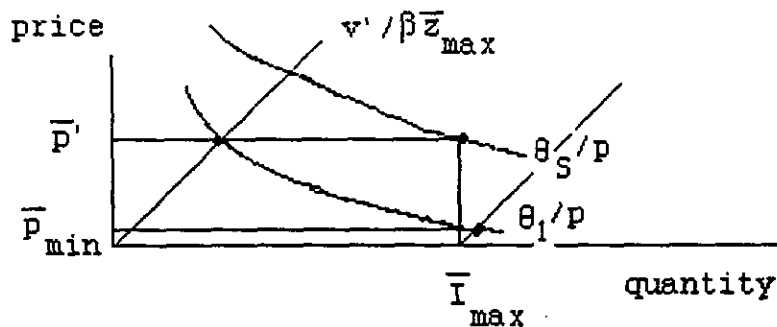

Figure 1

When $\omega \leq 1$, the following condition insures that inventories do not exceed I_{max}:

(30) $\qquad\qquad\qquad \bar{p}'\bar{z}_{min} > (1 - \delta).$

This condition implies that there is no supply to market S+1 when the price in the first market is above $\bar{p}'$. To show this we use the assumption that the highest marginal cost is unity. Therefore, the largest possible benefit from a unit in market S+1 (speculative inventories) is: $(1 - \delta)$. When the price in the current period's first market is $\bar{p}'$ prices in other current markets are higher (see Lemma 1) and the benefit from selling a unit in any current market is higher than the left hand side of (30). This means that when the first market price is higher than $\bar{p}'$, $k_{S+1} = 0$.

Thus, whenever speculation occurs, $p \leq \bar{p}'$ and Figure 1 implies that demand in the first market exceeds the entire current output and therefore, $I_{t+1} < I_t$. When the price in the first market is $p > \bar{p}'$ there is no speculation and therefore Lemma 1 and the market clearing conditions imply that the total supply over all markets must be smaller than θ_S/p and, using (29), this is less than I_{max}.

To prove existence of a stationary equilibrium, we consider the following set of functions:

$A = \{A = (\alpha_1, \ldots, \alpha_S, \omega) : \alpha_s \ (s = 1, \ldots, S)$ from $[0, \bar{I}_{max}]$ to $[0, 1/\bar{p}_{min}], \ \omega$ from $[0, \bar{I}_{max}]$ to $[0, 1]$ all are continuous and differentiable almost everywhere with $0 < \alpha_s^{'} \leq 1/\{\min_j (\theta_j - \theta_{j-1})\}$ and $-\sup(v^{"}) \leq \omega' < 0\}$.

<u>Theorem:</u> There exists a stationary equilibrium with $[\hat{\alpha}_1(I), \ldots, \hat{\alpha}_S(I), \hat{\omega}(I)] \in A$.

The outline of the proof is as follows: We choose functions from A and $I_t \in [0, I_{max}]$. We then solve for a temporary equilibrium. In particular, we relate r_s and $v'(\ )$ to I_t. We show that these

relationships define functions which are in A. Thus we create a mapping from A into itself. Since A is compact (by Ascoli's theorem) and convex, we can use Schauder's fixed point theorem to argue that the mapping has a fixed point, $\hat{A}$. Thus there exists a stationary equilibrium with $[\hat{\alpha}_1(I), \ldots, \hat{\alpha}_S(I), \hat{\omega}(I)] = \hat{A}(I)$.

Proof: We pick $A \in A$ and choose the price in the last market, p_S, arbitrarily. We compute prices and the demand in all $S + 1$ markets as a function of p_S and A under the assumption that the arbitrage conditions are satisfied. In particular, arbitrage condition (11') allows us to compute $p_1 z_1$ and therefore the supply decision (10'). This leads to supply and demand schedules as a function of p_S and their intersection is a temporary equilibrium.

We start with an algorithm to compute demand and prices. The first step is to deal with speculative inventories. Then we enter a recursion which at each stage s, starts with p_i, d_i, I^i_{t+1} and z_i for $i > s$, and computes p_s, d_s, I^s_{t+1} and z_s.

Speculative inventories must satisfy the Kuhn-Tucker condition (13'). We use (26) to write

$$(31) \qquad z_S = \sum_{s'=1}^{S} \Pi_s \cdot \sum_{j'=1}^{s'} \upsilon_{j'}^{s'} \alpha_j \cdot ((1-\delta) k_{s+1}))/\theta_S,$$

and we use the function $\omega(\ )$ for the next period's marginal cost, to rewrite (13') as:

$$(32) \qquad p_S \sum_{s'=1}^{S} \Pi_s \cdot \sum_{j'=1}^{s'} \upsilon_{j'}^{s'} \alpha_j \cdot ((1-\delta) k_{s+1}))/\theta_S \geq (1 - \delta) \omega((1-\delta) k_{s+1})$$

with equality if $k_{s+1} > 0$.

We denote the solution to (32) by $k_{s+1}(p_S; A)$. Whenever

$(1-\delta)k_{s+1}(p_S; A) > I_{max}$, we set: $k_{s+1}(p_S; A) = \bar{I}_{max}/(1-\delta)$. We use

$$z_S(p_S; A) = \sum_{s'=1}^{S} \Pi_s \cdot \sum_{j'=1}^{s'} v_j^{s'} \cdot \alpha_j \cdot ((1-\delta)k_{s+1}(p_S; A)))/\theta_S.$$

We now use (14'), p_S and $z_S(p_S; A)$ to compute the demand for

market S. This is denoted by $d_S(p_S; A)$. The level of inventories if

exactly S - 1 markets are opened is:

(33) $I^{S-1}(p_S; A) = \min\{(1-\delta)[(d_S(p_S; A)/p_S) + k_{s+1}(p_S; A)], \bar{I}_{max}\}.$

Thus, ignoring depreciation, inventories equal to the demand in the

markets that are not opened unless they are larger than I_{max}. Using

(33), we compute:

(34) $z_{S-1}(p_S; A) = (\Pi_{S-1}/q_{S-1})\sum_{s'=1}^{S} \Pi_s \cdot \sum_{j'=1}^{s'} v_j^{s'} \cdot \alpha_j \cdot (I^{S-1}(p_S; A))/\theta_{S-1}$

$+ (\Pi_S/q_{S-1})\sum_{s'=1}^{S} \Pi_s \cdot \sum_{j'=1}^{s'} v_j^{s'} \cdot \alpha_j \cdot ((1-\delta)k_{s+1}))/\theta_S.$

We can now compute p_{S-1}, from the arbitrage condition (12'):

(35) $p_{S-1}z_{S-1}(p_S; A) = \pi_S p_S z_S(p_S; A) + (1-\pi_S)(1-\delta)\omega(I^{S-1}(p_S; A)).$

We denote the solution to (35) by $p_{S-1}(p_S; A)$. By reapplying, (14')

and (33) - (35) we compute $p_{S-2}(p_S; A)$.

In general, given $p_s(p_S; A)$ and $z_s(p_S; A)$ we use (14') to

compute $d_s(p_S; A)$. We then use $p_i(p_S; A)$ and $d_i(p_S; A)$ for all

i > s-1, to compute:

(33') $I^{s-1}(p_S; A) =$

$\min\{(1-\delta)[\sum_{i>s-1} (d_i(p_S; A)/p_i(p_S; A)) + k_{s+1}(p_S; A)], \bar{I}_{max}\}.$

Using the computation of $I^s(p_S; A)$ for $s \geq j-1$ we get:

$$(34') \quad z_{j-1}(p_S; A) = \sum_{s=j-1}^{S} (\Pi_s/q_{j-1}) \sum_{s'=1}^{S} \Pi_s \cdot \sum_{j'=1}^{s'} v_{j'}^{s'} \cdot \alpha_j \cdot (I^s(p_S; A)))/\theta_s.$$

We can now compute $p_{s-1}(p_S; A)$ by:

$$(35') \qquad p_{s-1}z_{s-1}(p_S; A) = \pi_s p_s(p_S; A)z_s(p_S; A)$$
$$+ (1-\pi_s)(1-\delta)\omega(I^{s-1}(p_S; A)).$$

Lemma 3: $p_s(p_S; A)$ and $p_s(p_S; A)z_s(p_S; A)$ are both strictly increasing functions.

Proof: Let p_S increase. Since the α_s are increasing and ω is decreasing, the solution to (32), $k_{s+1}(p_S; A)$, is (weakly) decreasing in p_S. If k_{s+1} strictly decreases, then (32) holds with equality and since $\omega(\ )$ is strictly decreasing the value of $p_s z_s$ (the left hand side of (32)) must strictly increase. If k_{s+1} does not change, then z_s does not change and therefore $p_s z_s$ must strictly increase.

From (14') we know that d_S cannot increase. Using (33) we conclude that $I^{S-1}(\ )$ decreases and therefore $\omega(I^{S-1})$ increases. Using (35) we obtain that $p_{S-1}z_{S-1}$ increases. From (34) we infer that z_{S-1} decreases so that p_{S-1} must increase. The argument is then repeated for $S-2, S-3, \ldots, 1.$ $\square$

The aggregate demand is given by:

$$(36) \qquad d(p_S; A) = k_{s+1}(p_S; A) + \sum_s (d_s/p_s).$$

When p_S increases, all $p_S z_S$ increase (Lemma 3) and therefore (14') implies that d_S cannot increase. Since Lemma 3 also implies that all prices increase, it follows that the right hand side of (36) cannot increase. This implies that d( ) is a decreasing function.

The optimal labor supply is determined by:

(37) $$v'(L) = \beta p_1(p_S; A) z_1(p_S; A).$$

From this we get:

(38) $$L(p_S; A).$$

Since Lemma 3 implies that $p_1 z_1$ strictly increases as a function of p_S, the solution to (38) increases. This implies that L( ) is a strictly increasing function.

The total supply is:

(39) $$k(p_S, I_t; A) = L(p_S; A) + I_t.$$

Market clearing conditions require:

(40) $$d(p_S; A) = k(p_S, I_t; A).$$

Because $\bar{z}_{max}$ is finite, (14') implies that the nominal demand in the first market is strictly positive for sufficiently small values of p_1. Thus d( ) does not intersect the quantity axis. Since the supply is strictly increasing and d( ) is decreasing, this guarantees a unique solution to (40), which is denoted by $\hat{p}_S(I_t; A)$.
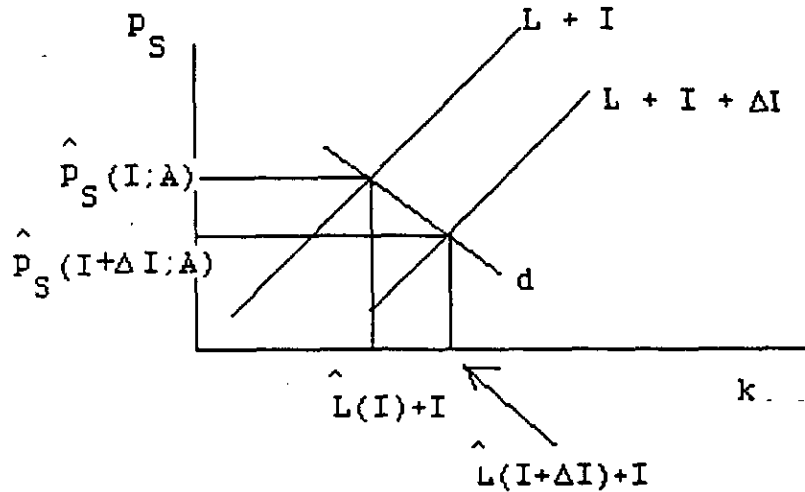
This is illustrated by Figure 2.

Figure 2

We now solve for the expected purchasing power of a normalized dollar held by buyers in market j, and the expected purchasing power of a normalized dollar earned by a seller in market j. These are:

(41)
$$\hat{r}_j =$$
$$= \max\{1/\hat{p}_j \ , \ \beta\Sigma_{s=j}^{S}(\Pi_s/q_j)\Sigma_{s'=1}^{S}\Pi_s \cdot \Sigma_{j'=1}^{S'}v_j^{s'} \cdot \alpha_j \cdot (I_{t+1}^S)/\theta_s\}.$$

where $\hat{x} = x(\hat{p}_S(I_t; A), A)$.

We now demonstrate that if $0 \leq I_t \leq \bar{I}_{max}$, then $0 \leq I_{t+1} \leq \bar{I}_{max}$. Lemma 2 implies that market 1 is always active and therefore:

(42)
$$I_{t+1} \leq I_t + \hat{L} - \theta_1/\hat{p}_1,$$

(With depreciation the inequality is strict.) It follows from the definition of $\bar{p}'$ that if $\hat{p}_1 < \bar{p}'$ then

$\hat{L} - \theta_1/\hat{p}_1 \leq 0$ and therefore (42) implies

$I_{t+1} \leq I_t$. If $\hat{p}_1 \geq \bar{p}^{\cdot}$ there is no speculation and therefore Lemma 1 implies:

(43) $\qquad\qquad\qquad d(\hat{p}_S; A) \leq \theta_S/\hat{p}_1 \leq \theta_S/\bar{p}^{\cdot} = \bar{I}_{max}.$

This and (40) leads to $k(\hat{p}_S, I_t; A) \leq \bar{I}_{max}$, which implies $I_{t+1} \leq \bar{I}_{max}$.

Thus, we solved for a temporary equilibrium for each choice of $A \in \mathcal{A}$ and for given I_t and showed that at the solution of the temporary equilibrium, the end-of-period inventories (I_{t+1}) cannot exceed $\bar{I}_{max}$.

We now turn to show that $[\hat{r}_1, \ldots, \hat{r}_S, v^{\cdot}(\hat{L})] \in A$. Since Lemma 2 says that all markets which open are active $d( )$ is continuous and differetiable almost everywhere. The same is true for the supply $k( )$ and therefore $\hat{p}_S(I_t; A)$ is continuous and differetiable almost everywhere. This implies that $\hat{r}_s = 1/p_s(\hat{p}_S(I_t; A); A)$ is continuous and differetiable almost everywhere.

We now show that $0 \leq \hat{r}_s \leq 1/\bar{p}_{min}$. The lower bound is obvious. For the upper bound notice that from (27) and from $I_t \leq \bar{I}_{max}$ it follows that $\hat{p}_1 \geq \bar{p}_{min}$. By Lemma 2, $\hat{r}_s = 1/\hat{p}_s$. Using Lemma 1 and $\hat{p}_1 \geq \bar{p}_{min}$ leads to: $1/\hat{p}_s \leq 1/\hat{p} \leq 1/\bar{p}_{min}$.

We now show that $v^{\cdot}(\hat{L}) \leq 1$. Since Lemma 2 says that in a temporary equilibrium all markets which open are active, it follows that $\beta\hat{p}_1\hat{z}_1 \leq 1$. Since $v^{\cdot} = \beta\hat{p}_1\hat{z}_1$ it follows that $v^{\cdot} \leq 1$.

We now show that the derivatives of $\hat{r}_s$ and $v^{\cdot}(\hat{L})$ with respect to I have the same bounds as the derivatives of α_s and ω. For this purpose let $\hat{L}(I) = L(\hat{p}_S[I; A]; A)$ denote the level of temporary equilibrium output when inventories are I. Note that an increase in

inventories by dI units will move the supply schedule k( ) to the right by exactly dI units (see Figure 5). Since d( ) is decreasing, $\hat{L}(I+dI) < \hat{L}(I) + dI$, and the increase in the temporary equilibrium level of supply, $dL = \hat{L}(I+dI) - \hat{L}(I)$, is smaller than dI. Thus

$$(44) \qquad -1 < \frac{dL}{dI} < 0 \text{ and } 0 < \frac{dk}{dI} < 1.$$

Since we showed that all prices are monotonically related to p_s, the supply to all markets must increase and therefore

$$(45) \qquad 0 < \frac{dk_s}{dI} < 1.$$

Using Lemma 2, we can write the market clearing condition as:

$$(46) \qquad (\theta_s - \theta_{s-1})r_s = k_s.$$

Differentiating (46) with respect to I leads to:

$$(47) \qquad 0 \leq (d\hat{r}_s/dI) = \frac{dk_s}{dI}/(\theta_s - \theta_{s-1}) \leq 1/(\min_j(\theta_j - \theta_{j-1})).$$

The condition $-1 < \frac{dL}{dI} < 0$, implies that

$$-\sup[v''] \leq \frac{dv'}{dI} = v''(\hat{L}(I))\frac{dL}{dI} \leq 0.$$

We have shown that $\hat{A}$ is in A and that I_t is in the domain of A. Thus we have mapped A into itself. Since A is convex and, by Ascoli's theorem, compact, we can use Schauder's fixed-point theorem to argue that the mapping has a fixed point. Thus there exists a stationary equilibrium with $[\hat{r}_1(I), ..., \hat{r}_S(I), v'(\hat{L}(I))] \in A.$ ☐

CONCLUDING REMARKS

The data generated by our model may look as if prices are initially rigid and then overshoot. A low realization of θ does not affect prices initially (in the period it occurs). But prices decline more than predicted by the naive quantity theory in the following period because the accumulated inventories reduce normalized prices.

It is natural to define real balances in our model as the money supply divided by an index of qouted prices in all markets, including prices in markets which do not open.[1] A low realization of θ is first assoicated with a decline in real balances because the ex-transfer money supply is relatively low. At the beginning of next period, inventories rise and normalized prices decline. This means that real balances go up. Thus, a monetary contraction is associated with a decline in real balances which is followed by an increase in real balances. Similarly, the initial reduction in consumption which occurs when the realization of θ is low, is followed, on average, by an increase in consumption.

Gertler and Gilchrist (1994), in an extensive study, observed that inventories tend to rise after a monetary contraction (a "Romer date") before returning to a steady-state level. The buildup of inventories which follows a monetary contraction tends to be larger and the adjustment seems to last longer in large firms than in smaller ones.

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[1] The consumer price index attempts to measure actual price offers and does not restrict attention to goods that were actually sold.

In our model, the behavior of inventories in response to an unanticipated monetary contraction (low realization of θ) is similar to that described in general terms by Gertler and Gilchrist (1994). Their main emphasis is on the different behavior of small and large firms with the latter having bigger and longer lasting deviations of inventories from their course. We believe that our model, with the proper extension, may also accomodate this difference. For if households have different time preference parameters, one would expect that the less patient households will supply markets which are more likely to open.[1] The missing link will then be to show that large firms are more likely to be the more patient ones. But this is plausible if large firms have easier access to credit markets.

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[1] A high discount rate makes inventories less valuable and therefore high-indexed markets relatively less attractive. The same holds for carrying costs. Eden and Horowitz (1994) show that in equilibrium, goods with higher carrying costs are supplied to lower indexed markets.

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