

# Indirect Estimation of Multifactor Continuous Time Term Structure Models\*

Vance L. Martin  
University of Melbourne

Adrian R. Pagan  
Australian National University

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## Abstract

Procedures for computing the parameters of a broad class of multifactor continuous time models of the term structure based on indirect estimation methods are proposed. The approach consists of simulating the unknown factors from a set of stochastic differential equations which are used to compute synthetic bond yields. The bond yields are calibrated with actual bond yields via an auxiliary model. The approach circumvents many of the difficulties associated with direct estimation of this class of models using maximum likelihood. In particular, the paper addresses the identification issues arising from singularities in the yields and spreads which tend not to be recognised in existing estimation procedures and thereby overcome potential misspecification problems inherent in direct methods. Indirect estimates of single and multifactor models are computed and compared with the estimates based on existing estimation procedures.

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\*All computations are performed using GAUSS 3.2. The optimization algorithm is BFGS in OPTMUM.

# 1 Introduction

This paper provides an alternative estimation strategy for computing parameter estimates of multifactor models of the term structure of interest rates for a broad class of specifications of the underlying set of stochastic differential equations (*SDE*) which drive the factors. The approach is to exploit the relationship that exists between factors and the term structure identified by Cox, Ingersoll and Ross (1985) (CIR hereafter), by using the indirect estimation methods of Gallant and Tauchen (1994) and Gouriéroux, Monfort and Renault (1993) to simulate the factors from the underlying SDEs to derive simulated bond yields which are calibrated with actual yields via an auxiliary model.

For the class of factor models adopted by CIR, as analytical expressions between the factors and bond yields exists, the simulated bond yields can be easily computed. For other specifications of the *SDEs*, in general analytical solutions of the CIR pricing equation do not exist. Given that there is strong evidence to suggest that the square root formulation of CIR is not always consistent with the data, for example Chan *et. al.* (1992), Brenner, Harjes and Kroner (1994), Koedijk, Nissen and Schotman and Wolff (1995) and Pagan, Hall and Martin (1995), a suitable extension is to allow for a general levels effect specification in the variance. The cost of adopting this generality is that no analytical expression relating the factors to bond yields exists; however solutions can be obtained numerically by using finite difference methods for solving ordinary and partial differential equations.<sup>1</sup>

Further generalizations of the *SDEs* are accommodated by the indirect estimator. The two-factor models derived by Longstaff and Schwartz (1992) and Edmister and Madan (1993), where one of the factors exhibits stochastic volatility, is easily handled as closed form solutions of the price of the asset exist. For non-independent multifactor models of the type considered by Duffie and Kan (1993), indirect estimation is also feasible as the price has an exponential affine form, although it is necessary to resort to numerical finite difference methods for solving a system of ordinary differential equations to

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<sup>1</sup>An exception is when there is no levels effect in the variance; see Jamshidian (1989). Given that the empirical evidence for a levels effect is overwhelming, this case does not appear to be especially interesting.

derive the weights in the pricing formula.

An advantage of the proposed indirect estimator is that it circumvents the singularity problems inherent in processes underlying the term structure.<sup>2</sup> The direct application of maximum likelihood methods as adopted by Brown and Dybvig (1986) for one-factor models, and Chen and Scott (1992, 1993) and Pearson and Sun (1994) in the case of multifactor models, requires specifying a density function for the factors. For the CIR class of term structure models where it is assumed that the factors follow square root processes, the appropriate density is the non-central chi-square distribution. As the dimension of the factors is less than the dimension of the set of yields belonging to the term structure, the approach consists of including a set of dummy measurement errors to enable the likelihood to be transformed from the unknown factors and measurement errors to the known interest rates. This is a potential source of misspecification and possibly explains the misspecification errors identified by Brown and Dybvig in estimating a one-factor model.

The rest of the paper proceeds as follows. An indirect estimator is outlined and applied in Section 2 to estimating single factor and multifactor CIR models of the term structure. These results provide a comparison with the direct estimation results of Chen and Scott (1993). The proposed algorithm, while it recognises the singularity problems that exist in term structure models, it does not take into account all of the potential singularities. This is rectified in Section 3 where the relationship between the singularity problem and the number of estimable parameters is made explicit. In doing so, issues such as testing the number of factors, and the relationship between the indirect estimator and the factor analysis approach adopted by Knez, Litterman and Scheinkman (1989), as well as by many others, are discussed. Procedures for performing indirect estimation of the parameters of more general multivariate continuous time factor processes underlying the term structure are outlined in Section 4. Section 5 provides some concluding comments

## 2 Estimating the CIR Class

This section provides a procedure for estimating multifactor term structure models where the factors follow square root processes and are independent. The proposed indirect estimator, while it does not take into account all of

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<sup>2</sup>The singularity problem at both empirical and theoretical levels is well documented; see Pagan, Hall and Martin (1995) for a recent review.

the potential singularities in term structure models, it does serve as a basis for comparison with the direct, maximum likelihood estimator. An indirect estimator which takes into account the “full” singularity features of term structure models is left for Section 3.

## 2.1 Basic definitions

Let  $Y_i(t)$ ,  $i = 1, 2, \dots, K$ , be a set of  $K$  factors at time  $t$ . The diffusion processes governing the movements of  $Y_i(t)$  over time are given by

$$dY_i(t) = \kappa_i (\theta_i - Y_i(t)) dt + \sigma_i \sqrt{Y_i(t)} dW_i(t) \quad (1)$$

where  $W_i(t)$  is a Wiener process with the property that  $dW_i(t) \sim N(0, dt)$ . For the moment the  $dW_i(t)$  are assumed to be independent. The parameter  $\kappa_i$  represents mean reversion, the long-run value of the  $i^{\text{th}}$  factor is given by  $\theta_i$ , and  $\sigma_i^2$  is the variance.

The price of a discount bond at time  $t$  that matures at time  $T$ ,  $P(t, T)$ , is determined by solving the partial differential equation

$$\frac{1}{2} \sum_{i=1}^K \sigma_i^2 Y_i(t) \frac{\partial^2 P}{\partial Y_i \partial Y_j} + \sum_{i=1}^K (\kappa_i \theta_i - \kappa_i Y_i(t) - \lambda_i Y_i(t)) \frac{\partial P}{\partial Y_i} + \frac{\partial P}{\partial t} - rP = 0 \quad (2)$$

subject to the boundary condition that the bond pays \$1 at the time of maturity  $T$

$$P(T, T) = 1 \quad (3)$$

The term  $\lambda_i Y_i$ , represents the risk premium, and  $r$  the nominal rate of interest which is also a function of the  $K$  factors.

The solution of (2) is derived by CIR, and has the exponential form

$$P(t, T) = \prod_{i=1}^K A_i \exp \left[ - \sum_{i=1}^K B_i Y_i(t) \right] \quad (4)$$

where

$$A_i = \left[ \frac{2\gamma_i \exp [0.5\tau (\kappa_i + \lambda_i + \gamma_i)]}{2\gamma_i + (\kappa_i + \lambda_i + \gamma_i) (\exp [\gamma_i \tau] - 1)} \right]^{\frac{2\kappa_i \theta_i}{\sigma_i^2}} \quad (5)$$

$$B_i = \left[ \frac{2 (\exp [\gamma_i \tau] - 1)}{2\gamma_i + (\kappa_i + \lambda_i + \gamma_i) (\exp [\gamma_i \tau] - 1)} \right] \quad (6)$$

and  $\tau = T - t$ , is the time to maturity, and  $\gamma_i = \sqrt{(\kappa_i + \lambda_i)^2 + 2\sigma_i^2}$ .

## 2.2 The Algorithm

To compute the parameters  $\phi = \{\kappa_i, \theta_i, \lambda_i, \sigma_i; i = 1, 2, \dots, K\}$ , by indirect estimation the following steps are adopted. In outlining the algorithm, calibration is chosen to be based on actual and simulated yields. An alternative approach is to use either bond prices or the log of bond prices, where the latter form is consistent with the transformation adopted by Chen and Scott (1993) in converting the unobserved variables to observed variables. In the case of zero coupon bonds, the relationship between prices and yields is simply

$$R_t = -\ln(P_t)/\tau \quad (7)$$

where  $\tau$ , is the maturity.

1. An initial set of parameter estimates is chosen:  $\phi^{(0)} = \{\kappa_i^{(0)}, \theta_i^{(0)}, \lambda_i^{(0)}, \sigma_i^{(0)}; i = 1, 2, \dots, K\}$ .
2. The *SDEs* in (1) are simulated using an Euler approximation

$$Y_{i,t+\Delta t} = Y_{i,t} + \kappa_i^{(0)} (\theta_i^{(0)} - Y_{i,t}) \Delta t + \sigma_i^{(0)} \sqrt{Y_{i,t}} u_{i,t} \quad (8)$$

where  $u_{i,t} \sim N(0, \Delta t)$ , to generate simulated time series on the  $K$  factors,  $Y_{i,t}$ ,  $i = 1, 2, \dots, K$ . The length of the time step is  $\Delta t$  which is chosen to be small.<sup>3</sup>

3. A set of  $N > K$ , simulated bond prices  $P_t^S(\tau_j)$ , with maturities  $\{\tau_1, \tau_2, \dots, \tau_N\}$ , are computed using (4) to (6)

$$P_t^S(\tau_j) = \prod_{i=1}^K A_i^{(0)} \exp \left[ - \sum_{i=1}^K B_i^{(0)} Y_{i,t} \right] \quad (9)$$

where the  $A_i^{(0)}$  and  $B_i^{(0)}$  terms signify that the parameters in the expressions (5) and (6) respectively, are evaluated at  $\phi^{(0)}$ . These are converted into simulated yields using (7)

$$R_t^S(\tau_j) = -\ln(P_t^S)/\tau_j \quad (10)$$

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<sup>3</sup>A natural barrier is used to ensure that  $Y_{i,t+\Delta t} \geq 0$ . This is achieved by setting  $Y_{i,t+\Delta t} = \kappa_i \theta_i$ , whenever the right-hand side of (8) is negative.

4. An auxiliary model based on a *VAR* is estimated for actual bond yields  $R_t(\tau_j)$ , with maturities  $\{\tau_1, \tau_2, \dots, \tau_N\}$ . The first order conditions of this model satisfy

$$\frac{\partial Q}{\partial \beta} (R_t(\tau_j), \hat{\beta}) = 0 \quad (11)$$

where  $Q()$  is the likelihood function of a *VAR*, and  $\hat{\beta}$  is the vector of *VAR* parameters based on the actual yields.

5. The first order conditions of the auxiliary model are evaluated at the parameter estimates of the *VAR*, but using the simulated bond yield data  $R_t^S(\tau_j)$

$$\frac{\partial Q}{\partial \beta} (R_t^S(\tau_j), \hat{\beta}) \quad (12)$$

6. The parameters associated with the factors are calibrated to satisfy the criterion

$$\hat{\phi} = \underset{\phi}{\text{Argmin}} \frac{\partial Q}{\partial \beta'} (R_t^S(\tau_j), \hat{\beta}) \Omega \frac{\partial Q}{\partial \beta} (R_t^S(\tau_j), \hat{\beta}) \quad (13)$$

where  $\Omega$  is a variance-covariance matrix given in *Gourieroux, Monfort and Renault (1993)*. In the case where the simulations are repeated  $H$  times,  $\partial Q / \partial \beta$  is computed as an average over these simulated paths. The objective function in (13) is minimized using the algorithm *BFGS* in *OPTMUM*.

## 2.3 Application to Yields

The proposed indirect estimator is now applied to estimating single and multifactor models of the term structure. For comparability with the *Chen and Scott (1993)* results, similar zero coupon bond rates are used; namely the three month  $R_{1,t}$ , six month  $R_{2,t}$ , five year  $R_{3,t}$ , and a rate equal to an unweighted average of the ten, fifteen and twenty year yields  $R_{4,t}$ . Thus the maturities are  $\tau = \{0.25, 0.50, 5.0, 15.00\}$ . The yields are taken from *Shiller and McCulloch (1990)*. The data are monthly, beginning in December 1946 and ending in December 1987, a sample of 483 observations. The four yields are displayed in Figure 1.

The indirect estimation results for the 1-factor, 2-factor and 3-factor CIR models are given in Table 1 based on the algorithm given above. A four

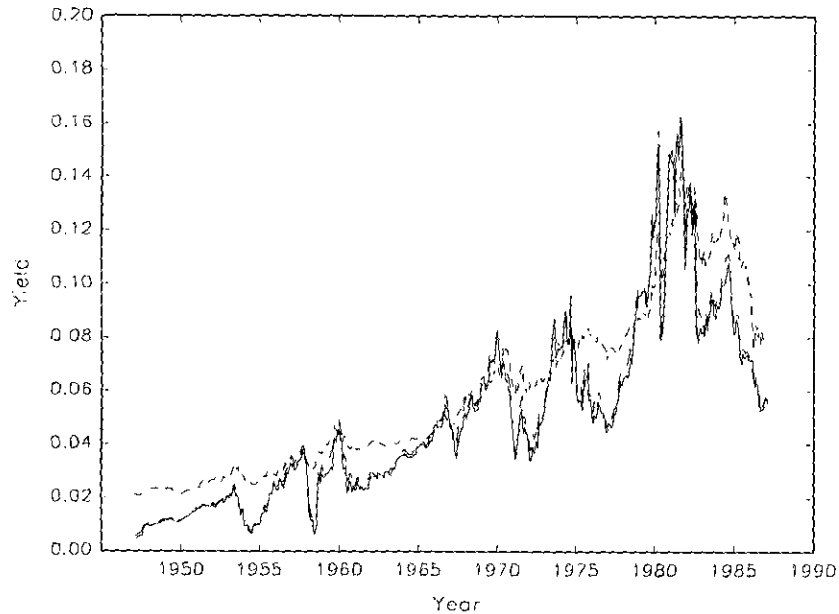


Figure 1: Yield data: December 1946 to December 1987

variate *VAR* with a constant and each interest rate lagged one period is chosen as the auxiliary model. Two variants of the auxiliary model are tried: one where there is an adjustment for time-varying variances by dividing each equation by the square root of the dependent variable, and another where there is no adjustment. The square root adjustment is motivated by the square root formulation of the *SDE* processes assumed to be driving the factors. As both auxiliary models yield similar results, only the results based on the square root adjustment are reported in Table 1.<sup>4</sup> The number of simulation paths is chosen as  $H = 100$ , and the time interval is set at  $\Delta t = 0.1$ . The initial value of the factors is computed as  $\min(R_{1,t}, R_{2,t}, R_{3,t}, R_{4,t}, t =$

<sup>4</sup>Another auxiliary model that would be appropriate would be to allow the *VAR* errors to exhibit a *GARCH* or even *EGARCH* structure following Pagan, Hall and Martin (1995) and Pastorello, Renault and Touzi (1994). This model has not been adopted as the strategy at this stage is to see how close the direct and indirect estimates are for the simplest of all auxiliary models.

1, 2, ..., T). The standard errors are computed using a Newey-West filter with 12 lags; see Gourieroux, Monfort and Renault (1993, pp.S112-3).<sup>5</sup>

Starting estimates used in the algorithm are based on the direct, MLE estimates reported by Chen and Scott (1993). For comparison these estimates are also given in Table 1. Overall the parameter estimates of the direct and indirect procedures are very similar in terms of sign, magnitude and statistical significance. Focussing on the 1-factor results, the main difference between the two sets of parameter estimates is the estimate of the risk premium parameter ( $\lambda_1$ ), which tends to be larger in magnitude using the indirect estimator than the direct estimator. However, inspection of the one-factor results shows that the indirect estimate of  $\kappa_1 + \lambda_1$ , the term which is important in pricing bonds, is  $0.4338 = 0.7016 - 0.2678$ , which compares favourably with the direct estimate of  $0.4243 = 0.4697 - 0.0454$ .

The direct and indirect parameter estimates of the 2-factor and 3-factor models are especially similar. In particular, both sets of results point a lack of additional information in the third factor. This suggests that the four yields used in the empirical analysis are captured arbitrarily well by at most two factors.

Overall these results are very encouraging as they show that it is quite simple to obtain parameter estimates of a multifactor *CIR* model without resorting to the use of a set of dummy errors to avoid the singularity problems inherent in the term structure when transforming from the unobserved variables to the observed variables.

### 3 Estimation with Singularities in the Spreads

A potential weakness with the indirect estimator proposed in the previous section is that it does not recognise all of the singularities in the model. From the expectations theory of the term structure, for a set of  $N$  interest rates there should be  $(N - 1)$  cointegrating vectors and one common trend. The cointegrating vectors are of the form that the cointegrating errors are

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<sup>5</sup>The weighting matrix in (13) is set at  $\Omega = I$ . Choosing  $\Omega$  as the optimal weighting matrix results in the algorithm getting stuck. This problem possibly reflects a more general problem, namely the use of gradient optimization routines to compute indirect estimates. This point is also observed by Pastorello *et. al.* (1993). While the use of a non-optimal weighting matrix causes some loss of efficiency in the parameter estimates, this loss should be small at least for the 3-factor model where the dimension of  $\phi$ , is similar to the dimension of the parameter vector of the auxiliary model, namely  $\beta$ .



the spreads. Further, the spreads in general, have a factor structure thereby reducing the dimension of the number of factors governing interest rates.

Letting  $S_t$  and  $R_t$ , denote vectors of spreads and interest rates respectively, the relationship between the two is

$$S_t = DR_t \quad (14)$$

where  $D$  is a  $(N - 1) \times N$  matrix of cointegrating vectors. In the case where there are  $M$  principal components  $Z_t$ , amongst the spreads, (14) can be written as

$$Z_t = AS_t = ADR_t \quad (15)$$

where  $A$  is a  $M \times (N - 1)$  matrix containing the normalized eigenvectors.

The structure of (15) suggests that it is appropriate to augment the indirect estimator algorithm as follows. First, the parameters of the auxiliary model are estimated using the principal components  $Z_t$  and not the actual yields or prices. This reduces the dimension of the VAR from  $N$  to  $M$ . Second, the simulated yields are converted into simulated principal components by using (15) with  $R_t$  replaced by  $R_t^S$ . The matrices  $D$ , and in particular,  $A$ , are the same for both actual and simulated data.

### 3.1 Singularities and Testing Factor Models

The structure of the term structure models discussed above has implications for performing tests on factor models. Consider the factor model investigated by Knez *et. al.* (1989)

$$Z_t = CY_t + \varepsilon_t \quad (16)$$

where  $Z_t$  is defined as above,  $Y_t$  is a set of  $K \leq M$  factors,  $C$  is  $M \times K$ , and  $\varepsilon_t$  is a vector of  $M$  independent errors with zero means.

The maximum number of parameters that can be estimated is  $M(M + 1)/2$ , which is the number of covariance terms of  $Cov(Z_t)$ . Now the total number of parameters in  $C$  that can be estimated is  $M \times K$ , which must satisfy

$$M \times K \leq M(M + 1)/2 \quad (17)$$

for identifiability. Putting  $K = M$ , with the exception of the one-factor model, shows that this inequality cannot be satisfied as it amounts to  $K \leq 1$ . Thus, for multivariate factor models some additional restrictions are needed. Letting  $L$  be the number of restrictions, then from (17) with  $K = M$

$$L \geq K(K - 1)/2 \quad (18)$$

In standard factor analysis, this is achieved by restricting  $C'C$  to a diagonal matrix as this results in  $K(K-1)/2$  restrictions. This implies that this class of models is exactly identified and therefore cannot be tested.

### 3.2 Application to Spreads

Table 2 contains the results of a principal components analysis on the spreads of eight Treasury bill rates. The maturities of the Treasury bills are 1 month, 2 month, 3 month, 4 month, 5 month, 6 month, 9 month and 1 year maturity, with the spreads being computed by subtracting the Federal funds rate from these rates. The data are monthly, beginning in January 1959 and ending in February 1991, a sample of 386 observations. This data set is chosen as it corresponds to a similar data set used by Knez, Litterman and Scheinkman (1989) who estimated a linear factor model for spreads. The eight spreads are displayed in Figure 2.

The normalized eigenvalues in Table 2 show that there is one dominant factor which explains 95.52% of the total sample variation. The second and third factors contribute an additional 3.6% and 0.67% respectively, to explaining total sample variation in the spreads.<sup>6</sup>

The structure of the normalized eigenvectors are displayed in Table 2. The first eigenvector represents the level of interest rates with all rates having roughly equal contribution. The second eigenvector has a bipolar representation which is possibly capturing the slope of the term structure. The third eigenvector displays the property that the greatest contribution comes from the two rates at opposite ends of the maturity spectrum, namely the one month and one year rates. The communalities in Table 2 show that the first principal component does not explain approximately 10% of variation in the one month and one year rates. Using the first two principal components the unexplained variance is reduced to between 1% and 3%, for these two rates. Using three principal components the unexplained variances are reduced to less than 0.4% for all rates. Given these results, and in keeping with the results of Knez *et. al.* (1989), factor models up to and including three factors, that is  $M = 3$  in (15), are estimated.

The indirect estimation results of single and multifactor models of the CIR class are given in Table 3. The maturity vector used to compute the eight

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<sup>6</sup>Estimates of the factor decomposition using the maximum likelihood factor analysis of Joreskog (1967) were also tried. These results are not reported as the algorithm was not able to find a local minimum for either the one, two or three factor models.

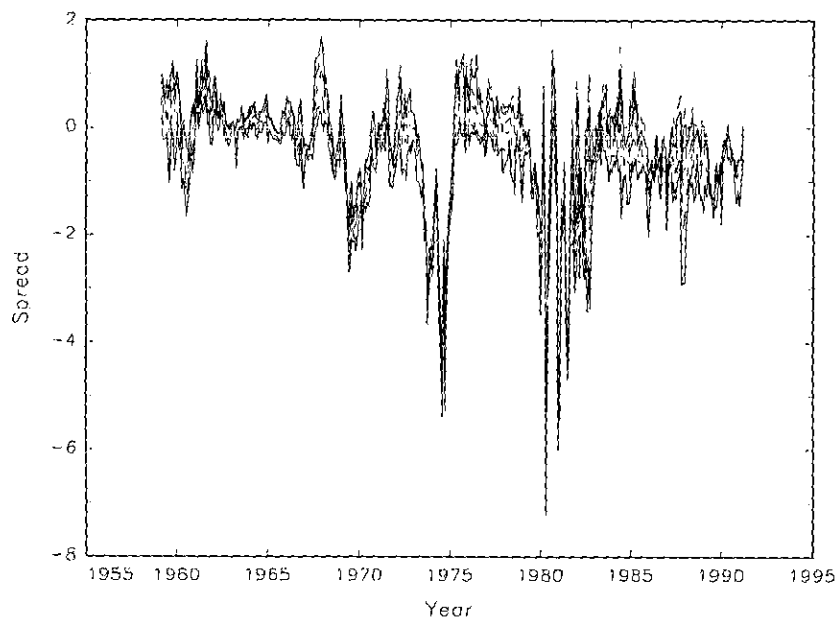


Figure 2: Spread data: January 1959 to February 1991

simulated bond rates is  $\tau = \{1/12, 2/12, 3/12, 4/12, 5/12, 6/12, 9/12, 1\}$ . The simulated spreads are computed by subtracting from each simulated bond rate a simulated rate with maturity of one day,  $1/252$ , so as to correspond to the Federal funds rate. The number of simulation paths is chosen as  $H = 100$ , and the time interval is  $\Delta t = 0.1$ . The initial value of the factors is chosen as the minimum value of the actual rates over the sample period. The auxiliary models are *VARs* with lag lengths of two for the two-factor and three-factor models, and an *AR(4)* for the one-factor model. All auxiliary models include a constant term.

A comparison of the parameter estimates for all factor model in Table 3 suggest that there is little information content contained in the second and third factors that is not already contained in the first factor.

## 4 Estimating a Broader Class of Models

### 4.1 Alternative variance specifications

The CIR class of factor models is attractive as analytical expressions for bond prices exist. To extend this class to allow for more general variance specifications it is necessary to resort to numerical solutions as analytical solutions are generally no longer available.

A generalization of the CIR square root model investigated by Chan *et al.* (1992) is to write the diffusion process governing movements in the factors as

$$dY_i(t) = \kappa_i (\theta_i - Y_i(t)) dt + \sigma_i Y_i(t)^{\alpha_i} dW_i(t) \quad (19)$$

where  $\alpha_i, i = 1, 2, \dots, K$ , is a set of additional parameter that need to be estimated. For the CIR class  $\alpha_i = 0.5, \forall i$ . The *pde* used to price bonds becomes

$$\frac{1}{2} \sum_{i=1}^K (\sigma_i^2 Y_i(t)^{2\alpha_i}) \frac{\partial^2 P}{\partial Y_i \partial Y_j} + \sum_{i=1}^K (\kappa_i \theta_i - \kappa_i Y_i(t) - \lambda_i Y_i(t)) \frac{\partial P}{\partial Y_i} + \frac{\partial P}{\partial t} - rP = 0 \quad (20)$$

For a one factor model  $K = 1$ , (20) can be solved using standard numerical procedures as applied to univariate parabolic partial differential equations. In this case (20) reduces to

$$\frac{1}{2} \sigma^2 Y(t)^{2\alpha} \frac{\partial^2 P}{\partial Y^2} + (\kappa \theta - \kappa Y(t) - \lambda Y(t)) \frac{\partial P}{\partial Y} + \frac{\partial P}{\partial t} - Y(t)P = 0 \quad (21)$$

where for simplicity the subscript  $i = 1$ , is dropped. The finite difference approximations to the derivatives are

$$\frac{\partial P}{\partial t} \simeq \frac{P_{i,j+1} - P_{i,j}}{k} \quad (22)$$

$$\frac{\partial P}{\partial Y} \simeq \frac{P_{i+1,j} - P_{i,j}}{h} \quad (23)$$

$$\frac{\partial^2 P}{\partial Y^2} \simeq \frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{h^2} \quad (24)$$

where  $k = dt$  and  $h = dY$ , and  $P_{i,j}$  represents the price at the  $j^{\text{th}}$  time step. Substituting (22) to (24) in (21) and rearranging gives an explicit representation of the *pde* which can be used to compute  $P_{i,j+1}$  given values

for  $P_{i-1,j}, P_{i,j}, P_{i+1,j}$ . For the implicit representation, the derivatives given by (23) and (24) are replaced by a simple unweighted average of the derivatives at time  $j$  and  $j + 1$ .<sup>7</sup>

To impose the end boundary condition that the bond price is \$1 at maturity, the partial differential equation is converted from time-space  $t$ , to maturity space  $\tau$ , by simply using the identity that  $\tau = T - t$ , to rewrite (21) as

$$\frac{1}{2}\sigma^2 Y(\tau)^{2\alpha} \frac{\partial^2 P}{\partial Y^2} + (\kappa\theta - \kappa Y(\tau) - \lambda Y(\tau)) \frac{\partial P}{\partial Y} - \frac{\partial P}{\partial \tau} - Y(\tau)P = 0 \quad (25)$$

The initial values are now chosen as  $P_{i,0} = 1$ , to ensure that the price of a bond at maturity is \$1. Finally, as there are no restrictions on the price of the bond at the boundary points  $P_{0,j}$ , where the factor equals zero, and  $P_{L,j}$  where  $L$  is chosen to be large to constitute a large value of the factor, it is appropriate to specify the boundary conditions in the finite difference solution as derivatives

$$-P_{0,j} = \frac{P_{1,j} - P_{-1,j}}{h} \quad (26)$$

$$(27)$$

$$-P_{R,j} = \frac{P_{L+1,j} - P_{L-1,j}}{h} \quad (28)$$

The use of a finite difference approximation to solve the *pde* is likely to be very slow as the grid of values of  $(Y, \tau)$  can be very large. For example, if the range of the factor is chosen as  $Y \in [0, 1]$ , and is  $\tau \in [0, 15]$ , for a bond with a maximum maturity of 15 years, the grid mesh is  $(1 + 1/h)(1 + 15/k)$ . Choosing  $h = k \approx 0.01$ , the number of grid points is 151601. Experience also shows that for unbounded problems, the numerical solution can become unstable at the endpoints as  $\tau$  increases. Potentially this problem can be circumvented by using a transformation to convert the price to lie in a closed interval following the methods suggested by Duffie and Kan (1993).

For multifactor models where there are no analytical solutions for the price, finite difference solutions are also possible. The price of a two-factor

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<sup>7</sup>Unlike the explicit method, the implicit solution requires that all values of  $P_{i,j+1}$  at the  $j + 1$  time step be computed simultaneously. The system of equations is tridiagonal which can be solved using efficient algorithms. The advantage of the implicit method is that it is globally stable for all values of  $h$  and  $k$ , unlike the explicit method which is stable only for a restricted range of values of  $h$  and  $k$ ; namely  $k/h^2 \leq 0.5$ .

model can be solved using the alternating direction implicit scheme. However, the computational time of these higher order factor models increases significantly.

## 4.2 Correlated factors

The solution of the bond prices derived above is based on the assumption that the factors are uncorrelated. To relax this assumption, the diffusion process is written as

$$dY(t) = \kappa(\theta - Y(t))dt + \sqrt{\Psi Y(t)}dW(t) \quad (29)$$

where  $Y(t)$  and  $W(t)$  are  $K$  dimensional processes containing the factors and Wiener processes respectively,  $\kappa$  and  $\theta$  are  $(K \times 1)$  vectors of parameters and  $\Psi$  is a  $(K \times K)$  variance-covariance matrix with the property that  $\Psi^{1/2}\Psi^{1/2} = \Psi$ , where  $\Psi^{1/2}$  is a unique positive definite matrix.

The partial differential equation used to price bonds is now

$$\frac{1}{2} \sum_{i=1}^K \sum_{j=1}^K \Psi_{ij} \sqrt{Y_i(t)Y_j(t)} \frac{\partial^2 P}{\partial Y_i \partial Y_j} + \sum_{i=1}^K (\kappa_i \theta_i - \kappa_i Y_i(t) - \lambda_i Y_i) \frac{\partial P}{\partial Y_i} + \frac{\partial P}{\partial t} - rP = 0 \quad (30)$$

In general analytical solutions do not exist and it is necessary to use numerical methods. For a two-factor model the alternating direction implicit scheme can be used to compute the prices numerically.

A formulation of the *SDE* that does admit an analytical solution for prices when factors are not independent is the affine *SDE* considered by Duffie and Kan (1993) and Frachot and Lesne (1993) is

$$dY_i(t) = \left( \alpha_i + \sum_{j=1}^K \beta_j Y_j(t) \right) dt + \sigma_{ii} \sqrt{\gamma_i + \sum_{j=1}^K \phi_j Y_j(t)} dW_i(t), \quad i, = 1, 2, \dots, K \quad (31)$$

where  $\gamma_k$  and  $\phi_k$  are scalars. The price is given by

$$P(t, T) = \exp \left[ A + \sum_{i=1}^K B_i Y_i(t) \right] \quad (32)$$

where  $A$  and  $B_i$ ,  $i, = 1, 2, \dots, K$ , are determined by solving a set of nonlinear ordinary differential equations numerically with the following boundary

conditions at  $\tau = 0$

$$\begin{aligned} A(0) &= 0 \\ B_i(0) &= 0, \quad \forall i \end{aligned} \tag{33}$$

to ensure that the price of the bond is \$1 at maturity.

### 4.3 Estimation without the price solution

The main computational difficulty with the indirect estimator for certain classes of problems arises when no analytical expression exists for the bond prices. This problem can be circumvented by replacing the simulated bond prices or yields, by the simulated factor(s). A drawback however, is that it is not possible to identify the risk premium parameters  $\lambda_i$ , as it does not appear in the *SDE* but only the *pde*. Thus if the problem is reinterpreted as pricing an asset using a “hypothetical” instantaneous rate that is generated from

$$dY_i(t) = (\delta_{0,i} - \delta_{1,i}Y_i(t)) dt + \sigma_i Y_i(t)^{\alpha_i} dW_i(t) \tag{34}$$

where  $\delta_{1,i} = \kappa_i + \lambda_i$ . It is possible to estimate the parameters in (34) by calibrating the simulated factors with the actual yields via an auxiliary model.

## 5 Conclusions

This paper has provided a framework for estimating a general class of continuous time, multifactor models of the term structure using the indirect estimator proposed by Gallant and Tauchen (1994) and Gourieroux, Monfort and Renault (1993). Special attention was devoted to estimating multifactor models of the class proposed by Cox, Ingersoll and Ross (1985), although procedures for estimating continuous time models characterised by more elaborate structures including levels effects, stochastic volatility and factor dependence, were also discussed.

Two empirical examples were given. The first example provided a comparison between the indirect estimates of 1-factor, 2-factor and 3-factor models of the term structure with the estimates obtained by Chen and Scott (1993) using a direct, MLE estimator. The second example provided estimates of multifactor models taking into account the full singular structure of interest rates arising from their cointegration properties as well as the

factor structure in the spreads. Overall the empirical results were very encouraging. The use of the indirect estimator also circumvented the potential misspecification problems inherent in direct estimation procedures which rely on specifying a set of dummy errors to avoid the singularity problems inherent in the term structure when transforming from the unobserved variables to the observed variables.



Table 1  
 Parameter estimates of the CIR model:  
 Yield data, December 1946 to December 1987  
 (abs. standard errors in brackets)

Parameter	Indirect Estimation			MLE (Chen and Scott)		
	1-factor	2-factor	3-factor	1-factor	2-factor	3-factor
$\kappa_1$	0.6979 (0.0300)	0.7165 (0.0458)	0.6567 (0.1612)	0.4697 (0.0543)	0.7660 (0.1513)	1.6331 (0.1655)
$\theta_1$	0.0526 (0.0026)	0.0541 (0.0194)	0.0966 (0.0358)	0.0618 (0.0074)	0.0321 (0.0064)	0.0324 (0.0029)
$\sigma_1$	0.1506 (0.0050)	0.1493 (0.0121)	0.1227 (0.0241)	0.0825 (0.0018)	0.1312 (0.0043)	0.1373 (0.0040)
$\lambda_1$	-0.2805 (0.0222)	-0.2784 (0.0438)	-0.2540 (0.0723)	-0.0454 (0.0564)	-0.1186 (0.1516)	-0.0317 (0.1504)
$\kappa_2$		0.0050 (0.0533)	0.0052 (0.0596)		0.0009 (0.0617)	0.0051 (0.2065)
$\theta_2$		0.0104 (0.1068)	0.0104 (0.1900)		0.0212 (1.4710)	0.0108 (0.4366)
$\sigma_2$		0.1065 (0.0084)	0.0468 (0.0220)		0.0531 (0.0017)	0.0755 (0.0017)
$\lambda_2$		-0.1091 (0.0611)	-0.1032 (0.0839)		-0.0415 (0.0618)	-0.1530 (0.2072)
$\kappa_3$			0.0083 (0.0143)			0.0062 (0.5259)
$\theta_3$			0.0082 (0.1517)			0.0091 (0.7747)
$\sigma_3$			0.1452 (0.0147)			0.1842 (0.0044)
$\lambda_3$			-0.0714 (0.0219)			-0.1373 (0.5260)

Table 2  
Principal components analysis:  
Spread data, January 1959 to February 1991

Interest rate (spread)	Loadings		
	1-factor	2-factor	3-factor
1-month	0.9415	-0.2975	0.1549
2-month	0.9746	-0.2157	-0.0214
3-month	0.9886	-0.1179	-0.0853
4-month	0.9958	-0.0262	-0.0818
5-month	0.9962	0.0494	-0.0581
6-month	0.9921	0.1072	-0.0328
9-month	0.9744	0.2178	0.0396
1-year	0.9540	0.2780	0.0971
	Communality		
	1-factor	2-factor	3-factor
1-month	0.8865	0.9750	0.9990
2-month	0.9499	0.9964	0.9969
3-month	0.9772	0.9911	0.9984
4-month	0.9916	0.9922	0.9989
5-month	0.9924	0.9948	0.9982
6-month	0.9843	0.9958	0.9969
9-month	0.9494	0.9968	0.9984
1-year	0.9100	0.9873	0.9968
Cumulative normalized eigenvalue	0.9552	0.9912	0.9979

Table 3  
 Parameter estimates of the CIR model:  
 Spread data, January 1959 to February 1991  
 (abs. standard errors in brackets)

Parameter	1-factor	2-factor	3-factor
$\kappa_1$	0.6973 (1.5055)	0.8674 (6.1998)	0.8464 (25.5247)
$\theta_1$	0.0200 (0.7830)	0.0254 (0.2573)	0.0244 (4.5590)
$\sigma_1$	0.3994 (0.9131)	0.3402 (3.1958)	0.3190 (1.1486)
$\lambda_1$	-0.0034 (10.8119)	-0.0253 (2.9627)	-0.0253 (22.1836)
$\kappa_2$		0.8454 (4.7807)	0.8045 (18.7503)
$\theta_2$		0.0063 (0.0258)	0.0062 (0.5115)
$\sigma_2$		0.5384 (1.4534)	0.5279 (3.0005)
$\lambda_2$		-0.1644 (2.8796)	-0.1656 (23.4945)
$\kappa_3$			0.6494 (18.6128)
$\theta_3$			0.0246 (3.9546)
$\sigma_3$			0.3194 (6.3431)
$\lambda_3$			-0.0306 (34.7496)

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One way to overcome this potential problem which offers relatively more stable solutions by circumventing the boundary problem of the *pde*, while at the sametime being computational more efficient, is motivated by the analytical solution of the CIR square root model. The solution of the CIR model is given by the exponential function in (4). The important characteristic of this solution is that the parameters  $A$  and  $B$ , are functions of  $\tau = T - t$ , only and not of the factor  $Y$ . It is this feature of the problem which enables an analytical solution to be obtained by solving a set of ordinary differential equations. Now consider writing the solution for the one factor model as

$$P(t, T) = A_1 (T - t, Y_1(t)) \exp [-B_1 (T - t, Y_1(t)) Y_1(t)] \quad (35)$$

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