Credit in a Random Matching Model With Private Information*

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January 1997

Abstract

We consider a random matching model without monetary exchange where agents have complete access to each others' histories. Exchange is motivated by risk sharing given random unobservable incomes. There is capital accumulation and an endogenous interest rate. The key feature of this environment is that information is mobile across locations even while goods are not. Optimal allocations in the dynamic private information environment resemble real-world

*We thank Ed Green and seminar participants at the Technion, the Atlanta Federal Reserve Bank, the Richmond Federal Reserve Bank, the Minneapolis Federal Reserve Bank, Indiana University, the Northwestern summer workshop 1996, the Midwest macro meetings 1996, Carnegie-Mellon University, the University of Rochester, and the University of Iowa for helpful comments and suggestions. Williamson is grateful to the National Science Foundation for financial support under grant SBR 93-08819.

credit arrangements in that there are credit balances, credit limits, and installment payments. The steady state has the property that there is a limiting distribution of expected utility entitlements with mobility and a positive fraction of agents who are credit constrained.

1. INTRODUCTION

Search environments with random matching have been used by Kiyotaki and Wright (1989, 1993), Williamson and Wright (1994) and Trejos and Wright (1995) in attempts to understand the role of money in exchange. It has been argued (e.g. Kiyotaki and Wright 1989) that the spatial frictions in these models preclude credit arrangements. That is, with a continuum of agents who are matched pairwise and at random in each period, no two agents can meet more than once in their lifetimes, and therefore intertemporal trade is impossible. However (see Aiyagari and Wallace 1991 and Kocherlakota 1996), what is critical to the absence of credit in a Kiyotaki-Wright search environment is not spatial frictions but the fact that agents do not have access to each other's trading histories. For example, in the environment considered by Kiyotaki and Wright (1989) where goods are indivisible and agents are specialized in production and consumption, subgame-perfect Nash equilibria which Pareto dominate all commodity money and fiat money equilibria exist when each agent has access to all other agents' complete trading histories.[1]

One might want to argue that spatial frictions help to motivate the lack of access to trading histories in search models. However, a problem is that in modern economies technology has evolved to the point where information can be moved between locations far more easily than most physical goods can be transported. In reality, credit transactions are typically not of the type where, effectively, an IOU is issued in exchange for some goods, with that IOU later being redeemed by the issuer. Typically, most of us conduct sequences of credit transactions with agents whom we never meet again, with settlement occurring through a centralized credit agency or check-clearing

[1]For example, in Kiyotaki and Wright's three-agent, three-commodity example, an efficient arrangement is for any agent to hold their production good until they meet an agent who consumes it, in which case the good is handed over, and another one is produced. This allocation can be supported as a Nash equilibrium.

mechanism.

The purpose of this paper is to study credit arrangements in a random matching environment in which agents' histories are public information. This matching setup differs from the typical monetary search environment in that there is a single good and consumers have random unobservable endowments and wish to share risk. As shown in the pioneering work of Townsend (1980), the desire to share risk in the presence of private information leads to a motive for intertemporal trade by tying future transfers to current transfers. Our model is closely-related to the environment studied by Green (1987), but we add random matching and a nonnegativity constraint on consumption, in addition to some other features. The approach we take is similar to what is done in the literature on dynamic private information (e.g. Green 1987, Spear and Srivastava 1987, Phelan and Townsend 1991, Atkeson and Lucas 1992, 1995, and Wang 1995), in that we analyze the allocation problem of a social planner who seeks to construct an efficient allocation subject to the constraints implied by private information.

In the model, there is a continuum of infinite-lived consumers who are matched with "stores" (i.e. locations) at random in each period. Consumption goods are produced from capital at a central location, and shipped out to each store at the beginning of each period, and the social planner uses unsold goods to produce capital which becomes productive in the next period. Consumption goods can not be transported across locations within the period. An interpretation of the physical environment and credit arrangement we consider is that it involves a firm simultaneously engaged in production, retail sales (through a large number of distinct retail outlets), and consumer finance.[2]

The economy here is very similar to that in Aiyagari (1994). The key differences are

[2]It could be argued that automobile manufacturers, which own some dealerships and also intermediate auto loans, provide a good fit to the environment we consider.

that there is random matching and allocations are (private information) constrained efficient in the economy studied here, whereas in Aiyagari (1994) all agents were together at all dates and the market structure and borrowing constraints were exogenously imposed. Indeed, ours is one of the first dynamic insurance economies with private information to include capital accumulation.[3] If the pairwise resource constraints implied by random matching turn out not to be binding (which can happen) then the efficient steady state allocation in our model would be a natural benchmark for the steady state allocation in Aiyagari (1994).

We show that efficient allocations can be determined by solving a set of recursive component planning problems, as in Atkeson and Lucas (1995). Then, we proceed to determine the characteristics of limiting distributions for this environment. A limiting distribution always exists, and it exhibits mobility, i.e. individual agents are mobile within the steady state distribution of wealth. The efficient allocation has features which resemble real-world credit systems. That is, agents have credit balances, and there are credit limits and installment payments. Further, and in spite of the fact that the marginal utility of consumption is infinite at zero and there is a positive probability of receiving a zero endowment for each consumer, there is a positive mass of agents who are credit-constrained in the steady state. This contrasts with results from incomplete markets models where imperfect consumption-smoothing is obtained by exogenously shutting down markets (e.g. Aiyagari 1994). However, as in Aiyagari (1994) we find that the interest rate is less than the time preference rate and that there is capital overaccumulation relative to a public information economy.

There are two novelties here. The first is that we study a pure credit arrangement in a random matching environment where there is no (social) role for monetary exchange. Second, we obtain a limiting distribution with mobility through an alter-

[3] See also Khan and Ravikumar (1996), which incorporates capital accumulation in a somewhat different way than is done here.

native to existing approaches in the literature.[4] In our model, the interest rate is endogenous, and there is capital accumulation. Allowing for capital accumulation rules out equilibrium interest rates which imply degenerate limiting distributions of expected utilities where all agents converge to the upper bound on expected utilities (here, the upper bound is implied by the resource constraints which come from random matching and immobility of resources across locations). Also, our economy satisfies "non-attainability of misery" as in Aiyagari and Alvarez (1995), which prevents a degenerate distribution where all agents are at the lower bound on expected utilities (given by the nonnegativity constraint on consumption and incentive compatibility).

The remainder of the paper is organized as follows. In Section 2, we describe the model, and in Section 3 we specify the problem the social planner solves to determine efficient allocations. Section 4 reformulates the component planning problem associated with the problem in Section 3 in terms of a Bellman equation. In Section 5, we use the Bellman equation to characterize the efficient allocation and the limiting distribution of expected utilities. Section 6 contains a discussion of the results and assumptions, and Section 7 is a conclusion.

[4]Some fixed interest rate private information economies have the property that the expected discounted utility of an arbitrarily large fraction of the population eventually becomes arbitrarily low (e.g. Green 1987), and there are related endogenous interest rate economies (e.g. Atkeson and Lucas 1992) where the wealth distribution continues to fan out over time. Nondegenerate limiting distributions of expected utilities with mobility are obtained by Atkeson and Lucas (1995) and Phelan (1995) by imposing a lower bound on expected utilities. In Atkeson-Lucas this lower bound is arbitrary, but Phelan makes the lower bound endogenous by supposing that long-term contracts are offered by firms to workers, and that workers can leave the contractual arrangement at any time and start a new contract with another firm.

2. THE MODEL

The population consists of a continuum of infinite-lived agents with unit mass, denoted *buyers*, each of whom has preferences given by

$$E_0(1-\beta)\sum_{t=0}^{\infty}\beta^t u(c_t),$$

where $0<\beta<1$, c_t is consumption, and $u(\cdot)$ is strictly increasing, strictly concave, and satisfies decreasing absolute risk aversion. Assume that $u(0)=0$ and $u'(0)=\infty$. Define $C(\cdot)$ by $C(u(c))=c$, that is $C(\cdot)=u^{-1}(\cdot)$. We then have $C(0)=0$ and $C'(0)=0$. There is a continuum of locations, denoted *stores,* also with unit mass, with a *seller* at each store. Stores are indexed by $i\in[0,1]$. Sellers will act as agents for the social planner in allocating consumption goods in each period.

Each period, buyers are matched pairwise and at random with stores. Thus, each buyer is matched with a seller. At the beginning of period t, the social planner has k_t units of capital available at a central location, where k_0 is given. Capital can be used by the planner to produce consumption goods at the beginning of the period, according to the production function $g(k_t)$. We assume that $g(\cdot)$ is strictly increasing, strictly concave, and continuously differentiable, with $g(0)=0$, $g'(0)=\infty$, and $g'(\infty)=0$. Capital depreciates by 100% after production. After consumption goods are produced at the central location, goods are transported costlessly to each store. Let x_t^i denote the quantity of consumption goods at store i. We assume that each store has a capacity constraint, that is $x_t^i\leq x^*$ for all i, where $x^*>0$. We also assume that

$$g(\hat{k})>x^*>g(\hat{k})-\hat{k}, \tag{1}$$

where $\hat{k}$ is the solution to

$$g'(\hat{k})=\frac{1}{\beta}. \tag{2}$$

The first inequality in (1) is designed to guarantee that transfers to buyers are limited by store capacity and not by available output. This assumption makes it possible to separate the problem of optimal capital accumulation from that of optimal transfers to buyers and makes the problem tractable. The second inequality in (1) is motivated by a consideration of the problem under public information regarding endowments. Obviously, full insurance is the natural outcome in this case. This inequality turns out to be a necessary condition for store capacity to be non-binding in attaining full insurance. Assumption (1) is discussed in more detail in Section 6.

We also assume that when the social planner sends consumption goods to stores, it is not yet known which buyers are matched with which stores for the period.

At the beginning of the period, a buyer receives an endowment, θ_t, which is an i.i.d. (across agents and time) draw from a probability distribution $F(\theta_t)$, where $\theta_t \geq 0$. The buyer's endowment is private information. After consumption goods arrive at the store and the buyer receives her endowment, the seller makes a transfer (which could be negative) to the buyer, where the transfer is determined by the buyer's reported history (recorded with the social planner) and the buyer's report of her current endowment shock. Consumption goods can not be moved across locations during the period, but at the end of the period any consumption goods not consumed by buyers are transported back to the social planner and converted, one-for-one, into capital.

3. EFFICIENT ALLOCATIONS

The social planner is given $\psi_0(w)$, the distribution of date 0 expected utilities across buyers, and k_0, the initial capital stock. If equal quantities of consumption goods are not transported to stores in each period, this can only introduce the possibility of more randomness in buyers' consumptions, and it cannot help incentives. Thus, there is no loss from considering only allocations where $x_t^i = x_t$ for all t and all $i \in [0, 1]$, and we can therefore drop i superscripts from the subsequent analysis. Also, the social planner will ship the largest quantity of goods possible, as this can only relax constraints in the planner's optimization problem. Thus, we have $x_t = \min[x^*, g(k_t)]$.

Let $\{\tau_t(w_0, \theta^t)\}_{t=0}^{\infty}$ denote the sequence of transfers received by a buyer from the sequence of sellers she meets, where $\theta^t = \{\theta_0, \theta_1, ..., \theta_t\}$ denotes the buyer's history of endowment shock reports to date t and w_0 is the buyer's date 0 expected utility entitlement.

Definition 1 *An allocation (τ, k) is a set of sequences $\{\tau_t(w_0, \theta^t)\}_{t=0}^{\infty}$, $\{k_t\}_{t=1}^{\infty}$, for each initial expected utility entitlement w_0, given k_0, which satisfies*

$$w_0 = E_0(1-\beta)\sum_{t=0}^{\infty}\beta^t u\left[\tau_t(w_0, \theta^t) + \theta_t\right] \tag{3}$$

for all w_0,

$$E_t\sum_{s=t}^{\infty}\beta^{s-t}u\left[\theta_s + \tau_s(w_0, \theta^s)\right] \geq u\left[\theta_t + \tau_t(w_0, \{\theta_0, \theta_1, ..., \theta_{t-1}, \theta'\})\right] \tag{4}$$

$$+E_t\sum_{s=t+1}^{\infty}\beta^{s-t}u\left[\theta_s + \tau_s(w_0, \{\theta_0, \theta_1, ..., \theta_{t-1}, \theta', \theta_{t+1}, ..., \theta_s\})\right],$$

for all w_0, and for all t,

$$-\theta_t \leq \tau_t(w_0, \theta^t) \leq x_t \tag{5}$$

for all w_0, t, and θ^t,

$$k_{t+1} \geq 0 \tag{6}$$

$$x_t = \min[x^*, g(k_t)] \tag{7}$$

for all t.

In the above definition, (3) is a promise-keeping constraint, (4) are temporary incentive compatibility constraints, (5) is the pairwise resource constraint, (6) is a nonnegativity constraint on capital, and (7) captures the capacity constraint on goods shipped to each store.

Definition 2 *An allocation* (τ, k) *attains* ψ_0 *with resource cost* $z \in R$ *if*

$$-g(k_t) + k_{t+1} + \int\int \tau_t(w, \theta^t) d\mu(\theta^t) d\psi_0(w) \leq z, \tag{8}$$

and (3)-(7) are satisfied.

In the above definition, $\mu(\theta^t)$ is the distribution of the history θ^t.

Definition 3 *An efficient allocation is a* (τ, k) *which attains* ψ_0 *with cost* z*, and there exists no other allocation which attains* ψ_0 *with cost less than* z*.*

Now, we follow Atkeson and Lucas (1995), in decentralizing the problem of determining efficient allocations by considering component planning problems. First suppose that there is a planner at the central location who starts with the initial capital stock, k_0, at the beginning of the first period. In each period, this planner produces given the existing capital stock, retains some output to accumulate capital for the succeeding period, retains an additional amount of output (denoted T_t) to pay for transfers to consumers, and sells the remaining output, facing the sequence of intertemporal prices $\{q_t\}_{t=0}^{\infty}$, where $q_t \in (0, 1)$ (i.e. $q_t = \frac{1}{1+r_t}$, where r_t is the one-period interest rate). That is, this planner can borrow and lend at market prices, and maximizes the present discounted value of profits, i.e. she chooses $\{k_t\}_{t=1}^{\infty}$ given k_0 to solve

$$\max\{(1 - q_0)[g(k_0) - T_0 - k_1] + \sum_{t=1}^{\infty}(1 - q_t)\prod_{s=0}^{t-1} q^s[g(k_t) - T_t - k_{t+1}]\}. \tag{9}$$

In addition to the planner at the central location, there is a planner associated with each initial expected utility entitlement w_0. At the beginning of each period, the planner at the central location ships $x_t = min[x^*, g(k_t)]$ units of consumption goods to each store. Then, after buyers have been randomly allocated to stores, the planner associated with w_0 receives a report from each of the buyers for whom she has responsibility, and makes a transfer to each. Any consumption goods not transferred to buyers are returned to the planner at the central location at the end of the period. The planner responsible for buyers with initial expected utility entitlement w_0 minimizes the cost of delivering w_0 given the price sequence $\{q_t\}_{t=0}^{\infty}$ and the sequence of shipments $\{x_t\}_{t=0}^{\infty}$ from the planner at the central location. That is, she chooses $\{\tau_t(w_0, \theta^t)\}_{t=0}^{\infty}$ to solve

$$\min\{(1-q_0)\int \tau_0(w_0,\theta_0)dF(\theta_0) + \sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1} q_s \int \tau_t(w_0,\theta^t)d\mu(\theta^t) \tag{10}$$

subject to (3)-(5).

Note that in (9)

$$T_t = \int\int \tau_t(w,\theta^t)d\mu(\theta^t)d\psi_0(w),$$

and will in general depend on $\{k_t\}_{t=0}^{\infty}$ via the resource constraints (5) and (7). This dependence is taken into account by the central planner in choosing a path for capital accumulation.

It is then straightforward to apply Theorem 1 on page 70 of Atkeson and Lucas (1995), to show that if there exists an allocation (τ, k), prices $\{q_t\}_{t=0}^{\infty}$, an initial distribution of expected utility entitlements $\psi_0(w)$, and aggregate resources z such that (τ, k) solves the above two minimization problems given $\{q_t\}_{t=0}^{\infty}$ (each planner minimizes cost given prices), (8) is satisfied with equality for all t (market clearing), and

$$1 - q_0 + \sum_{t=0}^{\infty}(1-q_t)\prod_{s=0}^{t-1} q_s < \infty,$$

then (τ, k) attains ψ_0 with resources z and is efficient.

A potential complication in using the above characterization of efficient allocations is that the total transfers $\{T_t\}$ appearing in (9) may depend on $\{k_t\}$. This makes it difficult to characterize an efficient $\{k_t\}$ sequence. A way around this difficulty is provided by the following proposition.

Proposition 1: Let the sequence $\{k_{t+1}^*\}_{t=0}^{\infty}$ satisfy

$$-1 + q_t + (1 - q_{t+1})q_t g'(k_{t+1}^*) = 0 \tag{11}$$

and

$$g(k_{t+1}^*) > x^*. \tag{12}$$

Let $\{\tau_t^*(w_0, \theta^t)\}_{t=0}^{\infty}$ minimize (10) and suppose that (8) is satisfied with equality for all t. Then (τ^*, k^*) attains ψ_0 with resources z and is efficient.

Proof. It is obvious that (τ^*, k^*) attains ψ_0 with resources z. Suppose that (τ^*, k^*) is not efficient. Then there exists $(\tilde{\tau}, \tilde{k})$ such that

$$-g(\tilde{k}_t) + \tilde{k}_{t+1} + \tilde{T}_t \leq \tilde{z} < z = -g(k_t^*) + k_{t+1}^* + T_t^*,$$

where

$$\tilde{T}_t = \int\int \tilde{\tau}_t(w, \theta^t) d\mu(\theta^t) d\psi_0(w),$$
$$T_t^* = \int\int \tau_t^*(w, \theta^t) d\mu(\theta^t) d\psi_0(w).$$

Therefore,

$$\begin{aligned} & (1 - q_0)(-g(k_0) + \tilde{k}_1 + \tilde{T}_1) + \sum_{t=1}^{\infty}(1 - q_t)\prod_{s=0}^{t-1} q_s(-g(\tilde{k}_t) + \tilde{k}_{t+1} + \tilde{T}_t) \qquad (13) \\ < \; & (1 - q_0)(-g(k_0) + k_1^* + T_1^*) + \sum_{t=1}^{\infty}(1 - q_t)\prod_{s=0}^{t-1} q_s(-g(k_t^*) + k_{t+1}^* + T_t^*) \end{aligned}$$

However, by virtue of (11) we must have

$$\begin{aligned} & (1 - q_0)(-g(k_0) + \tilde{k}_1) + \sum_{t=1}^{\infty}(1 - q_t)\prod_{s=0}^{t-1} q_s(-g(\tilde{k}_t) + \tilde{k}_{t+1}) \qquad (14) \\ \geq \; & (1 - q_0)(-g(k_0) + k_1^*) + \sum_{t=1}^{\infty}(1 - q_t)\prod_{s=0}^{t-1} q_s(-g(k_t^*) + k_{t+1}^*) \end{aligned}$$

because $\{k_{t+1}^*\}_{t=0}^{\infty}$ attains the minimum of the expression

$$(1-q_0)(-g(k_0)+k_1)+\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1}q_s(-g(k_t)+k_{t+1}).$$

Note that in addition to the first order conditions (11), the sequence of capital stocks $\{k_{t+1}^*\}_{t=0}^{\infty}$ also satisfies the transversality condition. This can be seen as follows. The pairwise resource constraints imply that

$$-\pi y \le T_t^* \le g(k_t^*).$$

Hence, (8) implies that

$$z \le k_{t+1}^* \le g(k_t^*)+\pi y + z.$$

By virtue of the restrictions on $g(\cdot)$ it follows that the sequence of capital stocks $\{k_t^*\}_{t=0}^{\infty}$ is bounded. Further, the assumption on the price sequence $\{q_t\}_{t=0}^{\infty}$ implies that

$$\lim_{t\to\infty}(1-q_t)\prod_{s=0}^{t-1}q_s = 0.$$

Therefore,

$$\lim_{t\to\infty}(1-q_t)\prod_{s=0}^{t-1}q_s k_{t+1}^* = 0,$$

which is the relevant transversality condition. Further,

$$\begin{aligned}
&(1-q_0)\int \tilde{\tau}_0(w_0,\theta_0)dF(\theta_0)+\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1}q_s\int \tilde{\tau}_t(w_0,\theta^t)d\mu(\theta^t) \qquad (15)\\
\ge\ &(1-q_0)\int \tau_0^*(w_0,\theta_0)dF(\theta_0)+\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1}q_s\int \tau_t^*(w_0,\theta^t)d\mu(\theta^t),
\end{aligned}$$

as $\{\tau^*\}$ attains the minimum in (10) and $\{\tilde{\tau}\}$ is a feasible choice for that problem. To check this we only need to verify that $\{\tilde{\tau}\}$ satisfies the pairwise resource constraints for the problem (10) for $t \ge 1$ with the sequence of capital stocks $\{k_{t+1}^*\}_{t=0}^{\infty}$. This is obvious because

$$-\theta_t \le \tilde{\tau}_t(w_0,\theta^t) \le \min[x^*, g(\tilde{k}_t)] \le x^* = \min[x^*, g(k_t^*)]$$

for $t \geq 1$ by virtue of (11). Now, integrating (15) with respect to $\psi_0(w_0)$ we have

$$(1-q_0)\tilde{T}_0 + \sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1} q_s \tilde{T}_t \geq (1-q_0)T_0^* + \sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1} q_s T_t^*. \tag{16}$$

However, adding (14) and (16), we see that the result contradicts (13). This contradiction establishes the desired result.□

This proposition is useful since it provides a way of characterizing an efficient sequence of capital stocks using (11), provided (12) holds, and an efficient sequence of transfers by solving problem (10). Note that by virtue of (12), the pairwise resource constraint for the problem (10) can be replaced by

$$-\theta_t \leq \tau_t(w_0, \theta^t) \leq x^*,$$

so that the $\{k_t\}$ sequence no longer enters problem (10).

In the next section we confine attention to steady states and use dynamic programming methods to solve problem (10) and thereby construct stationary efficient allocations.

4. BELLMAN EQUATION

Assume now that there are only two states, i.e. $\theta_t \in \{0, y\}$, where $\Pr[\theta_t = y] = \pi$, and $\Pr[\theta_t = 0] = 1 - \pi$, with $0 < \pi < 1$.[5] We will confine attention to steady states, where $q_t = q \in [\beta, 1)$ for all t. Let k_q be such that

$$g'(k_q) = \frac{1}{q}. \tag{17}$$

By virtue of assumptions (1) and (2) we have $g(k_q) > x^*$. Therefore, the sequence of capital stocks $k_t = k_q$ satisfies the conditions (11) and (12) in Proposition 1. Now, in a steady state, the other component planning problems can be specified as the

[5]The assumption that the low endowment is zero is without loss of generality.

solution to a Bellman equation, where $V_q(w)$ is interpreted as the minimum cost to the planner of delivering an expected utility entitlement of w to a given buyer, given q. That is, define $V_q(w)$ as follows.

$$\begin{aligned} V_q(w) &= \min\{(1-q)[\pi\tau_1(w) + (1-\pi)\tau_0(w)] \\ &\quad +q[\pi V_q(w_1(w)) + (1-\pi)V_q(w_0(w))]\} \end{aligned} \tag{18}$$

subject to

$$\pi[(1-\beta)u(y+\tau_1(w)) + \beta w_1(w)] + (1-\pi)[(1-\beta)u(\tau_0(w)) + \beta w_0(w)] = w, \tag{19}$$

$$(1-\beta)u(y+\tau_1(w)) + \beta w_1(w) \geq (1-\beta)u(y+\tau_0(w)) + \beta w_0(w), \tag{20}$$

$$(1-\beta)u(\tau_0(w)) + \beta w_0(w) \geq (1-\beta)u(\tau_1(w)) + \beta w_1(w), \tag{21}$$

$$\begin{aligned} -y &\leq \tau_1(w) \leq x^*, \\ 0 &\leq \tau_0(w) \leq x^*, \end{aligned} \tag{22}$$

$$w, w_0(w), w_1(w) \in [\underline{w}, \bar{w}],$$

where $\underline{w}= \pi u(y)$ and $\bar{w} = \pi u(y+x^*) + (1-\pi)u(x^*)$. Here, (19) is a promise-keeping constraint, (20) and (21) are incentive constraints, and (22) are the pairwise resource constraints, where x^* denotes the quantity of consumption goods shipped to each location in the steady state. The choice variables in the optimization problem on the right-hand side of the Bellman equation are $\tau_1(w)$, $\tau_0(w)$, $w_1(w)$, and $w_0(w)$. As noted earlier, the planner always has output $g(k_q)$ which exceeds the capacity constraint of stores given by x^*. Hence, it is x^* which appears in the pairwise resource constraints (22).

5. CHARACTERIZING THE STEADY STATE EFFICIENT ALLOCATION

Now that we have a recursive representation of the component planning problems, we can proceed to a derivation of the properties of the steady state efficient allocation. We consider the steady state where $z = 0$, that is, where the total net transfer to buyers is zero. Then, if $\psi_q(w)$ is the steady state distribution of expected utility entitlements across buyers, we must have

$$g(k_q) - k_q = \int [\pi \tau_{1,q}(w) + (1-\pi)\tau_{0,q}(w)] d\psi_q(w), \tag{23}$$

where the left-hand side of (23) is output per capita at the beginning of the period minus capital set aside at the end of the period, and the right-hand side of (23) is total transfers per capita. We can write (23) as

$$H_1(q) = H_2(q), \tag{24}$$

where

$$H_1(q) \equiv g(k_q) - k_q,$$

$$H_2(q) \equiv \int [\pi \tau_{1,q}(w) + (1-\pi)\tau_{0,q}(w)] d\psi_q(w).$$

We can think of solving for the steady state as follows. Given a price q, we can solve for the steady state quantity of capital, k_q, from (17). Then, we can solve (18) subject to (19)-(22) to obtain $V_q(w)$, $\tau_{1,q}(w)$, $\tau_{0,q}(w)$, $w_{1,q}(w)$, and $w_{0,q}(w)$. This then implies a dynamic stochastic path for w, and we can accordingly solve for $\psi_q(w)$. Then, we can ask whether (23) is satisfied and, if not, then try another q, etc.

For the analysis, it will be convenient to rewrite the Bellman equation by making the following change of variables. Let $u_1(w) = u(y + \tau_1(w))$ and $u_0(w) = u(\tau_0(w))$ so that $\tau_1(w) = C(u_1(w)) - y$, $\tau_0(w) = C(u_0(w))$ and $u(y + \tau_0(w)) = u(y + C(u_0(w)))$. We will also assume that if a buyer claims a positive endowment, i.e. $\theta = y$, then she must be able to show it. This assumption allows us to ignore the incentive constraint (21), so that we only need to worry about the incentive constraint (20).

With the above changes in notation and the added assumption, we can rewrite the Bellman equation as follows:

$$V_q(w) = \min \left\{ \begin{array}{c} (1-q)\left[\pi C(u_1(w)) + (1-\pi)C(u_0(w)) - \pi y\right] \\ +q\left[\pi V_q(w_1(w)) + (1-\pi)V_q(w_0(w))\right] \end{array} \right\}$$

subject to

$$\pi\left[(1-\beta)u_1(w) + \beta w_1(w)\right] + (1-\pi)\left[(1-\beta)u_0(w) + \beta w_0(w)\right] = w \qquad (25)$$

$$(1-\beta)u_1(w) + \beta w_1(w) \geq (1-\beta)u(y + C(u_0(w))) + \beta w_0(w) \qquad (26)$$

$$\begin{aligned} 0 &\leq C(u_1(w)) \leq y + x^* \qquad (27) \\ 0 &\leq C(u_0(w)) \leq x^* \\ w, w_1', w_0' &\in [\underline{w}, \bar{w}] \end{aligned}$$

The first step is to show that the cost function, $V_q(\cdot)$ is well-behaved, and then to obtain a characterization of the solution to the Bellman equation. This is done in the following proposition.

Proposition 2: (a) $V_q(\cdot)$ is strictly increasing, convex, and continuously differentiable.[6] (b) $u_1(w) \geq u_0(w), w_1(w) \geq w_0(w)$, and $u_1(w)$ and $w_1(w)$ are nondecreasing in w. (c) $u_0(\underline{w}) = 0$, $C(u_1(\underline{w}) < y$, $w_0(\underline{w}) =\underline{w}$, and $w_1(\underline{w}) >\underline{w}$. (d) For $w \in (\underline{w}, \bar{w})$, $w_0(w) < w$ and $\sup\{w : w_0(w) =\underline{w}\} >\underline{w}$.

Proof. (a) See Proposition 3.5 in Aiyagari and Alvarez (1995), p. 28. (b) If $u_0(w) > u_1(w)$ then set $u_0^*(w) = u_1^*(w) = \pi u_1(w) + (1-\pi)u_0(w)$ and we have $u_1(w) < u_1^*(w) =$

[6]The assumption that $u(\cdot)$ satisfies decreasing absolute risk aversion guarantees that the constraint set defined by (25)-(27) is convex without need to resort to lotteries.

$u_0^*(w) < u_0(w)$. This alternative policy satisfies the promise-keeping and incentive constraints. But, by convexity, the objective function decreases, a contradiction. Similarly, suppose $w_0(w) > w_1(w)$. Then set $w_0^*(w) = w_1^*(w) = \pi w_1(w) + (1-\pi)w_0(w)$, and we have $w_1(w) < w_1^*(w) = w_0^*(w) < w_0(w)$, and this works in the same way. For the rest, see Aiyagari and Alvarez (1995).□

From the above Proposition, we therefore know that buyers consume at least as much in the high-endowment state as in the low-endowment state, and that future expected utility entitlements are at least as large in the high-endowment state as in the low-endowment state. When the buyer's expected utility entitlement falls to the lower bound, $\underline{w}$, then if the buyer receives the low endowment, she receives a transfer of zero and a future expected utility entitlement of $\underline{w}$. However, if the buyer is at the lower bound and receives the high endowment, then she gets a negative transfer and an increase in her expected utility entitlement. Further, for w in some neighborhood of $\underline{w}$, if the buyer receives a low endowment then her future expected utility entitlement is set equal to $\underline{w}$. This result will play an important role in ensuring that there will be a positive fraction of credit constrained buyers in the steady state.

We can now proceed to examine the implications for the steady state distribution of expected utilities, $\psi(w)$, of alternative steady state prices, $q \in [\beta, 1)$. We first consider the case $q = \beta$. Let λ denote the Lagrange multiplier associated with the promise-keeping constraint (25), and μ the nonnegative multiplier associated with the incentive constraint (26). The first-order necessary condition for $w_1(w)$ is

$$-q\pi V_q'(w_1(w)) + \lambda\pi\beta + \mu\beta = 0, \text{ if } w_1(w) \in (\underline{w}, \bar{w}),$$

or

$$-q\pi V_q'(w_1(w)) + \lambda\pi\beta + \mu\beta > 0, \text{ if } w_1(w) = \bar{w}.$$

The envelope condition is

$$\lambda = V_q'(w)$$

Thus,

$$V_q'(w_1(w)) \le V_q'(w)\frac{\beta}{q} + \frac{\mu\beta}{q\pi}, \text{ with equality if } w_1(w) < \bar{w}.$$

Analogously, the first-order necessary condition for $w_0(w)$ can be written as

$$V_q'(w_0(w)) = V_q'(w) - \frac{\mu\beta}{q(1-\pi)} \text{ for } w_0(w) \in (\underline{w}, \bar{w}),$$

or

$$V_q'(w_0(w)) > V_q'(w)\frac{\beta}{q} - \frac{\mu\beta}{q(1-\pi)} \text{ if } w_0(w) = \underline{w}.$$

Proposition 3: If $V'(\bar{w}) = \infty$, then there exists $\hat{w}_1 < \bar{w}$ such that $u_1(w) = u(y+x^*)$ for $w \in [\hat{w}_1, \bar{w}]$.

Proof. Suppose $u_1(w) < u(y + x^*)$. Consider increasing $u_1(w)$ and lowering $w_1(w)$ in order to keep $(1-\beta)u_1(w) + \beta w_1(w)$ constant. This satisfies all the constraints. The change in the objective function is given by

$$\left[(1-q)C'(u_1(w)) - q\pi V_q'(w_1(w))\frac{(1-\beta)}{\beta}\right] du_1(w).$$

We know that when $w = \bar{w}$, $w_1(w) = \bar{w}$. Then, since $V'(\bar{w}) = \infty$, in a neighborhood of $w = \bar{w}$ the above expression must be negative, which is a contradiction.□

Proposition 4: If $V_q'(\bar{w}) = \infty$, then there exists $\hat{w}_2 < \bar{w}$ such that $u_0(w) = u(x^*)$ for $w \in [\hat{w}_2, \bar{w}]$.

Proof. Suppose $u_0(w) < u(x^*)$. Consider raising $u_0(w)$ and lowering $w_0(w)$ such that $(1-\beta)u_0(w) + \beta w_0(w)$ is held constant. This satisfies the promise-keeping constraint. It also satisfies the incentive constraint as the change in the right-hand side is given by

$$(1-\beta)[u'(y + C(u_0(w))) - 1]du_0(w) < 0$$

by concavity of $u(\cdot)$. The change in the objective function is given by

$$\left[(1-q)(1-\pi)C'(u_0(w)) - q(1-\pi)V_q'(w_0(w))\frac{(1-\beta)}{\beta}\right] du_0(w).$$

We know that when $w = \bar{w}$, $w_0(w) = \bar{w}$ and $V_q'(\bar{w}) = \infty$. Hence, in a neighborhood of $w = \bar{w}$, the above expression must be negative, which is a contradiction.□

Proposition 5: If $q = \beta$ then $V_q'(\bar{w}) < \infty$.

Proof. Suppose to the contrary that $V_q'(\bar{w}) = \infty$. Now, let $\hat{w} = \max[\hat{w}_1, \hat{w}_2]$. Then, for $w \in [\hat{w}, \bar{w}]$, we know that if $V_q'(\bar{w}) = \infty$ then $u_1(w) = u(y + x^*)$, $u_0(w) = u(x^*)$, and

$$w_1(w) = w_0(w) = \frac{w - (1-\beta)\bar{w}}{\beta} < w.$$

But the first-order necessary condition for $w_1(w)$ implies

$$V_q'(w_1(w)) = V_q'(w)\frac{\beta}{q} + \frac{\mu\beta}{q\pi} > V_q'(w),$$

which implies $w_1(w) > w$. This is a contradiction, hence $V_q'(\bar{w}) < \infty$.□

Now we will show that for $q = \beta$ the stochastic process of expected utility entitlements converges to $\bar{w}$ almost surely.

Proposition 6: Let $q = \beta$. Then $\{w_t\} \to \bar{w}$ a.s.

Proof. Suppose $w_1(\hat{w}) = \bar{w}$ for some $\hat{w} \in (\underline{w}, \bar{w})$. Since $w_1(w)$ is nondecreasing it follows that $w_1(w) = \bar{w}$ for $w \in [\hat{w}, \bar{w}]$. Consider $w \in (\underline{w}, \hat{w})$. If $w_1(w) \in (\underline{w}, \bar{w})$ then the first-order conditions for $w_1(w)$ imply that

$$\begin{aligned} V_q'(w_1(w)) &= V_q'(w)\frac{\beta}{q} + \frac{\mu\beta}{q\pi} \\ &= V_q'(w) + \frac{\mu}{\pi} \\ &> V_q'(w) \end{aligned}$$

Hence $w_1(w) > w$. If $w_1(w) = \bar{w}$ then again $w_1(w) > w$. Finally, we have already seen that $w_1(\underline{w}) > \underline{w}$. It follows that in a finite number of steps $\{w_t\} \to \bar{w}$.

Now suppose that $w_1(w) < \bar{w}$ for all $w \in (\underline{w}, \bar{w})$. Let $w \in (\underline{w}, \bar{w})$. Then $w_1(w) \in (\underline{w}, \bar{w})$ and the first-order condition for $w_1(w)$ yields

$$V_q'(w_1(w)) = V_q'(w) + \frac{\mu}{\pi}. \tag{28}$$

Further, $w_0(w) < \bar{w}$. If $w_0(w) \in (\underline{w}, \bar{w})$ then the first-order condition for $w_0(w)$ yields

$$\begin{aligned} V_q'(w_0(w)) &= V_q'(w)\frac{\beta}{q} - \frac{\mu\beta}{q(1-\pi)} \\ &= V_q'(w) - \frac{\mu}{1-\pi}. \end{aligned}$$

If $w_0(w) = \underline{w}$ then we have

$$V_q'(w_0(w)) \geq V_q'(w) - \frac{\mu}{1-\pi}.$$

Therefore

$$V_q'(w_0(w)) \geq V_q'(w) - \frac{\mu}{1-\pi} \tag{29}$$

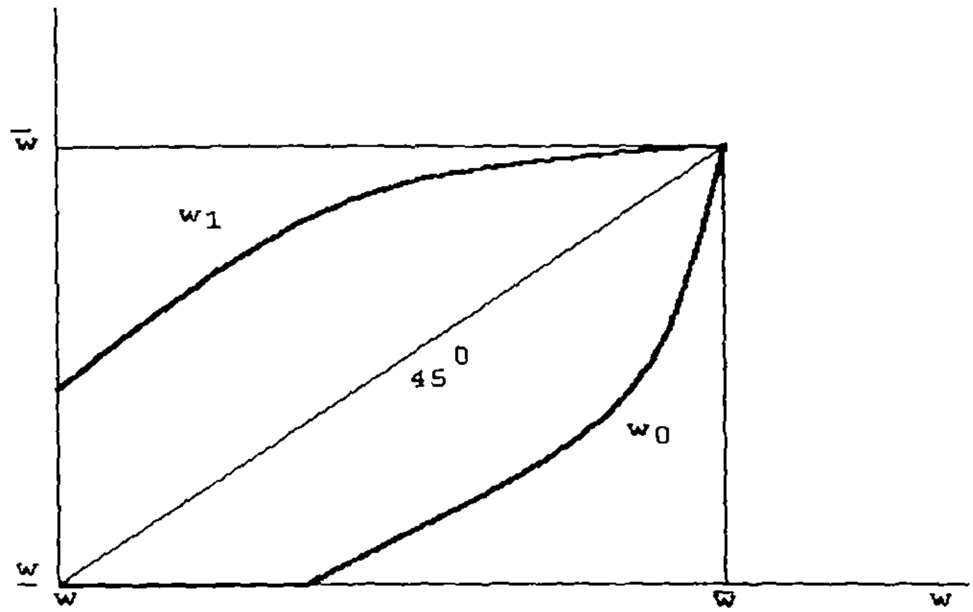
Combining (28) and (29) we have

$$\pi V_q'(w_1(w)) + (1-\pi)V_q'(w_0(w)) \geq V_q'(w).$$

Hence $\{V_q'(w_t)\}$ follows a sub-martingale which is bounded above since $V_q'(\bar{w}) < \infty$. Hence $\{V_q'(w_t)\}$ converges (a.s.), and it converges (a.s.) to $V_q'(\bar{w})$. Therefore, $\{w_t\} \to \bar{w}$ (a.s.).□

Thus, we have shown that when $q = \beta$, the limiting distribution of expected utilities is degenerate at $\bar{w}$ (see Figure 1). Therefore, in the limit each buyer receives a transfer of x^* in each state. We will therefore have $H_2(q) = x^* > H_1(q)$ for $q = \beta$ where the inequality follows from (1) and (2).

Figure 1: q = β



We will now analyze the stochastic process of expected utility entitlements for $1 > q > \beta$ and give conditions under which $H_2(q) < H_1(q)$ for some q. This result, together with continuity of $H_2(\cdot)$ will establish the existence of a $q^* \in (\beta, 1)$ such that $H_2(q^*) = H_1(q^*)$. The corresponding allocations are efficient and support the stationary distribution with zero cost. The first step in this process is to establish the following proposition.

Proposition 7: If $q > \beta$ then $V_q'(\bar{w}) = \infty$.

Proof. The first derivative with respect to $w_0(w)$ from the minimization problem on the right-hand side of the Bellman equation is as follows.

$$-q(1-\pi)V'(w_0') + \lambda(1-\pi)\beta - \mu\beta,$$

which is less than zero if $w_0(w) = \underline{w}$, greater than zero if $w_0(w) = \bar{w}$, and equal to zero if $w_0(w) \in (\underline{w}, \bar{w})$. We know that $w_0(\bar{w}) = \bar{w}$. By continuity we have $w_0(w) > \underline{w}$

for w in a neighborhood of $\bar{w}$. Therefore,

$$V_q'(w_0(w)) \leq \frac{\lambda\beta}{q} - \frac{\mu\beta}{q(1-\pi)},$$

with equality if $w_0(w) < \bar{w}$. Further, by the envelope condition

$$\lambda = V_q'(w).$$

Hence,

$$V_q'(w_0(w)) \leq V_q'(w)\frac{\beta}{q}.$$

If $V_q'(\bar{w}) < \infty$, then setting $w = w_0(w) = \bar{w}$, we have

$$V_q'(\bar{w}) \leq V_q'(\bar{w})\frac{\beta}{q} < V_q'(\bar{w}),$$

a contradiction. Therefore, $V_q'(\bar{w}) = \infty$.□

From Proposition (7), and Propositions (3) and (4), we then know that there exists some $\hat{w} < \bar{w}$ such that $u_1(w) = u(y+x^*)$ and $u_0(w) = u(x^*)$ for $w \in [\hat{w}, \bar{w}]$. Therefore, for $w \in [\hat{w}, \bar{w}]$ the Bellman equation becomes

$$V_q(w) = \min\left\{(1-q)x^* + q[\pi V_q(w_1(w)) + (1-\pi)V_q(w_0(w))]\right\}$$

subject to

$$(1-\beta)\bar{w} + \beta[\pi w_1(w) + (1-\pi)w_0(w)] = w,$$

$$\beta w_1(w) \geq \beta w_0(w).$$

Since $V_q(\cdot)$ is convex, without loss of generality we can set $w_1(w) = w_0(w) = \frac{w-(1-\beta)\bar{w}}{\beta}$, and

$$V_q(w) = (1-q)x^* + qV_q\left(\frac{w-(1-\beta)\bar{w}}{\beta}\right),$$

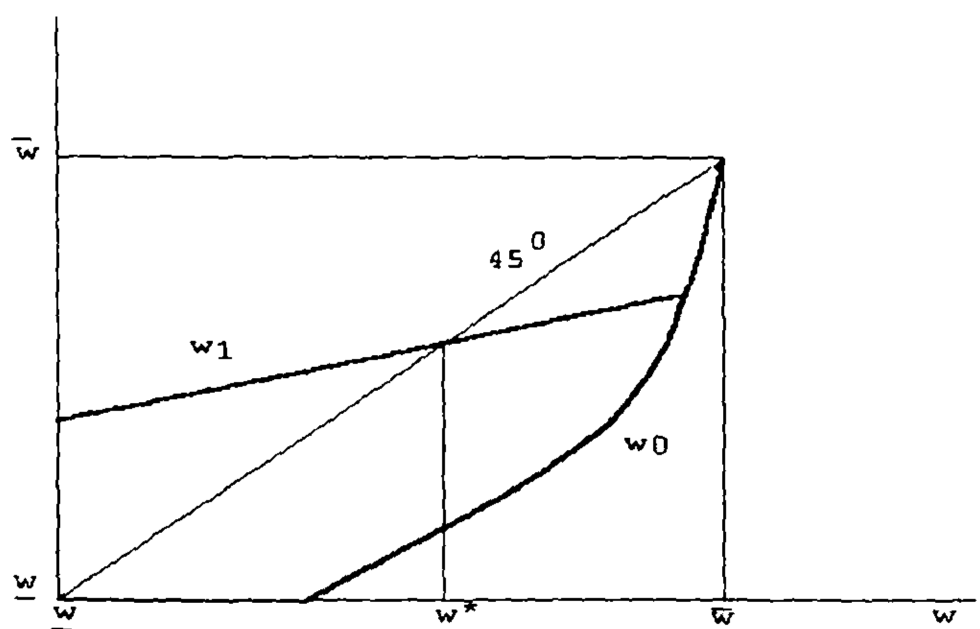
for $w \in [\hat{w}, \bar{w}]$. It follows that, for $w \in [\hat{w}, \bar{w}]$, $w_1(w) = w_0(w) = \frac{w-(1-\beta)\bar{w}}{\beta} < w$. We also know that $w_1(\underline{w}) > \underline{w}$. Further, if $w_0(w) = \underline{w}$ then $w_0(w) < w$ for $w > \underline{w}$. If $w_0(w) > \underline{w}$ then the first-order necessary condition for $w_0(w)$ implies

$$V_q'(w_0(w)) \leq V_q'(w)\frac{\beta}{q} - \frac{\mu\beta}{q(1-\pi)} \leq V_q'(w)\frac{\beta}{q} < V_q'(w),$$

which implies that $w_0(w) < w$ by convexity.

Thus, the graphs of $w_1(w)$, $w_0(w)$, $u_1(w)$ and $u_0(w)$ versus w look as in Figure 2. While the $w_1(w)$ curve need not intersect the 45^o line uniquely in $(\underline{w},\bar{w})$, there will be a smallest value $w^* > \underline{w}$ such that $w_1(w^*) = w^*$. Since $w_0(w) < w$ for $w \in (\underline{w},\bar{w})$, it follows that $[\underline{w},w^*]$ and $\{\bar{w}\}$ are the only ergodic sets. We will focus only on stationary distributions which put zero probability on $\{\bar{w}\}$. It is obvious that there exists a unique stationary distribution for $\{w_t\}$, and it exhibits mobility. When $q > \beta$, there is a tendency for the expected utility of a given buyer to drift down over time, since the planner is more patient than are buyers. However, the incentive structure in the optimal allocation will tend to push up (down) the expected utility of buyers with low(high) expected utility entitlements.

Figure 2: q = β



Now it is straightforward, along the lines of Atkeson and Lucas (1995),[7] to show that $H_2(q)$ is a continuous function on $[\beta, 1)$ (we omit the proof for brevity). What remains is to show that, for some $q \in (\beta, 1)$, $H_2(q) < H_1(q)$. We consider two cases below. Let $\hat{x} = g(k_1) - k_1$ where $g'(k_1) = 1$. Case 1 obtains when we have $x^* < \hat{x}$. Here, it is obvious that for q sufficiently close to unity we must have $H_2(q) < H_1(q)$. This is because $H_2(q) \leq x^*$ and $H_1(q) \equiv g(k_q) - k_q \rightarrow \hat{x}$ as $q \rightarrow 1$. This is summarized in the following proposition.

Proposition 8: If $x^* < \hat{x}$ then there exists $q \in (\beta, 1)$ such that $H_2(q) < H_1(q)$.

Proof. Obvious from the discussion above.

Now suppose we have case 2, where $x^* > \hat{x}$. Define $\hat{w}$, $\hat{c}_0$, and $\hat{\beta}$ as follows.

$$\hat{w} \equiv (1 - \beta)[\pi u(y + \hat{x}) + (1 - \pi)u(\hat{x})] + \beta \underline{w},$$

$$\pi u(y + \hat{c}_0) + (1 - \pi)u(\hat{c}_0) \equiv \hat{w},$$

$$\hat{\beta} \equiv \frac{\pi u'(y + \hat{c}_0)}{\pi u'(y + \hat{c}_0) + (1 - \pi)u(\hat{c}_0)}.$$

Note that $\hat{w} \in (\underline{w}, \bar{w})$ and $\hat{c}_0 \in (0, \hat{x})$.The former obtains because $0 < \hat{x} < x^*$ and

$$\underline{w} = \pi u(y) < \pi u(y + \hat{x}) + (1 - \pi)u(\hat{x}) < \pi u(y + x^*) + (1 - \pi)u(x^*) = \bar{w}.$$

The latter can be seen as follows. Let

$$h(c_0) = \pi u(y + c_0) + (1 - \pi)u(c_0).$$

Then,

$$h(0) = \underline{w} < \hat{w},$$

$$h(\hat{x}) = \pi u(y + \hat{x}) + (1 - \pi)u(\hat{x}) > \hat{w}.$$

[7] In particular, see Lemmas 10, 11, and 12, pp. 81-82 in Atkeson and Lucas (1995).

Further, $h(\cdot)$ is strictly increasing. Hence, there exists a unique $\hat{c}_0 \in (0, \hat{x})$ which satisfies $h(\hat{c}_0) = \hat{w}$.

We can now state a proposition giving conditions such that $H_2(q) < H_1(q)$ for some $q \in (\beta, 1)$ when case 2 obtains.

Proposition 9: Suppose $x^* > \hat{x}$. If $\beta < \hat{\beta}$ then there exists $q \in (\beta, 1)$ such that $H_2(q) < H_1(q)$.

Proof. Fix $w = \hat{w}$. Combining the promise keeping constraint (25) with the incentive compatibility constraint (26) at equality we have

$$\begin{aligned}\hat{w} &= (1-\beta)[\pi u(y + C(u_0(w))) + (1-\pi)u(C(u_0(w)))] + \beta w_0(w) \\ &\leq (1-\beta)[\pi u(y + C(u_0(w))) + (1-\pi)u(C(u_0(w)))] + \beta \hat{w}.\end{aligned}$$

Hence

$$\pi u(y + C(u_0(w))) + (1-\pi)u(C(u_0(w))) \geq \hat{w} = \pi u(y + \hat{c}_0) + (1-\pi)u(\hat{c}_0).$$

Therefore $C(u_0(w)) \geq \hat{c}_0$. Since the utility function $u(\cdot)$ is assumed to satisfy decreasing absolute risk aversion it is easy to verify that as a consequence we have

$$\hat{\beta} = \frac{\pi u'(y + \hat{c}_0)}{\pi u'(y + \hat{c}_0) + (1-\pi)u'(\hat{c}_0)} \leq \frac{\pi u'(y + C(u_0(w)))}{\pi u'(y + C(u_0(w))) + (1-\pi)u'(C(u_0(w)))}.$$

Therefore,

$$\beta < \frac{\pi u'(y + C(u_0(w)))}{\pi u'(y + C(u_0(w))) + (1-\pi)u'(C(u_0(w)))}. \tag{30}$$

Further,

$$\begin{aligned}\hat{w} &= (1-\beta)[\pi u(y + C(u_0(w))) + (1-\pi)u(C(u_0(w)))] + \beta w_0(w) \\ &\geq (1-\beta)[\pi u(y + C(u_0(w))) + (1-\pi)u(C(u_0(w)))] + \beta \underline{w}.\end{aligned}$$

It follows from the definition of $\hat{w}$ that $C(u_0(w)) \leq \hat{x}$. Therefore, $\tau_0 = C(u_0(w)) \leq \hat{x}$. Further, since $w_1(w) \geq w_0(w)$ we must have $u_1 \leq u(y + C(u_0(w)))$. Hence, $C(u_1(w)) \leq$

$y+C(u_0(w))$ and $\tau_1 = c_1 - y \leq C(u_0(w)) \leq \hat{x}$. It follows that total transfers associated with $w = \hat{w}$ are no greater than $\hat{x}$. We will now show that for q sufficiently close to unity we must have $w_1(w) < \hat{w}$. This will guarantee that the ergodic set for the stochastic process of $\{w_t\}$ will be contained in $[\underline{w}, \hat{w})$. Therefore, total transfers will be less than the transfers associated with $w = \hat{w}$. These transfers as shown above are no greater than $\hat{x} = g(k_1) - k_1$. It follows that $H_2(q) < \hat{x}$ for q sufficiently close to unity. Hence, we will have the desired result that $H_2(q) < H_1(q)$ for some $q \in [\beta, 1)$ because $H_1(q) \to \hat{x}$ as $q \to 1$. Now, to show that for q sufficiently close to unity we must have $w_1(w) < \hat{w}$ we proceed by contradiction. Suppose that there is $\varepsilon > 0$ such that $w_1(w) \geq \hat{w}$ for $q \in (1-\varepsilon, 1)$. Note that our assumptions imply that $0 < \tau_0 < x^*$ and $-y < \tau_1 < x^*$. The first obtains because $0 < \hat{c}_0 \leq C(u_0(w)) \leq \hat{x} < x^*$ and $\tau_0 = C(u_0(w))$. The second obtains because $u_1(w) \geq u_0(w) = u(C(u_0(w))) \geq u(\hat{c}_0) > 0$ which implies $C(u_1(w)) > 0$ in turn implying $\tau_1 = C(u_1(w)) - y \leq C(u_0(w)) \leq \hat{x} < x^*$. Therefore, the pairwise resource constraints are not binding. Hence, the first order conditions with respect to u_0 and u_1 together with the envelope condition can be use to derive the following equation.

$$V_q'(\hat{w}) = \frac{(1-\beta)[\pi\delta C'(u_1) + (1-\pi)C'(u_0)]}{(1-q)[\pi\delta + 1 - \pi]}$$

where

$$\delta = C'(u_0(w))u'(y + C(u_0(w))).$$

Further, the first-order condition for u_1 combined with that for $w_1(w)$ yields

$$\begin{aligned} C'(u_1(w)) &= \frac{(1-\beta)qV_q'(w_1(w))}{(1-q)\beta} \\ &\geq \frac{(1-\beta)qV_q'(\hat{w})}{(1-q)\beta} \\ &= \frac{q[\pi\delta C'(u_1(w)) + (1-\pi)C'(u_0(w))]}{\beta[\pi\delta + 1 - \pi]}. \end{aligned}$$

Dividing through by $C'(u_1)$, substituting for δ and rearranging we have

$$\begin{aligned} 1 &\geq \frac{q[\pi u'(y + C(u_0(w))) + (1-\pi)u'(C(u_1(w)))]}{\beta[\pi u'(y + C(u_0(w))) + (1-\pi)u'(C(u_0(w)))]} \\ &> \frac{q\pi u'(y + C(u_0(w)))}{\beta[\pi u'(y + C(u_0(w))) + (1-\pi)u'(C(u_0(w)))]}. \end{aligned}$$

However, by virtue of (30) the above can not hold for q sufficiently close to unity. This contradiction establishes that for q sufficiently close to unity we must have $w_1(w) < \hat{w}$.□

We can now state the following proposition.

Proposition 10: If either (i) $x^* < \hat{x}$ or (ii) $x^* > \hat{x}$ and $\beta < \hat{\beta}$ then there exists $q^* \in (\beta, 1)$ such that $H_2(q^*) = H_1(q^*)$.

Proof. Obvious.

The allocation and the stationary distribution associated with q^* are efficient and consistent with zero total net transfers from the planner.[8]

6. DISCUSSION OF RESULTS AND ASSUMPTIONS

The stochastic law of motion for $\{w_t\}$ (see Figure 2) has several of the features associated with a credit system, in particular credit balances, credit limits, and variable installments. We can interpret $w^* - \underline{w}$ as a buyer's credit limit, $w - \underline{w}$ as credit available, and $w^* - w$ as the current balance. If a buyer starts with $w = \underline{w}$ then he/she has no credit available. If no income is received then payment cannot be made, consumption is zero, and the buyer will continue to have no credit available. If the buyer receives positive income then a payment is made, the remainder is consumed, and

[8]The restriction on β required in case (ii) of Proposition 9 appears also in other dynamic models of insurance with private information. For example, the model of unemployment insurance in Atkeson and Lucas (1995) contains a similar restriction (see page 81) to guarantee the existence of efficient stationary allocations which support stationary distributions with zero transfers.

in the following period the buyer will have $w > \underline{w}$, that is the credit balance will be smaller than the credit limit. If the buyer continues to receive positive income then he/she continues to consume and to build up the credit balance until it reaches $w^* - \underline{w}$. Then, if the buyer receives no income, consumption is positive but the credit balance will be reduced. If the buyer continues to receive no income then he/she will eventually reach the credit limit with $w = \underline{w}$.

Note that the steady state distribution $\psi(w)$ will have the property that a positive mass of buyers is concentrated on the lower bound on expected utility entitlements, $\underline{w}$. That is, in the steady state there is a positive mass of agents who are credit-constrained. Now, suppose that we consider an alternative model with incomplete markets, for example Aiyagari (1994), with a borrowing constraint. In that model, if we had $u'(0) = \infty$ and a positive probability of obtaining a zero endowment in any period, as is the case here, then no agents would be credit-constrained. Thus, our private information model has some characteristics which are quite different from a model where markets are arbitrarily shut down.

Capital accumulation was introduced in this model so as to make the interest rate endogenous, which ultimately guarantees that $q > \beta$ and assures that the limiting distribution of expected utilities exhibits mobility. The implied interest rate is less than the time preference rate, and hence there is capital overaccumulation relative to the public information economy. Both of these features also obtain in incomplete markets models with borrowing constraints.

We now discuss the assumption of a capacity constraint x^* at stores and the restrictions (1) and (2). The restriction that at most x^* units of the consumption good can be shipped to each location is made for tractability. That is, if this constraint were not in place, and any quantity of goods could be shipped to each location, then the total output of goods available might be binding in the steady state, so that there would not be a straightforward determination of the steady state capital stock, as in

(17). The specific restrictions (1) and (2) guarantee that in any steady state $q > \beta$ and that transfers are limited by the capacity constraint x^* and not by available output. This permits a separation of the problem of choosing the capital stock from that of choosing transfers. That the specific nature of these assumptions guarantees $q > \beta$ should not be worrisome because if no capacity constraint were imposed then $q \leq \beta$ can never be a steady state.

This is because whenever $q \leq \beta$ the stationary distribution of expected utilities becomes degenerate at the highest feasible value.[9] Hence total transfers equal the highest feasible level of transfers which (in the absence of a capacity constraint) equal total output, i.e. $g(k)$. However, the associated cost is given by $g(k)-(g(k)-k) = k > 0$. Therefore, $q \leq \beta$ can never be consistent with a steady state with zero cost. Thus, the specific assumptions we made regarding the capacity constraint x^* to guarantee $q > \beta$ in a steady state do not taint our results concerning the nature of the stationary distribution.

The second inequality in (1) can also be motivated by a consideration of the problem under public information regarding endowments. It turns out that this inequality is a necessary condition for the capacity constraint to be non-binding and thereby permits the obvious full insurance outcome. This can be seen as follows.

While many full insurance steady state distributions exist under public information, we will focus on one where individuals have identical expected utility entitlements denoted w^*. The solution in this case is then given by $q = \beta$, $g'(\hat{k}) = \frac{1}{\beta}$, $c^* + \hat{k} = g(\hat{k}) + \pi y$, $w^* = u(c^*)$, $\tau_0 = c^*$, and $\tau_1 = c^* - y$. Here, $\hat{k}$ is the steady state capital stock, c^* is the constant level of consumption (i.e. there is full insurance), and τ_0 and τ_1 are the transfers in the bad and good states respectively. To make sure that the capacity

[9] We have shown that this applies for $q = \beta$, but it can be shown that this is also the case for $q < \beta$. Then $V_q'(\bar{w}) < \infty$ and there exists $\hat{w} \in (\underline{w}, \bar{w})$ such that $w_1(w) = \bar{w}$ for $w \in [\hat{w}, \bar{w}]$. Therefore, in a finite number of steps $\{w_t\} \rightarrow \bar{w}$.

constraint is not binding we need to have $\tau_0 = c^* \leq x^*$ that is $g(\hat{k}) - \hat{k} + \pi y \leq x^*$. Therefore, a necessary condition is $x^* > g(\hat{k}) - \hat{k}$, which is the second inequality in (1).

Finally, it is worth noting that the upper bound on transfers implied by random matching and the capacity constraint may or may not be binding in the steady state. That is, q^* and the corresponding steady state distribution ψ^* may be such that this constraint does not bind. In terms of Figure 2, this would happen if the transfer in the bad state when $w = w^*$ happens to be less than x^*. In such a case, the friction on resource movement implied by random matching does not matter and the allocation would continue to be efficient if all agents were assumed to be together at the central location at all times. Such an allocation would be a natural benchmark against which one could compare the steady state allocation with an arbitrary market structure and borrowing constraint studied in Aiyagari (1994).

7. SUMMARY AND CONCLUSION

Efficient steady state allocations in our random matching environment have features which resemble observed credit arrangements. That is, consumers visit a sequence of stores, receiving goods from the stores or transferring goods, with credit arrangements governed by a centralized credit agency. The key features of the steady state allocation can be interpreted in terms of a credit mechanism with credit balances, credit limits, and installment payments.

A novelty in the paper is that we obtain a steady state distribution of expected utility entitlements with mobility by modeling a capital accumulation economy with an endogenous interest rate. This assures that the social planner is more patient than consumers in the steady state. We do not impose any bounds on expected utility entitlements to obtain this result, as is done in other work (Atkeson and Lucas 1995, Phelan 1995). Indeed, ours is one of the first dynamic insurance economies with private information to include capital accumulation. If the friction on resource movement implied by random matching turns out not to matter, then the efficient steady state allocation in our model would be a natural and useful benchmark for comparing steady state allocations with those obtained subject to an arbitrary market structure and borrowing constraints as in Aiyagari (1994).

Future work will involve the introduction of monetary exchange into a similar environment. To do this requires that a friction be introduced which makes the credit mechanism imperfect, otherwise money will have no social role. A complication is that there are then two state variables in the social planner's optimization problem (the consumer's expected utility entitlement and her money balances) instead of one. We intend to use this environment to study optimal monetary policy.

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