

Asset Pricing when Risk Sharing is Limited by Default

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Abstract

We study the asset pricing implications of a multi-agent endowment economy where agents can default on debt. We build on the environment studied by Kocherlakota (1995) and Kehoe and Levine (1993). We present an equilibrium concept for an economy with complete markets and with endogenous solvency constraints. These solvency constraints prevent default, but at the cost of reduced risk sharing. We show that versions of the classical welfare theorems hold for this equilibrium definition. We characterize the pricing kernel, and compare it to the one for economies without participation constraints: interest rates are lower and risk premia depend on the covariance of the idiosyncratic and aggregate shocks.

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1. Introduction

Standard equilibrium asset pricing models have problems reproducing some of the basic facts in the data.¹ A promising direction for improvement along this dimension has been to maintain standard preferences and to allow for incomplete risk sharing across agents. Generally, in this class of models, financial markets are exogenously considered as incomplete. Compared to the representative agent model, these models have the attractive feature that agents cannot eliminate all idiosyncratic risk. For this reason, pricing kernels are generally more volatile than those of representative agent economies with the same aggregate consumption, which brings these models closer to the data. One drawback of studies following this approach, when compared with complete markets economies, is that they require more or less arbitrary assumptions about which set of securities is available. Model conclusions may in turn crucially depend on these assumptions. Another possible problem is tractability. Because finding equilibria of these models involves solving a complicated fixed point problem, it has been very difficult to analyze the case of many assets or many agents. In this paper, we study a class of models whose equilibrium, in general, entail limited risk sharing but that do not have some of these potential drawbacks.

Our approach for limiting risk sharing builds on work by Kehoe and Levine (1993) and Kocherlakota (1996).² These authors present and study efficient allocations in endowment economies where *participation constraints* ensure that agents would at no time be better off reverting permanently to autarchy. We show in our paper that these participation constraints can be modelled as portfolio constraints. We imagine a world, where, if agents default on some debt, they can be punished by seizing all the assets that they may own, but they cannot be punished by garnishing their labor income. In such an environment, risk sharing may be effectively reduced because agents with low income realizations can only borrow up to the amount they are willing to pay back in the future.

A main contribution of this paper is the introduction of a new equilibrium concept that emphasizes portfolio constraints: a competitive equilibrium with *solvency constraints*. Specifically, we focus on constraints that are tight enough to prevent default but allow as much risk sharing as possible. Except for the constraints, our equilibrium is identical to a Radner equilibrium with complete markets (*i.e.* a competitive equilibrium with a sequence of budget constraints).

¹The shortcomings of standard model versions are widely documented in studies such as Mehra and Prescott (1985), Cochrane and Hansen (1992).

²Earlier work in the sovereign debt literature by Eaton and Gersovitz (1981) first formalized the main idea of this approach.

These *endogenously* determined solvency constraints are agent and state specific and ensure that the participation constraints are fulfilled. This means that the amount of wealth that agents can carry to any particular date and event will never be small enough to make them choose to default and to revert to autarchy. At the same time, these constraints allow as much risk sharing as possible.

We show versions of the classical first and second welfare theorems for our equilibrium concept. Our decentralization differs from the one in Kehoe and Levine (1993) where the participation constraints are modelled as direct restrictions on the consumption possibility sets. We think that our decentralization relates more closely to the existing asset pricing literature that focuses on portfolio restrictions such as the work by He and Modest (1995) and Luttmer (1996) among others. Indeed, our decentralization provides a justification for such solvency constraints. Additionally, with our definition of equilibrium we obtain a simpler and a very intuitive representation of the prices of securities. In any event, both our equilibrium notion and that of Kehoe and Levine are closely related. We show the circumstances under which the equilibrium allocations are identical and the exact mapping between the two. This last point turns out to be important, since some results are easier to prove in one framework than in the other.

We use the welfare theorems to find conditions for the time discount factors, risk aversion, variance and persistence of endowments, under which autarchy is the only feasible allocation. These properties are important for quantitative predictions. Indeed, quantitative studies such as Mehra and Prescott (1984) and Cochrane and Hansen (1992) have shown that the same parameters are important for asset pricing results in representative agent models.

We present some properties of the pricing kernel. One-period contingent claims (Arrow securities) are priced by the agent with the highest marginal rate of substitution, which is the agent that is not constrained with respect to his holding of this asset. Thus the price of a contingent claim (an Arrow security) is equal to the highest marginal valuation across agents. Pricing of an arbitrary asset is accomplished by adding up the prices of the corresponding contingent claims (which, interestingly, does not need to coincide with the highest marginal valuation of the security across agents). As we mentioned before, this framework has two advantages for the purpose of asset pricing over the standard incomplete markets specifications. First, allocations do not depend on a particular arbitrary set of assets that is considered to be available. And second, with markets being complete, any security can be priced. Finally, we compare two properties of asset prices in an equilibrium with solvency constraints with the ones obtained in an identical economy, but without the solvency constraints. The first result is that interest rates are necessarily smaller in an economy with solvency constraints, in-

dependently of the precautionary motive (third derivative of the utility function), a feature emphasized in the literature. Second, we examine the following measure of the market price of risk: the excess return of a one-period security whose payments are a function of aggregate endowment. We show that unless relative endowment shocks are correlated with the aggregate shocks, this premium is the same that the one in an economy without solvency constraints. Third, we show as asset prices are determined by those agents with substantial individual risk to share. In particular, the presence of agents whose endowment is very similar to the aggregate endowment is irrelevant for asset pricing. Our companion paper (Alvarez and Jermann (1998)) contains further characterizations and a detailed analysis of quantitative properties.

The remainder of this paper is organized as follows. In section 2 we present the model environment. Section 3 presents some characteristics of efficient allocations. In Section 4 we introduce the competitive equilibrium with solvency constraints, show versions of the classical welfare theorems, relate the equilibrium concept to the one by Kehoe and Levine and analyze the pricing kernel. Section 5 concludes.

2. The Environment

We consider a pure exchange economy with I agents. Agents' endowments follow a finite state Markov process, agents' preferences are identical and given by time-separable expected discounted utility. We add to this simple environment participation constraints of the following form: the continuation utility implied by any allocation should be at least as high as the one implied by autarchy at any time and for any history.

Formally, we let $\{1, 2, \dots, I\} = I$, be the set of agents, with typical element i . We use $\{z_t\}$ to denote a Markov process with generic elements z , a member of the finite set Z , and transition probabilities given by matrix Π . We use $z^t = (z_0, z_1, z_2, \dots, z_t)$ to denote a length t history of z . The matrix Π generates conditional probabilities for histories that we denote as $\pi(z^t | z_0 = z)$. We use the symbol $\succeq$ for the partial order $z^{t'} \succeq z^t$ for $t' \geq t$ to indicate that $z^{t'}$ is a possible continuation of z^t , that is, that there exists a history z^s such that $z^{t'} = (z^t, z^s)$ for $s = t' - t$. We use the notation $\{c_i\}$ and $\{e_i\}$ for the stochastic process of consumption and endowment of each agent, hence $\{c_i\} = \{c_{i,t}(z^t) : \forall t \geq 0, z^t \in Z^t\}$. We define aggregate endowment as $e_t(z^t) \equiv \sum_{i \in I} e_{i,t}(z^t)$.

We assume that individual endowments are given by a function e_i , that depends only on z_t , so that $e_{i,t}(z^t) = e_i(z_t)$. We assume that $e_i(z) > 0$ for all i and z . We also assume that aggregate endowment, defined as $e(z) \equiv \sum_{i \in I} e_i(z)$,

is bounded for all $z \in Z$.

The utility for an agent corresponding to the consumption process $\{c\}$ starting at time t at history z^t is denoted by $U(c)(z^t)$ and is given by :

$$U(c)(z^t) \equiv \sum_{s=0}^{\infty} \sum_{z^{t+s} \in Z^{t+s}} \beta_{t,t+s}(z^{t+s-1}) u(c_{t+s}(z^{t+s})) \pi(z^{t+s}|z_t), \quad (2.1)$$

where u is the period utility and $\beta_{t,t+s}$ is a time discount factor. This implies that $U(\cdot)$ solves the recursion,

$$U(c)(z^t) = u(c(z_t)) + \beta_{t,t+1}(z_t) \sum_{z_{t+1} \in Z} U(c)(z^t, z_{t+1}) \pi(z_{t+1}|z_t).$$

We assume that $u : R_+ \rightarrow R$ is strictly increasing, strictly concave and C^1 . The multi-period time discount factor $\beta_{t,t+s+1}$ is defined recursively using the one period state contingent discount factor $\beta : Z \rightarrow (0, 1)$. Specifically, $\beta_{t,t}(z^{t-1}) \equiv 1$ for all z^{t-1} , $\beta_{t,t+1}(z^t) = \beta(z_t)$ for all z^t , and for $s > 1$, $\beta_{t,t+s+1} : Z^{t+s} \rightarrow [0, 1]$ satisfies

$$\beta_{t,t+s+1}(z^{t+s}) = \beta_{t,t+s}(z^{t+s-1}) \cdot \beta(z_{t+s}).$$

Note that the standard case with constant discount factor $\beta(z) = \bar{\beta}$ corresponds to $\beta_{t,t+s}(z^{t+s-1}) = \bar{\beta}^s$. As we will show below, letting the time discount factor to be state contingent, allows us to introduce stochastic growth as a special case within the same notational framework.

An allocation $\{c_i\}_{i \in I}$ is resource feasible if

$$\sum_{i=1}^I c_{i,t}(z^t) = e_t(z^t) \quad \forall t \geq 0, z^t \in Z^t, \quad (2.2)$$

and it satisfies the Participation Constraints if :

$$U(c_i)(z^t) \geq U(e_i)(z^t) \quad \forall t \geq 0, z^t \in Z^t. \quad (2.3)$$

Notice that $U(e_i)(z^t)$ depends only on z_t because $e_{i,t}(z^t)$ is a function of z_t .

Except for the state-contingent time discount factor our environment is a special case of the one by Kehoe and Levine (1993). In particular, we consider the case with one good and where the participation constraints have autarchy as the outside option.

2.1. Aggregate growth

Our specification of the time discount factor allows for aggregate growth to be introduced in the same fashion as for the specification of the aggregate endowment process in Mehra-Prescott (1985) and in much of the quantitative consumption based asset pricing literature. We show how an economy with stochastic growth, constant time discount factor and constant relative risk aversion can be expressed as an economy with constant aggregate endowment and state contingent discount factor, fitting the assumptions of the previous section.

Let

$$e_{t+1}(z^t, z_{t+1}) = e_t(z^t) \cdot \lambda(z_{t+1}) \quad \text{and} \quad e_{i,t}(z^t) = e_t(z^t) \cdot \epsilon_i(z_t) \text{ for } i \in \mathbf{I}$$

and define $\hat{e}_{i,t}(z^t) \equiv e_{i,t}(z^t) / e_t(z^t) = \epsilon_i(z_t)$ for all i so that $\hat{e}_t(z^t) = 1$ all z^t .

Assume a constant time discount factor β , then (2.1) satisfies

$$U(c_i)(z^t) = u(c_{i,t}(z^t)) + \beta \sum_{z_{t+1} \in Z} U(c_i)(z^t, z_{t+1}) \pi(z_{t+1}|z_t),$$

if additionally the period utility function is of the form $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some positive γ (for simplicity assume that $\gamma \neq 1$), defining $\hat{c}_{i,t}(z^t) \equiv c_{i,t}(z^t) / e_t(z^t)$, then $U(\cdot)$ satisfies

$$U(\hat{c}_i)(z^t) = \frac{(\hat{c}_{i,t}(z^t))^{1-\gamma}}{1-\gamma} + \hat{\beta}(z_t) \sum_{z_{t+1} \in Z} U(\hat{c}_i)(z^t, z_{t+1}) \hat{\pi}(z_{t+1}|z_t)$$

with probabilities and discount factor

$$\hat{\pi}(z'|z) \equiv \frac{\pi(z'|z) \cdot \lambda(z')^{1-\gamma}}{\sum_{z'} \pi(z'|z) \cdot \lambda(z')^{1-\gamma}} \text{ and } \hat{\beta}(z) \equiv \beta \cdot \sum_{z'} \pi(z'|z) \cdot \lambda(z')^{1-\gamma}.$$

Clearly, resource feasibility and participation constraints are satisfied for an allocation $\{c_i\}_{i \in \mathbf{I}}$ in an economy with aggregate growth $\lambda(\cdot)$ and constant discount factor β if and only if they are satisfied for the corresponding $\{\hat{c}_i\}_{i \in \mathbf{I}}$ allocation in the economy with constant aggregate endowment, discount factor $\hat{\beta}(\cdot)$ and probabilities $\hat{\pi}$. Moreover, the preference orderings are identical in the two corresponding economies.

3. Constrained efficient allocations

An allocation $\{c_i\}$ is feasible if it is resource feasible and satisfies the participation constraints. Constrained efficient allocations are feasible allocations that are not Pareto dominated by other feasible allocation.

Kocherlakota (1996) analyzes efficient allocations for a similar environment. Relative to his environment, we consider the case with state specific time discount factors, $I \geq 2$ agents and we allow for serially correlated shocks. Like in Kocherlakota (1996), efficient allocations may be characterized by risk sharing regimes of three types, depending on the parametric environment: Pareto efficiency in the usual unconstrained sense (full risk sharing), autarchy (no risk sharing) or limited risk sharing. In section (4.12) of our paper we will extend his characterization by showing under which conditions on the time discount factors, risk aversion, variance and persistence of endowments, autarchy is the only feasible allocation.

Given our focus on asset pricing, we first show a property of the marginal rates of substitution for efficient allocations that will be crucial for asset pricing. We show that an unconstrained agent has the highest marginal rate of substitution—and hence, all unconstrained agents equalize their marginal rates of substitution.

Proposition 3.1. *Let $\{c_i\}_{i \in I}$ be constrained efficient. If*

$$U(c_j)(z^t) > U(e_j)(z^t), \quad (3.1)$$

then

$$\beta(z) \frac{u'(c_{j,t+1}(z^t, z_{t+1}))}{u'(c_{j,t}(z^t))} \pi(z_{t+1}|z_t) = \max_{i \in I} \beta(z) \frac{u'(c_{i,t+1}(z^t, z_{t+1}))}{u'(c_{i,t}(z^t))} \pi(z_{t+1}|z_t).$$

4. Equilibrium, efficiency and asset prices

We define a competitive equilibrium with complete markets in Arrow securities and with solvency constraints. The solvency constraints prohibit agents from holding large amounts of contingent debt, hence preventing default. In general, these solvency constraints will be state-contingent, since the value of default (reverting to autarchy) varies with the state.³ We show the conditions under which

³In most of the quantitative asset pricing literature portfolio constraints are usually set exogenously and not as a function of the environment, see for instance Heaton and Lucas (1996) and Telmer (1993). Zhang (1997) computes numerical examples of an equilibrium subject to a borrowing constraint chosen so that agents are better off than in autarchy. Our model differs from Zhang's in that he exogenously limits agents to trade only one-period riskless bonds and that his borrowing constraint is neither time nor state dependent; additionally, there is no formal analysis of the relationship between equilibrium and efficient allocations. On the theory side, there is a substantial amount of work on how to model borrowing or solvency constraints in the infinite horizon incomplete markets model such as Levine and Zame (1996), Hernandez and Santos (1994), Santos and Woodford (1997) and Magill and Quinzii (1994). This work addresses a different question, namely what is the natural way to extend no Ponzi-game conditions to the incomplete market case.

efficient allocations can be decentralized as a competitive equilibrium with solvency constraints, and we show the conditions under which the first welfare theorem holds. We then use these results to characterize the extent of risk sharing, and analyze some properties of the asset pricing kernel.

4.1. Definition of competitive equilibrium with solvency constraints

Let $q_t(z^t, z')$ denote the period t , state z^t , price of one unit of the consumption good delivered at $t + 1$, contingent on the realization of $z_{t+1} = z'$, in terms of period t consumption goods. The holdings of agent i at t of this security are denoted by $a_{i,t+1}(z^t, z')$, and the lower limit on the holdings of agent i is denoted by $B_{i,t+1}(z^t, z')$. Following our notational convention, we use $\{q\}$, $\{a_i\}$ and $\{B_i\}$ for the corresponding stochastic process..

Definition 4.1. *An equilibrium with Solvency Constraints $\{B_i\}$ for initial conditions $\{a_{i,0}\}$ has quantities $\{c_i, a_i\}$ and prices $\{q\}$ such that for each i ,*

a. $\{c_i, a_i\}$ solves

$$J_{i,t}(a, z^t) = \max_{c, \{a_{z'}\}_{z' \in Z}} \left\{ u(c) + \beta(z) \sum_{z'} J_{i,t+1}(a_{z'}, (z^t, z')) \pi(z'|z_t) \right\} \quad (4.1)$$

$$e_{i,t}(z^t) + a = \sum_{z' \in Z} a_{z'} q_t(z^t, z') + c \quad (4.2)$$

$$a_{z'} \geq B_{i,t+1}(z^t, z') \quad \text{all } z' \in Z, \quad (4.3)$$

b.

$$\sum_{i \in I} c_{i,t}(z^t) = e_t(z^t), \quad \text{all } t, \text{ all } z^t$$

$$\sum_{i \in I} a_{i,t+1}(z^t, z') = 0, \quad \text{all } t, \text{ all } z^t, \text{ all } z'.$$

Notice that in problem (4.1) agents *do never contemplate the option of default*. Since the problem in (4.1) - (4.3) has a concave objective function and a convex feasible set the Euler and Transversality are sufficient to characterize the solution:

$$-u'(c_{i,t}(z^t)) q_t(z^t, z') + \beta(z) \pi(z'|z) u'(c_{i,t+1}(z^t, z')) \leq 0 \quad (4.4)$$

with equality if $a_{i,t+1} > B_{i,t+1}(z^t, z')$ and,

$$\lim_{t \rightarrow \infty} \sum_{z^t \in Z^t} \beta_{0,t}(z^{t-1}) u'(c_{i,t}(z^t)) \cdot [a_{i,t}(z^t) - B_{i,t}(z^t)] \cdot \pi(z^t|z_0) = 0. \quad (4.5)$$

Now we move to the analysis of the decision of default. We are interested in the following type of equilibrium with solvency constraints :

Definition 4.2. *An equilibrium with Solvency constraints that are not too tight is such that the solvency constraints satisfy*

$$J_{i,t+1} \left(B_{i,t+1} \left(z^{t+1} \right), z^{t+1} \right) = U \left(e_i \right) \left(z^{t+1} \right), \quad (4.6)$$

for all $t = 0, 1, \dots$ and for all $z^{t+1} \in Z^{t+1}$.

This condition ensures that solvency constraints prevent default by prohibiting agents from accumulating more contingent debt that they are willing to pay back. At the same time it allows agents as much insurance as possible. If the constraint binds and the continuation utility is strictly higher than the value of autarchy, the constraint could be relaxed a bit, without inducing the agent to default. Thus this conditions implies that

$$\begin{aligned} U(c_i)(z^t) &\geq U(e_i)(z^t) \quad \text{and} \\ U(c_i)(z^t) &= U(e_i)(z^t) \Leftrightarrow a_{i,t}(z^t) = B_{i,t}(z^t). \end{aligned} \quad (4.7)$$

Finally, we introduce the concept of *high implied interest rates*. This condition ensures finiteness of the value of endowment implied by a given allocation. Given an allocation $\{c_i^*\}$, for $i = 1, 2, \dots, I$, we define

$$q_t^*(z^t, z') \equiv \max_{i \in I} \left\{ \beta(z) \frac{u'(c_{i,t+1}^*(z^t, z'))}{u'(c_{i,t}^*(z^t))} \pi(z'|z) \right\}, \quad (4.8)$$

and

$$Q_0^*(z^t|z_0) = q_0^*(z_0, z_1) \cdot q_1^*(z_0, z_1, z_2) \cdots q_{t-1}^*(z^{t-1}, z_t). \quad (4.9)$$

Note, $Q_0^*(z^t|z_0)$ as the price of one unit of consumption delivered in state z^t in units of consumption of $t = 0$.

Definition 4.3. *The implied interest rates for the allocation $\{c_i^*\}$ are high if*

$$\sum_{t \geq 0} \sum_{z^t \in Z^t} Q_0^*(z^t|z_0) \left(\sum_{i \in I} c_{i,t}^*(z^t) \right) < +\infty. \quad (4.10)$$

4.2. Decentralizing optimal allocations: Second welfare theorem

In this section we characterize the allocations that can be supported as competitive equilibria with solvency constraints. In particular, we decentralize constrained efficient allocations, *i.e.*, we state a version of the second welfare theorem.

Proposition 4.4. *Given any allocation $\{c_i^*\}$ that satisfies : (a) resource feasibility at any time and event, (b) the participation constraints 2.3 for each agent at each time and event, (c) if the participation constraint of an agent does not bind then this agent has the higher marginal rate of substitution, *i.e.*, for all $t \geq 0$ and $z^t \in Z^t$, $z' \in Z$, then if for agent i*

$$U(c_i^*)(z', z^t) > U(e_i)(z', z^t) \implies \\ \beta(z) \frac{u'(c_{i,t+1}^*(z^t, z'))}{u'(c_{i,t}^*(z^t))} \pi(z'|z_t) = \max_{j \in I} \left\{ \beta(z) \frac{u'(c_{j,t+1}^*(z^t, z'))}{u'(c_{j,t}^*(z^t))} \pi(z'|z_t) \right\},$$

and, (d) the allocation has high implied interest rates, there exists (i) a process $\{B_i\}$, initial wealth $a_{i,0}$ and an asset holding process $\{a_i^*\}$ such that the plan $\{a_i^*, c_i^*\}$ is a competitive equilibrium for the solvency constraints $\{B_{i,t}\}$ and the initial wealth $a_{i,0}$. Moreover, (ii) the process for the solvency constraint $\{B_i\}$ can be chosen so that the solvency constraints for all agents satisfy 4.6 (are not too tight).

As a corollary of the previous proposition we get the second welfare theorem, since efficient allocations satisfy properties (a), (b) and (c) of proposition (4.4).

Corollary 4.5. *Any constrained efficient allocation that has high implied interest rates can be decentralized as a competitive equilibrium with solvency constraints where the constraints are not too tight.*

The connection between equilibrium allocations and efficient allocations is clear. In the equilibrium, solvency constraints that are not too tight take the place of the participation constraints of the efficient allocations. In equilibrium allocations, an agent whose solvency constraint is not binding has the highest marginal rate of substitution. In efficient allocations, an agent whose participation constraint does not bind has the highest marginal rate of substitution. Later we will show that for any constrained optimal allocation where there is some risk sharing the implied interest rates are high.

We finally establish that autarchy can always be decentralized as an equilibrium with solvency constraints that are not too tight.

Proposition 4.6. *The following*

$$\begin{aligned} c_{a,i,t}(z^t) &= e_{i,t}(z^t), \quad a_{a,i,t+1}(z^t, z') = B_{a,i,t}(z^t) = 0, \\ q_{a,t}(z^t, z') &\equiv \max_{i \in I} \beta(z) \frac{u'(e_{i,t+1}(z^t, z'))}{u'(e_{i,t}(z^t))} \pi(z'|z_t) \end{aligned} \quad (4.11)$$

for all $t \geq 0, z^t \in Z^t, z' \in Z$ and the initial conditions $a_{i,0} = 0, i \in I$ is an equilibrium with solvency constraint that are not too tight.

Notice that even though autarchy allocations can always be decentralized, in general, these are not constrained efficient allocations. Also notice that the implied interest rate in autarchy can be low, so that (4.10) may be violated.

4.3. 1st Welfare theorem

In this section we show the connection between the Kehoe-Levine (henceforth K-L) decentralization and an equilibrium with solvency constraints that are not too tight. First, we show that the implied Arrow prices in the K-L decentralization are equal to the highest marginal rate of substitution across agents, as in our equilibrium with solvency constraints that are not too tight. Second, we show that under weak conditions, the allocation of an equilibrium with solvency constraints corresponds to the allocations of a K-L equilibrium. An immediate consequence of these results is that we can use the first welfare theorem shown by K-L. And thus, we obtain a (qualified) version of the first welfare theorem for our equilibrium concept. That is, allocations corresponding to an equilibrium for solvency constraints that satisfy 4.6 (solvency constraints not too tight) and 4.10 (that have high implied interest rates) are constrained efficient.

Let us start setting up some notation. The problem for agent i in the K-L decentralization is

$$\max_{\{c_i\}} U(c_i)(z_0) \quad (4.12)$$

$$\text{s.t.} \quad : \quad p_0(c_i - e_i) \leq a_{i,0} \quad (4.13)$$

$$U(c_i)(z^t) \geq U(e_i)(z^t) \text{ for all } t \geq 0 \text{ and } z^t \in Z^t \quad (4.14)$$

where p_0 is a non-negative linear function. Thus, with the inequalities in (4.14) defining the consumption possibility set, an equilibrium is a standard Arrow-Debreu equilibrium, henceforth A-D.

Given p_0 , the dot product representation of the A-D prices is defined as follows. For any t and $z^t \in Z^t$ define

$$Q_0(z^t|z_0) \equiv p_0(\tilde{c})$$

where $\tilde{c}_s(z^s) = 0$ if $s \neq t$ and $z^s \neq z^t$ and otherwise $\tilde{c}_t(z^t) = 1$.

Definition 4.7. *The A-D prices are said to have a dot product representation if*

$$p_0(c) = \sum_{t \geq 0} \sum_{z^t \in Z^t} c_t(z^t) Q_0(z^t|z_0).$$

With this notation we can write the budget constraint of agent i as

$$\sum_{t \geq 0} \sum_{z^t \in Z^t} (c_{i,t}(z^t) - e_{i,t}(z^t)) Q_0(z^t|z_0) \leq a_{i,0}.$$

The corresponding Arrow prices are defined as

$$q_{0,t}(z^t, z') = \frac{Q_0(z^t, z'|z_0)}{Q_0(z^t|z_0)}.$$

Compared to our equilibrium concept the K-L equilibrium differs along two dimensions. First, they include the participation constraints in the consumption possibility sets, as opposed to our limits on borrowing (solvency constraints). Our approach makes the form of the pricing kernel immediate and allows us to relate directly to the literature on empirical asset pricing with solvency constraints such as the results of Luttmer (1996) and He and Modest (1995). Second, in the K-L decentralization agents are not given the option to default and walk away from their debts. That is, lifetime utility of consumption plans is limited within a lifetime budget constraint. Our equilibria with solvency constraints that are not too tight allow agents to default and break their budget constraint. Although agents have this option to default, they will prefer not to do so. We find this interpretation of the solvency constraint as enforcing individual rationality an attractive feature of our equilibrium concept.

The next proposition characterizes the Arrow prices in a K-L equilibrium.

Proposition 4.8. *Let $\{c_i\}, i = 1, \dots, I$, and $\{Q_0\}$ be the allocations and A-D prices corresponding to a K-L equilibrium. Let q_0 be the corresponding Arrow prices. Then*

$$q_{0,t}(z^t, z') = \max_{i \in I} \left\{ \beta(z) \frac{u'(c_{i,t+1}(z^t, z'))}{u'(c_{i,t}(z^t))} \pi(z'|z_t) \right\}, \quad (4.15)$$

and if

$$U(c_i)(z^t, z') > U(e_i)(z^t, z')$$

then

$$q_{0,t}(z^t, z') = \beta(z) \frac{u'(c_{i,t+1}(z^t, z'))}{u'(c_{i,t}(z^t))} \pi(z'|z_t).$$

Now we show that for any equilibrium with solvency constraints, if the implied interest rate are high and the constraints are not too tight, then the implied A-D prices and consumption allocations constitute a K-L equilibrium.

Proposition 4.9. Let $\{q, c_i, a_i\}$ be an equilibrium given the solvency constraints $\{B_i\}$ and the initial wealth a_{i0} . Assume that the A-D prices $\{Q\}$ implied by $\{q\}$ satisfy 4.10 (that the implied interest rates are high) and that the solvency constraints satisfy 4.6 (i.e. they are not too tight). Additionally, assume that for each $i \in \mathbf{I}$ there is a constant ξ_i such that for all t, z^t ,

$$|u(c_{i,t}(z^t))| \leq \xi_i \cdot u'(c_{i,t}(z^t)) \cdot c_{i,t}(z^t). \quad (4.16)$$

Then the consumption allocations $\{c_i\}$ and the A-D prices $\{Q_t\}$ constitute a K-L equilibrium.

Remark 1. The condition (4.16) is a joint requirement on the consumption allocation and the utility functions. It is satisfied automatically in several relevant cases. For instance, it holds for agent i if (a) $u(\cdot)$ has relative risk aversion different from one at zero consumption, i.e.:

$$\lim_{c \rightarrow 0} -\frac{c u''(c)}{u'(c)} \neq 1$$

which can be verified by repeated application of L'Hôpital's rule, or (b) $u'(0) < +\infty$, or (c) if consumption for agent i is uniformly bounded away from zero.

We obtain the first welfare theorem as a corollary of the previous proposition.

Corollary 4.10. Since a K-L equilibrium is a standard Arrow-Debreu equilibrium, its consumption allocation is efficient. Then by proposition (4.9) under the assumption of equation (4.16), an equilibrium with solvency constraints that are not too tight (i.e. they satisfy (4.6)) and with implied high interest rates (i.e. they satisfy 4.10) is efficient.

4.4. Solvency constraints and the extent of risk sharing

In this section we use the equivalence between a K-L equilibrium, equilibrium with solvency constraints that are not too tight and optimal allocations to address three issues. The first is to characterize the circumstances under which autarchy is the only feasible allocation. The second is to show that if an optimal allocation is different from autarchy, then it has high implied interest rates. The third is to show that solvency constraints are negative.

Proposition 4.11. *Consider the autarchy allocation, i.e., the one where the consumption process $\{c_i\}$ is equal to the endowment process $\{e_i\}$ for all i . If the implied interest rates for this allocation are high (i.e. satisfy 4.10) then autarchy is a constrained efficient allocation, and hence is the only feasible allocation.*

In the next proposition we give sufficient conditions so that autarchy is the only feasible allocation. For this proposition let

$$\Pi(\delta) \equiv \delta I + (1 - \delta) \Pi$$

where δ indexes the persistence of $\Pi(\delta)$.

Proposition 4.12. *Autarchy is the only feasible allocation in either of the four cases: (i) the time discount factor is sufficiently small (i.e. $\max_z \beta(z) \downarrow 0$), (ii) risk aversion is sufficiently small uniformly, (iii) the variance of the idiosyncratic shock is sufficiently close to zero ($\max_{i,z,\tilde{z}} |\epsilon_i(\tilde{z}) - \epsilon_i(z)| \rightarrow 0$), and (iv) the transition probability matrix is sufficiently close to identity, (i.e. $\delta \uparrow 1$).*

The previous proposition has clear implications for asset pricing. In particular, it suggests the type of parameter values needed so that idiosyncratic shocks generate high and volatile pricing kernels. These are the properties found in many empirical studies, such as Hansen and Cochrane (1992).

Next we show that for any constrained efficient allocation where some risk sharing is possible, the implied interest rates are high. This result complements our statement of the second welfare theorem, that uses as an assumption that the implied interest rates are high. These two results imply that efficient allocation with some risk sharing can be decentralized as an equilibrium with solvency constraints that are not too tight.

Proposition 4.13. *Let $\{c_i\}$ be a constrained efficient allocation. Assume that some risk sharing is possible, so that for each t , z^t , there is $z' \in Z$ such that one of the agent $j \in I$:*

$$U(c_j)(z^t, z') > U(e_j)(z^t, z'). \quad (4.17)$$

Then 4.10 is satisfied (i.e. the implied interest rates are high).

We end this section by showing that the solvency constraints are negative. Indeed, in the definition of an equilibrium with solvency constraints there is no requirement that the constraints are negative.⁴ We show that, for the cases of high interest rates, solvency constraints are effectively constraints on contingent borrowing *i.e.* they satisfy $B_{i,t+1} < 0$.

Proposition 4.14. *If $\{q^*, c_i^*, a_i^*, J_i^*, B_i^*, \}, a_{i,0}^*$ is an equilibrium with solvency constraints that are not too tight, where the implied interest rates are high, then there is an equilibrium with solvency constraints that are not too tight with $\{q^*, c_i^*, a_i^*, \bar{J}_i, \bar{B}_i, a_{i,0}^*\}$ where $\bar{B}_{i,t+1}(z^t, z') \leq 0$, with strict inequality if $U(c_i^*)(z^t) = U(e_i)(z^t)$ and $\exists z^s \succeq z^t$ $c_{i,s}^*(z^s) \neq e_{i,s}(z^s)$.*

4.5. Properties of asset returns in economies with solvency constraints

In this section we analyze some properties of the pricing kernels in economies with solvency constraints. We start by describing the pricing of securities that are more complex than Arrow securities. We then compare pricing implications in economies with and without participation constraints by considering marginal valuations, interest rates and risk premia. Finally we present a simple irrelevance result.

4.5.1. Pricing complex securities

Instead of allowing agents to trade only Arrow securities (one-period contingent claims) we want to let them trade any security, particularly multiperiod securities such as stocks and bonds. A straightforward extension of our framework allows us to do this. It turns out that the pricing of any security (under certain conditions stated below) can be obtained by pricing the corresponding portfolio of Arrow securities

We assume that at time t the set of securities that could have possibly been bought (or sold) at dates and states prior to t , z^t is given by $K_t(z^t)$. These securities have prices $q_{k,t}$ and may pay dividends $d_{k,t}$ at multiple dates. Analogously the set $\{K_{t+1}(z^t, z') : z' \in Z\}$ contains the set of securities that can be bought (or sold) at time and state t, z^t . In this case, agents face the following sequence of budget constraints for all $t = 0, 1, \dots$ and for all $z^t \in Z^t$:

$$\sum_{z' \in K_{t+1}(z^t, z')} a_{i,t+1,k'}(z^t) q_{k,t'}(z^t) + c_{i,t}(z^t)$$

⁴ A positive $B_{i,t+1}$ means that agent i has to save a minimum amount.

$$\leq \sum_{k \in K_t(z^t)} a_{i,t,k}(z^{t-1}) \left[q_{t,k}(z^t) + d_{t,k}(z^t) \right] + e_{i,t}(z^t).$$

To have the same budget set as in the case with Arrow securities we need two conditions: first, that this set of securities be rich enough so that markets are dynamically complete, and second, that the portfolio choice be restricted so that the value of an agent's portfolio be high enough in the different states. Specifically, for each period, a set of solvency constraints that limits the minimum value of next period's portfolio for each state is defined as :

$$\sum_{k' \in K_{t+1}(z^t, z')} \left[q_{t+1,k'}(z^t, z') + d_{t+1,k'}(z^t, z') \right] a_{i,t+1,k'}(z^t, z') \geq B_{i,t+1}(z^t, z')$$

for all $t = 0, 1, \dots$ and for all $z^t \in Z^t$. Clearly, every security is just a portfolio contingent claims and can be valued as such.

4.5.2. Prices and marginal valuations

Given the frictions introduced by solvency constraints, we find that the prices of securities with non-negative payoffs are generally higher than any of the agents' valuation. Superficially, this result may seem to imply an arbitrage opportunity, in the sense that the prices are too high for everyone, but recall that agents can short sell only limited amounts of securities.⁵

Assume that k' is a security available at time t , i.e. $k' \in K_{t+1}(z^t, z')$ and that

$$d_{t+s,k'}(z^{t+s}) \geq 0 \text{ for all } s > 0.$$

Let us denote the price of this security for the equilibrium with solvency constraints by $q_{t,k'}$ and the marginal valuation of agent i by $MV_{i,t,k'}$ where the marginal valuation is defined as

$$MV_{i,t,k'}(z^t) \equiv \sum_{s>0} \sum_{z^s \in Z^s} \beta_{t,t+s}(z^{t+s-1}) d_{t+s,k'}(z^t, z^s) \frac{u'(c_{i,t+s}(z^t, z^s))}{u'(c_{i,t}(z^t))} \pi(z^s | z_t).$$

This quantity measures the marginal change in utility, in terms of time t consumption, produced by an increase in the i^{th} agent's consumption that is proportional to the dividends of security k' at each future date. When agents are never constrained, or equivalently for the case of perfect insurance, we have that for all i ,

$$q_{t,k'}(z^t) = MV_{i,t,k'}(z^t).$$

In our environment we have the following result:

⁵ We thank George Constantinides for highlighting this point.

Lemma 4.15. *In an equilibrium with solvency constraints, when there is only limited risk sharing, then*

$$q_{t,k'}(z^t) \geq \max_{i \in I} MV_{i,t,k'}(z^t)$$

for all agents i , securities k' , time periods $t = 0, 1, \dots$, and states $z^t \in Z^t$. With strict inequality for an agent i if he is constrained at least once between t and $t + s$.

Notice that since we assume that the dividends are non-negative, increasing the holding of security k' can never lead to default in the future if the original plan did not already contemplate default. One may therefore be tempted to conclude that in equilibrium $q_{t,k'}(z^t) = \max_{i \in I} MV_{i,t,k'}(z^t)$. The misleading part of this conjecture is that this equality holds only for Arrow securities and not for general securities. The latter may pay in different states, but the pricing kernel is defined by the *max* across agents of the marginal rate of substitution, state by state. That is, the agent whose marginal rate of substitution is equal to the price of the Arrow security in a given state may not price the Arrow security in a different state.

4.5.3. Interest rates and security prices

For several relevant cases, we can show that interest rates are smaller in economies with solvency constraints than in corresponding economies without such constraints. Moreover, as opposed to the findings in some applications of incomplete markets economies, this effect does not rely on the precautionary savings motive (convexity of the marginal utility).

Proposition 4.16. *Arrow prices in an economy with solvency constraints are higher than in an otherwise identical economy without solvency constraints in either one of the following three cases : (i) $\sum_i \epsilon_i(z) = 1$, all z , i.e., constant aggregate endowment, (ii) $u(\cdot)$ has constant relative risk aversion, or (iii) $u(\cdot)$ is quadratic.*

We remind the reader that since the proposition holds for the case with constant relative risk aversion, using the mapping described in (2.1), it will also hold for the case with aggregate growth.

For the cases described in Proposition (4.16), the price of a one-period bond is higher in an economy with solvency constraints than in an otherwise identical economy without these constraints. It is strictly so, if one agent at least is constrained in each period. Regardless of this, the unconditional mean of the risk

free rate is strictly lower in the solvency constraints economy. By the same logic as in the proposition, in these cases, any security with non-negative payouts will have a weakly higher price in the solvency constraints economy compared to the corresponding representative agent one.

4.5.4. The premium for aggregate risk with idiosyncratic shocks

A major issue for any “heterogenous-agent” model of asset pricing is the mechanism through which idiosyncratic shocks can generate a risk premium for claims contingent on the aggregate shock, that is, the market risk premium. In this section we consider the endowment specification with aggregate growth, constant relative risk aversion and constant discount factor described in section (2.1). We derive sufficient conditions under which the economy with idiosyncratic shocks generates a risk premium on a one-period risky strip identical to the representative agent economy.

We specialize the endowment process so that $\lambda(\cdot)$ and, $\epsilon(\cdot) = \{\epsilon_i(\cdot)\}_{i \in I}$ are statistically independent and $\lambda(\cdot)$ is i.i.d. Assume that z can be written as $z = (x, y) \in Z = X \times Y$ and that λ and ϵ are functions of y and x , specifically: $\lambda : Y \rightarrow R$ and $\epsilon : X \rightarrow \Delta^I$, the I – th dimensional simplex.

Definition 4.17. We say that the aggregate shock is i.i.d. and independent of the idiosyncratic shock if there exists a probability distribution ϕ and a stochastic matrix ψ such that

$$\pi(z'|z) \equiv \pi((x', y') | (x, y)) = \phi(y') \cdot \psi(x'|x)$$

for all z, z' .

We consider the risk premium for one-period risky strips. A strip, is a security that pays a random dividend d_{t+1} only at one period. We consider a one-period strip whose payout is function only of the aggregate output, y_{t+1} , and hence whose price equals $\sum_{z_{t+1}} q_t(z^{t+1}) d_{t+1}(y_{t+1})$. Equation (4.18) defines the premium as the expected strip return divided by the risk free rate,

$$\left[\frac{\sum_{z_{t+1}} d_{t+1}(y_{t+1}) \pi(z_{t+1}|z_t)}{\sum_{z_{t+1}} q_t(z^{t+1}) d_{t+1}(y_{t+1})} \right] / \left[\frac{1}{\sum_{z_{t+1}} q_t(z^{t+1})} \right], \quad (4.18)$$

where $\left[1 / \sum_{z_{t+1}} q_t(z^{t+1}) \right]$ is the risk free rate. This is sometimes called the “multiplicative excess return of the one-period risky strip”. We choose this premium as opposed to the equity premium for its tractability. Indeed, the equity premium is a weighted sum of the entire infinite sequence of strips, one for each period.

Proposition 4.18. *If endowments and preferences are specified as in section 2.1 and if the aggregate shock is i.i.d. and independent of the idiosyncratic shock, then the multiplicative premium on a one-period risky strip with payout contingent on the aggregate output is the same in an economy with and without participation constraints.*

This proposition shows that dependence of the cross sectional distribution of earnings on the aggregate is required to obtain interesting results for excess returns.⁶ This result complements similar results by Mankiw (1986) and Constantinides and Duffie (1996) that were obtained in environments with exogenous asset market incompleteness.

4.5.5. Irrelevance of the average agent

The next result is important for thinking about quantitative implementation of this equilibrium concept. Consider first an equilibrium with solvency that are not too tight and add one agent to that economy. If the marginal rates of substitution under autarchy of this agent is smaller than the equilibrium prices, then there is an equilibrium for the expanded economy with the same prices. Additionally, the extra agent will be constrained all the time.

Proposition 4.19. *Let the processes $\{c_i^*, a_i^*, q^*, B_i^*\}$, and the initial conditions $a_{i,0}^*$ for $i = 1, 2, \dots, I$ be an equilibrium with solvency constraints that are not too tight. Add to the previous economy one agent with endowment process $\{e_{I+1}\}$. Assume that for all z^t and z_{t+1}*

$$\beta(z) \frac{u'(e_{I+1,t+1}(z^t, z_{t+1}))}{u'(e_{I+1,t}(z^t))} \pi(z_{t+1}|z_t) \leq q_t^*(z^t, z_{t+1}). \quad (4.19)$$

Then $\{c_i^, a_i^*, q^*, B_i^*\}$, for $i = 1, 2, \dots, I+1$ and the initial condition $a_{i,0}^*$ for $i = 1, 2, \dots, I+1$ is an equilibrium with solvency constraints that are not too tight where $\{c_{I+1}^*\} = \{e_{I+1}\}$, $\{a_{I+1}^*\} = \{0\}$ and $\{B_{I+1}^*\} = \{0\}$ and initial condition $a_{I+1,0}^* = 0$.*

The example in the next remark says that adding an extra agent whose endowment is perfectly correlated with aggregate endowment will have no effect on equilibrium prices.

⁶This result requires not only independence of the aggregate and idiosyncratic shocks, but also that the aggregate growth rates are i.i.d. Given that aggregate consumption has a very small forecastable component, this last assumption imposes a very mild restriction.

Remark 2. Let $\alpha > 0$ be a constant and $e_{I+1,t}(z^t) = \alpha \cdot e_t(z^t)$ for all t and z^t where e_t is the aggregate endowment for the economy with I agents. Assume that $u(\cdot)$ has constant relative risk aversion. Then, by using proposition (4.16) one obtains immediately that assumption (4.19) holds.

Agent $I + 1$ has no reason to participate in future insurance arrangements with the I agents, and thus he will default if he acquires any debt. By continuity, agents with individual endowment processes that are very similar to the aggregate endowment are not important for the determination of asset prices. Instead, only the agents with substantial idiosyncratic risk are key to determining asset prices.

5. Conclusions

We have presented a framework for analyzing asset prices where endogenous solvency constraints may end up limiting risk sharing. We have proposed a concept of a market equilibrium with explicit endogenous portfolio constraints. We derived the classical first and second welfare theorems and studied the pricing kernel that emerges in this setup. We derive some general properties of asset prices, for example we show that interest rates in the environment with solvency constraints are lower than in the corresponding representative agent economy, without the need for precautionary saving.

We view this paper as a first step in exploring asset pricing relationships in an economy where the possibility of default limits risk sharing. In a companion paper (Alvarez and Jermann (1998)) we have started examining the quantitative side more in depth. Given the characterization of the equilibrium allocations of this paper we are able to design simple and fast algorithms for computing efficient allocations. Applying these allows us to address several quantitative issues. We are interested in the first moments such as the mean risk free rate and the equity premium that have attracted so much attention in the recent literature on equilibrium asset pricing. We are also interested in the business cycle behavior of excess returns and the term structure that have been documented empirically.

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Appendix

Proof. Proposition (3.1)

The proof is based on a simple variational argument, so we only sketch it here. Let $i \neq j$ be the agent with the highest marginal rate of substitution. One can increase current consumption and decrease future consumption of agent j in state z_{t+1} so as to keep $U(c_j)(z^t)$ constant. This is feasible since there is slack in (3.1). By decreasing current consumption and increasing future consumption of agent i in state z_{t+1} while keeping material balance, $U(c_i)(z^t)$ increases since i 's marginal rate of substitution is the higher of the two, which is in contradiction to constrained efficiency. ■

Proof. Proposition (4.4)

The proof is by construction. Given the allocation, use 4.8 and 4.9 to define prices. Since by assumption the implied interest rates are high, we use the budget constraint (4.2) to construct asset holdings at each time and event so that $\{c_i^*\}$ is budget feasible. When an agent's marginal rate of substitution is not the highest one, let the solvency constraint equal the holding of the corresponding Arrow security. When an agent's marginal rate of substitution is the highest, we will define, as an intermediate step, solvency constraints $\{B_i^*\}$ such that

$$\tilde{B}_{i,t+1}(z^t, z') = - \sum_{z^s \succeq z^{t+1}} Q_s(z^s | z^{t+1}) e_{i,s}(z^s),$$

we will redefine the solvency constraints later. The sufficient Euler and Transversality conditions for individual maximization can now be checked. The Euler conditions (4.4) follows from the definition of the Arrow prices (4.8) and the assumption that the unconstrained agent has the highest marginal rate of substitution. Now we can check the transversality condition :

$$\begin{aligned} & \lim_{T \rightarrow \infty} \sum_{z^T \in Z^T} \beta_{0,T}(z^{T-1}) u'(c_{i,T}^*(z^T)) [a_{i,T}^*(z^T) - \tilde{B}_{i,T}(z^T)] \pi(z^T | z_0) \\ & \leq \lim_{T \rightarrow \infty} \sum_{z^T \in Z^T} \beta_{0,T}(z^{T-1}) u'(c_{i,T}^*(z^T)) \left[\sum_{z^s \succeq z^T} Q_s^*(z^s | z^T) c_{i,s}^*(z^s) \right] \pi(z^T | z_0) \\ & \leq \lim_{T \rightarrow \infty} \sum_{z^T \in Z^T} \beta_{0,T}(z^{T-1}) u'(c_{i,T}^*(z^T)) \left[\sum_{z^s \succeq z^T} Q_s^*(z^s | z^T) e_s(z^s) \right] \pi(z^T | z_0) \\ & \leq u'(c_{i,0}(z_0)) \lim_{T \rightarrow \infty} \sum_{z^T \in Z^T} Q^*(z^T | z_0) \sum_{z^s \succeq z^T} Q_s^*(z^s | z^T) e_s(z^s) = 0 \end{aligned}$$

where the first inequality follows since by construction $a_{i,T}^*(z^T) - \tilde{B}_{i,T}(z^T) = 0$

if the participation constraint binds or equal to $\sum_{z^s \succeq z^T} Q_s^* (z^s | z^T) c_{i,s}^* (z^s) \geq 0$ otherwise, the second because $c_{i,s}^* (z^s) \leq e_s (z^s)$ since the allocation is feasible, the third follows from the definition of Q_i^* 's in terms of product of the q_i^* and the relationship of the q_i^* 's with the marginal rate of substitution, and the final equality from the assumption that the implied interest rates are high.

Using the prices and solvency constraints we construct the function $J_{i,t}$, which is attained by $\{c_i^*\}$. Finally we use $J_{i,t}$ to redefine the values of the solvency constraints for the cases where an agent has the highest MRS, that is, when the constraints do not bind. In particular, we define the solvency constraints $\{B_i^*\}$ as so that they solve

$$J_{i,t+1} (B_{i,t+1}^* (z^t, z'), (z^t, z')) = U (e_i) (z^t, z'),$$

which has a solution since the $J_{i,t} (\cdot, z^t)$ are strictly decreasing. Clearly $\{c_i^*\}$ and $\{a_i^*\}$ still solve the problem (4.1) for the same prices and solvency constraints $\{B_i^*\}$, since the feasible set is smaller, but the original plan is still feasible. ■

Proof. Proposition (4.6)

It is immediate to verify that $\{c_{a,i}, a_{a,i}, q_a, B_{a,i}\}$ satisfy the sufficient conditions (4.4) and (4.5), resource feasibility and market clearing. Moreover $\{B_{a,i}\}$ are not too tight, since trivially $J_{i,t} (a_{a,i,t} (z^t), z^t) = J_{i,t} (B_{a,i,t} (z^t), z^t) = U (e_i) (z^t)$ for all t, z^t . ■

Proof. Proposition (4.8)

It follows from the following variational argument around the consumption allocation in a K-L equilibrium. For an arbitrary date ("the current date") allow consumption to decrease at the current date and to increase in the next at a particular state, so as to keep the same expenditure at the K-L prices. Chose the values of current and future consumption to maximize utility. Notice that the participation constraints at the future date will be satisfied since consumption can only increase in the future. Also, the participation constraints at the current date will be satisfied (at the maximum), since current continuation utility cannot decrease, given that no deviation is feasible for the modified problem. This problem has a strictly concave differentiable objective function and linear constraints, so the Kuhn-Tucker conditions are necessary. The result follows immediately from the Kuhn-Tucker conditions. ■

Proof. Proposition (4.9)

In an equilibrium, for given solvency constraints, the consumption allocation is feasible. If the solvency constraints are not too tight, the consumption allocations satisfy the participation constraints for each agent at all times and states. It only remains to be shown that given the A-D prices implied by $\{q\}$ the consumption

allocation maximizes utility subject to budget and participation constraints. It will suffice to find non-negative shadow values (multipliers) associated with the budget constraints and participation constraints and verify that they are a saddle. The following algorithm defines these multipliers as a function of the consumption allocation, Arrow prices and participation constraints. The multiplier on the A-D budget constraint (4.13) is set to

$$\zeta_i = \frac{u'(c_{i,0}(z_0))}{Q(z_0|z_0)} = u'(c_{i,0}(z_0)).$$

The multiplier for the participation constraints (4.14) at time $t = 0$ is $\eta_{i,0}(z_0) = 0$. The multipliers for the participation constraints (4.14) for all $t > 0$ and $z^t \in Z^t$ are defined recursively by

$$\beta_{0,t}(z^{t-1}) u'(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) = \zeta_i Q(z^t|z_0). \quad (5.1)$$

Finally, one verifies that these multipliers together with the consumption allocation are indeed a saddle. This is accomplished by verifying the first order conditions for a saddle. The fact that Arrow prices are defined as in (4.15) is used to show that the multipliers are non-negative.

We define the Lagrangian as follows :

$$L(\{\bar{c}_i\}, \{\bar{\zeta}_i\}, \{\bar{\eta}_i\}) \quad (5.2)$$

$$= U(\bar{c}_i)(z_0) + \bar{\zeta}_i \left[a_{i0} + \sum_{z^t \succeq z_0} Q(z^t|z_0) (e_i(z^t) - \bar{c}_{i,t}(z^t)) \right] \quad (5.3)$$

$$+ \sum_{z^t \succeq z_0} \beta_{0,t}(z^{t-1}) \bar{\eta}_{i,t}(z^t) [U(\bar{c}_i)(z^t) - U(e_i)(z^t)] \pi(z^t|z_0)$$

for any non-negative consumption plan $\{\bar{c}_i\}$ and non-negative multipliers $\bar{\zeta}_i, \{\bar{\eta}_i\}$.

First, by construction of the multiplier, it is straightforward to see that $\zeta_i, \{\eta_i\}$ minimizes $L(\{\zeta_i\}, \cdot, \cdot)$. Second, we turn to establish that $\{c_i\}$ maximizes $L(\cdot, \zeta_i, \{\eta_i\})$. In order to do so we interchange the order of summations, so as a technical requirement, we first check that each of the terms of the sum

$$\sum_{z^t \succeq z_0} \beta_{0,t}(z^{t-1}) \eta_{i,t}(z^t) U(e_i)(z^t) \pi(z^t|z_0) \text{ and } \sum_{z^t \succeq z_0} \beta_{0,t}(z^{t-1}) \eta_{i,t}(z^t) U(c_i)(z^t) \pi(z^t|z_0)$$

converges to the same finite limit.

Using the definitions of $U(c_i)(z^t)$, using (5.1) for the multipliers $\{\eta_i\}$, collecting terms on $u(c_{i,t}(z^t))$ and using the assumption (4.16), one obtains

$$\begin{aligned}
& \left| U(c_i)(z_0) + \sum_{t \geq 0} \sum_{z^t \in Z^t} \beta_{0,t}(z^{t-1}) \eta_{i,t}(z^t) U(c_i)(z^t) \pi(z^t|z_0) \right| \quad (5.4) \\
&= \left| \sum_{t \geq 0} \sum_{z^t \in Z^t} \beta_{0,t}(z^{t-1}) u(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) \right| \\
&\leq \xi_i \cdot u'(c_{i,0}(z_0)) \cdot \sum_{t \geq 0} \sum_{z^t \in Z^t} Q(z^t|z_0) c_{i,t}(z^t)
\end{aligned}$$

which is finite by the assumption that the implied interest rates are high.

By the previous arguments about the convergence of each component of the sums it is equivalent to verify that

$$\begin{aligned}
& U(c_i)(z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q(z^t|z_0) c_{i,t}(z^t) \right] + \sum_{z^t \succeq z_0} \beta_{0,t}(z^{t-1}) \eta_{i,t}(z^t) U(c_i)(z^t) \pi(z^t|z_0) \\
&\geq U(\bar{c}_i)(z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q(z^t|z_0) \bar{c}_{i,t}(z^t) \right] + \sum_{z^t \succeq z_0} \beta_{0,t}(z^{t-1}) \eta_{i,t}(z^t) U(\bar{c}_i)(z^t) \pi(z^t|z_0).
\end{aligned}$$

By collecting terms on $u(\bar{c}_{i,t}(z^t))$ we can write

$$\begin{aligned}
& \sum_{z^t \succeq z_0} \beta_{0,t}(z^{t-1}) u(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q(z^t|z_0) c_{i,t}(z^t) \right] \\
&\geq \sum_{z^t \succeq z_0} \beta_{0,t}(z^{t-1}) u(\bar{c}_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q(z^t|z_0) \bar{c}_{i,t}(z^t) \right].
\end{aligned}$$

By concavity and differentiability of u we have

$$u(\bar{c}_{i,t}(z^t)) \leq u(c_{i,t}(z^t)) + u'(c_{i,t}(z^t)) [\bar{c}_{i,t}(z^t) - c_{i,t}(z^t)] \quad (5.5)$$

and by definition of the $\{\eta_i\}$

$$\beta_{0,t}(z^{t-1}) u'(c_{i,t}(z^t)) \left[\sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) - \zeta_i Q(z^t|z_0) = 0. \quad (5.6)$$

Thus using the inequality (5.5), the equality (5.6) and rearranging, we obtain the desired inequality:

$$\begin{aligned}
& \sum_{z^t \succeq z_0} \beta_{0,t} (z^{t-1}) u(\bar{c}_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q(z^t|z_0) \bar{c}_{i,t}(z^t) \right] \\
& \leq \sum_{z^t \succeq z_0} \beta_{0,t} (z^{t-1}) u(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) \\
& \quad + \sum_{z^t \succeq z_0} \beta_{0,t} (z^{t-1}) u'(c_{i,t}(z^t)) [\bar{c}_{i,t}(z^t) - c_{i,t}(z^t)] \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) \\
& \quad - \zeta_i \left[\sum_{z^t \succeq z_0} Q(z^t|z_0) \bar{c}_{i,t}(z) \right] \\
& = \sum_{z^t \succeq z_0} \beta_{0,t} (z^{t-1}) u(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q(z^t|z_0) c_{i,t}(z) \right].
\end{aligned}$$

■

Proof. Proposition (4.11)

Recall that Proposition (4.6) shows that autarchy is an equilibrium with solvency constraints that are not too tight for $B_{i,t} = 0$ and $a_{i,0} = 0$. Then the result follows using the first welfare theorem (4.10) since by assumption the implied interest rates are high. ■

Proof. Proposition (4.12)

We need to show that under the given condition the autarchy-value of endowment converges to a finite limit. In autarchy, Arrow prices depend only on the current and past z , so they are Markov. Thus the value of the aggregate endowment satisfies a simple recursion. This recursion satisfies monotonicity and discounts at a rate that depends on assumptions (i) to (iv). We show below that this rate is strictly smaller than one for the limit values in (i) to (iv).

Let's denote the value of aggregate endowment evaluated at autarchy prices $q_a(z, z')$ conditional on current shock z as $A(z)$. This value satisfies the following recursion

$$\begin{aligned}
q_a(z, z') & \equiv \beta(z) \max_{i \in I} \left\{ \frac{u'(\epsilon_i(z'))}{u'(\epsilon_i(z))} \right\} \pi(z'|z) \\
A(z) & = \sum_{z' \in Z} q_a(z, z') [e(z') + A(z')]
\end{aligned}$$

or equivalently,

$$\begin{aligned}
A(z) &= \beta^*(z) \sum_{z' \in Z} [e(z') + A(z')] \pi^*(z'|z) \\
\beta^*(z) &\equiv \beta(z) \sum_{z' \in Z} \max_{i \in I} \left\{ \frac{u'(\epsilon_i(z'))}{u'(\epsilon_i(z))} \right\} \pi(z'|z) \\
\pi^*(z'|z) &\equiv \frac{\max_{i \in I} \left\{ \frac{u'(\epsilon_i(z'))}{u'(\epsilon_i(z))} \right\} \pi(z'|z)}{\sum_{z' \in Z} \max_{i \in I} \left\{ \frac{u'(\epsilon_i(z'))}{u'(\epsilon_i(z))} \right\} \pi(z'|z)}
\end{aligned}$$

Notice that the operator defined by that recursion is monotone and satisfies discounting at a rate less than one if $\beta^*(z) < 1$ for all z , in which case the value of aggregate endowment is finite.

We show that for arbitrary z , as we take each of the four limits, $\beta^*(z)$ tends to a number smaller than one. First, as $\max_z \beta(z) \rightarrow 0$ then $\max_z \beta^*(z)$ can be made arbitrarily small. For cases (ii) and (iii)

$$\max_{i \in I, z, \bar{z} \in Z} \frac{u'(\epsilon_i(\bar{z}))}{u'(\epsilon_i(z))} \rightarrow 1$$

hence $\beta^*(z) \rightarrow \beta(z)$. In case (iv) $\lim_{\delta \rightarrow 1} \pi^*(z'|z) = 1$ if $z' = z$ and 0 otherwise, and since $\frac{u'(\epsilon_i(z))}{u'(\epsilon_i(z))} = 1$, then $\beta^*(z) \rightarrow \beta(z) < 1$. ■

Proof. Proposition (4.13)

A K-L economy is a standard Arrow-Debreu convex economy, hence by the second welfare theorem any efficient allocation can be supported as a quasi-equilibrium by a linear function p_0 . By a straightforward adaptation of the argument in Proposition (4.8) under the assumed condition (4.17) the Arrow prices implicit in the supporting prices p_0 in a quasi equilibrium satisfy (4.15). Thus the Arrow prices are strictly positive. Let us denote the corresponding A-D prices by Q_0 . This establishes that agent j has a cheaper point in his consumption possibility set, and hence a quasi-equilibrium is an equilibrium. In a K-L equilibrium the value of the agent j endowment is finite under p_0 since preferences are strictly monotone. By monotonicity of the preferences, $\sum_{z^i \geq z_0} Q_0(z^i|z_0) c_i(z^i) \leq p_0(c)$, with equality if the prices have a dot product representation. Since $\epsilon_i > 0$ for all i is uniformly bounded, then the value of the endowment of each agent under Q_0 is finite, and hence the implied interest rates are high. ■

Proof. Proposition (4.14)

Consider an equilibrium $\{q^*, c_i^*, a_i^*, J_i^*, B_i^*\}$, $a_{i,0}^*$ with solvency constraints that are not too tight and with high interest rates. By Proposition (4.9) there is a K-L

equilibrium with the same Arrow prices, with prices p_0 given by the implied A-D prices. Define $\bar{J}_{i,0}(a, z_0)$ as the maximized value of Equation (4.12) subject to (4.13) and (4.14). Notice the following two properties of the function $\bar{J}_{i,0}$. First, $\bar{J}_{i,0}(0, z_0) \geq U(e_i)(z_0)$ because $c_i = e_i$ is budget feasible for $a_{i,0} = 0$. Second, if $\bar{J}_{i,0}(a, z_0) = U(e_i)(z_0)$, and if the optimal consumption $c_i \neq e_i$ then $a_{i,0} < 0$, because the objective function is strictly concave and the feasible set is convex. Since the continuation of a K-L equilibrium at node z^t is itself a K-L equilibrium then we can define price systems p_t and functions $\bar{J}_{i,t}(\cdot, z^t)$ for the economy starting at node z^t . These functions satisfy the same two properties that in the case of $t = 0$. Define $\bar{B}_{i,t+1}(z^{t+1})$ as the solution of $\bar{J}_{i,t+1}(\cdot, z^{t+1}) = U(e_i)(z^{t+1})$ for each z^{t+1} . It is immediate to verify that $\{q_t^*, c_{i,t}^*, a_{i,t+1}^*, \bar{J}_{i,t}, \bar{B}_{i,t+1}, a_{i,0}^*\}$ constitute an equilibrium with solvency constraints that are not too tight. Finally, from the two properties of the functions $\bar{J}$ we obtain the desired result for $\bar{B}_{i,t+1}$. ■

Proof. Lemma (4.15)

It follows directly from the fact that Arrow prices are equal to the highest marginal rate of substitution across agents. ■

Proof. Proposition (4.16)

Case (i): In the case with no solvency constraint all the marginal rate of substitution are equated, and because of constant aggregate endowment the ratio of the marginal utilities equals one. If in the economy with solvency constraints Arrow prices are smaller, then,

$$u'(c_{i,t+1}(z^{t+1})) < u'(c_{i,t}(z^t)),$$

for all i , then since u' is decreasing we arrive to $\sum_i c_{i,t+1}(z^{t+1}) > \sum_i c_{i,t}(z^t)$, a contradiction.

Case (ii), if in the economy with solvency constraints Arrow prices are smaller,

$$u'(c_{i,t+1}(z^{t+1})) / u'(c_{i,t}(z^t)) = \left[\frac{c_{i,t+1}(z^{t+1})}{c_{i,t}(z^t)} \right]^{-\gamma} < \left[\frac{e_{t+1}(z^{t+1})}{e_t(z^t)} \right]^{-\gamma}$$

which implies that,

$$\frac{\sum_{i \in I} c_{i,t+1}(z^{t+1})}{\sum_{i \in I} c_{i,t}(z^t)} < \frac{e_{t+1}(z^{t+1})}{e_t(z^t)}$$

a contradiction.

Case (iii): Denote by $c_{i,t}$, $c_{i,t+1}$ and $\bar{c}_{j,t}$, $\bar{c}_{j,t+1}$ the consumptions in the economies with solvency constraints and the one where the marginal rate of substitution are equated for z^t and z^{t+1} . If in the economy with solvency constraints Arrow prices are smaller, then for all i and j in I we have

$$u'(c_{i,t+1}) u'(\bar{c}_{j,t}) < u'(\bar{c}_{j,t+1}) u'(c_{i,t}).$$

Then, for each j by linearity of $u'(\cdot)$ adding across i and dividing by I ,

$$u' \left(\frac{e_{t+1}}{I} \right) u'(\bar{c}_{j,t}) < u'(\bar{c}_{j,t+1}) u' \left(\frac{e_t}{I} \right)$$

now adding across j and dividing by I ,

$$u' \left(\frac{e_{t+1}}{I} \right) u' \left(\frac{e_t}{I} \right) < u' \left(\frac{e_{t+1}}{I} \right) u' \left(\frac{e_t}{I} \right),$$

a contradiction. ■

Proof. Proposition (4.18)

We first prove an intermediate result for any constrained efficient allocation. Let $z^t \equiv (x^t, y^t)$. For all i , and for all z^t , $\hat{c}_{i,t}(z^t) \equiv \hat{c}_{i,t}(x^t, y^t)$ is the same for all y^t . Notice that in this case, by direct computation, β is constant, and thus $U(\hat{c}_i)(x^t, y^t)$ does not depend on y^t . Hence, neither the resource constraints, the participation constraints nor the discount factor depend on y^t . Thus, given the strict concavity of the utility function, it would not be efficient to make $\hat{c}_{i,t}(x^t, \cdot)$ depend on y^t .

Now we use the intermediate result to show the proposition. By the first welfare theorem and the representation of Arrow prices it suffices to analyze the marginal rate of substitution :

$$\left\{ \beta [\lambda(y_{t+1})]^{-\gamma} \phi(y_{t+1}) \right\} \cdot \left\{ \left(\frac{\hat{c}_{i,t+1}(x^{t+1}, y^{t+1})}{\hat{c}_{i,t}(x^t, y^t)} \right)^{-\gamma} \psi(x_{t+1}|x_t) \right\}$$

which can be written as the product of two functions. The first one depends on y_{t+1} , which is equal to the marginal rate of substitution for the economy without participation constraints, $\beta [\lambda(y_{t+1})]^{-\gamma} \phi(y_{t+1})$. The second one, $\left(\frac{\hat{c}_{i,t+1}(x^{t+1}, y^{t+1})}{\hat{c}_{i,t}(x^t, y^t)} \right)^{-\gamma} \psi(x_{t+1}|x_t)$, by the intermediate result, does not depend on y^{t+1} , and hence cancels out in the formula for the strip premium. ■

Proof. Proposition (4.19)

The allocation is clearly resource feasible and satisfies the participation constraints. It suffices to show that $\{c_i^*, a_i^*\}$ solve (4.1) given $\{q^*\}$ and $\{B_i\}$ and $a_{i,0}^*$ for $i = I+1$. It is immediate to verify that given (4.19) the Euler equations (4.4) are satisfied for all t and that since $a_{i,t+1}(z^{t+1}) = B_{i,t+1}(z^{t+1})$ for all z^{t+1} the Transversality condition (4.5) is satisfied. ■