Preliminary Version

Malthus to Solow

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1. Introduction

This paper provides a unified theory that can account for the secular properties, or "growth facts," of economic time series as observed in most, if not all, world economies prior to 1800 as well as the properties of time series observed in modern industrialized economies. Prior to 1800, living standards were roughly constant; *per capita* wage income, output and consumption did not grow. On the other hand, modern industrial economies enjoy seemingly endless growth in living standards. In addition, the theory provided in this paper explains the industrial revolution, which is the transition from an era when *per capita* incomes are stagnant to one with sustained growth. In our model, given that we have abstracted from institutions that may inhibit technology adoption, this transition is inevitable given positive rates of total factor productivity growth. In particular, at the time of the transition, there is no change in the structure of the economy (parameters describing preferences, technology and policy) and the equilibrium implied by our theory is unique.

The model we construct combines features of a modern Solow growth model with one that is Malthusian in spirit. The latter is a model that can account for the growth facts of a pre 1800 economy while the former describes properties of modern industrial economies. This is done by introducing two technologies into a standard general equilibrium growth model (the model of Diamond (1965)). The same good is produced by both technologies and total factor productivity grows exogenously in both sectors. One of these, denoted the Malthus technology, is a constant returns to scale production process with land, labor and reproducible capital as inputs. Here, we are effectively modeling production as taking place on family farms with marginal products equated across farms. An important feature is that land, unlike capital, is a fixed factor that cannot be produced and does not depreciate. The second, denoted the Solow technology, is similar to the first except that only labor and capital are used. Production in this sector can be interpreted as being carried out in factories. Marginal products are equated across factories and entry or exit occurs depending on the profitability of operating an additional factory.

We show that along the equilibrium growth path, only the Malthus technology is used in the early stages of development when the stock of useable knowledge is small. During these periods, assuming that population grows at the same rate as output, the living standard is constant. Eventually, as useable knowledge grows, the Solow technology begins to be used. That is, part of the available labor and capital are assigned to this sector. At this point, living standards begin to improve and population growth has less influence on the growth rate of *per capita* income. In the limit, the economy behaves like a standard Solow growth model, which displays many of the same secular properties as modern industrial economies.

The question addressed in this paper is related to that motivating Lucas (1998), but our approach differs sharply from his. Lucas is concerned with explaining differences in population dynamics that differentiate the pre-industrial revolution era from the modern era, and, as a result, endogenous fertility and human capital accumulation play central roles in his analysis. In particular, he constructs a model economy that, depending on the value of a certain parameter governing the private return to human capital accumulation, can exhibit either Malthusian or modern features. The approach taken by Lucas is related to that of Becker, Murphy and Tamura (1990), who also emphasize the importance of fertility and human capital investment choices. In their interpretation, the Mathusian and modern eras are different steady states of the same model. In both of these papers, the transition from an economy with stable living standards to one with growing living standards requires a change in the return to human capital accumulation. The transition from Malthus to Solow is not an equilibrium property of these models as it is in our theory.

Our approach is, however, quite similar to that of Laitner (1997). His model economy has two technologies that are essentially the same as ours, but the goods produced by these technologies are distinct. Hence, without additional restrictions on preferences which guarantee that Engle curves have a particular shape, one technology will not dominate as the economy develops. Our approach does not require these sort of assumptions.

The rest of this paper is organized as follows. In the next section we discuss some empirical facts concerning pre-industrial and post industrial economies. These facts are used later to test our theory. In section 3, the model economy is presented and an equilibrium is defined and characterized. The methods we use to compute an equilibrium path are described in section 4. We simulate the transition from Malthus to Solow in section 4, and compare our results with the facts discussed in section 2. Finally, some concluding comments are provided in section 5.

2. The Behavior of the English Economy from 1250-2000

The Period 1264-1800

The Malthusian model describes well the behavior of the English economy from 1250 to nearly 1800. In this period the real wages, and more generally the standard of living, displays no trend. Increases in the stock of useable knowledge that increased production possibilities gave rise to population growth, not to increases in standards of living. During this period there was a large exogenous shock that reduced the population significantly below trend for an extended period of time. The shock was the Black Death. For the hundred-year period that population was below trend, the real wage was significantly above its normal value. This observation is just what Malthusian theory predicts. We now review evidence supporting these statements.

Figure 1 plots the real wage for the period 1264-1800. The Phelps et al. (1956) price index we use to deflate money wages is far from ideal. The weights used are 80 percent for food, 7.5 percent for lighting and heating and 12.5 percent for textiles. During the period these categories of expenditures constituted most expenditures of the typical craftsman or laborer. This measure of the real wage in 1264 is the same as its average for the 1750-1800. There may have been a modest increase in the real wage due to a fall in the price of goods not in the market basket of consumables used by Phelps et al. (1956) and to increase in quality of goods. However, a modest increase over a 536-year period is an infinitesimal growth rate.

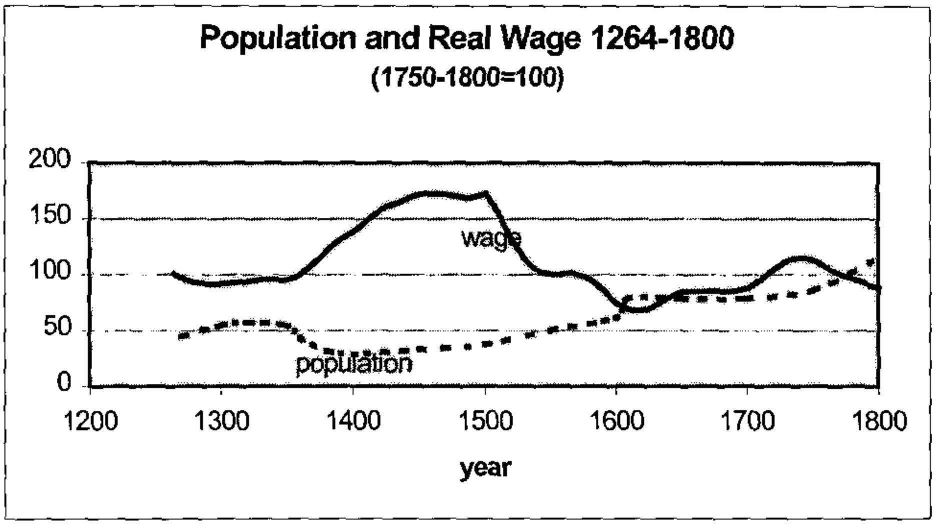

Figure 1

The behavior of population and real wage between 1410-1510 reveals dramatic opposite movements in the real wage and population. For this hundred-year period the real wage is more than 50 percent above its normal value while population is about a third below trend. This observation is in conformity with the Malthusian theory that a drop in the population due to plague will result in a high labor marginal product, and therefore real wage, until the population recovers.

Another prediction of Malthusian theory is that as population increases, land rents will rise. Figure 2 plots real land rents using the Phelps' et al. (1955) price index for consumables to deflate money rents. The observations are in striking conformity with the theory. When population was falling, land rents fell. When population increased, land rents increased.

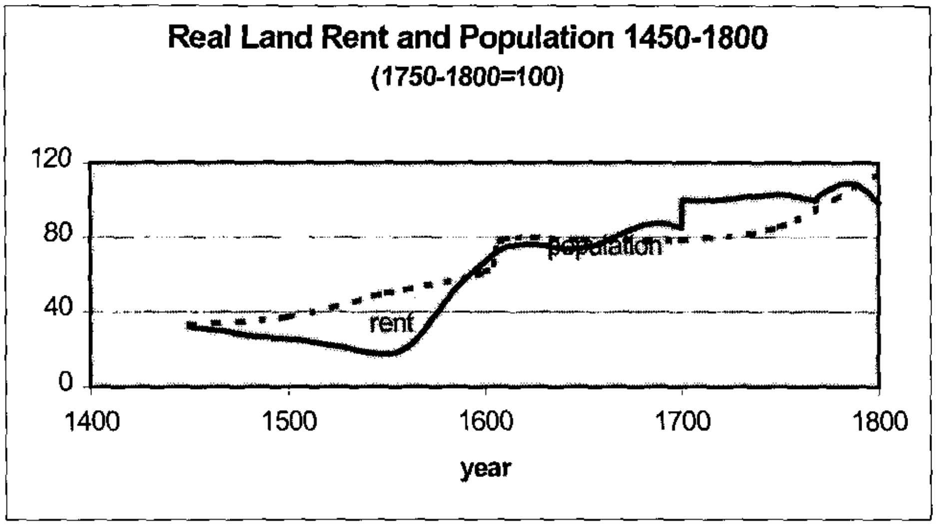

Figure 2

The period 1800-1989

Subsequent to 1800, the Solow model describes well the behavior of the English economy. Labor productivity, which moves closely with the real wage, grew at a steady rate. Population increases did not lead to falling standard of livings as the Malthusian theory predicts. This is documented in Table 1, which reports British labor productivity and population for selected years. The striking observation is that labor productivity increased by a factor of 22 between 1780 and 1989. This number may underestimate the increase in real wage because labor share of product probably increased a little over the period. More important reasons why the increase in real wage may be larger than this number are the difficulties in incorporating improvements in quality and new products in any cost of living index. Nordhaus (1997) has a dramatic example of this. Using lumens as a measure of lighting, Nordhaus finds that the price of lighting fell a thousand times more than conventional lighting price indexes find. Lighting in the nineteenth century was an important component of household consumption expenditures being almost 10 percent total expenditures. Nordhaus also finds that the price of lighting was essentially constant between 1264 and 1800.

Table 1

UK Productivity Levels

Year	GDP/Hour[1] (1985 $ U.S.)	Population[2] (Million)
1700	0.82[3]	8.4
1780	0.84	11.1
1820	1.21	21.2
1870	2.15	31.4
1890	2.86	37.5
1913	3.63	45.6
1929	4.58	45.7
1938	4.97	47.5
1960	8.15	52.4
1989	18.55	57.2

[1] Source of GDP/Hour: Maddison (1991 pp. 276 and 274-275).

[2] Maddison (1991, pp. 230-9 and p. 227).

[3] We added 5 percent to numbers with all of Ireland for years 1700, 1780 and 1820 to adjust for the fact that all of Ireland is included in these earlier data. The motivation for using 5 percent is that for years 1870, 1890 and 1913 Maddison reports data with and without the inclusion of Southern Ireland. UK labor productivity without Southern Ireland was 1.05 times the UK labor productivity with Southern Ireland.

The U.S. Economy 1870-1990

Another fact is that the value of farmland relative to the value of GDP has declined dramatically since 1870, the first year the needed census data are available, and the decline continues. Table 2 reports this ratio.

Table 2

U.S. Farm Land Value Relative to GDP[4]

Year	Percent
1870	88
1900	78
1929	37
1950	20
1990	9

The decline since 1929 would have been greater if large agriculture subsidies had not been instituted. The appropriate number from the point of view of our theory, where value is the present value of marginal products, is probably less than 5 percent in 1990. The decline in farmland value relative to annual GNP from 88 percent in 1870 to less than 5 percent in 1990 is a large decline.

3. Model Economy

Technology

The model economy studied is a one good two-sector version of Diamond's (1965) overlapping generations model. In the first production sector, which we denote the "Malthus" sector, capital, labor and land are combined to produce output. In the second sector, which we

[4] Sources: U.S. Department of Commerce , Bureau of the Census, 1975, *Historical Statistics of the United States Colonial Times to 1970, Part I* pp. 224, 255, and 462. Farmland values for 1990 was provided by Ken Erickson <erickson@mailbox.econ.ag.gov>. The 1870 value of land was obtained by taking 88 percent of the value of land plus farm buildings not including residences. In 1900 the value of agriculture land was 88 percent of the value of farm land plus structures.

will refer to as the "Solow" sector, just capital and labor are used to produce the same good. The production functions for the two sectors are as follows:

$$Y_{Mt} = \gamma^t_M K^{\phi}_{Mt} N^{\mu}_{Mt} L^{1-\phi-\mu}_{Mt} \tag{1}$$

$$Y_{St} = \gamma^t_S K^{\theta}_{St} N^{1-\theta}_{St} \tag{2}$$

Here, the subscript M denotes the Malthus and S denotes Solow. The variables Y_j, K_j, N_j, and L_j (j = M, S) refers to output produced, capital, labor and land employed in sector j, respectively. The parameters γ_M and γ_S are the total factor productivity growth factors in each sector and are each greater than 1.

Implicit behind these aggregate production functions are technologies for individual production units with the property that, given factor prices, the optimal unit size is small relative to the size of the economy and both entry and exit are permitted. The Malthus production unit can be thought of as a family farm. The Solow production unit, on the other hand, corresponds to a factory. Hence, land is an important factor of production in the Malthus production process and reproducible capital has relatively little importance. The Solow technology is one in which the roles of these two inputs are reversed. Consistent with this interpretation, we assume that $\theta > \phi$. In fact, in our formulation of the Solow technology, land, at least when interpreted as a fixed factor, does not affect production at all.

Output from either sector can be used for consumption or investment in capital. Capital is assumed to depreciate fully at the end of each period.[5] Hence, the resource constraint for the economy is given by,

$$C_t + K_{t+1} = Y_{Mt} + Y_{St}. \tag{3}$$

[5] In the calibration exercises carried out in this paper, we interpret a period to be 35 years. Hence, the assumption of 100 percent depreciation seems empirically plausible.

Land in this economy is in fixed supply; it cannot be produced and does not depreciate. We normalize the total quantity of land to be 1. In addition, land has no alternative use aside from being used for production in the Malthus sector.

Since the production functions exhibit constant returns to scale, we assume for analytical convenience that there is just one competitive firm operating in each sector. Given a wage rate (w), a rental rate for capital (r_K), and a rental rate for land (r_L), the firm in sector j solves the following problem:

$$\max\{Y_j - wN_j - r_K K_j - r_L L_j\}, \quad j = M, S, \tag{4}$$

subject to the production functions (1) and (2).

Preferences and Demographic Structure

Households live for two periods and have preferences that depend on consumption in each period of life. In particular, a young individual born in period t has preferences summarized by the following utility function:

$$U(c_{1t}, c_{2t+1}) = \log c_{1t} + \beta \log c_{2t+1}. \tag{5}$$

Here, c_{1t} is consumption of a young household in period t and c_{2t} is consumption of an old household born in period t-1.

The number of households born in period t is denoted by N_t, where

$$N_{t+1} = g(c_{1t})N_t. \tag{6}$$

Following Malthus (1798), we assume that the population growth rate depends on the standard of living, which we measure using consumption of a young household. The precise form of this functional relationship will be described in section 4.[6]

[6] A simple way to motivate a law of motion of this form is to allow young agents to choose how many children they have. Let n_{t+1} be the number of children chosen by a young household in period t and suppose that the utility function of a household is given by $U(c_{1t}, n_{t+1}) + \beta V(c_{2t+1})$, where U is increasing and concave in both arguments. In addition, suppose that n_{t+1} does not affect the budget constraint of the household. In this case, the optimality condition determining n_{t+1} is $U_2(c_{1t}, n_{t+1}) = 0$. This equation can be solved to obtain $n_{t+1} = g(c_{1t})$, which implies

The initial old (period t_0) in this economy are endowed with $k_{t_0} = K_{t_0} / N_{t_0-1}$ units capital and $\ell_{t_0} = 1 / N_{t_0-1}$ units of land. They rent the land and capital to firms and, at the end of the period, they sell the land to the young households. Each young agent is endowed with one unit of time that can be supplied as labor to the firm. The labor income earned by the young is used to finance consumption and the purchase of capital and land, the return from which will finance consumption when old. That is, the young households maximize (5) subject to the following budget constraints:

$$
\begin{aligned}
&c_{1t} + k_{t+1} + q_t l_{t+1} = w_t \\
&c_{2t+1} = r_{Kt+1} k_{t+1} + (r_{Lt+1} + q_{t+1}) l_{t+1}
\end{aligned}
\tag{7}
$$

The notation employed here is to use lower case letters, k and l, to denote the capital and land owned by a particular household and upper case letter, K and L ($L = 1$), to denote the total stock of capital and land available in the economy. The letter q denotes the price of land.

Competitive Equilibrium

Given N_{t_0}, k_{t_0} and l_{t_0} (where $N_{t_0-1} l_{t_0} = 1$), a competitive equilibrium in this economy consists of a sequences for $t \geq t_0$ of prices, $\{q_t, w_t, r_{Kt}, r_{Lt}\}$; firm allocations, $\{K_{Mt}, K_{St}, N_{Mt}, N_{St}, Y_{Mt}, Y_{St}\}$; and household allocations, $\{c_{1t}, c_{2t+1}, k_{t+1}, l_{t+1}\}$ such that:

1. Given the sequence of prices, the firm allocation solves the problems specified in equation (4).
2. Given the sequence of prices, the household allocation maximizes (5) subject to (7).
3. Markets clear: $K_{Mt} + K_{St} = N_{t-1} k_t$

$$
\begin{aligned}
&N_{Mt} + N_{St} = N_t \\
&N_{t-1} l_t = 1 \\
&Y_{Mt} + Y_{St} = N_t c_{1t} + N_{t-1} c_{2t} + N_t k_{t+1}
\end{aligned}
$$

4. $N_{t+1} = g(c_{1t}) N_t$

In characterizing an equilibrium, we make use of the following results:

that $N_{t+1} \equiv N_t n_{t+1} = g(c_{1t}) N_t$. We have found it convenient to model g as an exogenous function since we plan to calibrate the population dynamics to match historical data.

Proposition 1. For any wage rate w and capital rental rate r_K, it is profitable to operate the Malthus sector. That is, $Y_{St} > 0$ for all t.

Proof: Given w and r_K, solving problem (4) for the Malthus sector, maximum profits are equal to,

$$\Pi_M(w,r_K) = \gamma_M^{\frac{t}{1-\phi-\mu}}(1-\phi-\mu)\left(\frac{\phi}{r_K}\right)^{\frac{\phi}{1-\phi-\mu}}\left(\frac{\mu}{w}\right)^{\frac{\mu}{1-\phi-\mu}},$$

which is clearly positive for all t.

A similar argument applied to the Solow sector gives the following result:

Proposition 2. Given a wage rate w and capital rental rate r_K, Y_S is positive (maximized profits in the Solow sector are positive) if and only if,

$$\gamma_S^t > \left(\frac{r_K}{\theta}\right)^{\theta}\left(\frac{w}{1-\theta}\right)^{1-\theta}. \tag{8}$$

It follows that, given K, N and t, both sectors are operated in equilibrium (Y_M and Y_S will be positive) if and only if (8) is satisfied when w and r_K are set equal to their equilibrium values in an economy with only a Malthus technology.

Given an aggregate quantity of capital and labor equal to K and N, Proposition 2 is applied by first computing the Malthus-only wage and rental rate. These are,

$$\begin{aligned} w_t &= \mu\gamma^t K_t^{\phi} N_t^{\mu-1} \\ r_{Kt} &= \phi\gamma^t K_t^{\phi-1} N_t^{\mu} \\ r_{Lt} &= (1-\phi-\mu)\gamma^t K_t^{\phi} N_t^{\mu} \end{aligned} \tag{9}$$

If equation (8) is NOT satisfied, these are also the equilibrium wage and rental rates for the two-sector economy. If positive profits are in fact attainable in the Solow sector (equation (8) is satisfied), some resources will be moved to that sector and it will be operated.

By the First Welfare Theorem, an equilibrium allocation has the property that resources are efficiently allocated across the two sectors. Hence, when both sectors are operated the following problem must be solved:

$$Y(K,N,t)=\max_{K_S,N_S}\left\{\gamma_M^t(K-K_S)^{\phi}(N-N_S)^{\mu}+\gamma_S^t K_S^{\theta}N_S^{1-\theta}\right\}. \tag{10}$$

The constraint set is compact given that the choice variables K_S and N_S are constrained to closed intervals. This, and the fact the objective function is continuous, insures the existence of an optimum. The convexity of the constraint set and strict concavity of the objective insures uniqueness of the solution. In this case, the equilibrium wage and rental rates for the two-sector economy become,

$$\begin{aligned} w_t &= \mu\gamma_M^t K_{Mt}^{\phi}N_{Mt}^{\mu-1}=(1-\theta)\gamma_S^t K_{St}^{\theta}N_{St}^{-\theta} \\ r_{Kt} &= \phi\gamma_M^t K_{Mt}^{\phi-1}N_{Mt}^{\mu}=\theta\gamma_S^t K_{St}^{\theta-1}N_{St}^{1-\theta} \\ r_{Lt} &= (1-\phi-\mu)\gamma_M^t K_{Mt}^{\phi}N_{Mt}^{\mu} \end{aligned} \tag{11}$$

Given values for K, N and t, the above equations determine total output, the equilibrium wage rate and factor rental rates, and the allocation of resources across the two sectors, (K_M, K_S, N_M, N_S).

We now consider the household's optimization problem, which is to maximize (5) subject to (7). The first order necessary conditions for l_{t+1} and k_{t+1} can be arranged to yield the following expressions:

$$c_{1t}=\frac{w_t}{1+\beta} \tag{12}$$

$$q_{t+1}=q_t r_{Kt+1}-r_{Lt+1} \tag{13}$$

In addition, the budget constraints and market clearing conditions imply,

$$K_{t+1}=N_t(w_t-c_{1t})-q_t \tag{14}$$

Given values for t, K_t, N_t, and q_t, equations (11), (12), (6), and (14) are used to obtain w_t, r_{Kt}, r_{Lt} c_{1t} N_{t+1} and K_{t+1}. Next, given these results, (11) and (13) are used to obtain q_{t+1}. Given values for t_0, K_{t_0}, N_{t_0}, and q_{t_0}, this procedure can, in principle, be used to obtain a sequence of endogenous variables of any length. The first three of these initial conditions can be chosen arbitrarily, although specific criteria, which we describe in the next subsection, were employed in choosing values for these. The value of q_{t_0} is not arbitrary, however, and must be chosen so that the resulting sequence, $\{q_t\}_{t=t_0}^{\infty}$, does not diverge. That is, this sequence must have the properties of an equilibrium price sequence. We discuss the numerical procedure employed in computing an equilibrium path in section 4.

Malthus-Only Economy

Values for t_0 and K_{t_0} (N_0 is arbitrarily set equal to 1) are chosen, following actual historical experience, so that the artificial economy is initially using only the Malthus technology. In particular, K_{t_0} is set equal to the stock of capital at time t_0 along the asymptotic growth path of a one-sector economy with only a Malthus technology. This growth path is constructed so that individual consumption (c_{1t} and c_{2t}) is constant in the face of productivity growth ($\gamma_M > 0$). This requires that the growth rate of population be equal to $\gamma_M^{1/(1-\mu-\phi)}$. In addition, aggregate output, capital, consumption, the price of land, and the rental rate of land also grow at this rate. The wage rate and capital rental rate are constant. In this case, productivity increases translate directly into population increases and there is no improvement in individual living standards. This mimics the long run growth path (abstracting from plagues and other short run disturbances) that actual economies experienced for centuries prior to the industrial revolution.

Let c_{1M} be the value of c_{1t} along this Malthus-only asymptotic growth path. For the two-sector economy, we choose the function $g(c_{1t})$ in equation (6) so that $g(c_{1M}) = \gamma_M^{1/(1-\mu-\phi)}$. Since the wage and rental rate are constant along a Malthus only balanced growth path, the Solow technology will eventually be used as long as $\gamma_S > 1$ [see equation (8)]. Once this happens, living standards will begin to improve and the population growth rate will change according to the function $g(c_1)$ in equation (6). Over time, as the economy develops, the fraction of labor and capital assigned to the Malthus sector, along with the price of land, will asymptotically approach zero. At this point, the economy will behave essentially like a Solow-only economy. That is, the economy will converge to a growth path along which c_{1t}, c_{2t}, K_t/N_t, Y_t/N_t and w_t all grow at the rate $\gamma_S^{1/(1-\theta)} - 1$ and r_{Kt} is constant. Unlike the Malthus growth path, living standards continue to improve along this asymptotic path.

We summarize this discussion with the following proposition:

Proposition 3. Given that $g(c_{1M}) = \gamma_M^{1/(1-\mu-\phi)}$, there exists some date t such that equation (8) *is satisfied and the Solow technology is employed. The transition from Malthus to Solow is inevitable.*

4. Computational Issues

The set of equations discussed above are sufficient for determining the sequence of endogenous variables given initial values for the state variables *N* and *K*, as well as a value for the price, q_{t_0}. In this section, we describe how we obtain q_{t_0}, as well as an entire sequence $\{q_t\}_{t=t_0}^{t_n}$ that satisfies the definition of an equilibrium price sequence. This requires, in particular, choosing q_{t_0} so that q_t is nonnegative for all *t* and so that K_{t+1}, as determined by equation (14), does not fall below zero. The solution method that we employ is an iterative procedure designed to converge to a value of q_{t_0} that satisfies these properties. Hence, as long as an equilibrium exists, this solution procedure is able to approximate it subject to the accuracy of the computer. A proof that an equilibrium exists is given in the appendix.

The solution procedure used is a shooting algorithm that is implemented as follows. Setting K_{t_0} and N_{t_0} as described above and letting $q_{t_0} = \widetilde{q}_{t_0}$, where $\widetilde{q}_{t_0}$ is the steady state value along the Malthus-only growth path, we use equations derived in the previous section to obtain a sequence $\{\widetilde{q}_t, \widetilde{K}_t, \widetilde{N}_t\}_{t=t_0}^{T}$. The simulation stops (*T* is determined) once $\widetilde{q}_t$ either becomes negative or becomes so large that K_{t+1} is negative. If $\widetilde{q}_t$ becomes negative, we repeat the procedure with a larger value for $\widetilde{q}_{t_0}$. If $\widetilde{q}_t$ becomes too large, we repeat the procedure with a lower value. Through a process of iterating in this manner, we eventually bound the true value of q_{t_0} within a very small interval. The value of $\widetilde{q}_{t_0}$ that is obtained once the interval is sufficiently small is stored as the first element of the sequence $\{q_t\}_{t=t_0}^{t_n}$; that is, we set $q_{t_0} = \widetilde{q}_{t_0}$.

Although equation (13) could, in principle, be used to obtain the subsequent elements of this sequence, this procedure would magnify the numerical errors introduced by the finite accuracy of the computer. Hence, in order to correct as much as possible for these inaccuracies, we obtain the subsequent elements of the q sequence by following a procedure similar to that used to obtain the first element. In particular, we repeatedly form the sequence $\{\tilde{q}_t, \tilde{K}_t, \tilde{N}_t\}_{t=t_0+1}^{T}$ in order to narrow the upper and lower bound of q_{t_0+1} using the shooting algorithm just described. The values for $\tilde{K}_{t_0+1}$ and $\tilde{N}_{t_0+1}$ in these sequences are set equal to the values of $\tilde{K}_{t_0+1}$ and $\tilde{N}_{t_0+1}$ found in the final iteration of the previous step. The procedure is repeated until the entire sequence $\{q_t\}_{t=t_0}^{t_n}$ has been computed. Given this sequence, the remaining equilibrium prices and quantities can be computed using the equations described in the previous section.

An advantage of our shooting algorithm is that uniqueness or lack thereof can be determined for the particular economy being analyzed. This is the case because the search is one-dimensional over possible initial prices of land.[7] For the economies we studied, we find that for initial q outside a very narrow range, the generated paths become inconsistent with equilibrium. Given existence, equilibrium initial land price must be in this computationally determined narrow rage. Given continuity of the functions used in our shooting algorithm, equilibrium K_{t+1} and N_{t+1} have been determined to a high degree of accuracy.

4. Quantitative Findings

Calibration

Assuming that the economy is initially in a Malthus-only steady state, we simulate the equilibrium path for a number of periods until essentially all of the available capital and labor are employed in the Solow sector.[8] The model is calibrated so that (1) the Malthus-only economy is consistent with the growth facts describing the English economy prior to 1800; (2) the Solow-only economy matches the growth facts describing post World War II industrialized economies; (3) the population growth rate reacts to changing living standards as reported in Lucas (1998); and (4) the implied annual real interest rates are reasonable given available data.

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[7] The equilibrium conditions restrict $q_{t_0}^{*}$ to the closed interval $[0, w_{t_0}\beta/(1+\beta)]$.

[8] Proposition 1 implies that some fraction of total resources will always be employed in the Malthus sector, although this fraction can (and does in our simulations) converge to zero in the limit.

Requirement (1) is used to calibrate γ_M ϕ and μ; while (2) is used to assign values to γ_S and θ. Given that we did not have good data on factor shares for the earlier period, we chose to set labor's share equal to .6 in both sectors, which implies values for μ and θ equal to .6 and .4, respectively. We experimented with various low values for ϕ, and use a value of .1 in the experiment reported here. This implies that land's share in the Malthus-only economy is .3.

In the pre 1800 period, we found that population grew, on average, about .3 percent per year. In order that per capita income is constant in the Malthus-only economy, and given that a period in our model is interpreted to be 35 years, we set $\gamma_M = 1.032$. We calibrated γ_S to match the growth rate of *per capita* GDP in the postwar period. This led us to choose $\gamma_S = 1.518$.

We set the discount factor β equal to 1. This value implied annual interest rates that vary from about 2 percent in the Malthus era to between 4 and 4.5 percent in periods where the Solow technology is heavily used.

Lucas (1998) provides data on population growth rates along side *per capita* GDP for various regions of the world from 1750 to the present. From this we conclude that population growth rates increase linearly in c_1 from the Malthus steady state to a point where population is doubling each period (every 35 years). Over this period of rising population growth rates, living standards (c_1 in our model) double from the Malthus steady state. After this, the population growth rate decreases linearly in c_1 until living standards are 18 times what they were in the Malthus steady state. At higher c_1, population is assumed to remain constant. This gives us the following form for the function $g(c_1)$:

$$g(c_1) = \begin{cases} \gamma_M^{1/(1-\mu-\phi)}\left(2 - \frac{c_1}{c_{1M}}\right) + 2\left(\frac{c_1}{c_{1M}} - 1\right) & \textit{for } c_1 < 2c_{1M} \\ 2 - \frac{c_1 - 2c_{1M}}{16c_{1M}} & \text{for } 2c_{1M} \le c_1 \le 18c_{1M} \\ 1 & \text{for } c_1 > 18c_{1M} \end{cases} \tag{15}$$

The following figure graphs the function $g(c_1)$ against values of c_1/c_{1M} that were obtained in eight successive periods of our simulation:

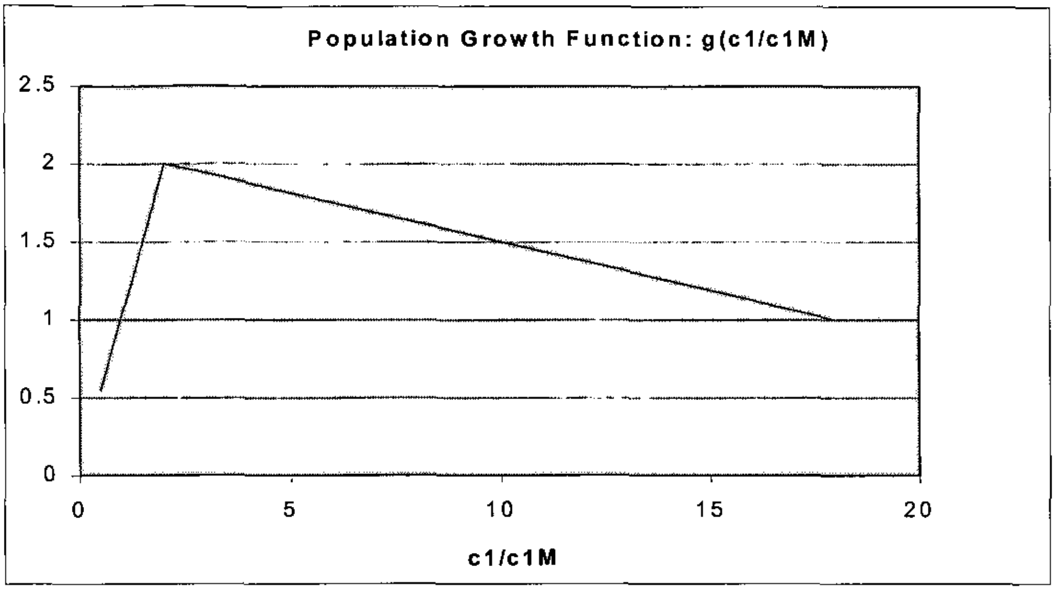

Figure 3

Findings

We simulated the economy beginning with period $t_0 = -5$ for eleven periods until the transition to the Solow technology was effectively complete. Figure 4 shows how the transition takes place by indicating the fraction of productive inputs (capital and labor) employed in the Malthus sector each period. The transition takes three generations until less than 1 percent of the resources are allocated to the Malthus sector.

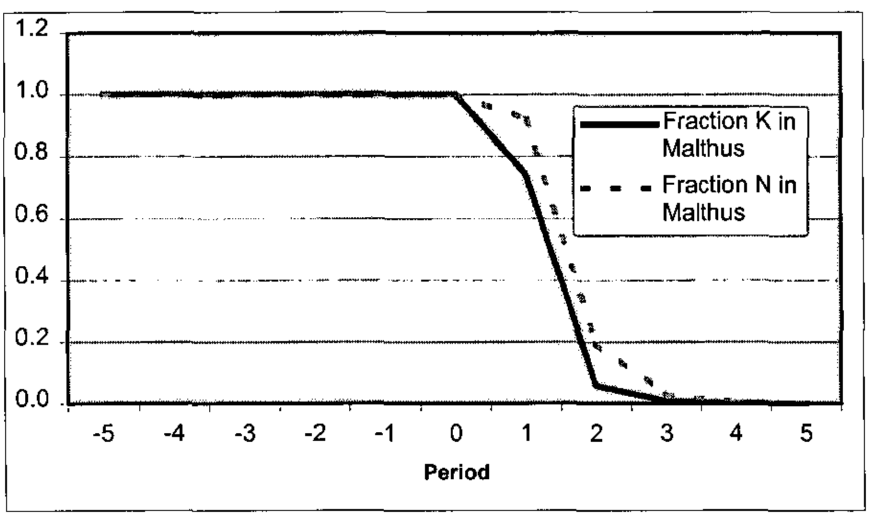

Figure 4

Only the Malthus technology is used from periods –5 to 0. During this time, as shown in Figure 5, output per worker remains constant. Once the industrial revolution begins in period 1, output per worker grows at increasingly higher rates. Eventually, the rate of growth will converge to the constant rate characterizing the steady state of a Solow-only economy.

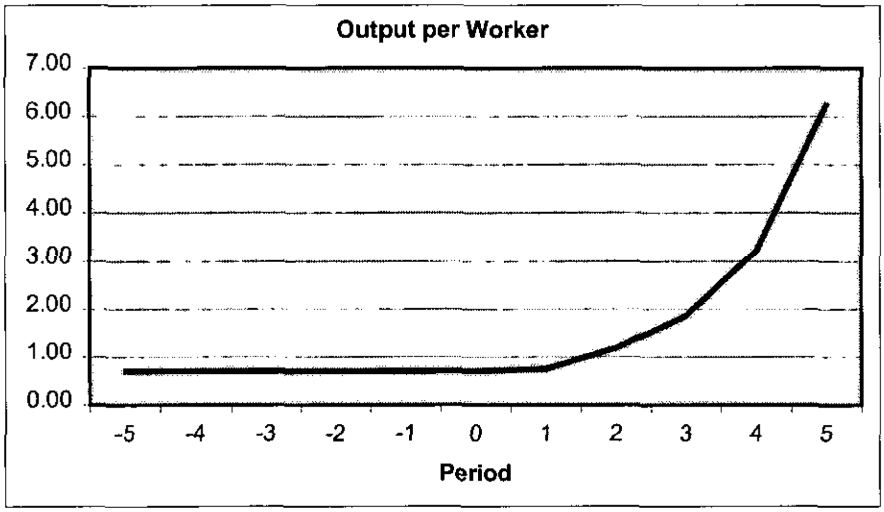

Figure 5

During the periods when only the Malthus technology is being used, population grows at the same rate as output and the wage stays constant. After period 0, population growth increases, and the real wage increases as well (see Figure 6, where the wage has been normalized to one in the Malthus steady state). This fits the pattern found in post 1800 England, as shown in Table 1.

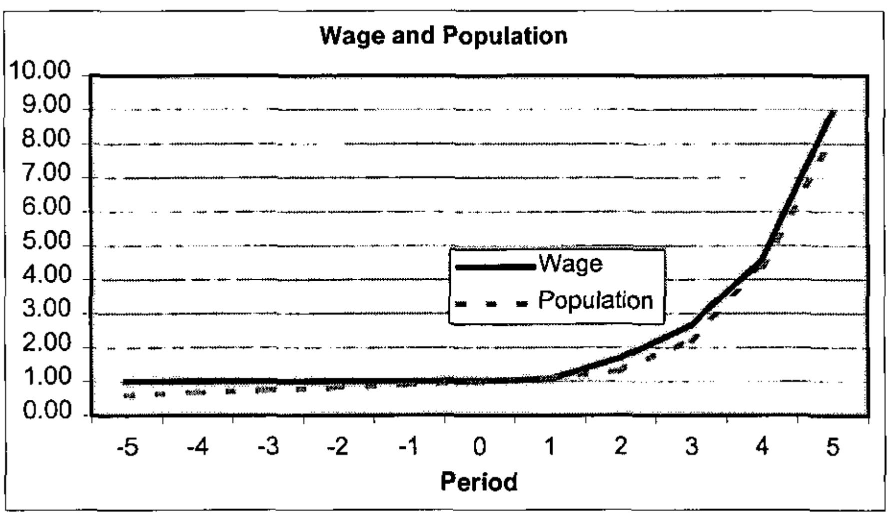

Figure 6

In addition, Figure 7 shows that the value of land relative to output decreases after the industrial revolution. Again, this is roughly consistent with the behavior of farmland values in the U.S. in the twentieth century (see Table 2).

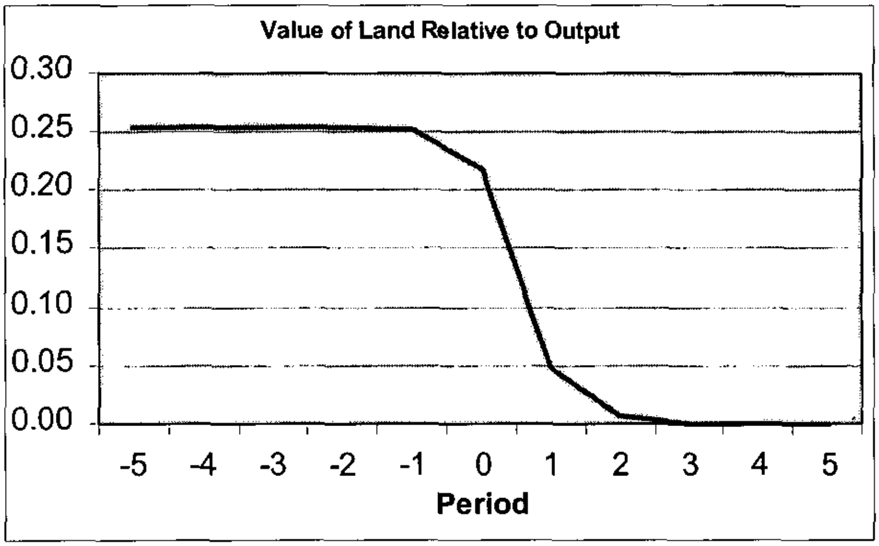

Figure 7

Finally, Figure 8 shows the population growth factor. It increases at the beginning of the industrial revolution to a maximum level with population doubling every 35 year period. Then the population growth rate declines as the standard of living increases until 5 periods or a 175 years after the start of the industrial revolution.

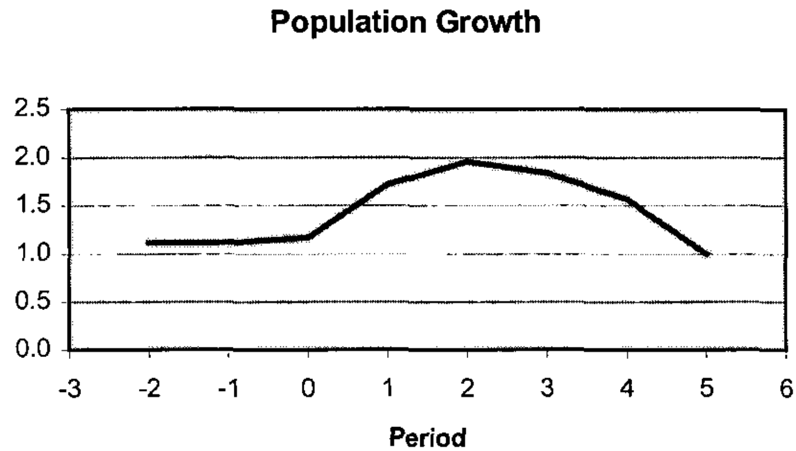

Figure 8

5. Conclusion

Most of the existing literature on economic growth is consistent with features of modern industrial economies, but inconsistent with the growth facts describing pre-industrial economies. This includes both models based on exogenous technical progress, such as Solow (1957), as well as more recent models with endogenous growth like Lucas (1988). There also exist examples of theories consistent with facts describing the early period, yet inconsistent with features of the later period [see Lucas (1998)]. In this paper we have presented a growth theory that is consistent with the growth facts from both periods. The transition from a pre-industrial to industrial economy, the industrial revolution, is a property of the equilibrium path associated with our theory.

We have chosen to cast our theory in the context of the overlapping generations model of Diamond (1965). We found this to be a natural way to model population growth in an equilibrium setting. We see no reason, however, why our results should not generalize to an infinite horizon context like that used in much of the growth literature—an optimal growth model with exogenous technology and population growth, for example. Our main results, as summarized in Propositions 1-3, depend primarily on properties of the technology and not at all on the length of an agent's life span.

Thomas Malthus theorized that when living standards improve, population growth rates increase. This feature is present in early stages of development as summarized in our function $g(c_1)$ in equation (15). However, population growth rates eventually fall and appear to level off as living standards improve after the industrial revolution. Our theory is silent as to why this occurs. Some economists, such as Lucas (1998), have argued that this may be due to a quantity-quality trade-off between the number of children a family produces versus the amount of human capital invested in each child. Other possibilities, perhaps more relevant in our context, include that the Solow technology might offer market opportunities that cause households to substitute out the home production sector into the market sector. That is, the same sorts of economic incentives that lured women into the workforce in the 1970's and 1980's may be responsible for the fall in population growth rates as living standards improve. We leave it to future work to explore these ideas.

Similarly, we have not explored the role policy might play in determining how quickly the Solow technology is adopted. For example, Parente and Prescott (1997) have studied how policy can affect the *level* of the total factor productivity parameter in the Solow technology. By keeping this parameter small, policy can affect when equation (8) is satisfied and, hence, when (if ever) the industrial revolutions occurs. The fact that the industrial revolution happened first in England in the early 19th century rather than in China, where the stock of useable knowledge may have actually been higher, is due perhaps to the institutions and policies in place in these two countries.

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Preliminary Version

Malthus to Solow

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We acknowledge the excellent research assistance provided by Daria Zakharova and helpful discussions with Tim Kehoe on the existence of an equilibrium in our model.

1. Introduction

This paper provides a unified theory that can account for the secular properties, or "growth facts," of economic time series as observed in most, if not all, world economies prior to 1800 as well as the properties of time series observed in modern industrialized economies. Prior to 1800, living standards were roughly constant; *per capita* wage income, output and consumption did not grow. On the other hand, modern industrial economies enjoy seemingly endless growth in living standards. In addition, the theory provided in this paper explains the industrial revolution, which is the transition from an era when *per capita* incomes are stagnant to one with sustained growth. In our model, given that we have abstracted from institutions that may inhibit technology adoption, this transition is inevitable given positive rates of total factor productivity growth. In particular, at the time of the transition, there is no change in the structure of the economy (parameters describing preferences, technology and policy) and the equilibrium implied by our theory is unique.

The model we construct combines features of a modern Solow growth model with one that is Malthusian in spirit. The latter is a model that can account for the growth facts of a pre 1800 economy while the former describes properties of modern industrial economies. This is done by introducing two technologies into a standard general equilibrium growth model (the model of Diamond (1965)). The same good is produced by both technologies and total factor productivity grows exogenously in both sectors. One of these, denoted the Malthus technology, is a constant returns to scale production process with land, labor and reproducible capital as inputs. Here, we are effectively modeling production as taking place on family farms with marginal products equated across farms. An important feature is that land, unlike capital, is a fixed factor that cannot be produced and does not depreciate. The second, denoted the Solow technology, is similar to the first except that only labor and capital are used. Production in this sector can be interpreted as being carried out in factories. Marginal products are equated across factories and entry or exit occurs depending on the profitability of operating an additional factory.

We show that along the equilibrium growth path, only the Malthus technology is used in the early stages of development when the stock of useable knowledge is small. During these periods, assuming that population grows at the same rate as output, the living standard is constant. Eventually, as useable knowledge grows, the Solow technology begins to be used. That is, part of the available labor and capital are assigned to this sector. At this point, living standards begin to improve and population growth has less influence on the growth rate of *per capita* income. In the limit, the economy behaves like a standard Solow growth model, which displays many of the same secular properties as modern industrial economies.

The question addressed in this paper is related to that motivating Lucas (1998), but our approach differs sharply from his. Lucas is concerned with explaining differences in population dynamics that differentiate the pre-industrial revolution era from the modern era, and, as a result, endogenous fertility and human capital accumulation play central roles in his analysis. In particular, he constructs a model economy that, depending on the value of a certain parameter governing the private return to human capital accumulation, can exhibit either Malthusian or modern features. The approach taken by Lucas is related to that of Becker, Murphy and Tamura (1990), who also emphasize the importance of fertility and human capital investment choices. In their interpretation, the Mathusian and modern eras are different steady states of the same model. In both of these papers, the transition from an economy with stable living standards to one with growing living standards requires a change in the return to human capital accumulation. The transition from Malthus to Solow is not an equilibrium property of these models as it is in our theory.

Our approach is, however, quite similar to that of Laitner (1997). His model economy has two technologies that are essentially the same as ours, but the goods produced by these technologies are distinct. Hence, without additional restrictions on preferences which guarantee that Engle curves have a particular shape, one technology will not dominate as the economy develops. Our approach does not require these sort of assumptions.

The rest of this paper is organized as follows. In the next section we discuss some empirical facts concerning pre-industrial and post industrial economies. These facts are used later to test our theory. In section 3, the model economy is presented and an equilibrium is defined and characterized. The methods we use to compute an equilibrium path are described in section 4. We simulate the transition from Malthus to Solow in section 4, and compare our results with the facts discussed in section 2. Finally, some concluding comments are provided in section 5.

2. The Behavior of the English Economy from 1250-2000

The Period 1264-1800

The Malthusian model describes well the behavior of the English economy from 1250 to nearly 1800. In this period the real wages, and more generally the standard of living, displays no trend. Increases in the stock of useable knowledge that increased production possibilities gave rise to population growth, not to increases in standards of living. During this period there was a large exogenous shock that reduced the population significantly below trend for an extended period of time. The shock was the Black Death. For the hundred-year period that population was below trend, the real wage was significantly above its normal value. This observation is just what Malthusian theory predicts. We now review evidence supporting these statements.

Figure 1 plots the real wage for the period 1264-1800. The Phelps et al. (1956) price index we use to deflate money wages is far from ideal. The weights used are 80 percent for food, 7.5 percent for lighting and heating and 12.5 percent for textiles. During the period these categories of expenditures constituted most expenditures of the typical craftsman or laborer. This measure of the real wage in 1264 is the same as its average for the 1750-1800. There may have been a modest increase in the real wage due to a fall in the price of goods not in the market basket of consumables used by Phelps et al. (1956) and to increase in quality of goods. However, a modest increase over a 536-year period is an infinitesimal growth rate.

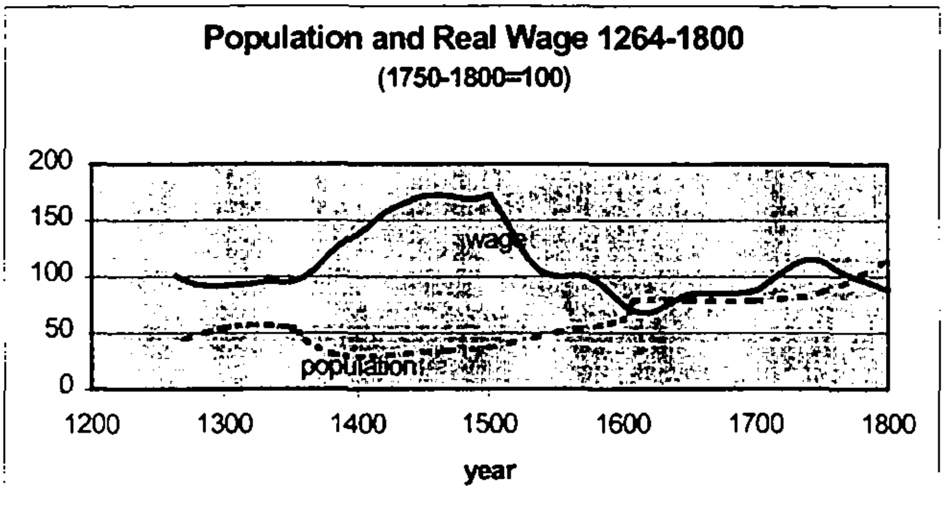

Figure 1

The behavior of population and real wage between 1410-1510 reveals dramatic opposite movements in the real wage and population. For this hundred-year period the real wage is more than 50 percent above its normal value while population is about a third below trend. This observation is in conformity with the Malthusian theory that a drop in the population due to plague will result in a high labor marginal product, and therefore real wage, until the population recovers.

Another prediction of Malthusian theory is that as population increases, land rents will rise. Figure 2 plots real land rents using the Phelps' et al. (1955) price index for consumables to deflate money rents. The observations are in striking conformity with the theory. When population was falling, land rents fell. When population increased, land rents increased.

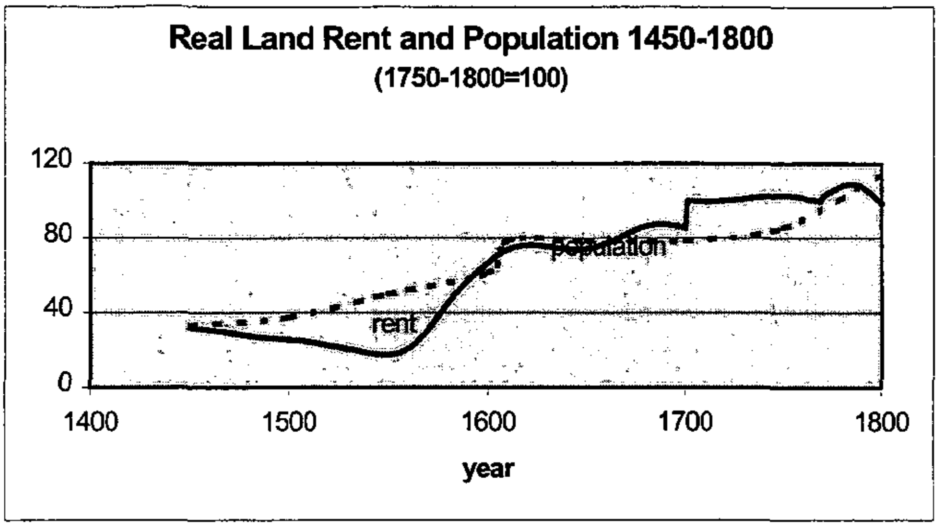

Figure 2

The period 1800-1989

Subsequent to 1800, the Solow model describes well the behavior of the English economy. Labor productivity, which moves closely with the real wage, grew at a steady rate. Population increases did not lead to falling standard of livings as the Malthusian theory predicts. This is documented in Table 1, which reports British labor productivity and population for selected years. The striking observation is that labor productivity increased by a factor of 22 between 1780 and 1989. This number may underestimate the increase in real wage because labor share of product probably increased a little over the period. More important reasons why the increase in real wage may be larger than this number are the difficulties in incorporating improvements in quality and new products in any cost of living index. Nordhaus (1997) has a dramatic example of this. Using lumens as a measure of lighting, Nordhaus finds that the price of lighting fell a thousand times more than conventional lighting price indexes find. Lighting in the nineteenth century was an important component of household consumption expenditures being almost 10 percent total expenditures. Nordhaus also finds that the price of lighting was essentially constant between 1264 and 1800.

Table 1

UK Productivity Levels

Year	GDP/Hour[1] (1985 $ U.S.)	Population[2] (Million)
1700	0.82[3]	8.4
1780	0.84	11.1
1820	1.21	21.2
1870	2.15	31.4
1890	2.86	37.5
1913	3.63	45.6
1929	4.58	45.7
1938	4.97	47.5
1960	8.15	52.4
1989	18.55	57.2

[1] Source of GDP/Hour: Maddison (1991 pp. 276 and 274-275).

[2] Maddison (1991, pp. 230-9 and p. 227).

[3] We added 5 percent to numbers with all of Ireland for years 1700, 1780 and 1820 to adjust for the fact that all of Ireland is included in these earlier data. The motivation for using 5 percent is that for years 1870, 1890 and 1913 Maddison reports data with and without the inclusion of Southern Ireland. UK labor productivity without Southern Ireland was 1.05 times the UK labor productivity with Southern Ireland.

The U.S. Economy 1870-1990

Another fact is that the value of farmland relative to the value of GDP has declined dramatically since 1870, the first year the needed census data are available, and the decline continues. Table 2 reports this ratio.

Table 2

U.S. Farm Land Value Relative to GDP[4]

Year	Percent
1870	88
1900	78
1929	37
1950	20
1990	9

The decline since 1929 would have been greater if large agriculture subsidies had not been instituted. The appropriate number from the point of view of our theory, where value is the present value of marginal products, is probably less than 5 percent in 1990. The decline in farmland value relative to annual GNP from 88 percent in 1870 to less than 5 percent in 1990 is a large decline.

3. Model Economy

Technology

The model economy studied is a one good two-sector version of Diamond's (1965) overlapping generations model. In the first production sector, which we denote the "Malthus" sector, capital, labor and land are combined to produce output. In the second sector, which we

[4] Sources: U.S. Department of Commerce , Bureau of the Census, 1975, *Historical Statistics of the United States Colonial Times to 1970, Part I* pp. 224, 255, and 462. Farmland values for 1990 was provided by Ken Erickson <erickson@mailbox.econ.ag.gov>. The 1870 value of land was obtained by taking 88 percent of the value of land plus farm buildings not including residences. In 1900 the value of agriculture land was 88 percent of the value of farm land plus structures.

will refer to as the "Solow" sector, just capital and labor are used to produce the same good. The production functions for the two sectors are as follows:

$$Y_{Mt} = \gamma_M^t K_{Mt}^{\phi} N_{Mt}^{\mu} L_{Mt}^{1-\phi-\mu} \tag{1}$$

$$Y_{St} = \gamma_S^t K_{St}^{\theta} N_{St}^{1-\theta} \tag{2}$$

Here, the subscript M denotes the Malthus and S denotes Solow. The variables Y_j, K_j, N_j, and L_j (j = M, S) refers to output produced, capital, labor and land employed in sector j, respectively. The parameters γ_M and γ_S are the total factor productivity growth factors in each sector and are each greater than 1.

Implicit behind these aggregate production functions are technologies for individual production units with the property that, given factor prices, the optimal unit size is small relative to the size of the economy and both entry and exit are permitted. The Malthus production unit can be thought of as a family farm. The Solow production unit, on the other hand, corresponds to a factory. Hence, land is an important factor of production in the Malthus production process and reproducible capital has relatively little importance. The Solow technology is one in which the roles of these two inputs are reversed. Consistent with this interpretation, we assume that $\theta > \phi$. In fact, in our formulation of the Solow technology, land, at least when interpreted as a fixed factor, does not affect production at all.

Output from either sector can be used for consumption or investment in capital. Capital is assumed to depreciate fully at the end of each period.[5] Hence, the resource constraint for the economy is given by,

$$C_t + K_{t+1} = Y_{Mt} + Y_{St}. \tag{3}$$

[5] In the calibration exercises carried out in this paper, we interpret a period to be 35 years. Hence, the assumption of 100 percent depreciation seems empirically plausible.

Land in this economy is in fixed supply; it cannot be produced and does not depreciate. We normalize the total quantity of land to be 1. In addition, land has no alternative use aside from being used for production in the Malthus sector.

Since the production functions exhibit constant returns to scale, we assume for analytical convenience that there is just one competitive firm operating in each sector. Given a wage rate (w), a rental rate for capital (r_K), and a rental rate for land (r_L), the firm in sector j solves the following problem:

$$\max\{Y_j - wN_j - r_K K_j - r_L L_j\}, \quad j = M, S, \tag{4}$$

subject to the production functions (1) and (2).

Preferences and Demographic Structure

Households live for two periods and have preferences that depend on consumption in each period of life. In particular, a young individual born in period t has preferences summarized by the following utility function:

$$U(c_{1t}, c_{2t+1}) = \log c_{1t} + \beta \log c_{2t+1}. \tag{5}$$

Here, c_{1t} is consumption of a young household in period t and c_{2t} is consumption of an old household born in period t-1.

The number of households born in period t is denoted by N_t, where

$$N_{t+1} = g(c_{1t})N_t. \tag{6}$$

Following Malthus (1798), we assume that the population growth rate depends on the standard of living, which we measure using consumption of a young household. The precise form of this functional relationship will be described in section 4.[6]

[6] A simple way to motivate a law of motion of this form is to allow young agents to choose how many children they have. Let n_{t+1} be the number of children chosen by a young household in period t and suppose that the utility function of a household is given by $U(c_{1t}, n_{t+1}) + \beta V(c_{2t+1})$, where U is increasing and concave in both arguments. In addition, suppose that n_{t+1} does not affect the budget constraint of the household. In this case, the optimality condition determining n_{t+1} is $U_2(c_{1t}, n_{t+1}) = 0$. This equation can be solved to obtain $n_{t+1} = g(c_{1t})$, which implies

The initial old (period t_0) in this economy are endowed with $k_{t_0} = K_{t_0} / N_{t_0-1}$ units capital and $\ell_{t_0} = 1 / N_{t_0-1}$ units of land. They rent the land and capital to firms and, at the end of the period, they sell the land to the young households. Each young agent is endowed with one unit of time that can be supplied as labor to the firm. The labor income earned by the young is used to finance consumption and the purchase of capital and land, the return from which will finance consumption when old. That is, the young households maximize (5) subject to the following budget constraints:

$$\begin{aligned} c_{1t} + k_{t+1} + q_t l_{t+1} &= w_t \\ c_{2t+1} &= r_{Kt+1} k_{t+1} + (r_{Lt+1} + q_{t+1}) l_{t+1} \end{aligned} \tag{7}$$

The notation employed here is to use lower case letters, k and l, to denote the capital and land owned by a particular household and upper case letter, K and L ($L = 1$), to denote the total stock of capital and land available in the economy. The letter q denotes the price of land.

Competitive Equilibrium

Given N_{t_0}, k_{t_0} and l_{t_0} (where $N_{t_0-1} l_{t_0} = 1$), a competitive equilibrium in this economy consists of a sequences for $t \geq t_0$ of prices, $\{ q_t, w_t, r_{Kt}, r_{Lt} \}$; firm allocations, $\{ K_{Mt}, K_{St}, N_{Mt}, N_{St}, Y_{Mt}, Y_{St} \}$; and household allocations, $\{ c_{1t}, c_{2t+1}, k_{t+1}, l_{t+1} \}$ such that:

1. Given the sequence of prices, the firm allocation solves the problems specified in equation (4).
2. Given the sequence of prices, the household allocation maximizes (5) subject to (7).
3. Markets clear: $K_{Mt} + K_{St} = N_{t-1} k_t$

$$N_{Mt} + N_{St} = N_t$$
$$N_{t-1} l_t = 1$$
$$Y_{Mt} + Y_{St} = N_t c_{1t} + N_{t-1} c_{2t} + N_t k_{t+1}$$

4. $N_{t+1} = g(c_{1t}) N_t$

In characterizing an equilibrium, we make use of the following results:

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that $N_{t+1} \equiv N_t n_{t+1} = g(c_{1t}) N_t$. We have found it convenient to model g as an exogenous function since we plan to calibrate the population dynamics to match historical data.

Proposition 1. For any wage rate w and capital rental rate r_K, it is profitable to operate the Malthus sector. That is, $Y_{St} > 0$ for all t.

Proof: Given w and r_K, solving problem (4) for the Malthus sector, maximum profits are equal to,

$$\Pi_M(w, r_K) = \gamma_M^{\frac{t}{1-\phi-\mu}}(1-\phi-\mu)\left(\frac{\phi}{r_K}\right)^{\frac{\phi}{1-\phi-\mu}}\left(\frac{\mu}{w}\right)^{\frac{\mu}{1-\phi-\mu}},$$

which is clearly positive for all t.

A similar argument applied to the Solow sector gives the following result:

Proposition 2. Given a wage rate w and capital rental rate r_K, Y_S is positive (maximized profits in the Solow sector are positive) if and only if,

$$\gamma_S^t > \left(\frac{r_K}{\theta}\right)^{\theta}\left(\frac{w}{1-\theta}\right)^{1-\theta}. \tag{8}$$

It follows that, given K, N and t, both sectors are operated in equilibrium (Y_M and Y_S will be positive) if and only if (8) is satisfied when w and r_K are set equal to their equilibrium values in an economy with only a Malthus technology.

Given an aggregate quantity of capital and labor equal to K and N, Proposition 2 is applied by first computing the Malthus-only wage and rental rate. These are,

$$\begin{aligned}
w_t &= \mu\gamma^t K_t^{\phi} N_t^{\mu-1} \\
r_{Kt} &= \phi\gamma^t K_t^{\phi-1} N_t^{\mu} \\
r_{Lt} &= (1-\phi-\mu)\gamma^t K_t^{\phi} N_t^{\mu}
\end{aligned} \tag{9}$$

If equation (8) is NOT satisfied, these are also the equilibrium wage and rental rates for the two-sector economy. If positive profits are in fact attainable in the Solow sector (equation (8) is satisfied), some resources will be moved to that sector and it will be operated.

By the First Welfare Theorem, an equilibrium allocation has the property that resources are efficiently allocated across the two sectors. Hence, when both sectors are operated the following problem must be solved:

$$Y(K,N,t)=\max_{K_S,N_S}\left\{\gamma_M^t(K-K_S)^{\phi}(N-N_S)^{\mu}+\gamma_S^t K_S^{\theta}N_S^{1-\theta}\right\}. \tag{10}$$

The constraint set is compact given that the choice variables K_S and N_S are constrained to closed intervals. This, and the fact the objective function is continuous, insures the existence of an optimum. The convexity of the constraint set and strict concavity of the objective insures uniqueness of the solution. In this case, the equilibrium wage and rental rates for the two-sector economy become,

$$\begin{aligned}
w_t &= \mu\gamma_M^t K_{Mt}^{\phi}N_{Mt}^{\mu-1}=(1-\theta)\gamma_S^t K_{St}^{\theta}N_{St}^{-\theta}\\
r_{Kt} &= \phi\gamma_M^t K_{Mt}^{\phi-1}N_{Mt}^{\mu}=\theta\gamma_S^t K_{St}^{\theta-1}N_{St}^{1-\theta}\\
r_{Lt} &= (1-\phi-\mu)\gamma_M^t K_{Mt}^{\phi}N_{Mt}^{\mu}
\end{aligned} \tag{11}$$

Given values for K, N and t, the above equations determine total output, the equilibrium wage rate and factor rental rates, and the allocation of resources across the two sectors, (K_M, K_S, N_M, N_S).

We now consider the household's optimization problem, which is to maximize (5) subject to (7). The first order necessary conditions for l_{t+1} and k_{t+1} can be arranged to yield the following expressions:

$$c_{1t}=\frac{w_t}{1+\beta} \tag{12}$$

$$q_{t+1}=q_t r_{Kt+1}-r_{Lt+1} \tag{13}$$

In addition, the budget constraints and market clearing conditions imply,

$$K_{t+1}=N_t(w_t-c_{1t})-q_t \tag{14}$$

Given values for t, K_t, N_t, and q_t, equations (11), (12), (6), and (14) are used to obtain w_t, r_{Kt}, r_{Lt} c_{1t} N_{t+1} and K_{t+1}. Next, given these results, (11) and (13) are used to obtain q_{t+1}. Given values for t_0, K_{t_0}, N_{t_0}, and q_{t_0}, this procedure can, in principle, be used to obtain a sequence of endogenous variables of any length. The first three of these initial conditions can be chosen arbitrarily, although specific criteria, which we describe in the next subsection, were employed in choosing values for these. The value of q_{t_0} is not arbitrary, however, and must be chosen so that the resulting sequence, $\{q_t\}_{t=t_0}^{\infty}$, does not diverge. That is, this sequence must have the properties of an equilibrium price sequence. We discuss the numerical procedure employed in computing an equilibrium path in section 4.

Malthus-Only Economy

Values for t_0 and K_{t_0} (N_0 is arbitrarily set equal to 1) are chosen, following actual historical experience, so that the artificial economy is initially using only the Malthus technology. In particular, K_{t_0} is set equal to the stock of capital at time t_0 along the asymptotic growth path of a one-sector economy with only a Malthus technology. This growth path is constructed so that individual consumption (c_{1t} and c_{2t}) is constant in the face of productivity growth ($\gamma_M > 0$). This requires that the growth rate of population be equal to $\gamma_M^{1/(1-\mu-\phi)}$. In addition, aggregate output, capital, consumption, the price of land, and the rental rate of land also grow at this rate. The wage rate and capital rental rate are constant. In this case, productivity increases translate directly into population increases and there is no improvement in individual living standards. This mimics the long run growth path (abstracting from plagues and other short run disturbances) that actual economies experienced for centuries prior to the industrial revolution.

Let c_{1M} be the value of c_{1t} along this Malthus-only asymptotic growth path. For the two-sector economy, we choose the function $g(c_{1t})$ in equation (6) so that $g(c_{1M}) = \gamma_M^{1/(1-\mu-\phi)}$. Since the wage and rental rate are constant along a Malthus only balanced growth path, the Solow technology will eventually be used as long as $\gamma_S > 1$ [see equation (8)]. Once this happens, living standards will begin to improve and the population growth rate will change according to the function $g(c_1)$ in equation (6). Over time, as the economy develops, the fraction of labor and capital assigned to the Malthus sector, along with the price of land, will asymptotically approach zero. At this point, the economy will behave essentially like a Solow-only economy. That is, the economy will converge to a growth path along which c_{1t}, c_{2t}, K_t/N_t, Y_t/N_t and w_t all grow at the rate $\gamma_S^{1/(1-\theta)} - 1$ and r_{Kt} is constant. Unlike the Malthus growth path, living standards continue to improve along this asymptotic path.

We summarize this discussion with the following proposition:

Proposition 3. Given that $g(c_{1M}) = \gamma_M^{1/(1-\mu-\phi)}$, *there exists some date t such that equation* (8) *is satisfied and the Solow technology is employed. The transition from Malthus to Solow is inevitable.*

4. Computational Issues

The set of equations discussed above are sufficient for determining the sequence of endogenous variables given initial values for the state variables N and K, as well as a value for the price, q_{t_0}. In this section, we describe how we obtain q_{t_0}, as well as an entire sequence $\{q_t\}_{t=t_0}^{t_n}$ that satisfies the definition of an equilibrium price sequence. This requires, in particular, choosing q_{t_0} so that q_t is nonnegative for all t and so that K_{t+1}, as determined by equation (14), does not fall below zero. The solution method that we employ is an iterative procedure designed to converge to a value of q_{t_0} that satisfies these properties. Hence, as long as an equilibrium exists, this solution procedure is able to approximate it subject to the accuracy of the computer. A proof that an equilibrium exists is given in the appendix.

The solution procedure used is a shooting algorithm that is implemented as follows. Setting K_{t_0} and N_{t_0} as described above and letting $q_{t_0} = \tilde{q}_{t_0}$, where $\tilde{q}_{t_0}$ is the steady state value along the Malthus-only growth path, we use equations derived in the previous section to obtain a sequence $\{\tilde{q}_t, \tilde{K}_t, \tilde{N}_t\}_{t=t_0}^{T}$. The simulation stops ($T$ is determined) once $\tilde{q}_t$ either becomes negative or becomes so large that K_{t+1} is negative. If $\tilde{q}_t$ becomes negative, we repeat the procedure with a larger value for $\tilde{q}_{t_0}$. If $\tilde{q}_t$ becomes too large, we repeat the procedure with a lower value. Through a process of iterating in this manner, we eventually bound the true value of q_{t_0} within a very small interval. The value of $\tilde{q}_{t_0}$ that is obtained once the interval is sufficiently small is stored as the first element of the sequence $\{q_t\}_{t=t_0}^{t_n}$; that is, we set $q_{t_0} = \tilde{q}_{t_0}$.

Although equation (13) could, in principle, be used to obtain the subsequent elements of this sequence, this procedure would magnify the numerical errors introduced by the finite accuracy of the computer. Hence, in order to correct as much as possible for these inaccuracies, we obtain the subsequent elements of the q sequence by following a procedure similar to that used to obtain the first element. In particular, we repeatedly form the sequence $\{\widetilde{q}_t, \widetilde{K}_t, \widetilde{N}_t\}_{t=t_0+1}^{T}$ in order to narrow the upper and lower bound of q_{t_0+1} using the shooting algorithm just described. The values for $\widetilde{K}_{t_0+1}$ and $\widetilde{N}_{t_0+1}$ in these sequences are set equal to the values of $\widetilde{K}_{t_0+1}$ and $\widetilde{N}_{t_0+1}$ found in the final iteration of the previous step. The procedure is repeated until the entire sequence $\{q_t\}_{t=t_0}^{t_n}$ has been computed. Given this sequence, the remaining equilibrium prices and quantities can be computed using the equations described in the previous section.

An advantage of our shooting algorithm is that uniqueness or lack thereof can be determined for the particular economy being analyzed. This is the case because the search is one-dimensional over possible initial prices of land.[7] For the economies we studied, we find that for initial q outside a very narrow range, the generated paths become inconsistent with equilibrium. Given existence, equilibrium initial land price must be in this computationally determined narrow rage. Given continuity of the functions used in our shooting algorithm, equilibrium K_{t+1} and N_{t+1} have been determined to a high degree of accuracy.

4. Quantitative Findings

Calibration

Assuming that the economy is initially in a Malthus-only steady state, we simulate the equilibrium path for a number of periods until essentially all of the available capital and labor are employed in the Solow sector.[8] The model is calibrated so that (1) the Malthus-only economy is consistent with the growth facts describing the English economy prior to 1800; (2) the Solow-only economy matches the growth facts describing post World War II industrialized economies; (3) the population growth rate reacts to changing living standards as reported in Lucas (1998); and (4) the implied annual real interest rates are reasonable given available data.

[7] The equilibrium conditions restrict q_{t_0} to the closed interval $[0, w_{t_0}\beta/(1+\beta)]$.

[8] Proposition 1 implies that some fraction of total resources will always be employed in the Malthus sector, although this fraction can (and does in our simulations) converge to zero in the limit.

Requirement (1) is used to calibrate γ_M ϕ and μ; while (2) is used to assign values to γ_S and θ. Given that we did not have good data on factor shares for the earlier period, we chose to set labor's share equal to .6 in both sectors, which implies values for μ and θ equal to .6 and .4, respectively. We experimented with various low values for ϕ, and use a value of .1 in the experiment reported here. This implies that land's share in the Malthus-only economy is .3.

In the pre 1800 period, we found that population grew, on average, about .3 percent per year. In order that per capita income is constant in the Malthus-only economy, and given that a period in our model is interpreted to be 35 years, we set $\gamma_M = 1.032$. We calibrated γ_S to match the growth rate of *per capita* GDP in the postwar period. This led us to choose $\gamma_S = 1.518$.

We set the discount factor β equal to 1. This value implied annual interest rates that vary from about 2 percent in the Malthus era to between 4 and 4.5 percent in periods where the Solow technology is heavily used.

Lucas (1998) provides data on population growth rates along side *per capita* GDP for various regions of the world from 1750 to the present. From this we conclude that population growth rates increase linearly in c_1 from the Malthus steady state to a point where population is doubling each period (every 35 years). Over this period of rising population growth rates, living standards (c_1 in our model) double from the Malthus steady state. After this, the population growth rate decreases linearly in c_1 until living standards are 18 times what they were in the Malthus steady state. At higher c_1, population is assumed to remain constant. This gives us the following form for the function $g(c_1)$:

$$g(c_1) = \begin{cases} \gamma_M^{1/(1-\mu-\phi)}\left(2 - \dfrac{c_1}{c_{1M}}\right) + 2\left(\dfrac{c_1}{c_{1M}} - 1\right) & \text{for } c_1 < 2c_{1M} \\ 2 - \dfrac{c_1 - 2c_{1M}}{16c_{1M}} & \text{for } 2c_{1M} \le c_1 \le 18c_{1M} \\ 1 & \text{for } c_1 > 18c_{1M} \end{cases} \tag{15}$$

The following figure graphs the function $g(c_1)$ against values of c_1/c_{1M} that were obtained in eight successive periods of our simulation:

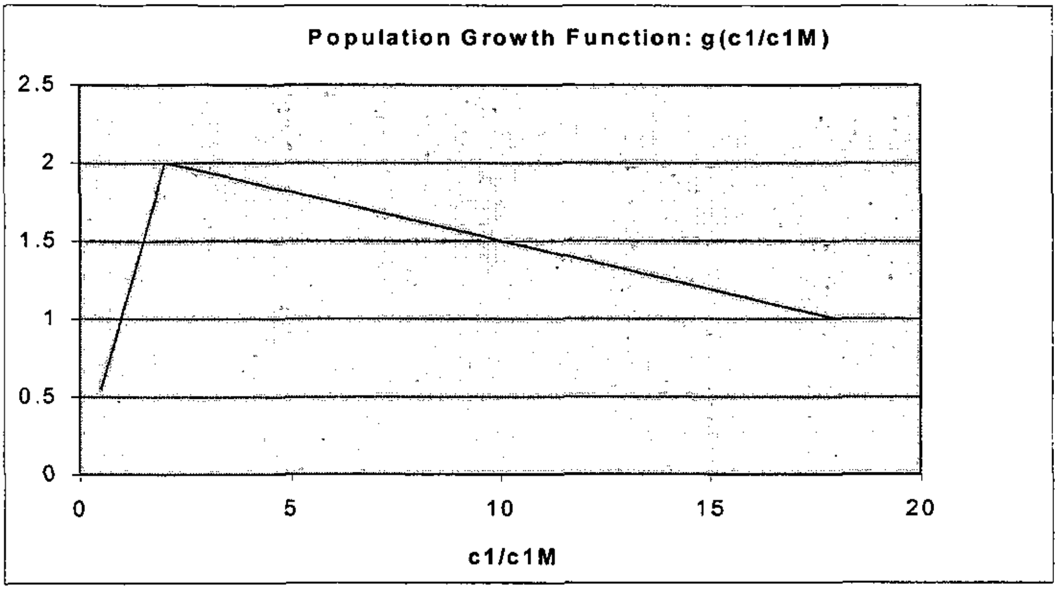

Figure 3

Findings

We simulated the economy beginning with period $t_0 = -5$ for eleven periods until the transition to the Solow technology was effectively complete. Figure 4 shows how the transition takes place by indicating the fraction of productive inputs (capital and labor) employed in the Malthus sector each period. The transition takes three generations until less than 1 percent of the resources are allocated to the Malthus sector.

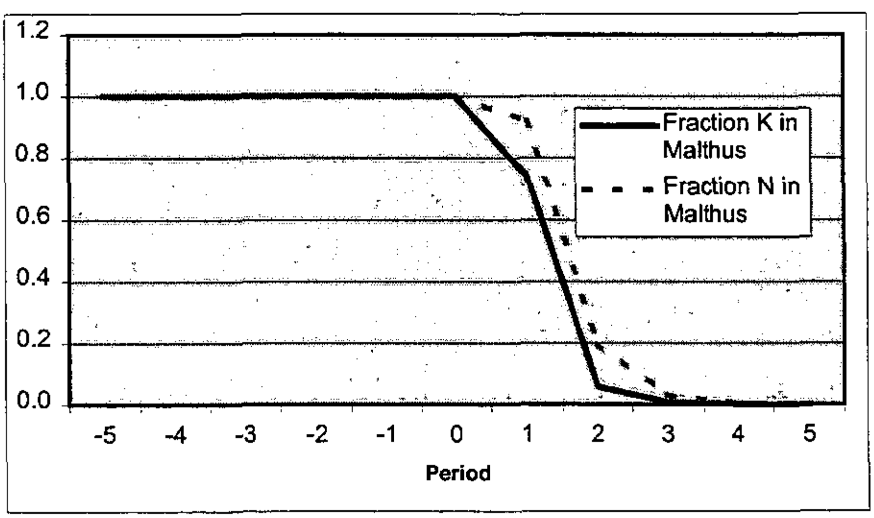

Figure 4

Only the Malthus technology is used from periods –5 to 0. During this time, as shown in Figure 5, output per worker remains constant. Once the industrial revolution begins in period 1, output per worker grows at increasingly higher rates. Eventually, the rate of growth will converge to the constant rate characterizing the steady state of a Solow-only economy.

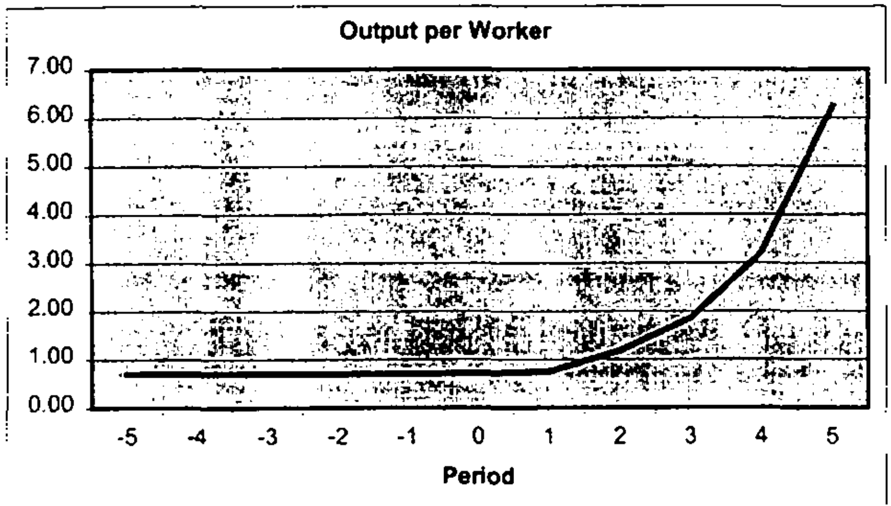

Figure 5

During the periods when only the Malthus technology is being used, population grows at the same rate as output and the wage stays constant. After period 0, population growth increases, and the real wage increases as well (see Figure 6, where the wage has been normalized to one in the Malthus steady state). This fits the pattern found in post 1800 England, as shown in Table 1.

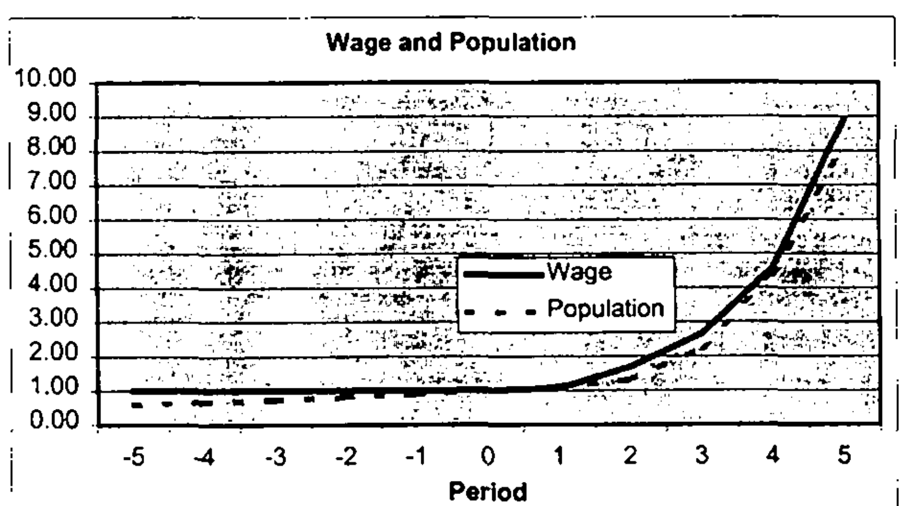

Figure 6

In addition, Figure 7 shows that the value of land relative to output decreases after the industrial revolution. Again, this is roughly consistent with the behavior of farmland values in the U.S. in the twentieth century (see Table 2).

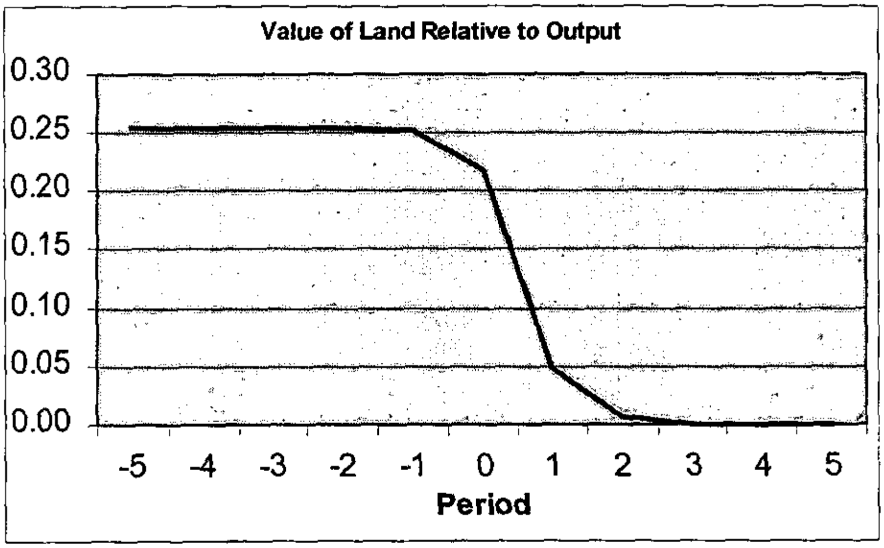

Figure 7

Finally, Figure 8 shows the population growth factor. It increases at the beginning of the industrial revolution to a maximum level with population doubling every 35 year period. Then the population growth rate declines as the standard of living increases until 5 periods or a 175 years after the start of the industrial revolution.

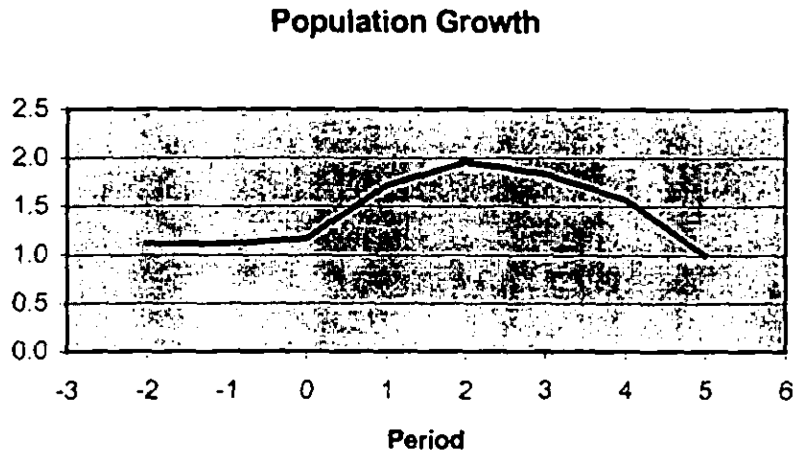

Figure 8

5. Conclusion

Most of the existing literature on economic growth is consistent with features of modern industrial economies, but inconsistent with the growth facts describing pre-industrial economies. This includes both models based on exogenous technical progress, such as Solow (1957), as well as more recent models with endogenous growth like Lucas (1988). There also exist examples of theories consistent with facts describing the early period, yet inconsistent with features of the later period [see Lucas (1998)]. In this paper we have presented a growth theory that is consistent with the growth facts from both periods. The transition from a pre-industrial to industrial economy, the industrial revolution, is a property of the equilibrium path associated with our theory.

We have chosen to cast our theory in the context of the overlapping generations model of Diamond (1965). We found this to be a natural way to model population growth in an equilibrium setting. We see no reason, however, why our results should not generalize to an infinite horizon context like that used in much of the growth literature—an optimal growth model with exogenous technology and population growth, for example. Our main results, as summarized in Propositions 1-3, depend primarily on properties of the technology and not at all on the length of an agent's life span.

Thomas Malthus theorized that when living standards improve, population growth rates increase. This feature is present in early stages of development as summarized in our function $g(c_1)$ in equation (15). However, population growth rates eventually fall and appear to level off as living standards improve after the industrial revolution. Our theory is silent as to why this occurs. Some economists, such as Lucas (1998), have argued that this may be due to a quantity-quality trade-off between the number of children a family produces versus the amount of human capital invested in each child. Other possibilities, perhaps more relevant in our context, include that the Solow technology might offer market opportunities that cause households to substitute out the home production sector into the market sector. That is, the same sorts of economic incentives that lured women into the workforce in the 1970's and 1980's may be responsible for the fall in population growth rates as living standards improve. We leave it to future work to explore these ideas.

Similarly, we have not explored the role policy might play in determining how quickly the Solow technology is adopted. For example, Parente and Prescott (1997) have studied how policy can affect the *level* of the total factor productivity parameter in the Solow technology. By keeping this parameter small, policy can affect when equation (8) is satisfied and, hence, when (if ever) the industrial revolutions occurs. The fact that the industrial revolution happened first in England in the early 19th century rather than in China, where the stock of useable knowledge may have actually been higher, is due perhaps to the institutions and policies in place in these two countries.

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