Risk taking by entrepreneurs

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Abstract

Entrepreneurs bear substantial risk, but empirical evidence shows no sign of a positive premium. This paper develops a theory of endogenous entrepreneurial risk taking that explains why self-financed entrepreneurs may find it optimal to invest into risky projects offering no risk premium. The model has also a number of implications for firm dynamics supported by empirical evidence, such as a positive correlation between survival, size, and firm age.

1 Introduction

Entrepreneurs bear substantial risk. According to recent estimates[1], to compensate for the extra risk entrepreneurial returns (return to private equity) should exceed public equity by at least 10 percent. Yet the evidence shows no signs of a positive premium.[2] A number of hypotheses have been offered to explain this puzzle, all of them based on the idea that entrepreneurs have a different set of preferences (e.g. risk tolerance or overoptimism.) This paper provides an alternative theory of endogenous entrepreneurial risk-taking that does not rely on individual heterogeneity.

The key ingredients in our theory are borrowing constraints, the existence of an outside opportunity and endogenous risk choice. A self-financed entrepreneur chooses every period how much to invest in a project, which is chosen from a set of alternatives. All available projects offer the same expected return but a different variance. After returns are realized, the entrepreneur decides whether to exit and take the outside opportunity (e.g. become a worker) or to stay in business.

The possibility of exit creates a nonconcavity in the entrepreneurs' continuation value: for values of wealth below a certain threshold, the outside opportunity gives higher utility; for higher wealth levels, entrepreneurial activity is preferred. Risky projects provide lotteries over future wealth that eliminate this nonconcavity and are particularly valuable to entrepreneurs with wealth levels close to this threshold. As the level of wealth increases, entrepreneurs invest in less risky projects.

It is the relatively poor entrepreneurs that decide to take more risk. At the same time, due to self-financing, they invest less in their projects than richer entrepreneurs. Correspondingly, the model implies that survival rates of the business are positively correlated with business size. Moreover, if agents enter entrepreneurship with relatively low wealth levels (as occurs in a case with endogenous entry that we study), our model also implies that young businesses exhibit lower survival rates. It also appears that, conditional on survival, small (younger) firms grow faster than larger (older) ones. All these

[1] These calculations assume standard levels of risk aversion (CRRA=2). See Heaton and Lucas (2000).

[2] Moskowitz and Vissing-Jorgensen (2002) estimate the return to entrepreneurial investment using data from SCF (Survey of Consumer Finances) and FFA/NIPA(the Flow of Funds Accounts and National Income and Product Accounts) and report that the average return to all private equity is similar to that of the public market equity index.

implications are supported by strong empirical evidence from the literature on firm dynamics (see, e.g. Evans 1987, Dunne, Roberts and Samuelson 1989 and Davis and Haltiwanger 1992).

In order to stress the role of risk taking, our model allows entrepreneurs to choose completely safe projects with the same expected return. All exit in our model occurs precisely because low wealth entrepreneurs purposively choose risk. If risky projects were not available, no exit would occur.

As mentioned above, three features are key to our model: the existence of an outside opportunity, financial constraints and the endogenous choice of risk. Many papers consider some of these features separately, but as far as we know ours is the first that considers all of them together. Discrete occupational choices appear in several papers, following Lucas (1978). Borrowing constraints have been considered in several recent papers (Gomes 2001, Albuquerque and Hopenhayn 2002, Clementi and Hopenhayn 2002) and is consistent with the empirical evidence presented in Evans and Jovanovic (1989), Gertler and Gilchrist (1994) Fazzari, Hubbard and Petersen (1988) and others. The use of lotteries to convexify discrete choice sets was introduced in the macro literature by Rogerson (1988).

A number of papers address the question of which agents decide to become entrepreneurs. All these models rely on some source of heterogeneity. The classical work in this field is a general equilibrium model by Kihlstrom and Laffont (1979), where it is assumed that agents differ in their degrees of risk aversion. Obviously the least risk averse agents are selected into entrepreneurship, which is assumed to be a risky activity. In a recent paper, Cressy (1999) points out that different degrees of risk aversion can be the result of differences in wealth. In particular, if preferences exhibit decreasing absolute risk aversion (DARA), wealthier agents become entrepreneurs. The same happens in the occupational choice model described in the paper, but due to the presence of borrowing constraints.

The empirical regularities on firm dynamics have been explained in models by Jovanovic (1982) and Hopenhayn (1992) and Ericson and Pakes (1995). These models rely on exogenous shocks to firms' productivities and selection. In Jovanovic the source is learning about (ex-ante) heterogeneity in entrepreneurial skills. In Hopenhayn survival rates and the dynamics of returns are determined by an exogenous stochastic process of firms' productivity shocks and the distribution of entrants. In Ericson and Pakes the shocks affect the outcome of investments made by firms.

In contrast to the studies listed above, we do not assume any heterogeneity in risk aversion (as in Kihlstrom and Laffont), or in the returns to entrepreneurial activity (as in Jovanovic). In our setup risk taking is a voluntary decision of agents and not an ex ante feature of the available technology (as in Kihlstrom and Laffont, and Cressy). In contrast to Hopenhayn (1992), we endogenize the stochastic process that drives firm dynamics.

The paper is organized as follows. Section 2 describes the basic model of entrepreneurial risk choice. In this section the outside opportunity is described by a function of wealth with some general properties. This section gives the core results of the paper. Section 3 gives a detailed occupational choice model that endogenizes the outside value function. There is entry and exit from employment to entrepreneurship. We explore conditions under which risk taking occurs in equilibrium and provide benchmark computations to assess its value.

2 The Model

2.1 The Environment

The entrepreneur is an infinitely lived risk averse agent with time separable utility $u(c)$ and discount factor β. Assume $u(c)$ is concave, strictly increasing and satisfies standard Inada conditions. The entrepreneur starts a period with accumulated wealth w. At the beginning of each period he first decides whether to continue in business or to quit and get an outside value $R(w)$, which is an increasing and concave function of his wealth. Entrepreneurs are self-financed and while in business face the following set of investment opportunities.

There is a set of available projects with random return $\widetilde{A}k$, where k is the amount invested. Entrepreneurs must choose one of these projects and the investment level $k \leq w$. All projects offer the same expected return $E\widetilde{A} = A$, but different levels of risk. We assume the expected return $A > 1/\beta$. The distribution of a project's rates of return is concentrated in two points, $x \leq y$. (As shown later, this assumption is without loss of generality.) If the low return x is realized with probability $1 - p$, the average return is $A = (1 - p)x + py$, and the high return y may be expressed as

$$y = x + \frac{A - x}{p} \geq A. \tag{1}$$

Thus, we will identify every available project by the value of the lower return x and the probability of the higher return p. Denote by $\Omega_2(A)$ the set of available projects [3],

$$\Omega_2(A) = \{(x,p)|x \in [0,A], p \in [0,1]\}.$$

If $x = A$ or $p = 1$ the project is safe, delivering the return A for sure; for all other values of x and p the project is risky. The existence of riskless projects that are not dominated in expected return is obviously an extreme assumption. It is convenient for technical reasons and it helps to emphasize the point that risk taking is not necessarily associated with higher returns.

Intuitively, risk taking in this set up occurs due to the presence of the outside opportunity. If risky projects are not available, the value of an active entrepreneur with current wealth w is defined by the standard dynamic problem[4]

$$V_l(w) = \max_k \{u(w-k) + \beta V_l(Ak)\}. \tag{2}$$

If $R(w)$ and $V_l(w)$ have at least one intersection, the value of the entrepreneur with the option to quit is a non-concave function $\max\{R(w), V_l(w)\}$. This nonconcavity suggests that a lottery on wealth levels could be welfare improving. As will be seen, in the absence of such lottery, an entrepreneur may find it beneficial to invest in a risky project.

If risk taking is possible, an entrepreneur with current wealth w that decides to stay in business, picks a project $(x,p) \in \Omega_2(A)$ and the amount of wealth $k \in [0,w]$ invested into this project. Given that the entrepreneur has no access to financing, consumption will equal $w-k$. By the beginning of the following period the return of the project is realized, giving the entrepreneur wealth yk in case of success or xk in case of failure. At this stage the entrepreneur must decide again whether to continue in business or to quit and take the outside value.

Letting $V(w)$ denote the value for an entrepreneur with wealth w at the begging of the period (exit stage), the value $V_E(w)$ at the investment stage

[3]Subindex 2 corresponds to the number of mass points of the payoffs'distribution

[4]Note that the the return in (2) is unbounded (due to $A\beta > 1$), so we must assume that the agents' utility function $u(c)$ is such that the solution to (2) exists. This is true for a general class of the utility functions, including CRRA.

is given by:

$$\begin{aligned} V_E(w) &= \max_{k,x,p}\{u(w-k)+\beta[pV(yk)+(1-p)V(xk)]\}, \\ s.t. \qquad y &= x+\frac{A-x}{p}, \end{aligned} \tag{3}$$

In turn, the agent's initial value and exit decision are given by:

$$V(w) = \max\{V_E(w), R(w)\}. \tag{4}$$

We will call (3)-(4) the *optimal risk choice problem* (ORCP). Its solution gives the entrepreneur's exit decision, consumption path and project risk choice. The latter is the main focus of our work. An entrepreneur who chooses $p < 1$ invests into a risky project. The risk of business failure is larger for smaller values of p. As we show below, risk taking decreases with the level of wealth while total investment increases. Using the scale of the project (i.e. total investment) as a measure of business size, the model implies that smaller firms take more risk and face higher failure rates.

2.2 The Solution

This section characterizes the solution to the entrepreneurial choice problem. We divide the problem in three steps: 1) project risk choice; 2) consumption/investment decision and 3) exit decision. A sketch of the main features of the solution is given here. More details and proofs are provided in the appendix.

2.2.1 Project risk choice

Let k denote the total investment in the project. The expected payoff is then Ak, independently of the level of risk chosen. Figure 1 illustrates this decision problem. If the end-of-period wealth is below w_E, the entrepreneur will quit and take the outside option; if it is above he will stay in business. The continuation value $V(Ak)$ is thus given by the envelope of the two concave[5] functions, $R(w)$ and $V_E(w)$. As a consequence of the option to exit, this value is not a concave function in end-of-period wealth.

[5]The outside value $R(w)$ is concave by assumption. Lemma 1 establishes the concavity of $V_E(w)$.

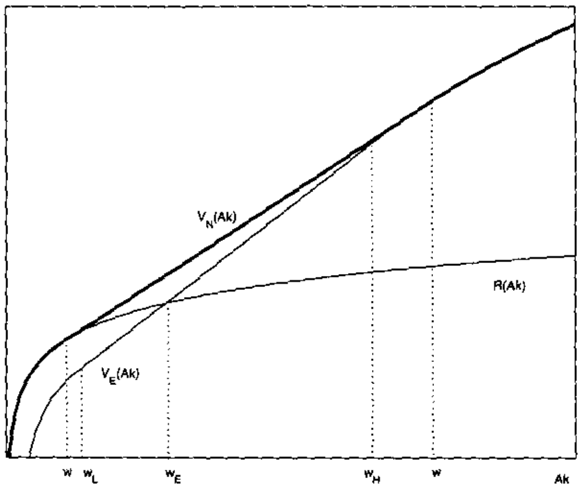

Figure 1: End-of-period expected value $V_N(Ak)$ of entrepreneur

The choice of project risk is used to randomize end-of-period wealth on the two points $\underline{w}$ and $\overline{w}$ depicted in Figure 1, giving an expected value that corresponds to the concave envelope of the two value functions considered.[6] Let $V_N(Ak)$ denote this function:

$$V_N(Ak) = \begin{cases} R(Ak) & \text{for } Ak \leq \underline{w}, \\ R(\underline{w}) + (Ak - \underline{w}) / (\overline{w} - \underline{w}) (V(\overline{w}) - R(\underline{w})) & \text{for } \underline{w} < Ak < \overline{w}, \\ V(Ak) & \text{for } Ak \geq \overline{w}. \end{cases}$$

As shown in the figure, depending on the level of investment k, we may distinguish three cases: If $Ak \leq \underline{w}$, it is optimal not to randomize and exit in the following period. In case $Ak \geq \bar{w}$, it is also optimal to invest in the safe project. Finally, if $\underline{w} < Ak < \bar{w}$, it is optimal to randomize between the two endpoints.

More formally, this choice is implied by the first order conditions for the

[6]The figure assumes that $R(w)$ and $V_E(w)$ have a unique intersection point. This obviously depends on the outside value function. In section 3 we derive this outside value function from a model of entrepreneurial choice and show that the single crossing property holds.

dynamic problem of the entrepreneur (3):

$$\begin{aligned} (x): \qquad & V'(yk) = V'(xk), \\ (p): \qquad & V'(yk) = \frac{V(yk) - V(xk)}{yk - xk}. \end{aligned} \tag{5}$$

These two equations say that the possible project's payoffs must coincide with the tangent points $\underline{w}$ and $\overline{w}$. Thus the optimal randomization is accomplished by choosing the project with $x = \underline{w}/k$, $y = \overline{w}/k$ and $p = (Ak - \underline{w}) / (\overline{w} - \underline{w})$. Note that the probability of the high payoff ("success") increases linearly with the scale of the project k.

2.2.2 Consumption/Investment choice

Letting w denote the wealth of the entrepreneur and since projects are self-financed, the level of consumption $c = w - k$. The consumption/investment decision is the solution to the following problem:

$$V_E(w) = \max_k u(w - k) + \beta V_N(Ak). \tag{6}$$

$V_N(Ak)$ denotes the expected value of an entrepreneur by the beginning of the next period if currently he invests k in the optimally chosen project. As shown before, it is the concave envelope of $V(w)$ and $R(w)$.The following lemma states that V_N and thus V_E are concave functions.

Lemma 1 *(concavity of $V_E(w)$)*
The possibility of investing into risky projects makes $V_E(w)$ concave.

We proceed to characterize the consumption/savings decision. The first order conditions for problem (6) are given by:

$$u'(w - k) = \beta A V_N'(Ak),$$

where

$$V_N'(Ak) = \begin{cases} R'(Ak) & \text{for } Ak \leq \underline{w} \\ R'(\underline{w}) = V_E'(\bar{w}) & \text{for } \underline{w} < Ak < \bar{w} \\ V'(Ak) & \text{for } Ak \geq \bar{w}. \end{cases}$$

The above first order conditions imply that consumption is constant at a level c^* given by $u'(c^*) = \beta A R'(\underline{w})$ when optimal investment Ak falls in the risk taking region, $\underline{w} < Ak < \bar{w}$. This corresponds to initial wealth levels w such that $w_L < w < w_H$, where $w_L = \underline{w}/A + c^*$ and $w_H = \bar{w}/A + c^*$. In this region, investment $k = w - c^*$ increases linearly with the agent's wealth and the probability of a successful realization increases. Outside this region, there is no risk taking and consumption and investment increase with wealth.

The above conditions also imply that once the wealth of the entrepreneur surpasses the threshold w_H, it grows continuously, remaining above $\bar{w}$ forever after. From that point on, there is no more risk taking. This is a special feature of the model explained by the existence of riskless projects and the absence of risk premia. In a more realistic setup, firms could recur in the set $w_L < w < w_H$ after a series of bad shocks.

A sharper characterization of the value of the entrepreneur $V_E(w)$ follows from the above comments. This value coincides with the value of a risk-free entrepreneur $V_l(w)$ for $w \geq w_H$; is linear in the intermediate range $w_L < w < w_H$; coincides with the value of the entrepreneur that invests into a safe project and quits in the following period for $w \leq w_L$. Note that if risky projects were not available, active entrepreneurs would face two options - either to stay or exit- at the beginning of the following period. Risk taking increases the entrepreneur's utility by eliminating this nonconcavity in the continuation value.

2.2.3 The optimal exit decision

The entrepreneur exits when $R(w) > V_E(w)$. A sufficient condition for this region to be nonempty is that $(1-\beta) R(0) > u(0)$. This condition is satisfied when the outside option includes some other source of income. When $R(w)$ crosses $V_E(w)$ at a unique point w_E, as in the example considered in section 3, this becomes the threshold for exit.

Suppose exit is given by a threshold policy with cutoff value w_E. Three situations may arise: *(i)* $w_E \leq w_L$; *(ii)* $w_L < w_E < w_H$ and *(iii)* $w_H < w_E$. For the last case, risk-taking would not be observed since entrepreneurs would exit once they are in the risk-taking region. In the other two cases risk-taking is observed. In case *(ii)*, the entrepreneur invests in a risky project, exits if it fails and stays forever if it succeeds. There is an upper bound on the probability of failure given by $(1 - p(w_E)) < 1$. In contrast, in case *(i)* there is no upper bound on project failure.[7]

[7]The example given in section 3 suggests that while case *(i)* is atypical, the other two

2.2.4 Characterization of the solution

The following Proposition summarizes the results derived in this section.

PROPOSITION 1 *Suppose the entrepreneur selects projects from the class $\Omega_2(A)$ with an expected return $A > 1/\beta$. Suppose the outside value of the entrepreneur $R(w)$ is concave. If $R(w)$ and $V_E(w)$ have a unique intersection point w_E, then there exist wealth levels $w_L < w_H$ such that:*

(i) Entrepreneurs exit if $w \leq w_E$ and stay if $w > w_E$;

(ii) Letting $w_* = \max\{w_L, w_E\}$ and $w^* = \max\{w_H, w_E\}$:

(a) entrepreneurs invest in safe projects and stay in business forever if $w \geq w^*$;

(b) invest in risky projects if $w \in (w_*, w^*)$ and stay in business the following period with probability $p(w) = (Ak(w) - \underline{w})/(\overline{w} - \underline{w})$;

(c) invest in safe projects if $w \leq w_*$ and exit in the following period.

(iii) If an entrepreneur chooses a risky project (i.e. $w \in (w_*, w^*)$), the probability of survival $p(w)$ and the level of investment $k(w)$ are increasing in w, while consumption $c(w)$ is constant.

The previous Proposition has some immediate implications for firm dynamics. In the following, we measure a firm's size by the level of its investment k.

CORROLLARY 1 *(i) Survival probability increases with firm size (ii) Conditional on survival, smaller firms have higher growth rates.*

The above results assume a single crossing of the functions $R(w)$ and $V_E(w)$. In case of multiple crossings, there will be more than one region of risk-taking. Within each of these regions, total investment will increase and the risk of failure decrease with wealth.

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cases may occur.

2.3 Extending the Class of Projects

In the above analysis we assume that the only projects available to entrepreneurs have returns concentrated in two points. In this section we show that this restriction is without loss of generality..

Let $\Omega(A) = \{\lambda | \int d\lambda = 1 \text{ and } \int z d\lambda(z) = A\}$. This is the set of all probability distributions of returns with mean A. Obviously, the class $\Omega_2(A)$ considered earlier is a subset of $\Omega(A)$. Thus, if we assume the entrepreneur chooses a project from $\Omega(A)$, all projects $(x, p) \in \Omega_2(A)$ are still available to him.

The following Proposition gives our main result in this section.

PROPOSITION 2 *Suppose the outside value of the entrepreneur $R(w)$ satisfies the assumptions of Proposition 1. Let the entrepreneur choose any project from $\Omega(A)$, where $\beta A > 1$. Then the distribution of returns of the project chosen is concentrated in two points, so the entrepreneurial decision is identical to the one described in Proposition 1.*

The proof of Proposition 2 is very intuitive. The decision problem (3) of the active entrepreneur is now given by:

$$\begin{aligned} V_E &= \max_{k,\lambda}\{u(w-k) + \beta \int V(zk)d\lambda(z)\}, \\ \text{s.t.:} &\quad \int d\lambda = 1 \text{ and } \int z d\lambda(z) = A. \end{aligned} \tag{7}$$

Together with the exit decision (4) it forms a well defined dynamic programming problem which has a unique solution.

If $V_E(w)$ coincides with the value function (3) found in the previous section, the value of the entrepreneur $V(w)$ is a piecewise concave function over the intervals $(0, w_E)$ and $(w_E, +\infty)$ (this follows from Lemma 1). For any given distribution of returns λ, Let x_λ and y_λ be the expected returns on the intervals $(0, w_E)$ and $(w_E, +\infty)$, respectively. Let $p_\lambda = \lambda(w_E, +\infty)$, the probability of the upper set of returns. Consider an alternative project that pays either x_λ (with probability $1 - p_\lambda$) and y_λ (with probability p_λ). Given that the value function is concave in the two regions considered, the expected return of this project is at least as high as the original one.

3 An Example: Occupational Choice Model

In Section 2 no interpretation was provided for the outside value. In this Section we endogenize $R(w)$ in a simple occupational choice model, study conditions under which risk taking will and will not occur, and provide some simulation results.

3.1 The Set Up of the Model

The decision problem of the entrepreneur is defined by (3) and (4) of the previous section. An entrepreneur becomes a worker if he exits from business. Workers receive wage $\phi > 0$ every period and save in a risk free bond to smooth consumption over time. The rate of return to the risk-free bond is r. We assume that $1/\beta \leq 1 + r < A$. This assumption, combined with self-financing, implies that only relatively wealthy agents are willing to become entrepreneurs.

At the beginning of every period a worker gets randomly "hit with an idea" that allows him to become an entrepreneur. The probability of this event is $0 \leq q \leq 1$. If the worker chooses to become an entrepreneur he receives no wage income. If the worker decides not to enter entrepreneurship, his situation becomes identical to that of a worker who was not faced with this opportunity. Let $R(w)$ denote the value of the worker conditional on not becoming an entrepreneur in the current period. Prior to the realization of the shock, the value to the worker $R_c(w)$ is given by

$$R_c(w) = (1 - q)R(w) + q \max\{V_E(w), R(w)\}. \tag{8}$$

This value defines the continuation value of the agent who is a worker in the current period. If the worker does not enter entrepreneurship, he must decide how much to save in the risk free bond, so his value $R(w)$ is given by:

$$R(w) = \max_a \{u(w + \phi - a) + \beta R_c((1 + r)a)\}. \tag{9}$$

The above two equations, together with (3) and (4) fully characterize the behavior of the agents in this discrete occupational choice model.

3.2 The Solution

The worker becomes an entrepreneur only if: (i) he gets an opportunity; and (ii) his current wealth level is such that $V_E(w) \geq R(w)$. Denote by w_E the lowest wealth level at which workers are willing to enter entrepreneurship, $V_E(w_E) = R(w_E)$, which determines the entry threshold rule for the workers. Since there is no entry or exit cost, w_E determines the exit threshold rule for the entrepreneurs. In the general setup, the entrepreneurial investment decision was described in Proposition 1, which requires concavity of $R(w)$ and single crossing of $R(w)$ and $V_E(w)$. Below we show that although these properties not necessarily hold, the results of Proposition 1 are still valid.

Lemma 2 *(Characterization of $R(w)$)*
Let $1/\beta \leq 1+r < A$, $R(w)$ solves (3), (4), (8) and (9), and $a(w)$ determines the worker's optimal rule of saving. Let $\widehat{R}(w)$ be the concave envelope of $R(w)$. Then

(i) if $q > 0$ then $R(w)$ is not concave;

(ii) if $w \leq w_E$ then the wealth profile of the worker increases over time, i.e. $(1+r)a(w) > w$;

(iii) if $R(w) = \widehat{R}(w)$ then $R((1+r)a(w)) = \widehat{R}((1+r)a(w))$;

(iv) if $R(w)$ is replaced with $\widehat{R}(w)$ in (4) then the behavior of the entrepreneurs investing in the projects with strictly positive probability of survival does not change.

The main implication of Lemma 2 is that although the value of the worker $R(w)$ is not concave, we may use its concave envelope $\widehat{R}(w)$ in order to describe the behavior of entrepreneurs. Nonconcavity of $R(w)$ is driven by the presence of the kink in the worker's continuation value function that necessarily occurs in w_E. Since below w_E the worker chooses an increasing in time wealth profile in order to benefit from becoming an entrepreneur in future, the kink in w_E is recursively reproduced onto the lower values of wealth. Therefore $R(w)$ is piecewisely concave to the left of w_E.

The behavior of a risk taking entrepreneur randomizing between exiting and staying in business in the next period is determined by the tangent points of $R(w)$ and $V_E(w)$ with their common tangent line. Obviously, these tangent points, as well as the tangent line itself, do not shift if instead of $R(w)$ its concave envelope $\widehat{R}(w)$ is used. Constructing and analyzing the concave envelope of $R(w)$ is useful for one more reason. The part (iii) of Lemma 2 says that if the current value of the worker falls into the strictly concave part of $\widehat{R}(w)$, his continuation value also belongs to the strictly concave part of $\widehat{R}(w)$. This property will be used later on to derive the uniqueness of the observed entry point w_E.

In the absence of risk taking opportunities the continuation value of the entrepreneur is given by $\max\{R(w), V_E(w)\}$. Apart from the kink in every point where $R(w) = V_E(w)$, this function has a number of kink points generated by nonconcavity of $R(w)$. Since risk taking allows entrepreneur to eliminate all this kinks, two types of risk takers may be potentially observed: those who randomize between exiting and staying in the next period, and those who exit for sure.

In the following Lemma we prove that there exists only one region of randomization with a positive probability of survival, i.e. that $\widehat{R}(w)$ and $V_E(w)$ have a unique intersection. At the same time, we cannot eliminate multiple intersection between $R(w)$ and $V_E(w)$, and that is why randomization followed by exit in the following period (unconditional on the realized payoff) may occur. However, this type of randomization places the worker's next period value on the strictly concave part of $\widehat{R}(w)$, and thus, by (iii) of Lemma 2, the sequence of all his future values will also fall into a strictly concave part, and no randomization with zero probability of business survival will be needed. Therefore, assuming without the loss of generality that initial wealth of every worker falls into a strictly concave part of $\widehat{R}(w)$ (or equal to zero), we conclude that only risk taking with positive probability of survival may be observed. Correspondingly, the cutoff entry wealth level is unique.

Lemma 3 *(Entry rule)*

(i) There exist a unique w_E such that $\widehat{R}(w_E) = V_E(w_E)$ and $\widehat{R}(w) > V_E(w)$ for $w < w_E$;

(ii) no entry is observed below w_E.

Figure 2 depicts the value functions previously defined. The intersection of $R(w)$ and $V_E(w)$ (solid thin lines) determines the entry wealth level w_E. For $w > w_E$, $V_E(w) > R(w)$, so every worker chooses entrepreneurship whenever this option is available to him. Since this occurs with probability q, $R_c(w)$ is a linear combination of $V_E(w)$ and $R(w)$ for $w \geq w_{NE}$. If $w \leq w_E$, the worker does not enter entrepreneurship, independently of the realized opportunity, so $R_c(w) = R(w)$ in this region. For simplicity we depicted the

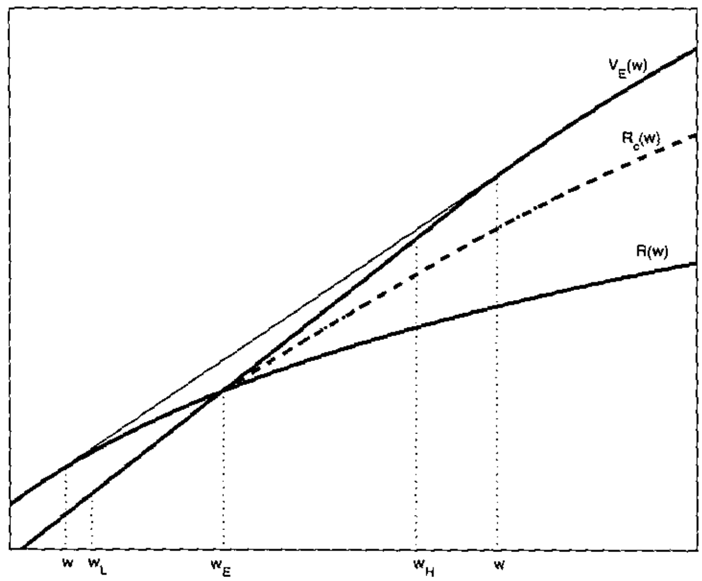

Figure 2: Value functions' allocation in the occupational choice model.

worker's current and continuation values $R(w)$ and $R_c(w)$ concave to the left of w_E, although a number of kinks occurs in this region.

Now we may use Lemmas 2 and 3 to characterize the behavior of the agents' in this occupational choice economy:

PROPOSITION 3 *If entry, exit and investment choice is defined by (3), (4), (8), and (9), then there exist* $0 < \underline{w} < w_E$ *and* $0 < w_H < \overline{w}$ *such that*

(i) workers with wealth levels $w > w_E$ *enter into entrepreneurship with probability* q*;*

(ii) entrepreneurs exit from business if $w \leq w_E$ *and stay otherwise;*

(iii) entrepreneurs with wealth levels $w_E \leq w \leq \max\{w_E, w_H\}$ *invest into risky projects, survival rates* $p(w)$ *of their businesses are bounded away from zero and increase with* w*, investment* $k(w)$ *also increases, while consumption* $c(w)$ *stays constant;*

(iv) entrepreneurs with wealth levels $w > \max\{w_E, w_H\}$ *invest into fully safe projects and stay in business forever; their investment* $k(w)$ *and consumption* $c(w)$ *increase in* w*.*

From Proposition 3 it follows that if an entrepreneur enters with wealth levels $w < w_H$, he invests in a risky project, obtaining either $\underline{w}$ or $\overline{w}$ at the beginning of the following period, depending on the realization of the project's return. If the low return is realized, the entrepreneur exits in the next period with wealth $\underline{w} < w_E$, otherwise he invests into a fully safe project from next period on. The probability $p(w)$ of the high return determines the survival probability of the establishment. Those entrepreneurs who enter with higher levels of wealth choose higher $p(w)$ and thus are more likely to stay in business.

3.3 Risk Taking

Risk taking not necessarily occurs in this environment. In particular, if the entry wealth level w_E exceeds the upper bound of the randomization region (w_H), risky investments will never be chosen. In the environment described above this happens if $q = 1$, i.e., if there is no uncertainty about being able to enter entrepreneurship.

PROPOSITION 4 *There exist $0 \leq \overline{q} < 1$ such that risk taking does not occur if $q \geq \overline{q}$.*

The result in the above Proposition is driven by the agents' desire to smooth consumption over time. In the absence of uncertainty, the worker correctly foresees his continuation value $\max\{R(w), V_E(w)\}$ and thus chooses a savings policy such that the downward jump in consumption at the moment he enters entrepreneurship is small. Correspondingly, the kink in the value function at the point of entry is so small that randomizations are not beneficial.

In contrast, in the presence of an uninsurable shock to entrepreneurial opportunities, the continuation value and optimal savings policy prior to the shock realization change after the resolution of this uncertainty. If the ex-post desired investment increases compared to its ex-ante desired level , current consumption would obviously go down. The possibility of risk taking allows an entrepreneur to decrease the size of this downward jump in consumption. In particular, as a consequence of the outside opportunity, the entrepreneur consumes more than the safe investment policy suggests - actually, as much as the entrepreneur with wealth level w_H does - and the rest of his wealth invests in risky projects. In the following period, independent of the project's payoff, he raises consumption up to $\bar{c}$ such that $u'(\bar{c}) = R'(\underline{w}) = V_E'(\overline{w})$. And only the future path of consumption will depend on the realized return of the risky project. Finally, only entrepreneurs with relatively low wealth levels use this consumption smoothing mechanism - because it is only for them that the outside opportunity provides the necessary insurance in case of project failure.

3.4 A Partial Case: $\beta(1+r)=1$

In the case of $\beta(1+r) = 1$ all the previous results imply but more may be said about entry threshold rule and the properties of risk taking. First of all, note that if entry into entrepreneurship is not possible ($q = 0$), the worker's wealth and consumption stay constant over time. The presence of entrepreneurial opportunity in future stimulates worker's wealth profile to grow until it reaches the wealth level at which opening business is efficient.

The allocation of the value functions associated with this partial case is illustrated on Figure 4 (in the end of the paper). It is easy to verify (directly follows from the proof of Lemma 2) that that $\widehat{R}_c(w)$ is now linear in the interval $(0, w_E)$. Then, obviously, there exist no common tangent line to $R(w)$ and $V_E(w)$, and thus risk taking entrepreneurs choose the corner solution $x = 0$ and end up with wealth $\underline{w} = 0$ if their businesses fail. Correspondingly, the value function $V_E(w)$ of the entrepreneur is linear to the left of w_H.

Since Lemma 2 applies, $\widehat{R}(w)$ and $V_E(w)$ have single intersection. If risk taking occurs ($w_H > w_E$) then at the intersection point $V_E'(w_E)$ exceeds $R_c'(w_E)$ and, correspondingly, the linear part of $V_E(w)$ is steeper than the linear part of $\widehat{R}(w)$. On the other hand, in the proof of Lemma 2 it is shown that the worker with wealth w_0^* (the closest to w_E kink point) saves more than $w_E/(1+r)$ for the next period. Consequently, $\lim_{w \to w_0^*+} R(w) < R_c'(w_E) < V_E'(w_E) = V_E'(w_0)$. Therefore, no intersection of $V_E(w)$ and $R(w)$ may occur in the neighborhood of w_0^*. Similarly, no intersection may occur in the neighborhoods of the lower kink points. That is why, no entry into entrepreneurship occurs below w_E, independently of the initial workers' wealth distribution.

3.5 The Numerical Example

The following numerical example illustrates the implications of the preceding theoretical analysis. We use the following parameter values: $\beta = 0.98$, r is

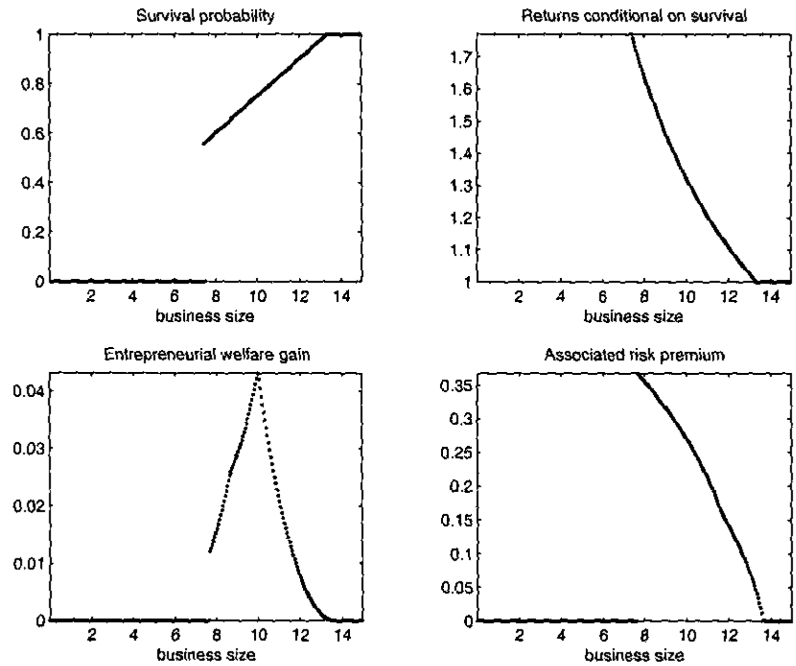

Figure 3: Optimal project choice, $u(c) = \ln(c)$, $q = 0.2$.

equal to the inverse of the rate of time preference $1/\beta$, and the expected entrepreneurial return A is 10 percent higher than the return to the risk-free bond. In this example we choose a logarithmic utility function $u(c) = \ln(c)$ and later consider how the behavior of entrepreneurs changes as the coefficient of relative risk aversion increases. Since receiving an opportunity to open business is a random event, entry into entrepreneurship may occur at any wealth level above w_E. The two top plots of Figure 3 depict survival probability and return conditional on survival as a function of business size. As was summarized in Corollary 1, larger establishments are more likely to survive, but experience lower rates of returns. Note that in this economy exit from entrepreneurship occurs only due to the presence of risk taking: if the risky projects are not available, the homogeneity of expected project's returns together with condition $1/\beta \leq 1 + r < A$ implies that all entrepreneurs continue operating their businesses once the entry decision has been made.

The bottom left plot of Figure 3 presents the welfare gain that risk taking entrepreneurs obtain due to the availability of risky projects. Most of all benefit those entrepreneurs who enter with the wealth level equal to the cutoff entry wealth in the absence of risky projects. For these agents the welfare gain is fairly high: they would lose 4.3 percent of their life-time consumption if the risky investment were not available.

The bottom right graph plots the amount of risk premium needed to compensate for risk taking if the outside opportunity were not available. In this example, the poorest entrepreneur bearing the largest risk would require 37% of risk premium for the investment they make if they were not able to use an outside opportunity as an insurance instrument.

Obviously, the behavior of entrepreneurs depends on the arrival rate of the entrepreneurial opportunity. In accordance with Proposition 4, if opening a business is an event that may be planned long in advance, the worker organizes his consumption path in such a way that the downward jump in consumption at the moment of entering entrepreneurship is small, and thus risk taking is not beneficial. However, if the decision to become an entrepreneur is made unexpectedly, risky investment allow entrepreneur to adjust his consumption path gradually. Table 1 shows how the amount of risk taken and the required risk premium change if entry into entrepreneurship gets more predictable (q increases).

Table 1: Risk taking and required risk premium

q	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\min p(w)$	0.31	0.55	0.71	0.80	0.87	0.93	0.96	0.99	-
max RP	0.49	0.37	0.30	0.26	0.20	0.12	0.06	0.01	-

The first raw of Table 1 reports the lowest survival rate that is observed in the economy. If the opportunity to enter entrepreneurship is a rare event ($q = 0.1$), workers facing this opportunity at relatively low wealth levels decide to enter and invest into very risky projects, exiting with probability $1 - p(w) = 0.69$ in the following period. If the risky projects were not available, the poorest entrepreneur would require a risk premium of approximately half of the expected entrepreneurial return to compensate for the amount of risk he takes. As the probability of the entrepreneurial opportunity increases, less risk taking occurs. This happens because as q increases, workers make better predictions about their future optimal investment level, and thus when the entrepreneurial opportunity arrives it leads to smaller jumps in consumption, decreasing entrepreneur's incentives for risk taking. According to Proposition 4, there exists a maximum $\overline{q}$ such that no one makes risky investment if entry into entrepreneurship is possible with probability higher than $\overline{q}$. In the simulated example $\overline{q} \approx 0.81$.

4 Final Remarks

Entrepreneurship is risky, but there appears to be no premium to private equity. Any theory addressing this puzzle must rely, directly or indirectly, on a positive -or at least neutral- attitude towards risk. Earlier papers in this area assume directly that entrepreneurs have a lower degree of risk aversion. In our paper, the indirect utility function of the entrepreneur has a nonconcave region, where riskiness is desired. However, this nonconcavity is created by the existence of an outside opportunity so it does not rely on assumptions about preferences for risk.

As a theory of risk taking, our model has specific implications. The combination of the outside option and financing constraints imply a desire for risk at low wealth levels, close to the exit threshold. As a consequence, risk taking decreases with the level of wealth, giving rise to the positive correlation between size (measured by investment) and survival found in the data. This is an implication of our theory that would be hard to derive just from the heterogeneity of preferences. As an example, Cressy (2000) justifies risk-taking by entrepreneurs assuming that higher wealth makes agents less risk averse. A consequence of this assumption is that larger firms should take more risk and thus exhibit more variable growth, which is counter to the data.

Entrepreneurs in our model are self-financed. This is obviously an extreme form of borrowing constraint. A recent paper by Clementi and Hopenhayn (2002), derive borrowing constraints as part of an optimal lending contract in the presence of moral hazard. The nonconvexity due to an outside (liquidation) option is also present in their model and there is a region where randomization is optimal.

We have chosen to keep our model stylized in order to get sharper results. As a downside, our model has some special unrealistic features. Most notably, risk-taking occurs only once; if the outcome is favorable, the entrepreneur takes no further risk and stays in business forever. These results follow from the possibility of choosing projects with arbitrary risk levels (including a fully safe one) and equal returns. Risk taking could last for more than one period if the variance of returns was bounded above. On the other hand, a lower bound on project risk or a return/risk trade-off, generates the possibility of future exit by firms that are currently outside the randomization region.

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5 Appendix

Proof of Proposition 1:
We develop the proof recursively. Assume that the value function $V_E(w)$ in the right hand side of (4) is concave, has unique intersection with $R(w)$, and to the right of $\overline{w}$ coincides with the determined in (2) value function $V_l(w)$ (where $\overline{w}$ denotes the tangent point between $V_E(w)$ and the common tangent line drawn to $V_E(w)$ and $R(w)$). To complete the proof we must show that these assumptions imply that (i)-(iii) of the Propositions are satisfied and that similar properties hold for the entrepreneurial value function endogenously determined in (3).

If the entrepreneur chooses a risk-free project, his value function solves the following dynamic problem:

$$\begin{aligned} V_F(w) &= \max[\max_{k_1}\{u(w-k_1)+\beta R(Ak_1)\}, \max_{k_2}\{u(w-k_2)+\beta V_E(Ak_2)\}] \\ &= \max\{V_1(w), V_2(w)\}. \end{aligned} \tag{10}$$

Obviously, defined in this way $V_F(w)$ is not concave, although each of the two functions in the right hand side of (10) is concave. Denote by w_H and w_L the wealth levels at which $Ak_1(w_H) = \overline{w}$ and $Ak_2(w_L) = \underline{w}$. Obviously, $R(\underline{w}) < V_E(\overline{w})$ implies that $V_1(w_L) < V_2(w_H)$ and $V_F(w) = V_l(w)$ for $w \geq w_H$ by definition of $V_F(w)$ and w_H. By the first order conditions, $V_1'(w_L) = \beta AR'(\underline{w}) = \beta AV_E'(\overline{w}) = V_2'(w_H)$, and thus $u(w_L - k_2(w_L)) = u(w_H - k_1(w_H))$. Therefore,

$$\begin{aligned} \frac{V_1(w_H)-V_2(w_L)}{w_H - w_L} &= \frac{\beta V_E(\overline{w}) - \beta R(\underline{w})}{k_1(w_H) - k_2(w_L)} = \beta A \frac{V_E(\overline{w}) - R(\underline{w})}{\overline{w} - \underline{w}} \\ &= \beta AV_E'(\overline{w}) = \beta AR'(\underline{w}) = V_1'(w_H) = V_2'(w_L). \end{aligned}$$

The above implies that the line drawn through $(w_L, V_1(w_L))$ and $(w_H, V_2(w_H))$ is tangent to both $V_1(w)$ and $V_2(w)$. Consequently, $V_1(w_L) > V_2(w_L)$ and $V_1(w_H) < V_2(w_H)$. Moreover, this common tangent line is unique because if there exist another common tangent line with the correspondent tangent points w' and w'' then, by uniqueness of intersection of $V_E(w)$ and $R(w)$, $AK_1(w') = \underline{w}$ and $Ak_2(w'') = \overline{w}$. The latter, by concavity of $R(w)$ and $V_E(w)$, implies that $w' = w_L$ and $w'' = w_H$.

If the entrepreneur decides to invest in a risky project, the first order conditions (5) must satisfy with equality. Single crossing of $V_E(w)$ and $R(w)$

implies that $xk = \underline{w}$ and $yk = \overline{w}$. Then from (1) it follows that the probability of the realization of high payoff $p = (Ak - \underline{w})/(\overline{w} - \underline{w})$ is an increasing function of entrepreneurial wealth, i.e. (iii) of the Proposition is proven. By the first order condition with respect to k, the value function of the entrepreneur is linear if investment into a risky project is optimal: $V_E'(w) = u'(w-k) = \beta A V_E'(\overline{w})$. This implies that by choosing a risky project the entrepreneur is able to eliminate a nonconcavity in $V_F(w)$, and therefore risky investments are made only if the current wealth of the entrepreneur falls into (w_L, w_H). Note that since $w_H < \overline{w}$ and $\beta A > 1$ entrepreneurs with wealth level $w > \overline{w}$ invest in a risk free project and stay in business forever. Moreover, the condition $R(\underline{w}) > V_E(\underline{w})$ is necessary for $\underline{w}$ to be a tangent point with the common tangent line to $R(w)$ and $V_E(w)$, thus entrepreneurs exit if $w = \underline{w}$. This proves (ii) of the Proposition.

Now we verify that the assumptions made in the first paragraph hold. Concavity of $V_E(w)$ is established above. Next, $V_E(w) = V_l(w)$ for $w \geq \overline{w}$ since $V_E(w) = V_F(w) = V_l(w)$ for $w \geq w_H$ and $w_H \leq \overline{w}$. As to the uniqueness of intersection of $R(w)$ and $V_E(w)$, additional assumptions on $R(w)$ are to be made. The necessary condition would be a single crossing property for $R(w)$ and $V_l(w)$ - quite a standard assumption. If the latter holds, the multiple intersection of $R(w)$ and $V_E(w)$ could occur only if $R(w)$ has more than one intersection point with the function $V_1(w)$. Since the shape of $V_1(w)$ is determined by the shape of $R(w)$, whether or not single crossing property is satisfied for $R(w)$ and $V_E(w)$ depends on the properties of $R(w)$, which so far has not been endogenized. That is why (i) of the Proposition holds only if exogenously chosen $R(w)$ is such that $R(w)$ and $V_E(w)$ have unique intersection. Q.E.D.

Proof of Lemma 2:

First, we make a number of assumptions about the properties of the entrepreneurial value function $V_E(w)$: (A1) $V_E(w)$ is concave; (A2) $V_E(w) = V_l(w)$ for $w \geq \overline{w}$, where $V_l(w)$ is defined in (2). In the proof of Lemma 3 we show that these assumptions are indeed satisfied.

(i) Assume that $R(w)$ is concave.
It is straightforward to verify that $R(w)$ and $V_E(w)$ have at least one intersection point: (a) $R(0) \geq u(\phi)/(1-\beta) > \lim_{w \to u} u(w)/(1-\beta) = V_E(0)$; (b) if $R(w) > V_E(w)$ for all $w \geq 0$ then $R_c(w) = R(w)$ and, consequently, $R(w) = u(\phi + (1-\beta)w)/(1-\beta)$. Using assumption (A2)

it is easy to verify that for $w > \max\{\overline{w}, \phi/(\beta - 1/A)\}$ the inequality $V_E(w) = V_l(w) > u(\phi + (1-\beta)w)/(1-\beta) = R(w)$ holds, which leads to the contradiction and implies that at least one intersection point of $R(w)$ and $V_E(w)$ exists.

Then from concavity of both $R(w)$ and $V_E(w)$ it follows that the defined in (8) value function $R_c(w)$ is not concave, which in turns implies that the defined in (9) function $R(w)$ is not concave either.

(ii)-(iii) If the value function $V_E(w)$ is known, the equations (8) and (9) define a standard dynamic programming problem that has a unique solution. To find this solution it is enough to construct the value functions $R(w)$ and $R_c(w)$ and verify that they satisfy (8) and (9).

It is straightforward to show that there exist w_E such that $R(w_E) = V_E(w_E)$ and $R(w) < V_E(w)$ for $w > w_E$. This implies that $R'(w_E) < V_E'(w_E)$, from which, by definition of $R_c(w)$ in (8), it follows that $R'(w_E) < R_c'(w_E)$. Using assumption (A1), the concavity of $R(w)$ and $R_c(w)$ over $[w_E, +\infty)$ is recursively established. Therefore, letting $a(w)$ denote the optimal saving policy associated with (9), we conclude that $(1+r)a(w_E) > w_E$. Thus there exist a wealth level $w_0 < w_E$ such that $(1+r)a(w_0) = w_E$ and $(1+r)a(w) > w_E$ for $w > w_0$ (as depicted on Figure 5).

Define a function $R_0(w)$ that coincides with $R(w)$ for $w \geq w_0$, is continuously differentiable in w_0, and is a straight line for $w < w_0$. Since $R_0'(w_0) \geq \lim_{w \to w_E+} R_c'(w)$, there exist a common tangent line to $R_0(w)$ and $R_c(w)$ with the correspondent tangent points $\underline{w_0} \in (w_0, w_E)$ and $\overline{w_0} \in (w_E, +\infty)$, for which the following equalities hold:

$$R_0'(\underline{w_0}) = R_c'(\overline{w_0}) = \frac{R_c(\overline{w_0}) - R_0(\underline{w_0})}{\overline{w_0} - \underline{w_0}}. \tag{11}$$

Define another function:

$$R_1(w) = \max_{a_1}\{u(w + \phi - a_1) + \beta R_0((1+r)a_1)\}. \tag{12}$$

Obviously, $R_1(w_0) \geq R_0(w_0)$ since the optimal in (9) saving level $a_0(w_0) = w_E/(1+r)$ is available in (12). However, this consumption/saving allocation is not optimal for (12) because the first order condition holds with strict inequality: $u'(w_0 + \phi - a_0(w_0)) =$

$\beta(1+r)R_c'(w_E) > \beta(1+r)R_0'(w_E)$. Thus the optimal level of savings $a_1(w_0)$ must fall below w_E. Consequently, $R_1(w_0) > R_0(w_0)$.

Denote by w_0' the wealth level at which the payoff to the optimal savings associated with the problem (12) equals to w_E, $(1+r)a_1(w_0') = w_E$. From $(1+r)a_1(w_0) < w_E$ it follows that $w_0' > w_0$. Saving level $a_1(w_0')$ is also feasible in the maximization problem (9), so $R_1(w_0') < R_s(w_0')$, where the strict inequality occurs because the first order and the envelope conditions to (9) are not satisfied. This implies that there exist $w_0^* \in (w_0, w_0')$ such that $R_1(w_0^*) = R_0(w_0^*)$ and $a_1(w_0^*) < w_E/(1+r) < a_0(w_0^*)$.

Let w_1 denote the wealth level at which optimal in (12) saving $a_1(w_1)$ equals to $w_0/(1+r)$. Since $R_1'(w_1) > R_0'(w_0)$, there exist a common tangent line to $R_1(w)$ and $R_s(w)$ with the tangent points $\underline{w_1} \in (w_1, w_0^*)$ and $\overline{w_1} \in (w_0^*, w_E)$ correspondingly. If $(c_1(\underline{w_1}), a_1(\underline{w_1}))$ and $(c_0(\overline{w_1}), a_0(\overline{w_1}))$ stand for the correspondent consumption/saving allocation, then $c_1(\underline{w_1}) = c_0(\overline{w_1})$ because the slopes in $\underline{w_1}$ and $\overline{w_1}$ are equal, and thus:

$R_0'(\overline{w_1}) = R_1'(\underline{w_1}) = \frac{R_0(\overline{w_1}) - R_1(\underline{w_1})}{\overline{w_1} - \underline{w_1}} = \beta(1+r)\frac{R_c((1+r)a_0(\overline{w_1})) - R_0((1+r)a_1(\underline{w_1}))}{(1+r)a_0(\overline{w_1}) - (1+r)a_1(\underline{w_1})}$.
Now, using the first order conditions for (9) and (12) we conclude that $(1+r)a_0(\overline{w_1})$ and $(1+r)a_1(\underline{w_1})$ solve (11), and consequently $(1+r)a_0(\overline{w_1}) = \overline{w_0}$ and $(1+r)a_1(\underline{w_1}) = \underline{w_0}$. This implies that if $w < \underline{w_1}$ then $(1+r)a_1(w) < \underline{w_0}$, as well as if $w > \overline{w_1}$ then $(1+r)a_0(w) > \overline{w_0}$. Finally, since $R_1'(\underline{w_1}) \leq R_0'(\underline{w_0})$, the function $\max\{R_1(w), R_0(w), R(w)\}$, together with the common tangent lines (passing through $\underline{w_1}$, $\overline{w_1}$, and $\underline{w_0}$, $\overline{w_0}$) forms a concave function over $(w_1, +\infty)$.

Determine a sequence of functions $\{R_n(w), n \geq 1\}$ in a recursive way:

$$R_n(w) = \max_{a_n}\{u(w + \phi - a_n) + \beta R_{n-1}((1+r)a_n)\}, \qquad (13)$$

and define

$$R(w) = \max\{R_0(w), R_1(w), ..., R_n(w), ...\}. \qquad (14)$$

If $R(w)$ has a unique intersection with $V_E(w)$ at the point w_E then, obviously, $R(w)$ solves (9) and (8), and the following properties hold: (1) if $w \in (w_{n+1}^*, w_n^*)$ then $(1+r)a_{n+1}(w) \in (w_n^*, w_{n-1}^*)$, i.e. (ii) of the Lemma holds; (2) if $w \in (\overline{w_{n+1}}, \underline{w_n})$ then $(1+r)a_{n+1}(w) \in (\overline{w_n}, \underline{w_{n-1}})$, i.e. (iii) of the Lemma holds.

In Lemma 3 we show that $\widehat{R}(w)$ (the concave envelope of $R(w)$) and $V_E(w)$ satisfy a single crossing property, although intersections of $R(w)$ and $V_E(w)$ may potentially occur within the intervals $(\underline{w_n}, \overline{w_n})$. If this happens, the shape of $R(w)$ changes within $(\underline{w_{n+1}}, \overline{w_{n+1}})$, but the shape of $\widehat{R}(w)$ and the properties (1) and (2) remain unchanged.

(iv) The last statement of the Lemma is directly implied by the fact that the concave envelope on $\max\{R(w), V_E(w)\}$ coincides with the concave envelope on $\max\{\widehat{R}(w), V_E(w)\}$. Q.E.D.

Proof of Lemma 3:

(i) We construct the proof by making the recursive argument: assuming the $V_E(w)$ is such that $\widehat{R}(w)$ and $V_E(w)$ have unique intersection point we show that the similar property holds for the value of the entrepreneur $V_E(w)$ endogenously defined in (3). At the same time, we verify recursively that assumptions (A1) and (A2) hold.

Denote by w_E the largest point at which $R(w_E) = V_E(w_E)$ holds. Note that $R(w) < V_E(w)$ for $w < w_E$. By construction of $R(w)$ in the proof of Lemma 2, $R(w_E) = \widehat{R}(w_E)$. If w_E is the unique intersection of $\widehat{R}(w)$ and $V_E(w)$, then $\widehat{R}(w)$ has the shape that was described in details above. In particular, the property (iii) of Lemma 2 holds.

As in Proposition1, the possibility of risk taking allows entrepreneur to eliminate all nonconcavities in his next period's value. Thus, after the decision about the riskiness of the project has been made, the expected continuation value of the entrepreneur is given by the concave envelope of $\max\{R(w), V_E(w)\}$, which obviously coincides with the concave envelope of $\max\{\widehat{R}(w), V_E(w)\}$. Thus $V_E(w)$ is concave and assumption (A1) holds. Applying similar reasoning as in the proof of Proposition 1 and using assumption (A2), we imply that the current value of the entrepreneur is a concave envelope on $\max\{V_s(w), V_l(w)\}$, where

$$V_s(w) = \max_{k_s}\{u(w - k_s) + \beta\widehat{R}(Ak_s)\}. \tag{15}$$

Note that due to nonconcavity of $R(w)$ there exist intervals of wealth within which $V_s(w)$ is linear. If the value of the entrepreneur falls into one of these intervals, he chooses to invest in a risky project in order to eliminate nonconcavity in $R(w)$, but, independently of the realization of the project's return, the entrepreneur quits in the following period.

Since $\widehat{R}(w)$ and $V_E(w)$ have unique intersection and both functions are concave, there is only one randomization region of the next period wealth, $(\underline{w}, \overline{w})$, in which the probability of business survival is positive ($\underline{w}$ and $\overline{w}$ are the tangent points of $\widehat{R}(w)$ and $V_E(w)$ with their common tangent line).

Denote by w_L and w_H the wealth levels at which $Ak_s(w_L) = \underline{w}$ and $Ak_l(w) = w_H$, where $k_l(w)$ and $k_s(w)$ denote the optimal saving decisions in the problems (2) and (15). Then the sequence of arguments similar to the one we used in the proof of Proposition 1 implies that w_L and w_H are the tangent points of $V_S(w)$ and $V_l(w)$ with their common tangent line. Obviously, $V_E(w) = V_l(w)$ for $w > w_H$, so assumption (A2) holds.

Let us evaluate $V_s(w_L)$. Since $Ak_s(w_L) = \underline{w}$, $V_s(w_L) = u(c_s(w_L)) + \beta\widehat{R}(\underline{w})$, and by the first order condition $u'(c_s(w_L)) = \beta A\widehat{R}'(\underline{w})$. By construction of $\widehat{R}(w)$, there exists $w' < \underline{w}$ such that $\widehat{R}(w') = R(w')$ and the optimal savings of the worker at w' are such that $(1+r)a(w') = \underline{w}$. Then $\widehat{R}(w') = u(c_R(w')) + \beta\widehat{R}(\underline{w})$, and $u'(c_R(w')) = \beta(1+r)\widehat{R}'(\underline{w})$. Since $1+r < A$, $u'(c_R(w')) < u'(c_s(w_L))$, and, consequently, $c_R(w') > c_s(w_L)$. Therefore, $V_s(w_L) < \widehat{R}(w') < \widehat{R}(\underline{w})$. Note also that this property implies that there exists a positive lower bound on the probability of survival of the businesses with risky investment.

Since $V_S(w_L) < \widehat{R}(\underline{w})$ and $V_E(w)$ is linear in the interval (w_L, w_H), the multiple intersection of $\widehat{R}(w)$ and the derived $V_E(w)$ may occur only if $V_s(w)$ and $\widehat{R}(w)$ have a multiple intersection as it is shown on Figure 6. (Remember that $V_s(0) = u(0) + \beta R(0) = u(0) + \beta u(\phi)/(1-\beta) < u(\phi)/(1-\beta) = R(0)$.) If this happens, there exist w_1 and w_2 such that $V_s'(w_1) = \widehat{R}'(w_2) = (\widehat{R}(w_2) - V_s(w_1))/(w_2 - w_1)$ and $V_s(w_1) < \widehat{R}(w_2)$. Since $w_1 < w_L$, the continuation wealth of the entrepreneur with the current wealth w_1 is equal to $w_1' = Ak_s(w_1) < \underline{w}$. Letting w_2' denote the continuation wealth of the worker at w_2 and using (iii) of Lemma 2, $\beta A\widehat{R}'(w_1') = V_s'(w_1) = \widehat{R}'(w_2) = \beta(1+r)\widehat{R}'(w_2')$. Thus $\widehat{R}'(w_1') < \widehat{R}'(w_2')$, which by concavity of $\widehat{R}(w)$ implies that $\widehat{R}(w_1') > \widehat{R}(w_2')$. Therefore, since agents' consumption levels at w_1 and w_2 coincide, the inequality $V_s(w_1) > \widehat{R}(w_2)$ must hold, which contradicts to the properties of w_1 and w_2. Therefore, $V_E(w)$ and $\widehat{R}(w)$ have unique intersection in w_E.

(ii) The sketch of the proof of the second part is described in the paragraph preceding Lemma 3. Q.E.D.

Proof of Proposition 4:
If $q = 1$ then $R_c(w) = V_E(w)$ for $w \geq w_E$ and $V_E(w) = V_l(w)$ for $w \geq \overline{w}$. By construction of $R(w)$ in the proof of Lemma 2, the tangent point $\underline{w}$ falls into the same concave part of $R(w)$ where w_E belongs. Denote by w' the optimal continuation wealth of the worker whose current wealth level is w. Since $R'(w_E) > V_E'(w_E)$ and $\beta(1+r) \geq 1$, we conclude that $w_E' > \underline{w}' \geq \overline{w}' > w_E$ (see Figure 7).

Let w_0 be the wealth level at which a risk-free entrepreneur invests $k(w_0) = w_E'/A$ in the project. Then the first order conditions and the inequality $1 + r \leq A$ imply that the worker at w_E consumes more than entrepreneur at w_0. Since the continuation value of both agents equals to $V_E(w_E')$, the entrepreneur at w_0 is worse off than the worker at w_E, $R(w_E) > V_E(w_E)$.

Finally, since $w_E' > \overline{w}$, the wealth level w_H, at which a risk-free entrepreneur invests $\overline{w}/A$, is smaller than w_0, implying that $w_H < w_E$. This means that no entry occurs within the risk taking interval (w_L, w_H), and thus no risky investment is made if $q = 1$. By continuity, risk taking does not occur if q is large enough. Q.E.D.

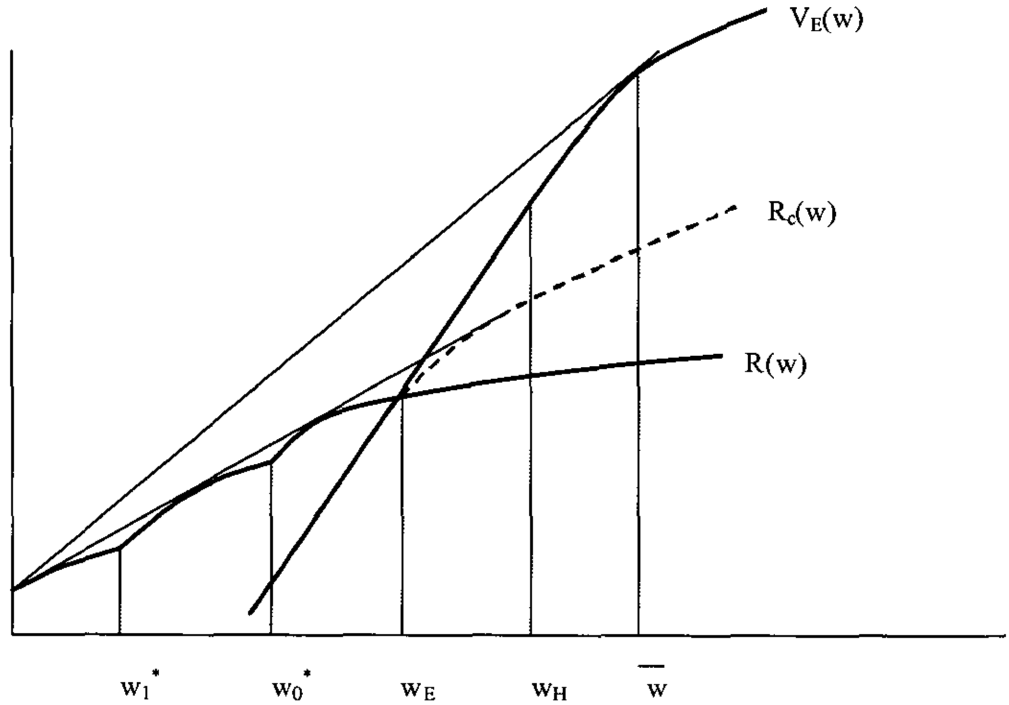

Figure 4: Value function allocation for beta*(1+r)=1

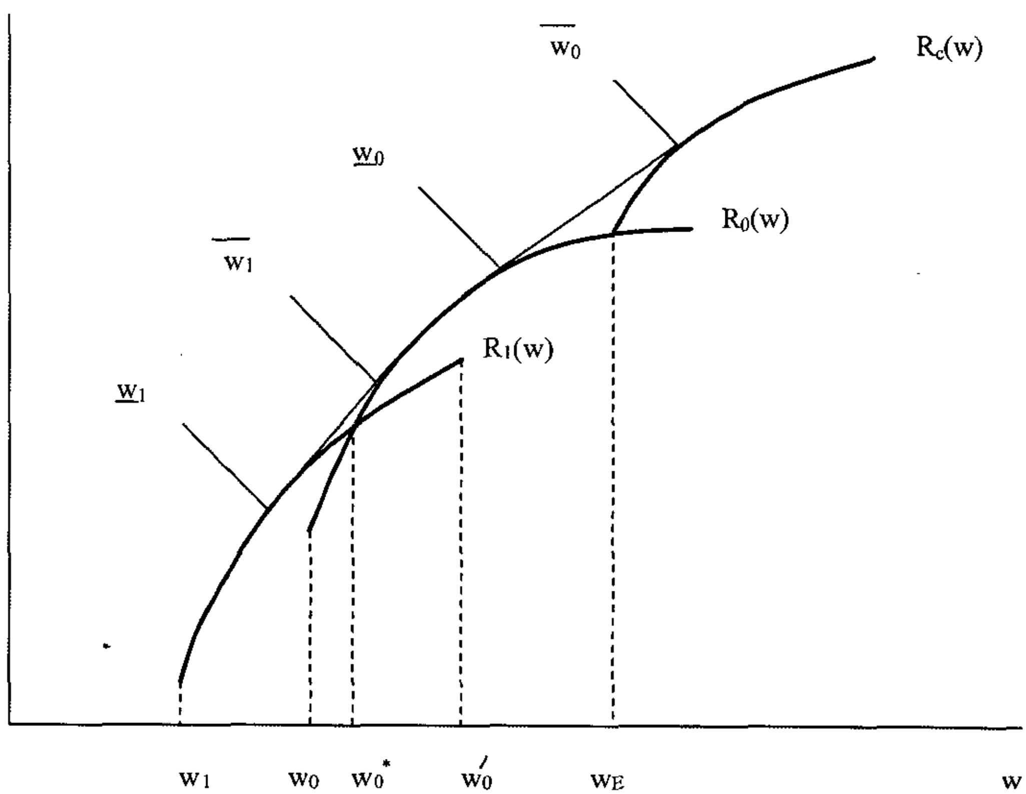

Figure 5: R(w) endogenously determined in the occupational choice model

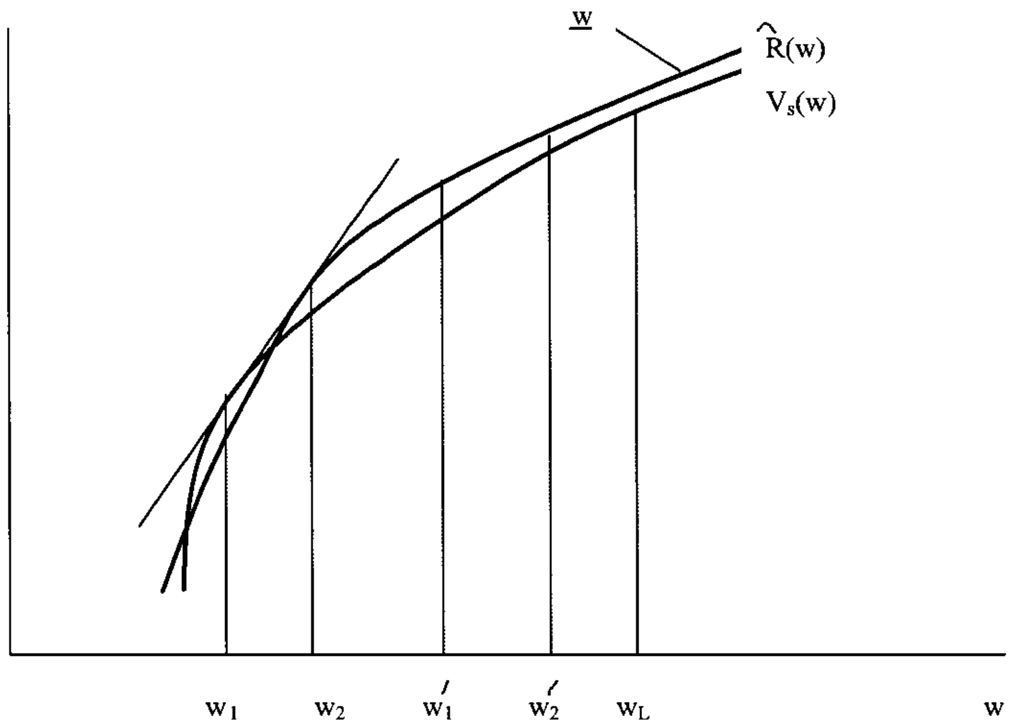

Figure 6: Single crossing of $V_s(w)$ and $\hat{R}(w)$

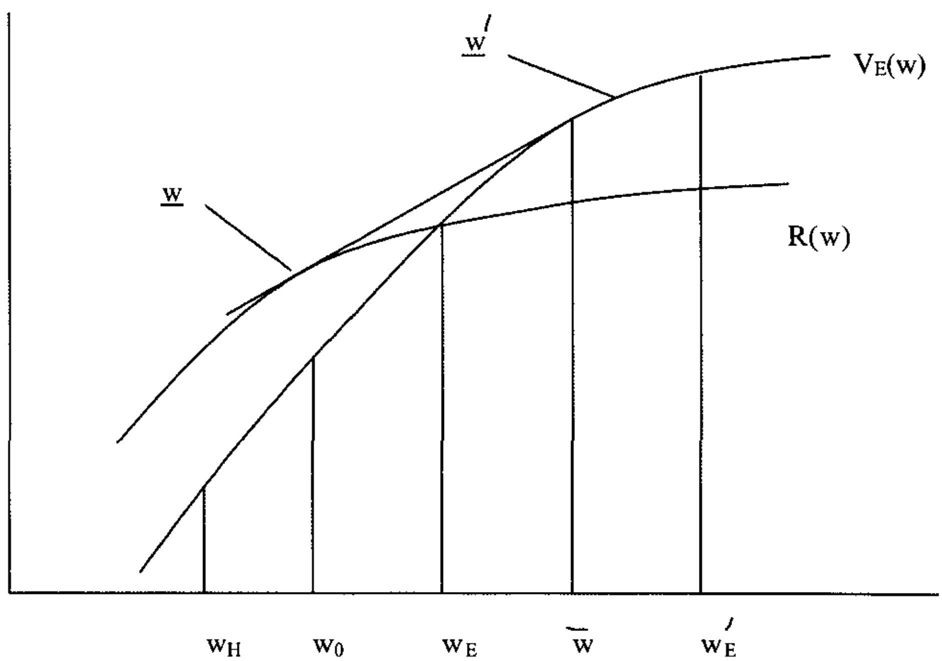

Figure 7: No risk taking in case of q=1