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Sophisticated Monetary Policies*

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ABSTRACT

The Ramsey approach to policy analysis finds the best competitive equilibrium given a set of available instruments. This approach is silent about unique implementation, namely designing policies so that the associated competitive equilibrium is unique. This silence is particularly problematic in monetary policy environments where many ways of specifying policy lead to indeterminacy. We show that sophisticated policies which depend on the history of private actions and which can differ on and off the equilibrium path can uniquely implement any desired competitive equilibrium. A large literature has argued that monetary policy should adhere to the Taylor principle to eliminate indeterminacy. Our findings say that adherence to the Taylor principle on these grounds is unnecessary. Finally, we show that sophisticated policies are robust to imperfect information.

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Frank Ramsey proposed a now classic approach to policy analysis under commitment. This approach specifies the set of instruments available to policymakers. The *Ramsey problem* is to find the competitive equilibrium that maximizes social welfare with the given set of instruments. Barro (1979), Lucas and Stokey (1983) among many others have extended this approach to situations with uncertainty by specifying the instruments as functions of exogenous events. This extension has made the approach very useful in addressing policy questions in macroeconomics[1].

While the Ramsey approach has been very useful in characterizing the best competitive outcomes, by itself it is not an operational guide to policy in the sense that it does not tell policy-makers how to conduct policy for all possible histories. An approach that is an operational guide would specify policies for every history and would describe what the corresponding outcomes will be for every history. Here we extend the language of Chari and Kehoe (1993) to an environment in which the policymaker has commitment in order to specify policies after every history and to describe continuation outcomes after every history. In our approach, we allow policies to depend on the history of past actions by private agents and allow them to differ on and off the equilibrium path. We label such policies *sophisticated policies* and label the resulting equilibrium a *sophisticated equilibrium*.

In many macroeconomic models, especially monetary models, many ways of specifying policies lead to indeterminacy and therefore could lead to undesirable outcomes. In such models the question of pressing interest is, Can policies be specified as operational guides which tell policymakers what to do after every history and which lead a desired outcome to be the unique equilibrium outcome? If we can, then we say the policy *uniquely implements* the desired outcome.

In this paper we study two standard monetary economies: a simple model with one period price setting and a model with Calvo price setting. The main contribution in this paper is to show that, under sufficient conditions, any outcome of a competitive equilibrium can be uniquely implemented by appropriately chosen sophisticated policies.

Our findings are of particular relevance to monetary policy. The simplest way of applying the Ramsey approach to policy-making is to find the desired competitive equilibrium, which specifies policy as a function of exogenous events, and use that function for policy. This way of describing policies is problematic in monetary policy environments in which policy-makers use short-term interest rates as their principal monetary policy instrument. The reason is that, since at least the early work of Sargent and Wallace (1975), researchers have been aware that policies which make interest rates functions only of exogenous events can lead to indeterminate outcomes. Such policies could lead to the best equilibrium but they could also lead to undesirable outcomes including hyperinflation and outcomes with excessive volatility due to sunspot-like fluctuations. In this sense, policies which lead to indeterminacy are risky and researchers generally agree that such policies should be avoided.

This concern with the risks arising from indeterminacy has led to a substantial literature which is aimed at finding policy rules which eliminate indeterminacy. (See, for example, McCallum 1979 and for some recent work see Woodford 2003.) The recent literature argues that interest rate rules should follow the *Taylor principle:* interest rates should rise more than one-for-one with a rise in inflation rates relative to target inflation. The reasoning is that an interest rate rule that obeys the Taylor principle leads to unique outcomes while a rule that violates it leads to indeterminacy. A related literature argues that the undesirable inflation experiences of the 1970s in the United States was due in large part to the failure of monetary policy to obey the Taylor principle in that time period. (See, for example, Clarida, Gali, and Gertler (2000).)

Our finding that sophisticated policies can implement any equilibrium outcome implies that such policies can implement outcomes which violate the Taylor Principle along the equilibrium path as well as those which obey such a principle along the equilibrium path. Since sophisticated policies can uniquely implement any desired competitive equilibrium it follows that adherence to the Taylor principle is not needed for uniqueness. Moreover, our findings imply that historical evidence that policy violated the Taylor principle in some periods does not necessarily imply that such policy was unwise.

Our paper is also related to previous papers that address problems of indeterminacy in monetary economies (see Wallace 1981, Obstfeld and Rogoff 1983, Benhabib, Schmitt-Grohe, Uribe 2001, Christiano and Rostagno 2001, King, 2001, and Svensson and Woodford, 2005). Read at face value these papers pursue a different approach to implementation that we refer to as *implementation via nonexistence.* The idea is to specify policies so that for all outcomes but the desired one, no competitive equilibrium exists. This approach is quite different from the approach taken here as well as the approach taken in the microeconomic literature on implementation. The general idea of implementation via nonexistence pursued in various contexts has been criticized in both the macroeconomic and the microeconomic literature. In the macroeconomic literature Kocherlakota and Phelan (1999), Buiter (2002), Ljungqvist and Sargent (2004), and Bassetto (2005) criticize this general idea in the context of the fiscal theory of the price level and Cochrane (2007) criticizes in the context text of the literature on monetary policy rules. In the microeconomic literature, Jackson (2001) criticizes a similar approach to implementation.

We agree with those who argue that this approach trivializes the implementation problem. To see why, consider the following policy. If private agents choose the desired outcome, continue with the desired policy. If private agents deviate from the desired outcome: then forever after set government spending at a high level and taxes to zero. Clearly, any deviation leads to nonexistence of equilibrium and hence we trivially have implementation via nonexistence. Our approach, in contrast, insists that policies be specified so that a competitive equilibrium exists following any deviation. We achieve implementation in the traditional microeconomic sense, namely by specifying policies which provide incentives for agents not to deviate–not by nonexistence. In our approach policies are specified so that even if all other agents deviate an individual agent has no incentive to do so.

Despite our criticisms of implementation via nonexistence, the papers in this literature make two valuable contributions. The first is the idea of *regime-switching*. This idea dates back to at least Wallace (1981) and has been used by Obstfeld and Rogoff (1983), Christiano and Rostagno (2001), and Benhabib, Schmitt-Grohe, Uribe (2002). The basic idea in, say, Benhabib, Schmitt-Grohe, Uribe (2002) is that if the economy embarks on an undesirable path some combination of monetary and fiscal policy switches regimes in such a way that the government's budget constraint is violated and no equilibrium exists.

We use regime-switching in some of our sophisticated policies. We show that under sufficient conditions switching from an interest regime to a money regime after deviations can uniquely implement any desired outcome. In contrast to the existing literature, however, we do not achieve implementation via nonexistence, but rather by structuring the money regime so as to discourage deviations in the first place.

The second idea is that of the *King rule*, namely an interest rate policy which makes the difference between the interest rate and its target is a linear function of the difference between inflation and its target, with a coefficient greater than 1. This idea dates back to at least King (2001) and has been used by Svensson and Woodford (2004). Restricting attention to equilibria in which all variables are bounded, this rule implements equilibria via nonexistence: any deviation of outcomes from the target cause the outcomes to become unbounded thus lead to nonexistence of (bounded) equilibria.

As we show, in the simple model the King rule cannot uniquely implement any outcomes and in the Calvo model it cannot implement some unbounded outcomes. Interestingly, this rule, appropriately translated, leads to unique implementation of bounded equilibria in the Calvo model. But, again, our implementation works by discouraging deviations not via nonexistence.

The basic idea of the construction is the same in both models. Consider constucting central bank policies that uniquely implement a desired competitive equilibrium. Along the equilibrium path choose the policies to be those given by the desired competitive equilibrium. Structure the policies off the equilibrium path, referred to as *reversion* policies, to discourage deviations. Specifically, if the average choice of private agents deviates from those in the desired equilibrium, choose the reversion policies so that the optimal choice, or *best response,* of each individual agent is different from the average choice.

Our construction requires that best responses be *controllable*, in the sense that policies can be found which ensure that, following any deviation, the best response of any individual private agent is different from the average choice of the private agents. Controllability imples that reversion policies can be constructed so that no deviation is optimal and hence that the desired equilibrium is uniquely implemented. A sufficient condition for controllability is that policies can be found so that the continuation equilibrium is unique and varies with policy. The latter requirement typically holds so that if policies can be found under which the continuation equilibrium is unique (somewhere) then we have unique implementation (everywhere). This sufficient condition suggests a simple way to state the general message of our paper: uniqueness somewhere generates uniqueness everywhere.

Our reversion policies have an important property: they are not extreme in any sense. Indeed, they simply bring inflation back to the desired path and do not threaten the private economy with dire outcomes following deviations.

One concern with our construction is that it apparently relies on the idea that the central bank perfectly observe private agents' actions and thus can detect any deviation. We show that our results are robust to imperfect information about private agents' actions.

Here we propose one way to eliminate indeterminacy under interest rate rules. For some other proposed resolutions to the indeterminacy issue, see the work of Bassetto (2002) and Adao, Correia, and Teles (2006).

1. A Simple Model with One-Period Price Stickiness

We begin by illustrating the basic idea of our construction of sophisticated policies using the simple model with one-period price setting because the dynamical system associated with the competitive equilibrium is very simple and this simplicity allows us to focus on the strategic aspects of sophisticated policies. In a subsequent section we investigate a New Keynesian model with multiperiod price setting.

The model we use to analyze the optimal choice of a monetary policy instrument is a modified version of the basic sticky price model with a New Classical Phillips curve (as in Woodford 2003, Chap. 3, Sec. 1.3). Here, in order to make our results comparable to those in the literature, we describe a simple, linearized version of the model. In Appendix A, we describe the general equilibrium model that when linearized gives rise to the equilibrium conditions studied here. Our implementation result holds in the nonlinear model as well.

A. The Determinants of Output and Inflation

Consider a monetary economy populated by a large number of identical, infinitely lived consumers, a continuum of producers, and a central bank. Each producer uses labor to produce a differentiated good on the unit interval. Producers $i \in [0, \alpha]$ are *flexible price* producers, and producers $i \in [\alpha, 1]$ are *sticky price* producers.

The timing within a period is as follows. At the beginning of the period, sticky price producers set their prices after which the government chooses its monetary policy, either by setting interest rates or by choosing the quantity of money. Shocks η_t and ν_t are then realized.

At the end of the period, flexible price producers set their prices, and consumers make their decisions. We interpret the shock η_t as a *flight to quality* shock that affects the attractiveness of government debt relative to private claims and the shock ν_t as a *velocity* shock.

Here we develop necessary conditions for an equilibrium and then, in the next section, formally define an equilibrium. Here and throughout we express all variables in log-deviation form. In particular, this way of expressing variables implies that none of our equations will have constant terms.

Consumer behavior in this model is summarized by an intertemporal Euler equation and a cash-in-advance constraint. We can write the linearized Euler equation as

$$(1) \qquad y_t = E_t\left[y_{t+1}\right] - \psi\left(i_t - E_t\left[\pi_{t+1}\right]\right) + \eta_t,$$

where y_t is aggregate output, i_t is the nominal interest rate, η_t is an i.i.d. mean zero shock with variance $var(\eta)$, and $\pi_{t+1} = p_{t+1} - p_t$ is the inflation rate from time period t to $t+1$, where p_t is the aggregate price level. The parameter ψ determines the intertemporal elasticity, and E_t denotes the expectations of a representative agent given that agent's information in period t, which includes the shock η_t.

The cash-in-advance constraint, when first-differenced, implies that the relationship between inflation π_t, money growth μ_t, and output growth $y_t - y_{t-1}$ is given by a quantity equation of the form

$$(2) \qquad \pi_t = \mu_t - (y_t - y_{t-1}) + \nu_t.$$

where ν_t is an i.i.d. mean zero shock with variance $var(\nu)$.

We turn now to producer behavior. The aggregate price level p_t is a linear combination of the prices p_{ft} set by the flexible price producers and the prices p_{st} set by the sticky price producers and is given by

$$(3) \qquad p_t = \int_0^\alpha p_{ft}(i) + \int_\alpha^1 p_{st}(i).$$

The optimal price set by an individual flexible price producer i satisfies

$$(4) \qquad p_{ft}(i) = p_t + \gamma y_t,$$

where the parameter γ is the elasticity of the equilibrium real wage with respect to output and is referred to in the literature as *Taylor's* γ. The optimal price set by sticky price producer i satisfies

$$(5) \qquad p_{st}(i) = E_{t-1}\left[p_t + \gamma y_t\right],$$

where E_{t-1} denotes expectations at the beginning of period t before the shock η_t is realized. Using language from game theory, we can think of equations (4) and (5) as akin to the best responses of each flexible and sticky price producer given their beliefs about the aggregate price level and aggregate output. Equations (1)–(5) completely describe the simple model.

In this model, the flexible price producers are uninteresting strategically, in that their expectations about the future have no influence on their decisions. Their prices are set mechanically according to the static considerations reflected in (4). In all that follows, equation (4) will hold on and off the equilibrium path, and we can think of $p_{ft}(i)$ as being residually determined by (4) and substitute out for $p_{ft}(i)$ from these equations. To do so, substitute (4) into (3) and solve for p_t to get

$$(6) \qquad p_t = \kappa y_t + \frac{1}{1-\alpha}\int_{\alpha}^{1} p_{st}(i),$$

where $\kappa = \alpha\gamma/(1-\alpha)$. Now the system is summarized by (1), (2), (5), and (6).

We follow the literature and express the sticky price producers' decisions in inflation rates rather than price levels. To do so, let $x_t(i) = p_{st}(i) - p_{t-1}$ and rewrite (5) as

$$(7) \qquad x_t(i) = E_{t-1}\left[\pi_t + \gamma y_t\right].$$

We find it convenient to define

$$(8) \qquad x_t = \frac{1}{1-\alpha}\int_{\alpha}^{1} x_t(i)\ di$$

to be the average price set by the sticky price producers relative to the aggregate price level in period $t-1$. We can then rewrite (6) as

$$(9) \qquad \pi_t = \kappa y_t + x_t.$$

For later use, note that the economy is summarized by (1), (2), and (7)–(9), so that when checking whether a constructed outcome is a competitive equilibrium, we will need only to check whether these equations are satisfied.

In the following lemma, we show how this economy produces the key features of a New Classical Phillips curve along the equilibrium path in which

(10) $x_t(i) = x_t.$

(See the discussion below for what happens following deviations from the equilibrium path.)

Lemma 1. Any allocations that satisfy (7)−(9) and (10) satisfy (i)

(11) $x_t = E_{t-1}\pi_t,$

(ii) $E_{t-1}y_t = 0$, and (iii) the New Classical Phillips curve:

(12) $\pi_t = \kappa y_t + E_{t-1}\pi_t$

where $\kappa = \alpha\gamma/(1-\alpha)$.

Proof. To prove (ii), substitute (9) into (7). Integrating both sides of the resulting equation from α to 1 and using (8) yields (ii). Taking expectations of both sides of (9) and using (ii) yields (i) and (iii). *Q.E.D.*

Note that when it sets monetary policy, the central bank chooses to operate under either a money regime or an interest rate regime. In the money regime, the central bank sets μ_t, and the interest rate is residually determined from (1) after the realization of the shock η_t. In the interest rate regime, the central bank sets i_t, and money growth is residually determined from (2) after the realization of the shock η_t. Of course, in both regimes, equations (1) and (2) hold.

B. Competitive Equilibrium

We define a notion of competitive equilibrium in the spirit of Barro (1979) and Lucas and Stokey (1983). In this equilibrium allocations, prices, and policies are all defined as functions of the history of exogenous events $s^t = (s_0, \ldots, s_t)$, where $s_t = (\eta_t, \nu_t)$.

The actions of the sticky price producers, inflation, and output can be summarized by $\{x_t(s^{t-1}), \pi_t(s^t), y_t(s^t)\}$. In terms of the policies we find it convenient to let the regime choice as well as the policy choice within the regime as $\delta_t(s^{t-1}) = (\delta_{1t}(s^{t-1}), \delta_{2t}(s^{t-1}))$ where the first argument $\delta_{1t}(s^{t-1}) \in \{M, I\}$ denotes the regime choice, money (M) or interest rates (I), and the second argument denotes the policy choice within the regime, either money growth $\mu_t(s^{t-1})$ or interest rates $i_t(s^{t-1})$. If the money regime is chosen at t the interest rate is determined residually at the end of the period while if the interest rate regime is chosen at t then the money growth rate is determined residually at the end of the period. Let $a_t(s^t)$ denote the allocations, prices, and policies in this competitive equilibrium.

A collection of allocations, prices, and policies $a_t(s^t) = \{x_t(s^{t-1}), \pi_t(s^t), y_t(s^t), \delta_t(s^{t-1})\}$ is a *competitive equilibrium* if it satisfies (1), (2), (9), and (11).

C. Sophisticated Equilibrium

We now turn to sophisticated equilibrium. This definition is very similar to that of the definition of competitive equilibrium except that we allow allocations, prices, and policies to be functions of the history of both aggregate private actions and policies as well as the history of exogenous events.

We make two observations before we turn to our formal definition. First our definition of sophisticated equilibrium simply specifies policy rules by the central bank and does not require any form of optimality by the central bank. We specify sophisticated policies in this manner to show that our result regarding unique implementation does not depend on the objectives of the central bank. One way of thinking of our sophisticated policies is that the policies are specified at the beginning of period 0 and then the central bank is is committed to follow them.

Second, the only interesting private agents in this model are the sticky price producers. Their behavior at the beginning of period t depends on what they expect the government to do and what other sticky price producers do. The flexible price producers are described by a simple static rule (4). The behavior of the consumers and the flexible price producers is summarized by an intertemporal Euler equation (1) and the cash-in-advance constraint (2).

We turn now to defining the histories that private and the central bank confront when they make their decisions. The public events that occur in a period are, in chronological order, $q_t = (x_t; \delta_t; s_t; y_t, \pi_t)$. Letting h_t denote the history of these events from period 0 up to and including those in period t, we have that $h_t = (h_{t-1}, q_t)$ for $t \geq 1$ and $h_0 = q_0$. As a matter of notational convenience, we focus on perfect public equilibria in which the central bank's strategy is a function of only the public history.

The public history faced by the sticky price producers at the beginning of period t when they set their prices is h_{t-1}. A strategy for the sticky price producers is a sequence of rules $\sigma_s(i) = \{x_t(i, h_{t-1})\}$ for choosing wages for every possible public history, while average prices by these producers are given by $\sigma_x = \{x_t(h_{t-1})\}$.

The public history faced by the central bank when it sets its regime and either its money growth or interest rate policy is $h_{gt} = (h_{t-1}, x_t)$. A strategy for the central bank $\{\delta_t(h_{gt})\}$ is a sequence of rules for choosing the regime as well as the policy within the regime, either $\mu_t(h_{gt})$ or $i_t(h_{gt})$.

If the money regime is chosen in period t $\big(\delta_{1t}(h_{gt})$ specifies $M\big)$, then interest rates $i_t\big(h_{yt}\big)$, output $y_t\big(h_{yt}\big)$,and inflation rates $\pi_t\big(h_{yt}\big)$ are determined residually from (1), (2), (9), and (11) after the relevant shocks are realized, where here $h_{yt} = (h_{t-1}, x_t; M, \mu_t; s_t)$ is the history that determines output, inflation, and interest rates in the current period.

If, instead, in period t the interest rate regime is chosen $\big(\delta_{1t}(h_{gt})$ specifies $I\big)$, then the money growth rate $\mu_t\big(h_{yt}\big)$ as well as output $y_t\big(h_{yt}\big)$ and inflation $\pi_t\big(h_{yt}\big)$ are determined residually from (1), (2), (9), and (11) after the relevant shocks are realized, where here $h_{yt} = (h_{t-1}, x_t; I, i_t; s_t)$ is the history that determines output, inflation, and money growth in the current period.

We let σ_g denote the strategy of the central bank consisting of the regime choice and the policies under that regime. At the end of period t, output and inflation are determined as functions of the relevant history h_{yt} according to the rules $y_t\big(h_{yt}\big)$ and $\pi_t\big(h_{yt}\big)$. We let $\sigma_y = \big\{y_t\big(h_{yt}\big)\big\}$ and $\sigma_\pi = \big\{\pi_t\big(h_{yt}\big)\big\}$ denote the sequence of output and inflation rules.

A *sophisticated equilibrium* given the policies here is a collection of strategies $(\sigma_s(i), \sigma_x, \sigma_g)$ and output and inflation rules (σ_y, σ_π) such that given the other strategies and rules, $\sigma_s(i)$ is optimal for all histories in the sense that

$$x_t(i, h_{t-1}) = E\left[\pi_t(h_{yt}) + \gamma y_t(h_{yt})\right], \tag{13}$$

the aggregate choices σ_x are related to the individual choices $\sigma_s(i)$ according to

$$x_t(h_{t-1}) = \frac{1}{1-\alpha} \int_\alpha^1 x_t(i, h_{t-1})\, di, \tag{14}$$

the Phillips curve is given by

$$(15) \quad \pi_t(h_{yt}) = \kappa y_t(h_{yt}) + x_t(h_{t-1})$$

and (1) and (2) are satisfied in the manner described above.

In light of condition (14) and the observation that given (σ_g, σ_x), output, inflation, and the residually determined policy are mechanically given by (1), (2), and (9), we summarize a sophisticated equilibrium by (σ_g, σ_x). Note for later, from Lemma 1 that

$$(16) \quad x_t(h_{t-1}) = E[\pi_t | h_{t-1}].$$

Associated with each sophisticated equilibrium $\sigma = (\sigma_g, \sigma_x)$ are the particular stochastic processes for outcomes that occur along the equilibrium path, called *sophisticated outcomes*. These sophisticated outcomes can be generated from the strategies in the standard recursive fashion. These outcomes can then be written as a function of the history of exogenous events $s^t = (s_0, \ldots, s_t)$, where $s_t = (\eta_t, \nu_t)$. These (on the equilibrium path) outcomes include allocations $a(\sigma) = \{x_t(s^{t-1}; \sigma), \pi_t(s^t; \sigma), y_t(s^t; \sigma), \delta_t(s^{t-1})\}$. We call an allocation $a(\sigma)$ associated with a sophisticated equilibrium σ a *sophisticated outcome.* The following lemma is an immediate consequence of the definitions of competitive equilibrium and sophisticated outcomes.

Lemma 2. (Equivalence between competitive equilibria and sophisticated outcomes.) A sophisticated outcome is a competitive equilibrium and for any given competitive equilibrium there exists a sophisticated policy which supports the competitive equilibrium as a sophisticated outcome.

Equilibrium with Sophisticated Policies

We now show that any competitive equilibrium in which the central bank uses interest rates as its instrument can be uniquely implemented with sophisticated policies. Later we show that when the central bank uses money as its instrument, unique implementation is trivial.

The basic idea behind our construction is that the central bank starts by picking any competitive equilibrium allocations and sets its interest rate policy on the equilibrium path consistent with this equilibrium. The central bank then constructs its policy off the equilibrium path so that any deviations from these allocations would never be optimal for the deviating agent. In so doing, the constructed sophisticated policies support the chosen allocations as the unique equilibrium allocations.

In our construction, we find it convenient to consider sophisticated policies with *one-period reversion to money.* Under these policies, the central bank discourages deviations by switching to a money regime for one period, and for the rest of the off-the-equilibrium-path policies, it uses the continuation of what it would have done on the equilibrium path. In particular, after a deviation, the central bank switches to a level of the money supply which generates the same expected inflation as in the original equilibrium. (Of course, we could have chosen many other values that also would discourage deviations, but we found this value to be the most intuitive one.[2]) Having the central bank switch to a money regime, instead of another interest rate in an interest rate regime after a deviation is convenient because as we show in the following lemma outcomes are uniquely determined under a money regime.

Lemma 3. (Controllability of Best Responses) For any history h_{gt}, if the central bank chooses the money regime with money growth μ_t, then output y_t and inflation π_t are uniquely determined and given by

$$y_t = \frac{\mu_t + \nu_t + y_{t-1} - x_t}{1+\kappa} \tag{17}$$

$$\pi_t = \kappa y_t + x_t. \tag{18}$$

Proof. The proof is immediate from substituting (2) into (9) and recalling that y_{t-1} and x_t are in the history h_{gt}. *Q.E.D.*

Note that this lemma applies to histories h_{gt} which have been generated off the equilibrium path as well as on it. In particular, it applies to histories in which the sticky price producers' choice of inflation x_t represents a deviation from their strategies (and does not equal their expectations of inflation.)

We use this lemma to construct sophisticated policies that uniquely support any competitive outcome. To do so fix a desired competitive equilibrium outcome path $\big(x_t^*(s^{t-1}),$ $\pi_t^*(s^t),\ y_t^*(s^t)\big)$ together with central bank policies $i_t^*(s^{t-1})$. Consider the following trigger-type policy that supports these outcomes as unique equilibria: If sticky price producers choose

x_t at t to coincide with the desired outcomes $x_t^*(s^{t-1})$, then let central bank policy in period t be $i_t^*(s^{t-1})$. If not and these producers deviate to some $\tilde{x}_t(s^{t-1}) \neq x_t^*(s^{t-1})$, then for that period t, let the central bank switch to a money regime with money growth set so that the expected inflation for that period equals the expected level of inflation in the original equilibrium, namely, $x_t^*(s^{t-1})$. To determine the required level of money growth, use (17) and (18) to calculate that the required level of money growth is given by

$$(19) \quad \tilde{\mu}_t = \tilde{x}_t(s^{t-1}) - y_{t-1} + \frac{1+\kappa}{\kappa}\left[x_t^*(s^{t-1}) - \tilde{x}_t(s^{t-1})\right].$$

>From period $t+1$ on along this deviation path, let the central bank use what it would have done if there had been no deviation. From period $t+1$ on along the equilibrium path, let the central bank continue on with the analog of the policies just described.

We use these policies to establish the following proposition:

PROPOSITION 1. UNIQUE IMPLEMENTATION WITH SOPHISTICATED POLICIES Any competitive equilibrium outcome in which the central bank uses interest rates as its instrument can be implemented as a unique equilibrium with sophisticated policies with one-period reversion to money.

Proof. Consider the sophisticated policies described above, and suppose that in period t the sticky price producers deviate to $\tilde{x}_t(s^{t-1}) \neq x_t^*(s^{t-1})$. Then the central bank sets money growth according to (19), and the resulting inflation, by construction, is $\pi_t^*(s^t)$, and the resulting output is

$$(20) \quad \tilde{y}_t = \frac{\tilde{\mu}_t + \nu_t + y_{t-1} - \tilde{x}_t(s^{t-1})}{1+\kappa},$$

where we have used (17). Substituting for $\tilde{\mu}_t$ from (19) gives that

$$(21) \quad E_{t-1}\tilde{y}_t = \frac{1}{\kappa}\left[x_t^*(s^{t-1}) - \tilde{x}_t(s^{t-1})\right].$$

We need to show that given these levels of inflation and output, a sticky price producer will not find it optimal to make this deviation. That is, the sticky price producer will set $x_t(i)$ to some value other than $\tilde{x}_t$. From (7), we can see that the best response of a sticky price producer is

$$(22) \quad x_t(i) = E_{t-1}\left[x_t^*(s^{t-1}) + \gamma\tilde{y}_t\right],$$

where we have used the fact that $\tilde{\mu}_t$ is constructed to generate a level of inflation equal to $x_t^*(s^{t-1})$. Combining (21) and (22), we have that the best response of the sticky price producer is

$$x_t(i) = x_t^*(s^{t-1}) + \frac{1}{\kappa}\left[x_t^*(s^{t-1}) - \tilde{x}_t(s^{t-1})\right],$$

since $\kappa > 0$, clearly $x_t(i) \neq \tilde{x}_t(s^{t-1})$ whenever $\tilde{x}_t(s^{t-1}) \neq x_t^*(s^{t-1})$. That is, an individual sticky price producer will never find it optimal to follow the deviation $\tilde{x}_t(s^{t-1})$ whenever $\tilde{x}_t(s^{t-1})$ is indeed a deviation from $x_t^*(s^{t-1})$. *Q.E.D.*

The logic of the proof of the proposition makes clear that in order for reversions to money to uniquely implement equilibrium outcomes, sophisticated policies must have a key *controllability* property: following a deviation, the central bank can choose policies so as to make it not optimal for an individual price-setter to cooperate with the deviation. A sufficient condition for this property is that an individual price-setter's best response is uniquely determined by (and monotone in) the money growth rate. The construction of money growth given in (19) shows that in the simple model, after a deviation monetary policy can be chosen in such a way that the best response of any individual price-setter can be controlled.

A simple way of describing our unique implementation result is that uniqueness of best responses under some regime guarantees unique implementation of any desired outcome. Note that if the variance of the money shock v_t is large, all of the outcomes under money are undesirable. Nevertheless, the money regime is useful as an off-equilibrium commitment that helps support desirable outcomes along the equilibrium path under interest rate regimes.

Finally, as a technical aside, note from the proof of Proposition 1 that we do not need uniqueness for all money growth policies. Rather, all we need is that for every deviation we can find a monetary policy which induces a best response correspondence that does not include the deviation.

So far we have focused on implementing competitive outcomes when the central bank uses interest rates as its instrument. From Lemma 3 is immediately follows that it is trivial to uniquely implement competitive outcomes in which the central bank uses money as its instrument. Clearly, we can use a simple generalization of Proposition 3 to uniquely implement competitive equilibrium in which the central bank uses interest rates in some periods and money in other periods: in periods in which the monetary regime specifies interest rates, use sophisticated policies with reversion to money while in periods in which the monetary regime specifies money, make the money growth independent of the decisions of private agents.

Necessity of Regime Switching for Unique Implementation

We now turn to a common way of modeling policies, referred to as *restricted policies.* Such policies are restricted to be the same on and off the equilibrium path and are typically assumed to be linear functions of private agents' actions. Here we show that any interest rate policies that are linear functions of actions and shocks that the central bank has observed lead to a continuum of equilibria. Hence, such policies cannot uniquely implement any desired outcome. In this sense, in order to uniquely implement any desired outcome the central bank must switch from an interest rate regime to a money regime following deviations.

Consider a class of restricted policies of the form

$$(23) \quad i_t = \bar{\imath}_t + \sum_{s=0}^{\infty} \phi_{xs} x_{t-s} + \sum_{s=1}^{\infty} \phi_{ys} y_{t-s} + \sum_{s=1}^{\infty} \phi_{\pi s} \pi_{t-s}$$

where $\bar{\imath}_t$ can depend upon the history of stochastic events $\{\eta_s\}_{s=0}^{t-1}$. Notice that policies of this kind are linear feedback rules on variables in the central bank's history. We can then can establish the following result.

PROPOSITION 2. INDETERMINACY OF EQUILIBRIUM UNDER RESTRICTED POLICIES The linear competitive equilibria with interest rate rules of the linear feedback form (23) have outcomes of the form

$$(24) \quad x_{t+1} = i_t + c\eta_t,\ \pi_t = x_t + \kappa(1+\psi c)\eta_t,\ \text{and } y_t = (1+\psi c)\eta_t.$$

For every feedback rule the economy has a continuum of competitive equilibria indexed by the parameter c and by x_0.

Proof. In order to verify that the outcomes which satisfy (24) are part of an equilibrium, we need to check that they satisfy (1), (9), and (16). That they satisfy (16) follows by taking expectations of both sides of the equation $\pi_t = x_t + \kappa(1+\psi c)\eta_t$. Substituting for x_{t+1} from (24) and i_t from (23) into (1), we obtain that $y_t = (1+\psi c)\eta_t$, as required by (24). Inspecting the expressions for π_t and y_t in (24) shows that they satisfy (9). *Q.E.D.*

This proposition shows that if the central bank follows an interest rate regime in all periods for all histories, the economy has a continuum of competitive equilibria. In this sense, unique implemention requires regime switching.

The class of linear feedback rules in (23) includes a popular specification of the Taylor rules of the form as

$$(25) \quad i_t = \bar{\imath}_t + \phi E_{t-1}\pi_t + bE_{t-1}y_t.$$

When the parameter $\phi > 1$, such policies are said to satisfy the *Taylor principle*, namely, that the central bank should raise its interest rate more than one-for-one with increases in inflation. When $\phi < 1$, such policies are said to violate that principle.

Of course, the Taylor rule is not a well-defined function of histories until we fill in how expectations are formed. To do so we begin with a simple lemma. The lemma shows that under any interest rate rule, the expected inflation rate is uniquely determined by the policy, but the realized inflation rate may not be.

Lemma 4. In any history h_{t-1},

$$(26) \quad E\left[y_t|h_{t-1}\right] = 0.$$

If that history gives rise to an interest rate regime, then

$$(27) \quad E\left[\pi_{t+1}|h_{t-1}\right] = i_t(h_{gt}),$$

where $h_{gt} = \big(h_{t-1}, x_t(h_{t-1})\big)$.

Proof. Note that (26) is simply a restatement of part (ii) of Lemma 1. Taking expectations of the Euler equation (1) with respect to h_{t-1} gives that

$$(28) \quad E\left[y_t|h_{t-1}\right] = E\left[y_{t+1}|h_{t-1}\right] - \psi\big(i_t(h_{t-1}) - E\left[\pi_{t+1}|h_{t-1}\right]\big).$$

Using the law of iterated expectations gives that $E\left[y_{t+1}|h_{t-1}\right] = 0$. From (28) we then have (27), that $E\left[\pi_{t+1}|h_{t-1}\right] = i_t(h_{gt})$. *Q.E.D.*

>From this lemma we know that $E\left[y_t|h_{t-1}\right] = 0$. Since $E\left[\pi_t|h_{t-1}\right] = x_t$, policies of the Taylor rule form can be written as

$$(29) \quad i_t = \bar{\imath}_t + \phi x_t.$$

Thus, policies of the Taylor rule form (29) are linear feedback rules of the form (23) and thus lead to indeterminacy, regardless of the value of ϕ. For every $\phi \geq 1$ the economy has a continuum of unbounded equilibria indexed by c and $x_0 \geq 0$ as well as a unique bounded equilibrium with $c = 0$ and $x_0 = 0$. For $\phi < 1$, all the equilibria are bounded.

To discuss boundedness, it is useful to substitute from (29) into the first equation in (24) to obtain a difference equation in expected inflation

(30) $\quad x_{t+1} = \bar{\imath}_t + \phi x_t + c\eta_t.$

If $\phi \geq 1$, then clearly $x_t = 0$ when $c = 0$ and $x_0 = 0$ and is unbounded otherwise. If $\phi < 1$, then clearly x_t is bounded.

We now show that rules of the form specified in (23) include rules of the form discussed by King (2001) and Svensson and Woodford (2005) given by

(31) $\quad i_t = i_t^* + \phi(E_{t-1}\pi_t - E_{t-1}\pi_t^*)$

where i_t^* and π_t^* can depend upon the history of stochastic events. The idea behind a *King rule* of the form (31) is i_t^* and π_t^* are the interest rates and inflation rates that the central bank desires to implement uniquely. From Lemma 4 the King rule can be written in the form

(32) $\quad i_t = i_t^* + \phi(x_t - x_t^*).$

Clearly, such a rule is of the linear feedback rule form (23).

In the literature, researchers often restrict attention to equilibria in which inflation is bounded. Here we argue that equilibria in which inflation is unbounded cannot be dismissed in this model on logical grounds. Equilibria in which inflation explodes are perfectly reasonable because the explosion in inflation is associated with an explosion in the money supply. To see this association, suppose that policy is described by a Taylor rule of the form (29) with $\bar{\imath} = 0$ and $\phi > 1$ and, for simplicity, suppose that $\eta_t = \nu_t = 0$ for all t. Using (26), we know that $y_t = 0$ for all t, and hence, from (2) the growth of the money supply is given by

(33) $\quad \mu_t = x_t = \phi^t x_0.$

Thus, in these equilibria, inflation explodes because money growth explodes. Each equilibrium is indexed by a different initial value of the endogenous variable x_0. This endogenous variable depends solely on expectations of future policy and is not pinned down by any initial condition or transversality condition.

The idea that the central bank's printing of money at an ever-increasing rate leads to a hyperinflation is at the core of most monetary models. In these equilibria, inflation does not arise from the speculative reasons analyzed by Obstfeld and Rogoff (1983) but from the conventional money printing reasons analyzed by Cagan (1956). In this sense, the theory predicts for perfectly standard and sensible reasons that if the central bank follows a Taylor rule that satisfies the Taylor principle, then the economy can suffer from any one of a continuum of very undesirable paths for inflation.

Now consider an economy with the stochastic shocks. When $\phi \geq 1$, the economy has two kinds of indeterminate equilibrium. In one kind, $c = 0$ and expected inflation grows in a deterministic fashion. In the other kind, $c \neq 0$ and expected inflation grows in a stochastic fashion with mean growth rate ϕ. When $\phi < 1$, the economy has a continuum of bounded equilibria. In one kind, $c = 0$ and expected inflation gradually reverts to 0. In the other kind, $c \neq 0$ and expected inflation fluctuates and its mean value reverts to 0. The intuitive idea behind the multiplicity of stochastic equilibria in Proposition 2 associated with $c \neq 0$ is that interest rates pin down only expected inflation and not the state-by-state realizations indexed by the parameter c.

In Proposition 2, we focused on linear competitive equilibria which can be described as time-invariant linear functions of the shocks. Clearly, there are other competitive equilibria in which the coefficients of the allocation rules depend on time t as well as the history of the shocks. There are also competitive equilibria in which the allocations depend on exogenous sunspots. Our theorems about supporting competitive equilibrium outcomes with sophisticated policy rules apply equally well to all of these equilibria.

Extension to Interest Elastic Money Demand

To keep the exposition simple we have assumed that money demand is interest inelastic. This feature of the model implies that if a money regime is adopted in some period t then the equilibrium outcomes in that period are uniquely determined by the money growth rate in that period. This uniqueness under a money regime is what allows us to use one-period reversion to a money regime to support any desired competitive equilibrium.

Now consider economies with interest elastic money demand. For such economies consider sophisticated policies which specify an interest regime along the equilibrium path and an infinite reversion to a money regime following a deviation. Such policies can uniquely implement any desired outcome if best responses are controllable. A sufficient condition for such controllability is that competitive equilibria are unique with a suitably chosen money regime. As with inelastic money demand the uniqueness under a money regime is what allows us to use reversions to a money regime to support any desired competitive equilibrium.

A sizable literature has analysed the uniqueness of competitive equilibrium under money growth policies. Obsteld and Rogoff (1983) and Woodford (1996) provide sufficient conditions for uniqueness of competitive equilibria in such a circumstance. For example, Obstfeld and Rogoff consider a money-in-the-utility function model with preferences of the form $u(c) + v(m)$ where m is real balances and show that a sufficient condition for uniqueness under a money regime is

$$\lim_{m \to 0} m v'(m) > 0.$$

These authors focused attention on flexible price models but their results can be readily extended to our simple sticky price model. Indeed, their sufficient conditions apply unchanged to a deterministic version of our simple sticky price model. The reason is that our model without shocks is effectively identical to a flexible price model. Hence, under appropriate sufficient conditions our implementation results extend to environments with interest elastic money demand.

2. A Model with Staggered Price-Setting

We turn now to a version of the simple model laid out above with staggered price-setting. The main point of this section is to show that our primary result, namely, that sophisticated policies can implement uniquely any equilibrium allocation, carries through to this widely used setting. To make this point in the simplest fashion, we abstract from aggregate uncertainty. We first show that, along the lines of the argument for our simple sticky price model, we can uniquely implement any desired outcome with an infinite reversion to a money regime following a deviation. We then show that, under sufficient conditions, we can also unique implement any desired outcome with policies that use interest rate regimes both on and off the equilibrium path.

We then turn to the implications of our analysis for the Taylor principle. The common interpretation, stressed by Taylor (1993) and Clarida, Gali, and Gertler (2000) among others is that it refers to the comovements of interest rates and inflation rates along the equilibrium path. Under this interpretation the stochastic processes for interest rates and inflation rates satisfy this principle if, on average, along the equilibrium path a rise in inflation rates is associated with a more than one for one rise in interest rates. We show that the Taylor principle, interpreted in this fashion, is neither necessary nor sufficient for either determinacy or efficiency.

A more subtle interpretation, stressed by King (2001), Svensson and Woodford (2005), and Cochrane (2007), is that the Taylor principle is a prescription for what the central bank will do following a deviation from the equilibrium path. Under this interpretation the Taylor principle describes the commitments of the central bank to its behavior off the equilibrium path. In particular, if inflation rates rise relative to their level on the equilibrium path the central bank commits to raising interest rates more than one for one with the rise in inflation. The Taylor principle, interpreted in this fashion, as a statement about off the equilibrium path behavior has no relation to the efficiency of the equilibrium. We do show, however, under sufficient conditions a policy rule that obeys the Taylor principle off the equilibrium path can ensure determinacy, at least of bounded equilibria.

A. Setup

We begin by setting up the model. We show that sophisticated policies with reversion to money can implement any competitive equilibrium uniquely and then show that sophisticated policies with reversion to interest rates can implement any competitive equilibrium uniquely.

Consider, then, a model with no aggregate uncertainty in which prices are set in a staggered fashion as in Calvo (1983). At the beginning of each period, a fraction $1 - \alpha$ of producers are randomly chosen and allowed to reset their prices. After that, the central bank makes its decisions, and then, finally, consumers make their decisions. This economy has no flexible price firms. The nonlinear economy is described in Appendix A.

The linearized equations for this model are similar to those in the simple model. The Euler equation (1) and the money growth equation (2) are the same except that there are no shocks, η_t, ν_t. The price set by a producer which is permitted to reset its price is given by the analog of (5), which is

$$(34) \quad p_{st}(i) = (1-\alpha\beta)\left[\sum_{r=0}^{\infty}(\alpha\beta)^{r-t}(\gamma y_r + p_r)\right].$$

Here also Taylor's γ, is the elasticity of the equilibrium real wage with respect to output. Letting p_{st} denote the average price set by firms that are permitted to reset their prices in period t, this equation can be rewritten recursively as

$$(35) \quad p_{st}(i) = (1-\alpha\beta)\left[\gamma y_t + p_t\right] + \alpha\beta p_{st+1},$$

together with a type of transversality condition $\lim_{T\to\infty}(\alpha\beta)^T p_{sT}(i) = 0$. The aggregate price equation can be written as

$$(36) \quad p_t = \alpha p_{t-1} + (1-\alpha)p_{st}.$$

To make our analysis parallel to the literature, we express the decisions of the sticky price producers in terms of the inflation rate rather than prices. Letting $x_t(i) = p_{st}(i) - p_{t-1}$, with some manipulation, we can rewrite (35) as

$$(37) \quad x_t(i) = (1-\alpha\beta)\gamma y_t + \pi_t + \alpha\beta x_{t+1}.$$

We can also rewrite (36) as

$$(38) \quad \pi_t = (1-\alpha)x_t,$$

where x_t is the average across i of $x_t(i)$.

The transversality-type condition can be rewritten in terms of inflation rates as

$$(39) \quad \lim_{T\to\infty}(\alpha\beta)^T x_t(i) = 0.$$

In equilibrium, since $x_t(i) = x_t$ and (38) holds, this restriction is equivalent to

$$(40) \quad \lim_{T\to\infty}(\alpha\beta)^T \pi_t = 0.$$

In the following lemma, we show that this economy produces the key features of a New Keynesian Phillips curve along the equilibrium path in which

(41) $x_t(i) = x_t.$

Lemma 5. Any allocations that satisfy (37)–(41) also satisfy the New Keynesian Phillips curve

(42) $\pi_t = \kappa y_t + \beta \pi_{t+1},$

where $\kappa = (1-\alpha)(1-\alpha\beta)\gamma/\alpha$.

Proof. To prove (42), substitute for x_t using (38) and (41) into (37). Collecting terms yields (42). *Q.E.D.*

We then have that a competitive equilibrium must satisfy (1), (2), (40) and (42). In addition to these conditions, we now argue that a competitive equilibrium must satisfy two boundedness conditions. Such conditions are controversial in the literature. Standard analyses of New Keynesian models impose strict boundedness conditions, namely that both output and inflation must be bounded above and below in any reasonable equilibrium. Cochrane (2007) has forcefully criticized this practice, arguing that any boundedness conditions must have a solid economic rationale. Here we provide a rationale for two such conditions. In our view, there are solid arguments for requiring that output y_t is bounded above so that

(43) $y_t \leq \bar{y}$ for some $\bar{y}$

and inflation is bounded below so that

(44) $\pi \geq \underline{\pi}$ for some $\underline{\pi}$.

The rationale for output being bounded above is that in this economy there is a finite amount of labor to produce the output. The rationale for requiring that inflation is bounded below comes from the restriction that nominal interest must be nonnegative. (Note that even though the real value of consumer's holdings of bonds must satisfy a tranversality condition, this condition does not impose any restrictions on the paths of y_t and π_t. The reason is that in our nonlinear model the government has access to lump sum taxes so that government debt can be arbitrarily chosen to satisfy any transversality condition.)

We think of the boundedness conditions (43) and (44) as being minimal. These bounds allow for outcomes in which y_t, the log of output, falls without bound (so that the level of output converges to zero). They also allow for outcomes in which inflation rates rise without bound. For completeness, we provide conditions under which our implementation result holds with stricter and weaker boundedness conditions below.

With these restrictions, a *competitive equilibrium* is a sequence of inflation rates and output which satisfy the deterministic versions of (1), (2), as well as (40), (42), (43) and (44). Clearly, this definition is analogous to that for a deterministic version of the competitive equilibrium in the simple model. The definition of a sophisticated equilibrium is also analogous to that in the simple model. It should be clear that the equivalence of competitive equilibria and sophisticated outcomes, as in Lemma 2, holds here.

We now turn to unique implementation of competitive equilibrium by sophisticated policies.

B. Sophisticated Policies

We now show that any competitive equilibrium can be uniquely implemented with sophisticated policies. The basic idea behind our construction is that the central bank starts by picking any competitive equilibrium allocations and sets its policy on the equilibrium path consistent with this equilibrium. The central bank then constructs its policy off the equilibrium path so that any deviations from these allocations would never be a best response for any individual price setter. In so doing, the constructed sophisticated policies support the chosen allocations as the unique equilibrium allocations.

To prove our implementation result we need to find a policy of the central bank such that if all other producers but one choose a particular deviation, it is optimal for the one producer to choose a price different from the particular deviation. If such policies can be found we say the best responses are *controllable.* As we discuss below a sufficient condition for controllability is that the continuation equilibrium is unique.

With Reversion to a Money Regime

In our construction of sophisticated policies with reversion to a money regime, we find it convenient to consider sophisticated policies with *infinite reversion to money.* Under these policies, along the equilibrium path the central bank chooses the prescribed interest rates i_t^*. If, instead, sticky price producers deviate by setting $\tilde{x}_t \neq x_t^*$, then the central bank switches to a money regime with money growth set so that the profit-maximizing value of $x_t(i)$ is such that $x_t(i) \neq \tilde{x}_t$.

To illustrate the details of our construction of monetary policy following a deviation, we suppose that in the nonlinear economy preferences are given by $U(c, l) = \log c + b(1 - l)$, where c is consumption and l is labor supply, so that in the linearized economy Taylor's γ equals one. We also suppose that after a deviation the central bank reverts to a constant money supply $m = \log M$. With a constant money supply, it is convenient to use the original formulation of the economy in which we use price levels rather than inflation rates. The cash-in-advance constraint implies that $y_r + p_r = m$ for all r so that with $\gamma = 1$, (34) reduces to

$$(45) \quad p_{st}(i) = (1 - \alpha\beta) \left[\sum_{r=0}^{\infty} (\beta\alpha)^{r-t} m \right] = m.$$

That is, if after a deviation the central bank chooses a constant level of the money supply m then sticky price producers optimally choose their prices to be m.

We can use (45) to show how a sophisticated policy with infinite reversion to money deters deviations. To do so, consider a history in which price-setters in period t deviate from p_{st}^* to $\tilde{p}_{st}$. Clearly, (45) implies that for any history, the central bank can effectively control the best response of any price-setter by the appropriate choice of monetary policy. Specifically, the central bank can make it optimal for an individual price-setter to choose $p_{st}(i) \neq \tilde{p}_{st}$.

The following proposition then follows immediately.

Proposition 4. Unique Implementation with Reversion to Money Suppose that $\gamma = 1$. Then any competitive equilibrium, that is any sequence of inflation and output that satisfies the deterministic versions of (1), (2), (40), (42), (43) and (44), can be implemented as a unique equilibrium with sophisticated policies with an infinite reversion to money.

The logic of the proof of the proposition again makes clear that in order for reversions to money to uniquely implement equilibrium outcomes, sophisticated policies must have a controllability property. Equation (45) makes clear that the best response of each individual price-setter is controllable.

With Reversion to an Interest Rate Regime

We turn now to constructing our policies in which the central bank chooses interest rates both on and off the equilibrium path. We prove a lemma that gives conditions under which the continuation equilibrium under these policies is unique. We use this construction and our lemma to prove our main result regarding unique implementation.

Construction of Sophisticated Policies We construct policies to support an arbitrary competitive equilibrium outcome $\{x_t^*, \pi_t^*, y_t^*, i_t^*\}_{t=0}^{\infty}$. In our construction we need to define policies for all histories in which private agents may or may not have deviated. Note that we do not define policies for histories in which the central bank alone has deviated. (Doing so is straightforward but unneccesary because we assume the central bank can commit to its policies.)

Consider first histories along the equilibrium path, that is, histories in which there has never been a deviation. Such histories can be written as $h_{gs} = (h_{s-1}^*, x_s^*, \pi_s^*)$ where $h_{s-1}^* = \{x_t^*, \pi_t^*, y_t^*, i_t^*\}_{t=0}^{s-1}$. For such histories let the central bank chooses the prescribed interest rates (i_t^*).

Consider next histories in which the first deviation $\tilde{x}_s$ occurs in some period s, that is, $h_{gs} = (h_{s-1}^*, x_s, \pi_s)$ but $x_s \neq x_s^*$. We now construct policies that discourage individual price setters from joining in this deviation, that is, we construct policies so the optimal price chosen by an individual price setter $x_s(i)$ differs from $\tilde{x}_s$.

In period s, as in all periods, inflation and the aggregate price setting choice are mechanically linked by (38). This mechanical link means we can equally well think of the deviation in terms of inflation or the price setting choice. It is convenient to express the deviation in terms of inflation. Thus, we let $\tilde{\pi}_s = (1-\alpha)\tilde{x}_s$ denote the inflation associated with the deviation $\tilde{x}_s$. Let the sophisticated policy specify reversion to a modified Taylor rule of the form

$$(46) \quad i_s = \left\{ \begin{array}{c} \phi\tilde{\pi}_s \text{ if } \tilde{\pi}_s \neq 0 \\ \bar{\imath}_s \text{ if } \tilde{\pi}_s = 0 \end{array} \right\}.$$

where the setting of the parameter ϕ is described below and $\bar{\imath}_s$ is some nonzero number.

Consider next histories h_{gs} in which a deviation has occured in some period $t < s$. For such histories let the policy be given by (46).

Note that in our construction the policies chosen in the period with the first deviation coincide with policies in periods subsequent to the deviation except when the first deviation is to $\tilde{\pi}_s = 0$. The reason is that under the ϕ that we choose for (46) the continuation equilibrium will imply that $\pi_s = 0$ so that $x_s(i) = 0$. To discourage a deviation to $\tilde{\pi}_s = 0$ we need to choose a policy that makes $x_s(i) \neq 0$. A policy that sets $\bar{\imath}_s \neq 0$ ensures that $x_s(i) \neq 0$.

Uniqueness of Continuation Equilibrium In order to show the our constructed policies uniquely implement the desired outcome we show that after a deviation in period s, the best responses in period s are controllable. To do so we first show that the continuation equilibrium from $s+1$ onward is unique.

Given our construction in all periods $t \geq s+1$ the central bank is using a Taylor rule of the form

$$(47) \quad i_t = \phi\pi_t.$$

We show that the Taylor rule parameter ϕ can be chosen in such a way that the continuation equilibrium is uniquely given by $y_t = \pi_t = 0$ for all $t \geq s+1$.

To verify uniqueness of the continuation equilibrium for an appropriate choice of ϕ, we begin by solving (1), (42), and (47) without imposing the transversality-like condition (40) (or any boundedness conditions). To do so, we substitute out i_t in (1), using (47), to get

$$(48) \quad y_{t+1} + \psi\pi_{t+1} = y_t + \psi\phi\pi_t,$$

which together with (42) defines a dynamical system. Letting $z_t = (y_t, \pi_t)'$, with some manipulation we can stack these equations to give $z_{t+1} = Az_t$, where

$$A = \begin{bmatrix} a & b \\ \frac{-\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

and where $a = 1 + \kappa\psi/\beta$, $b = \psi(\phi - 1/\beta)$. The solutions to this system are

$$(49) \quad y_t = \lambda_1^{t-s-1}\omega_{1s+1} + \lambda_2^{t-s-1}\omega_{2s+1} \text{ and } \pi_t = \lambda_1^{t-s-1}(\frac{\lambda_1 - a}{b})\omega_{1s+1} + \lambda_2^{t-s-1}(\frac{\lambda_2 - a}{b})\omega_{2s+1}$$

where $\lambda_1 < \lambda_2$, the eigenvalues of A, are given by

$$(50) \quad \lambda_1, \lambda_2 = \frac{1}{2}\left(\frac{1+\kappa\psi}{\beta} + 1\right) \pm \frac{1}{2}\sqrt{\left(\frac{1+\kappa\psi}{\beta} - 1\right)^2 - 4(\phi - 1)\frac{\kappa\psi}{\beta}}$$

and $\omega_{1s+1} = \left[(\frac{\lambda_2 - a}{b})y_{s+1} - \pi_{s+1}\right]/\Delta$, $\omega_{2s+1} = \left[(\frac{a-\lambda_1}{b})y_{s+1} + \pi_{s+1}\right]/\Delta$, where Δ is the determinant of A. Here and throughout we restrict attention to values of $\phi \in [0, \phi_{\max}]$, where $\phi_{\max}$ is the largest value of ϕ that yields real eigenvalues. (That is, at $\phi_{\max}$ the discriminant in (50) is zero.)

For a continuation outcome to be part of an equilibrium outcome, it must satisfy the transversality-like condition (40) and the boundedness conditions (43), (44) as well as (49). The restrictions imposed by the transversality condition (40) on the solutions described in (49) can be derived by substituting for π_t in (40), using (49), to get

$$(51) \quad \lim_{T\to\infty}(\beta\alpha)^T\left[\lambda_1^T(\frac{\lambda_1 - a}{b})\omega_{1s+1} + \lambda_2^T(\frac{\lambda_2 - a}{b})\omega_{2s+1}\right] = 0.$$

The boundedness conditions can be rewritten using (49) as

$$(52) \quad y_s = \lambda_1^{t-s-1}\omega_{1s+1} + \lambda_2^{t-s-1}\omega_{2s+1} \le \bar{y} \text{ and } \pi_s = \lambda_1^{t-s-1}(\frac{\lambda_1 - a}{b})\omega_{1s+1} + \lambda_2^{t-s-1}(\frac{\lambda_2 - a}{b})\omega_{2s+1} \ge \underline{\pi}.$$

The "initial" conditions $\omega_{1s+1}, \omega_{2s+1}$ satisfying (51) and (52) determine the continuation outcomes from (49).

We now develop conditions such that there exists a Taylor rule coefficient ϕ under which the only solution to (49) that satisfies our transverality and boundedness conditions has $\omega_{1s+1} = \omega_{2s+1} = 0$. It is easy to see from (50) that if $\phi < 1$, the smaller eigenvalue λ_1 is less than one while if $\phi > 1$ then λ_1 is greater than one. Clearly, then no $\phi < 1$ will guarantee uniqueness because with $\lambda_1 < 1$, a continuum of values of ω_{1s+1} satisfying (51) and (52) exists. Notice also that not all $\phi > 1$ will yield uniqueness. For some values of $\phi > 1$ there will be a continuum of solutions that have the property the inflation converges to infinity, the level of output is bounded (the log of output converges to negative infinity) and the transversality condition is satisfied. Hence, for such values we will have a continuum of equilibria. These considerations imply that the Taylor coefficient ϕ we seek must be larger than one and have the property that under it all solutions to (49) with either $\omega_{1s+1} \neq 0$ or $\omega_{2s+1} \neq 0$ violate either our the transversality condition or our boundedness condition and hence will not be continuation competitive equilibria.

We now develop a lemma which shows that under the condition

$$(53) \quad \alpha(1 + \kappa\psi) > 1$$

there is some value of $\bar{\phi}$ greater than one such that after any history, if the central bank switches to a Taylor rule with $\phi \in (1, \bar{\phi})$ the resulting continuation is unique. That is, under (53), the initial conditions $\omega_{1s+1}, \omega_{2s+1}$ satisfying (51) and (52) are unique, and equal to 0, for a range of values of the Taylor coefficient ϕ greater than 1.

The idea of the proof is that we eliminate the "large root indeterminacy" associated with the initial condition ω_{2s+1} using the transversality condition. This condition requires that prices not diverge to infinity faster than $(1/\beta\alpha)^t$ in that $(\beta\alpha)^t\pi_t$ must converge to zero. Given the form of inflation in (49), this condition requires that $(\beta\alpha\lambda_2)^t\omega_{2s+1}$ converge to zero. In the appendix we show that under (53), $\beta\alpha\lambda_2 > 1$ for $\phi \in (1, \bar{\phi})$ so that $\omega_{2s+1} = 0$.

We eliminate the "small root indeterminacy" associated with the initial condition ω_{1s+1} using the boundedness condition. To develop this argument suppose that $\omega_{2s+1} = 0$. The form of output and inflation in (52) implies that if $\lambda_1 > 1$ and $(\lambda_1 - a)/b > 0$, then both output and inflation converge to infinity (when $\omega_{1s+1} > 0$) or both converge to minus infinity (when $\omega_{1s+1} < 0$). In the former case output is unbounded above and in the latter case inflation is unbounded below. In the appendix we show that if $\phi \in (1, \bar{\phi})$, $\lambda_1 > 1$ and $(\lambda_1 - a)/b > 0$. Hence $\omega_{1s+1} = 0$. We then conclude that for $\phi \in (1, \bar{\phi})$, $\omega_{1s+1} = \omega_{2s+1} = 0$ so that $y_t = \pi_t = 0$ for $t \geq s + 1$.

Consider then the following lemma which is proved in Appendix B.

Lemma 6. Suppose (53) is satisfied. Then there exists some value of $\bar{\phi} > 1$ such that if the central bank chooses a reversion policy of the Taylor rule form with $\phi \in (1, \bar{\phi})$ then the resulting continuation is unique from $s + 1$ onwards and the associated output and inflation rates are zero in all periods $t \geq s + 1$ where the deviation occurs in period s.

We now use Lemma 6 to show that the policies following a deviation, parameterized by ϕ and $\bar{\imath}_s$, can be chosen so that the best response $x_s(i)$ of an individual price-setter is unique and controllable. Let ϕ be chosen so that $\phi \in (1, \bar{\phi})$ and satisfies $\phi \neq 1/(1-\alpha)\kappa\psi$. From (37) and (38), the best response $x_s(i)$ given the inflation $\tilde{\pi}_s$ induced by the deviation is

$$x_s(i) = (1-\alpha\beta)\gamma y_s + \frac{\tilde{x}_s}{1-\alpha} + \alpha\beta x_{s+1}. \tag{54}$$

Note that $x_{s+1} = 0$ because Lemma 6 implies that for all periods after the one with the deviation, output and inflation are zero; that is, $y_r = \pi_r = x_r = 0$ for all $r \geq s+1$. Next note that substituting $y_{s+1} = \pi_{s+1} = 0$ into the Euler equation (1) gives that $y_s = -\psi i_s$. Using both of these results, we can rewrite (54) as

$$x_s(i) = -(1-\alpha\beta)\gamma\psi i_s + \frac{\tilde{x}_s}{1-\alpha}. \tag{55}$$

Using (38) and the form of the sophisticated policy which implies that $i_s = \phi\tilde{\pi}_s$, we can rewrite (55) as

$$x_s(i) = \frac{1-\alpha(1-\alpha)\kappa\psi\phi}{1-\alpha}\tilde{x}_s.. \tag{56}$$

The condition that $\phi \neq 1/(1-\alpha)\kappa\psi$ implies that $x_s(i) \neq \tilde{x}_s$ unless $\tilde{x}_s = 0$.

If $\tilde{x}_s = 0$, then recall that the policy rule specifies that i_s is some nonzero number, so that, using (55), $x_s(i)$ is not equal to zero. We then have proved the following proposition.

Proposition 5. Unique Implementation with Reversion to Taylor Rules Suppose (53) is satisfied. Then a sophisticated policy indexed by $\bar{\imath}_s$ and ϕ with $\bar{\imath}_s \neq 0$ and $\phi \in (1, \bar{\phi})$ and satisfying $\phi \neq 1/(1-\alpha)\kappa\psi$ uniquely implements any competitive equilibrium outcome.

The basic idea of our construction is that by reverting to a Taylor rule with ϕ in the determinate region, the central bank uniquely pins down the continuation values of output and inflation from $s+1$ on. By varying the policy in period s, the central bank can uniquely control any best response and thereby discourage any deviation. Thus, here, as before, sophisticated policies can be used to control best responses.

In our proof we used one particular set of policies off the equilibrium path to discourage deviations, but many others will also discourage deviations. For example, for histories off the equilibrium path we could instead have used policies of the form

$$i_t = \bar{\imath}_t + \phi \pi_t \tag{57}$$

where $\bar{\imath}_t$ is an exogenous bounded deterministic sequence and $\phi \in (1, \bar{\phi})$. These policies off the equilibrium path along with policies that specify $i_t = i_t^*$ on the equilibrium path with $\pi_t = \pi_t^*$ can uniquely implement outcomes with interest rates i_t^*, inflation π_t^*, and the associated output y_t^*.

If (53) is violated, it can be shown that there is indeterminacy under interest rate rules for all $\phi \in [0, \phi_{\max}]$. For such economies, sophisticated policies with specify reversion to Taylor rules do not uniquely implement outcomes. It may still be possible to uniquely implement outcomes by specifying reversion to money rules.

The King Rule We now show that the King rule can implement bounded equilibria but does not implement all equilibria. To that end consider policies of the King rule form

$$i_t(h_{gt}) = i_t^* + \phi(x_t - x_t^*) \tag{58}$$

where x_t is an element of the history $h_{gt} = (h_{t-1}, x_t, \pi_t)$ where $h_{t-1} = \{x_s, \pi_s, y_s, i_s\}_{s=0}^{t-1}$.

This rule specifies behavior both on and off the equilibrium path. For histories on the equilibrium path $x_t = x_t^*$ so that $i_t(h_{gt}) = i_t^*$. For histories off the equilibrium path this rule specifies $i_t(h_{gt}) = i_t^*$ whenever $x_t = x_t^*$ and (58) for histories with $x_t \neq x_t^*$.

To show that, in our context, the King rule can implement bounded equilibria we replace the levels of variables in our dynamical system with their deviations from the target and mimic our proof above.

To see that the King rule cannot necessarily implement unbounded equilibria consider the following desired outcome

$$y_t^* = 0, \ \pi_t^* = \frac{\pi_{t-1}^*}{\beta}.$$

Under the King rule there are a continuum of equilibria of the form:

$$y_t = y_t^* + \lambda_1^t \omega_{10}$$

$$\pi_t = \pi_t^* + \lambda_1^t (\frac{\lambda_1 - a}{b})\omega_{10}$$

The reason is that $\lambda_1 < 1/\beta$ so that there is a continuum of values of $\omega_{10} \leq 0$ which satisfy tranversality and boundedness. The basic idea is the second term in π_t goes to minus infinity slower than the first term goes to plus infinity. With $\omega_{10} \leq 0$, y_t goes to minus infinity and π_t goes to plus infinity. Such a path violates neither our boundness conditions nor our transversality condition.

An interesting feature of this rule is only current deviations affect the current setting of policy and past deviations have no effect on the current setting of policy. In this sense, this rule forgives past deviations. If there is a deviation in some period s the economy returns to the original equilibrium path in the period after the deviation. Note that our formulation gives outcomes following deviations which are quite different from those in the literature on implementation via nonexistence. In our formulation following a deviation, inflation does not explode but rather returns to the original equilibrium path. Nonetheless, our translation of the King policy achieves implementation by discouraging deviations from the equilibrium path. In contrast, in the literature's formulation following a deviation inflation explodes and no equilibrium exists. Hence, the literature's formulation of the King policy achieves implementation via nonexistence.

Other Views on Bounds So far we have considered one view on bounds. Since the issue of what bounds to impose is controversial, we discuss other views briefly. Adding bounds reduces the region of indeterminacy and expands the region of determinacy. These bounds increase the applicability of these policies, but reduce their need. As the region of determinacy expands, the range of parameter values for which sophisticated policies can be used for unique implementation also expands. As the region of indeterminacy shrinks, however, the range of parameter values for which sophisticated policies are needed shrinks.

Strict Bound View Consider first the standard view in the literature, namely, the *strict bound* view. In this view, only outcomes that are bounded both above and below are considered reasonable. Under this view the range of Taylor rules coefficients which yield uniqueness expands to include all values of $\phi \in (1, \phi_{\max})$. To see the expansion in the range, note from (50) that $\lambda_1 > 1$ when $\phi > 1$. Since $\lambda_2 \geq \lambda_1 > 1$ for $\phi > 1$, (49) and the boundedness conditions imply that $\omega_{1s+1} = \omega_{2s+1} = 0$. Hence, the continuation equilibrium is unique for all $\phi > 1$. Here, we can choose the Taylor rule parameter in a reversion to any value of $\phi > 1$; hence, the analog of Proposition 5 holds even for parameter values that violate (53). Clearly, the strict bound view expands the applicability of sophisticated policies by expanding the range of ϕ such that the equilibrium is determinate relative to our view. It also reduces the range for which these policies are needed.

No Bound View Another view is that neither transversality nor boundness conditions should be imposed. Under this view reversion to interest rate regimes cannot achieve implementation but reversion to a money regime can.

C. Welfare and the Taylor Principle

We have shown that adherence to the Taylor principle is not necessary nor sufficient for the unique implementation of a desired equilibrium outcome. Here we ask whether efficient outcomes satisfy the Taylor principle in the sense that an observer of the efficient outcome path who regressed interest rates on inflation rates would find a regression coefficient greater than one. We show that whether this regression coefficient is larger or smaller than one has little to do with efficiency.

To make this question interesting we need to add stochastic shocks to the model. We follow much of the literature in adding a *cost-push* shock, namely a stochastic shock to the New Keynesian Phillips curve so that it is of the form

$$(59) \quad \pi_t = \beta E_t \pi_{t+1} + \kappa y_t + u_t$$

where u_t is i.i.d., has mean zero, and is the cost-push shock. Following the exposition in Woodford (2003, Ch. 6) the efficient allocations solve

$$(60) \quad \min \frac{1}{2} \sum_{t=0}^{\infty} E_0 \beta^t [\pi_t^2 + \frac{\kappa}{\theta} y_t^2]$$

subject to (59) where we have assumed that the economy has no distortions in the steady state. (The basic idea is the monopoly distortion is offset by a constant subsidy to labor.) Here θ

is a parameter that determines the elasticity of substitution $1/(1-\theta)$ between differentiated goods as well as the steady state markup $1/\theta$.

Following Woodford (2003, p. 489) it is easy to derive the efficient allocations. Given these allocations a regression of interest rates on inflation has a regression coefficient of

$$\frac{cov(i_t, \pi_t)}{var(\pi_t)} = [\theta - 1] \frac{\mu_1(1-\mu_1)}{2} \tag{61}$$

where $0 < \mu_1 < 1$ is the smaller root of the characteristic equation

$$\beta\mu^2 - (1 + \beta + \kappa\theta)\mu + 1 = 0.$$

Clearly, as θ ranges from 1 to infinity the regression coefficient ranges from 0 to infinity. Hence, the magnitude of this coefficient has little to do with efficiency.

3. Trembles and Imperfect Information

Thus far we have shown that any equilibrium outcome can be implemented as a unique equilibrium with sophisticated policies. In our equilibrium, deviations in private actions lead to changes in the regime. This observation leads one to ask how sophisticated policies should be constructed if we allow for trembles in private decisions or if the central bank can monitor private decisions only imperfectly.

A. Trembles

Consider first allowing for trembles in private decisions by supposing that the actual price chosen by a price-setter, $x_t(i)$, differs from the intended price, $\hat{x}_t(i)$, by an additive error $\varepsilon_t(i)$, so that

$$x_t(i) = \hat{x}_t(i) + \varepsilon_t(i).$$

If $\varepsilon_t(i)$ is independently distributed across agents, then it simply washes out in the aggregate and is irrelevant. Even if $\varepsilon_t(i)$ is correlated across agents, say, because it has both an aggregate and an idiosyncratic component, our argument goes through unchanged if the central bank can observe the aggregate component, say, with a random sample of prices.

B. Imperfect Information

More interesting is a situation in which the central bank has imperfect information about prices. We consider two formulations of imperfect information. In the first, labeled *imperfect monitoring*, the central bank observes the aggregate action of price-setters x_t with probability q and observes nothing with probability $1-q$. In the second, labeled *measurement error*, the central bank observes the actions of price setters with symmetric measurement error. Of course, if the central bank could see some other variable perfectly, such as output or interest rates on private debt, then it could infer what the private agents did. In this sense, we think of these setups as ones that gives the central bank minimal amounts of information relative to what actual central banks have.

We will show that with imperfect monitoring we can exactly implement any desired outcome while with measurement error we can implement outcomes which are close to the desired outcomes when the measurement error is small.

Imperfect Monitoring

Consider the imperfect monitoring formulation. We restrict attention to deviations which generate bounded paths for inflation, with the rationale that the central bank can easily figure out if the economy is on an unbounded path.

We prove the following proposition in Appendix B:

PROPOSITION 7. UNIQUE IMPLEMENTATION WITH IMPERFECT MONITORING. If the detection probability q is sufficiently high, so that

$$\frac{1}{1-q} > 1 + \beta q + (1-q)\kappa\psi, \tag{62}$$

then sophisticated policies with infinite reversion to money can uniquely implement any competitive equilibrium outcome. Under condition (53) and (62), sophisticated policies with reversion to interest rates can uniquely implement any equilibrium outcome

The sophisticated policies we use to prove this result are as follows. If the central bank detects a deviation, then it switches to a suitably chosen policy that yields uniqueness. Such a policy could be either a reversion to a money regime or a reversion to an interest rate regime in the determinate region. With such policies in place, it is easy to work out the dynamical system following undetected deviations. If the detection probability satisfies (62), then the dynamical system has a unique solution, so that the best response is controllable.

Notice that for any values of the other parameters, there is always a detection probability strictly less than one that satisfies (62).

Suppose next that the central bank perfectly monitors prices every K periods. An argument similar to that in Proposition 7 can then be used to obtain unique implementation. The essential idea behind both this result and that in Proposition 7 is that indeterminacy arises in the New Keynesian model because the associated dynamical system lacks a terminal condition. Periodic monitoring provides the needed terminal condition and probabilistic monitoring acts a form of discounting that effectively provides a terminal condition.

Measurement Error

Next we turn to the measurement error formulation. In this formulation the central bank observes

$$(63) \quad \tilde{x}_t = x_t + \varepsilon_t$$

where ε_t is i.i.d. over time and has mean zero and variance σ_ε^2. Consider supporting some desired bounded outcome path. We consider monetary policies of the King rule form (58). We have already shown that the King Rule uniquely implements bounded equilibria when the economy has no measurement error. Here the King rule can be written as

$$(64) \quad i_t(h_{gt}) = i_t^* + \phi(\tilde{x}_t - x_t^*) = i_t^* + \phi(x_t - x_t^*) + \phi\varepsilon_t$$

It is easy to show that in this economy with measurement error the best response of any indidvual price setter is identical to that in the economy without measurement error. The reason is that the best response depends only the expected values of future variables. Given that the measurement error ε_t has mean zero these expected values are unchanged. It follows that the unique equilibrium in this economy with measurement error has $x_t = x_t^*$. The realized value of output y_t, however, fluctuates around the target value y_t^*. From the Euler equation the realized value of output is given by $y_t = y_t^* - \psi\phi\varepsilon_t$. Clearly, as the size of the measurement error ε_t goes to zero, the outcomes converge to the desired outcomes. We have established the following proposition.

Proposition 8. Approximate Implementation with Measurement Error. As the variance of the measurement error approaches zero, sophisticated policies of the King rule form yield outcomes with converge to the desired outcomes.

4. Concluding Remarks

We have defined and illustrated what we have called *sophisticated policies* for monetary economies and have shown how they can uniquely implement any competitive outcome. The logic of our arguments should extend to other applications, including analyses of financial crises, fiscal policy and so on.

The main message of this paper is that in designing policy we should follow the Ramsey approach to determine the best competitive equilibrium, and then check whether best responses are controllable. If they are, then sophisticated policies of the kind we have constructed can uniquely implement the Ramsey outcome. If they are not, then policymakers have no choice but to accept indeterminacy. We have shown that this way of thinking about implementation makes the Taylor principle irrelevant.

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Notes

[1]An extensive literature has used the Ramsey approach to discuss optimal monetary policy. See, among others, Chari, Christiano, and Kehoe (1996), Schmitt-Grohe and Uribe (2004),Siu (2004), and Corrreia, Nicolini, and Teles (2008).

[2]We choose this part of the policy in order to make it abundantly clear that after a deviation the central bank is not doing anything exotic, such as producing a hyperinflation. Rather, in an intuitive sense, the central bank is simply getting the economy back on the track it had been on before the deviation threatenned to shift it in another direction.

5. Appendix A: The Nonlinear Economies

Here we describe the nonlinear economies that when linearized give the equilibrium conditions described in the body of this work.

A. *The Simple Sticky Price Model*

This model is a monetary economy populated by a large number of identical, infinitely lived consumers, flexible price and sticky price intermediate good producers, final good producers, and a government. In each period t, the economy experiences one of finitely many events $s_{t.}$ We denote by $s^t = (s_0, \ldots, s_t)$ the history of events up through and including period t. The probability, as of period zero, of any particular history s^t is $g(s^t)$. The initial realization s_0 is given.

The timing within a period is as follows. At the beginning of the period, sticky price producers set their prices and the govenment chooses its monetary policy, either by setting interest rates or by choosing the quantity of money. The event s_t is then realized. At the end of the period, flexible price producers set their prices, and consumers and final good producers make their decisions. The event s_t is associated with a *flight to quality* shock $(1 - \tau(s^t))$ that affects the attractiveness of government debt relative to private claims.

In each period t, the commodities in this economy are labor, a consumption good, money, and a continuum of intermediate goods indexed by $i \in [0, 1]$. The technology for producing final goods from intermediate goods at history s^t is

$$(65) \quad y(s^t) = \left[\int y(i, s^t)^\theta \, di\right]^{\frac{1}{\theta}},$$

where $y(s^t)$ is the final good and $y(i, s^t)$ is an intermediate good of type i. The technology for producing each intermediate good i is simply

$$(66) \quad y(i, s^t) = l(i, s^t)$$

where $l(i, s^t)$ is the input of labor.

Intermediate good producers behave as imperfect competitors. Fraction α of intermediate good producers have flexible prices in that they set their prices in period t after the realization of the shock s_t. Fraction $1 - \alpha$ have sticky prices, in that they set their prices in period t before the realization of the shock s_t. Let $P_f(i, s^t)$ denote the price set by a flexible price producer $i \in [0, \alpha]$, and let $P_s(i, s^t)$ denote the price set by a sticky price producer $i \in [\alpha, 1]$.

Final good producers behave competitively. In each period t, they choose inputs $y(i, s^t)$, for all $i \in [0, 1]$, and output $y(s^t)$ in order to maximize profits given by

$$\text{(67)} \quad \max \; P(s^t)y(s^t) - \int_0^\alpha P_f(i, s^t)y(i, s^t)\, di - \int_\alpha^1 P_s(i, s^{t-1})y(i, s^t)\, di$$

subject to (65), where $P(s^t)$ is the price of the final good in period t. Solving the problem in (67) gives the input demand functions

$$\text{(68)} \quad y^d(i, s^t) = \left[\frac{P(s^t)}{P(i)}\right]^{\frac{1}{1-\theta}} y(s^t),$$

where $P(i)$ is the price charged by the intermediate good producer i. The zero profit condition implies that

$$\text{(69)} \quad P(s^t) = \left[\int_0^\alpha P_f(i, s^t)^{\frac{\theta}{\theta-1}}\, di + \int_\alpha^1 P_s(i, s^{t-1})^{\frac{\theta}{\theta-1}}\, di\right]^{\frac{\theta-1}{\theta}}.$$

Using (66), we can see that the problem faced by the flexible price producers is to choose $P_1(i, s^t)$ in order to maximize

$$\text{(70)} \quad \left[P_f(i, s^{t-1}) - W(s^t)\right] y^d(i, s^t)$$

subject to (68), where $W(s^t)$ is the nominal wage rate. The resulting optimal price is given as a markup over the nominal wage rate

$$\text{(71)} \quad P_s(i, s^t) = \frac{1}{\theta} W(s^t).$$

The problem faced by the sticky price producers is to choose $P_s(i, s^{t-1})$ in order to maximize

$$\text{(72)} \quad \sum_{s^t} Q(s^t|s^{t-1}) \left[P_2(i, s^{t-1}) - W(s^t)\right] y^d(i, s^t)$$

subject to (68), where $Q(s^t|s^{t-1})$ is the price of a dollar at s^t in units of a dollar at s^{t-1}. The resulting optimal price is given as a markup over weighted expected marginal costs

$$\text{(73)} \quad P_s(i, s^{t-1}) = \frac{1}{\theta} \frac{\sum_{s^t} Q(s^t|s^{t-1})P(s^t)^{\frac{1}{1-\theta}} W(s^t)y(s^t)}{\sum_{s^t} Q(s^t|s^{t-1})P(s^t)^{\frac{1}{1-\theta}} y(s^t)}.$$

We turn now to the consumers. The consumer side of the economy is a variant of the standard cash-in-advance formulation, as in Lucas (1992), with two modifications. First, we assume that the government pays interest on wages at the private market interest rate. This modification ensures that the consumer's first-order condition for labor supply is undistorted as in the cashless economies of Woodford (2003). Second, we allow for flight to quality shocks that affect government debt relative to private debt.

Consumer preferences are given by

$$(74) \quad \sum_{t=0}^{\infty} \sum_{s^t} \beta^t g(s^t)\, U(c(s^t), l(s^t)),$$

where $c(s^t)$ and $l(s^t)$ are consumption and labor. In each period $t = 0, 1, \ldots$, consumers face a cash-in-advance constraint in which purchases of consumption goods are constrained by their holdings of nominal money balances $M(s^t)$ according to

$$(75) \quad P(s^t)c(s^t) = M(s^t)$$

as well as a sequence of budget constraints

$$(76) \quad M(s^t)+\frac{B(s^t)}{R(s^t)} = R_p(s^{t-1})(1+\tau_l)W(s^{t-1})\ell(s^{t-1})+\left[1-\tau(s^{t-1})\right] B(s^{t-1})+T(s^t)+\Pi(s^t),$$

where $B(s^t)$ is government debt with price $1/R(s^t), R_p(s^t)$ is the rate of return on private debt, $\Pi(s^t)$ is the nominal profits of the intermediate good producers, τ_l is a subsidy to labor income and $T(s^t)$ is nominal transfers and where the right side of (76) is given in period 0. The subsidy τ_l is set, as is standard in the literature, to undo the inefficiency in a steady state due to monopoly power Specifically, $(1 + \tau_l) = 1/\theta$. Note that we have imposed that the cash-in-advance constraint holds with equality.

The consumer's problem is to maximize utility, subject to the cash-in-advance constraint, the budget constraint, and borrowing constraints, $B(s^{t+1}) \geq \bar{B}$, for some large negative number $\bar{B}$. For notational simplicity, we have suppressed decisions on holdings of private state-contingent debt with price $Q(s^t|s^{t-1})$ and private state-uncontingent debt with the private market interest rate $R_p(s^t)$. Clearly, $\frac{1}{R_p(s^t)} = \sum_{s_{t+1}} Q\left(s^{t+1}|s^t\right)$ and

$$Q\left(s^{t+1}|s^t\right) = \beta g(s^{t+1}|s^t)\frac{U_c(s^{t+1})P(s^t)}{U_c(s^t)P(s^{t+1})}.$$

The first-order conditions for the consumer's problem imply that

$$-\frac{U_l(s^t)}{U_c(s^t)} = \frac{(1+\tau_l)W(s^t)}{P(s^t)}$$

$$(77) \quad \frac{1}{R(s^t)} = \left[1-\tau(s^t)\right] \sum_{s^{t+1}} \beta g(s^{t+1}|s^t) \frac{U_c(s^{t+1})}{U_c(s^t)} \frac{P(s^t)}{P(s^{t+1})}.$$

If we log-linearize this economy, then we can obtain the equations in the body for the simple model. Setting $(1+\tau_l) = 1/\theta$,we obtain the quadratic approximation to welfare used in the text.

B. *The Model with Staggered Price-Setting*

This model is nearly identical to the simple model above. The main differences are that in this new model there are no flexible price producers and each producer can reset prices in each period with probability $1-\alpha$.

The problem of a producer who is allowed to reset is to

$$\max_{P_s(s^t)} \sum_{r=t}^{\infty} \sum_{s^r} \alpha^{r-t} Q(s^r|s^t) \left[P_s(s^t) C_s(s^r) - W(s^r) C_s(s^r)\right]$$

subject to

$$C_s(s^t) = \left(\frac{P_s(s^t)}{P(s^t)}\right)^{-\theta} C(s^t).$$

The first-order conditions imply that

$$P_s(s^t) = \frac{\theta}{\theta-1} \frac{\sum_{r=t}^{\infty} \sum_{s^r} \alpha^{r-t} Q(s^r|s^t) \frac{W(s^r)}{P(s^r)} \left(\frac{1}{P(s^r)}\right)^{-\theta-1} C(s^r)}{\sum_{r=t}^{\infty} \sum_{s^r} \alpha^{r-t} Q(s^r|s^t) \left(\frac{1}{P(s^r)}\right)^{-\theta} C(s^r)}.$$

The consumer side of the model is identical to that in the simple model. This staggered price-setting model when linearized gives the equilibrium conditions described in the body.

6. Appendix B: Proofs of Lemma 6 and Propositions 6 and 7

Here we prove Lemma 6 and Propositions 6 and 7. We will use the following to help prove Lemma 6 and the propositions. Let $\lambda_1(\phi)$ and $\lambda_2(\phi)$ be defined from (50).

Lemma A. The smaller eigenvalue $\lambda_1(\phi)$ is increasing in ϕ and the larger eigenvalue $\lambda_2(\phi)$ is decreasing in ϕ. Furthermore, for all $\phi \in [1, 1/\beta)$ the smaller eigenvalue satisfies $\lambda_1(\phi) > 1$ and $\lambda_1(\phi) - a)/b > 0$..

Proof. From (50), we have that the smaller eigenvalue is

$$(78) \quad \lambda_1(\phi) = \frac{1}{2}\left(1+\frac{1+\kappa\psi}{\beta}\right) - \frac{1}{2}\sqrt{\left(1+\frac{1+\kappa\psi}{\beta}\right)^2 - 4\frac{(1+\phi\psi\kappa)}{\beta}}.$$

and the larger eigenvalue is

$$(79)\quad \lambda_2(\phi) = \frac{1}{2}\left(1 + \frac{1+\kappa\psi}{\beta}\right) + \frac{1}{2}\sqrt{\left(1 + \frac{1+\kappa\psi}{\beta}\right)^2 - 4\frac{(1+\phi\psi\kappa)}{\beta}}.$$

Clearly, λ_1 is an increasing function of ϕ and λ_2 is a decreasing function of ϕ.

To prove that $\lambda_1(\phi) > 1$ for $\phi \in [1, 1/\beta)$ note that

$$\lambda_1(1) = \frac{1}{2}\left(\frac{1+\kappa\psi}{\beta} + 1\right) - \frac{1}{2}\sqrt{\left(\frac{1+\kappa\psi}{\beta} - 1\right)^2} = 1$$

while

$$\lambda_1\left(\frac{1}{\beta}\right) = \frac{1}{2}\left(1 + \frac{\kappa\psi}{\beta} + \frac{1}{\beta}\right) - \frac{1}{2}\sqrt{\left(1 + \frac{\kappa\psi}{\beta} + \frac{1}{\beta}\right)^2 - \frac{4}{\beta}(1 + \frac{\kappa\psi}{\beta})}$$

$$= \frac{1}{2}\left(1 + \frac{\kappa\psi}{\beta} + \frac{1}{\beta}\right) - \frac{1}{2}\sqrt{\left(1 + \frac{\kappa\psi}{\beta} - \frac{1}{\beta}\right)^2} = \frac{1}{\beta}$$

so that $\lambda_1(\phi) > 1$ for all $\phi \in (1, 1/\beta)$.

Next, to prove that $\lambda_1(\phi) - a)/b > 0$ for $\phi \in [1, 1/\beta)$ note that straightforward algebra gives

$$\lambda_1(\phi) - \left(1 + \frac{\kappa\psi}{\beta}\right) = \left\{ \begin{array}{c} -\frac{\kappa\psi}{\beta} < 0 \text{ for } \phi = 1 \\ \frac{1}{\beta} - (1 + \frac{\kappa\psi}{\beta}) < 0 \text{ for } \phi = \frac{1}{\beta} \end{array} \right\}.$$

Since $a = 1 + \kappa\psi/\beta$ and $b = \phi - 1/\beta$, we have shown that $(\lambda_1(\phi) - a)/b > 0$ for all $\phi \in [1, 1/\beta)$. *Q.E.D.*

A. Lemma 6

Recall that Lemma 6 is the following.

Lemma 6. Suppose (53) is satisfied. Then there exists some value of $\bar{\phi} > 1$ such that if the central bank chooses a reversion policy of the Taylor rule form with $\phi \in (1, \bar{\phi})$ then the resulting continuation is unique and the associated output and inflation rates are zero in all periods $t \geq s+1$ where the deviation occurs in period s.

Proof of Lemma 6. We now develop sufficient conditions under which the initial conditions $\omega_{1s+1}, \omega_{2s+1}$ satisfying (51) and (52) are unique, and equal to 0, for a range of values of the Taylor coefficient ϕ greater than 1.

We eliminate the large root indeterminacy by finding values for the Taylor rule parameter ϕ so that the transversality condition rules out paths for inflation that explode at rate λ_2 so that equilibria must have $\omega_{2s+1} = 0$. To see how we find such values, let ϕ^* be defined by

(80) $\quad \beta\alpha\lambda_2(\phi^*) = 1$

if $\beta\alpha\lambda_2(\phi_{\max}) \leq 1$, and by $\phi_{\max}$ if there is no value of $\phi \in [0, \phi_{\max}]$ for which $\beta\alpha\lambda_2(\phi) = 1$. We now show that under (53), $\phi^* > 1$. To see this note from (79) that $\lambda_2(1) = (1 + \kappa\psi)/\beta$ so that $\beta\alpha\lambda_2(1) = \alpha(1 + \kappa\psi)$ which by (53) is greater than one. Since $\lambda_2(\phi)$ is decreasing it follows that if $\beta\alpha\lambda_2(\phi^*) = 1$ is satisfied for some point ϕ^* in $[1, \phi_{\max}]$ then $\phi^* > 1$. If no such point exists then $\phi^* = \phi_{\max}$ which is also greater than 1. Either way $\phi^* > 1$. Hence, $\beta\alpha\lambda_2(\phi) > 1$ for all $\phi \in [0, \phi^*)$ and the transversality condition, written as (51), is satisfied only if $\omega_{2s+1} = 0$ for all $\phi \in [0, \phi^*)$.

We eliminate small root indeterminacy by finding values for the Taylor rule parameter such that the smaller root $\lambda_1(\phi)$ is larger than one and the coefficient on the initial condition on the small root ω_{1s+1}, namely $(\lambda_1(\phi) - a)/b > 0$. For such values of ϕ the bound on output in (52) requires that $\omega_{1s+1} \leq 0$ and the bound on inflation in (52) requires that $\omega_{1s+1} \geq 0$ so that $\omega_{1s+1} = 0$. From Lemma A we have that the required interval is $[1, 1/\beta)$ because for all $\phi \in [1, 1/\beta)$ we have that $\lambda_1(\phi) > 1$ and $(\lambda_1(\phi) - a)/b > 0$.

Combining the two parts of the argument we have that if ϕ satisfies both $\phi \in [0, \phi^*)$ and $\phi \in [1, 1/\beta)$ then both large root indeterminacy and small root indeterminacy are eliminated. The intersection of these intervals is contained in $(1, \bar{\phi})$ where

$$\bar{\phi} = \min[\phi^*, \frac{1}{\beta}].$$

Since $\phi^* > 1$ and $1/\beta > 1$, clearly $\bar{\phi} > 1$. In sum, we have shown that that there exists a $\bar{\phi} > 1$ such that for $\phi \in (1, \bar{\phi})$ the initial conditions for the dynamical system $\omega_{2s+1} = \omega_{1s+1} = 0$ that starts after the deviation. Hence from (49) we have that $y_t = \pi_t = 0$ for all $t \geq s + 1$. *Q.E.D.*

B. Proposition 6

Recall that Proposition 6 is the following:

Proposition 6. Rules Satisfying the Taylor Principle are Inefficient The outcomes under a Taylor rule of the form (29) with $\phi > 1$ are dominated by outcomes of an equilibrium with $\phi = 0$.

We begin by working out the stochastic processes for y_t and π_t implied by the dynamical system. For notational simplicity we write η_{2t} as simply η_t. We begin with the dynamical system that arises with $\phi = 0$. We can write this system as

$$y_t = E_t y_{t+1} + \psi \pi_{t+1} + \eta_t \tag{81}$$

$$\pi_{t+1} = \beta E_t \pi_{t+2} + \kappa E_t y_{t+1}. \tag{82}$$

We solve this system using the method exposited by Lubik and Schorfeide (2003). In solving this system, it is convenient to let $u_t = \pi_{t+1}$ and to let the forecast errors be defined by $\varepsilon_{yt} \equiv y_t - E_{t-1} y_t$ and $\varepsilon_{ut} \equiv u_t - E_{t-1} u_t$. After some manipulation, we can write (81) and (82) as

$$E_t z_{t+1} = \Gamma E_{t-1} z_t + \Psi \eta_t + \Pi \varepsilon_t, \tag{83}$$

where $z_t = [E_t y_{t+1}, E_t u_{t+1}]'$, $\varepsilon_t = [\varepsilon_{yt}, \varepsilon_{ut}]'$and

$$\Gamma = \begin{bmatrix} 1 & -\psi \\ -\frac{\kappa}{\beta} & \frac{\kappa\psi+1}{\beta} \end{bmatrix}, \quad \Psi = \begin{bmatrix} -1 \\ \frac{\kappa}{\beta} \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & -\psi \\ -\frac{\kappa}{\beta} & \frac{\kappa\psi+1}{\beta} \end{bmatrix}. \tag{84}$$

Let $J \wedge J^{-1} = \Gamma$ be the Jordan decomposition of Γ. Letting $w_t = J^{-1} E_t z_{t+1}$, we can write this system as

$$w_t = \wedge w_{t-1} + J^{-1} \Psi \eta_t + J^{-1} \Pi \varepsilon_t,$$

with eigenvalues, $\lambda_1 \leq \lambda_2$,

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(1 + \frac{1+\kappa\psi}{\beta}\right) \pm \frac{1}{2}\sqrt{\left(1 - \frac{1+\kappa\psi}{\beta}\right)^2 + 4\frac{\kappa\psi}{\beta}},$$

and eigenvectors

$$J = \begin{bmatrix} 1 & 1 \\ (1-\lambda_1)/\psi & (1-\lambda_2)/\psi \end{bmatrix}.$$

It is immediate that since $\kappa\psi > 0$, $0 \leq \lambda_1 < 1 < \lambda_2$, so that there exists a continuum of solutions. More precisely, since the number of explosive roots, here 1, is less than the number of expectation errors, here 2, the system has one degree of indeterminacy.

The best outcome clearly has bounded output and inflation, so that we need to choose both the initial condition on ω_{20} and the shocks so as to never put weight on the explosive root. These restrictions can be summarized by a condition on the deterministic component of the system

$$(85) \quad [J^{-1}E_t z_{t+1}]_{2\cdot} = 0$$

and a condition on the stochastic component

$$(86) \quad [J^{-1}\Psi]_{2\cdot}\eta_t + [J^{-1}\Pi]_{2\cdot}\ \varepsilon_t = 0.$$

where $[A]_{2\cdot}$denotes the second row of matrix A. With some algebra, we can write these conditions as

$$(87) \quad \frac{\lambda_1 - 1}{\psi} E_{t-1}y_t + E_{t-1}u_t = 0$$

and

$$(88) \quad \left(\frac{1-\lambda_1}{\psi} + \frac{\kappa}{\beta}\right)(\eta_t - \varepsilon_{yt}) + \left(1 - \lambda_1 + \frac{1}{\beta}(\kappa\psi + 1)\right)\varepsilon_{ut} = 0.$$

For later use, let D be defined from (88) so that

$$(89) \quad \varepsilon_{ut} = D(\nu_t - \varepsilon_{yt}).$$

Since we have chosen w_{2t} to be identically zero, we can write the solution to the system as

$$(90) \quad w_{1t} = \lambda_1 w_{1t-1} + \left[J^{-1}\Psi\right]_1 \eta_t + \left[J^{-1}\Pi\varepsilon_t\right]_1.$$

Recall that

$$w_{1t} = \left[J^{-1}E_t z_{t+1}\right]_1 = \frac{1}{\Delta}\left(\frac{1-\lambda_2}{\psi}E_t y_{t+1} - E_t u_{t+1}\right).$$

Using (87) in (90), we have, after some manipulation, that

$$
\begin{aligned}
E_t u_{t+1} &= \lambda_1 E_{t-1} u_t + \left(\frac{\lambda_2 - 1}{\lambda_1 - 1} - 1\right)^{-1} \left(\frac{\lambda_2 - 1}{\psi} - \frac{\kappa}{\beta}\right) \eta_t \\
&\quad + \left(\frac{\lambda_2 - 1}{\lambda_1 - 1} - 1\right)^{-1} \left(\frac{1 - \lambda_2}{\psi} + \frac{\kappa}{\beta}\right) \varepsilon_{yt} + \left(\frac{\lambda_2 - 1}{\lambda_1 - 1} - 1\right)^{-1} \left(\lambda_2 - 1 - \frac{\kappa\psi + 1}{\beta}\right) \varepsilon_{ut}
\end{aligned}
$$

and

$$
\begin{aligned}
E_t y_{t+1} &= \lambda_1 E_{t-1} y_t \\
&\quad + \frac{\psi}{1 - \lambda_1} \left(\frac{\lambda_1 - 1}{\lambda_2 - \lambda_1}\right) \left[\left(\frac{\lambda_2 - 1}{\psi} - \frac{\kappa}{\beta}\right)(\eta_t - \varepsilon_{yt}) + \left(\lambda_2 - 1 - \frac{\kappa\psi + 1}{\beta}\right) \varepsilon_{ut}\right]
\end{aligned}
$$

Using (89), we can write this latter equation as

(91) $\quad E_t y_{t+1} = \lambda_1 E_{t-1} y_t + F(\eta_t - \varepsilon_{yt}),$

where

$$
F = \frac{b}{\lambda_2 - \lambda_1} \left[\left(-\frac{\lambda_2 - a}{b} - \frac{\kappa}{\beta}\right) + \left(\lambda_2 - a - \frac{1}{\beta}(\kappa\psi + 1)\right) D\right]
$$

and ε_{yt} is a free random variable which captures the stochastic indeterminacy of the system. The solution for y_{t+1} is, then,

$$
y_{t+1} = E_t y_{t+1} + \varepsilon_{yt+1},
$$

where $E_t y_{t+1}$ is given by (91). Using (87), we have that

(92) $\quad u_{t+1} = \dfrac{1 - \lambda_1}{\psi} E_t y_{t+1} + D(\eta_{t+1} - \varepsilon_{yt+1}).$

Consider now the proof of Proposition 6.

Proof. We compute welfare using

(93) $\quad E_0 \sum \beta^t \left(\gamma y_t^2 + \pi_t^2\right)$

and the system described above. To do so, we assume that y_0 is drawn from the invariant distribution, so that from (91) we have that

(94) $\quad \text{var} E_t y_{t+1} = \left(\dfrac{F^2}{1 - \lambda^2}\right) \text{var}(\varepsilon_t - \eta_{yt}).$

>From the definition of the forecast error η_{yt+1}, we have that

$$(95) \quad \mathrm{var}(y_{t+1}) = \mathrm{var}E_t y_{t+1} + \mathrm{var}(\varepsilon_{yt+1})$$

while from (92) we have that

$$(96) \quad \mathrm{var}(u_{t+1}) = \left(\frac{1-\lambda_1}{\psi}\right)^2 \mathrm{var}E_t y_{t+1} + D^2 \mathrm{var}(\eta_{t+1} - \varepsilon_{yt+1}).$$

Using (94)–(96), we have that (93) is proportional to

$$(97) \quad \mathrm{var}(\eta_{t+1} - \varepsilon_{yt+1}) \left(\frac{F^2}{1-\lambda^2} \left[\gamma + \left(\frac{1-\lambda_1}{\psi}\right)^2 \right] + D^2 \right) + \gamma \mathrm{var}(\varepsilon_{yt+1})$$

Choose $\varepsilon_{yt} = A\eta_t$. Then (97) is proportional to

$$(1-A)^2 \left(\frac{F^2}{1-\lambda^2} \left[\gamma + \left(\frac{1-\lambda_1}{\psi}\right)^2 \right] + D^2 \right) + A^2.$$

The $\phi > 1$ solution corresponds to $A = 1$. Since $\left(\frac{F^2}{1-\lambda^2} \left[1 + \left(\frac{1-\lambda_1}{\psi}\right)^2 \right] + D^2 \right) \neq \infty$, it is clear that $A = 1$ is not optimal. *Q.E.D.*

C. *Proposition 7*

Recall that Proposition 7 is the following:

PROPOSITION 7. Under (53), if the detection probability q is sufficiently high, so that

$$(98) \quad \frac{1}{1-q} > 1 + \beta q + (1-q)\kappa\psi,$$

then sophisticated policies can uniquely implement any competitive equilibrium.

Proof. Consider the sophisticated policies of the form used in the proof of Proposition 5 except that the reversion phase is triggered only if a deviation is detected. By construction there is a unique equilibrium continuation following a detection.

We now show that when there is no detection there is also a unique equilibrium. Consider then the dynamical system when a deviation occurs but is not detected. For notational simplicity, imagine that the deviation occurs in period 0. In period t, the deviation at zero is detected with probability q. Let y_t^m and π_{t+1}^m denote output and inflation under the reversion policy when the period 0 deviation is first detected in period t. Choose the reversion polcy so that $y_t^m = \pi_{t+1}^m = 0$ for all $t \geq 1$. The resulting system is, then,

$$(99) \quad y_t = (1-q)\left(y_{t+1} + \psi \pi_{t+1}\right)$$

$$(100) \quad \pi_t = (1-q)\left(\beta \pi_{t+1} + \kappa y_t\right).$$

A sequence of output and inflation is part of a continuation equilibrium if and only if it satisfies (99), (100), (40), (43) and (44). Letting $z_t = (y_t, \pi_t)'$, with some manipulation we can stack these equations to give $z_{t+1} = A' z_t$, where

$$A' = \begin{bmatrix} a' & b' \\ \frac{-\kappa}{\beta} & \frac{1}{\beta(1-q)} \end{bmatrix}$$

and where $a' = 1/(1-q) + \kappa\psi/\beta$, $b' = -\psi/(\beta(1-q))$. The solutions to this system are

$$(101) \quad y_t = \lambda_1^t \omega_1' + \lambda_2^t \omega_2' \quad \text{and} \quad \pi_t = \lambda_1^t \left(\frac{\lambda_1 - a'}{b'}\right)\omega_1' + \lambda_2^t \left(\frac{\lambda_2 - a'}{b'}\right)\omega_2'$$

for $t \geq 1$.

The eigenvalues of the transition matrix A' are

$$\lambda_1, \lambda_2 = \frac{1}{2(1-q)} + \frac{1}{2\beta}\left(\kappa\psi + \frac{1}{1-q}\right) \pm \left[\left(\frac{1}{2(1-q)} + \frac{\kappa\psi}{2\beta}\right)^2 + \frac{1}{4(1-q)^2\beta^2} + \frac{\kappa\psi}{(1-q)\beta^2}\right]^{1/2}$$

With some algebra, it can be shown that condition (98), implies that $\lambda_2 \geq \lambda_1 > 1$ so that the paths of output and inflation given by (101) do not have bounded indeterminacy. Thus, the only possible solutions are $y_t = \pi_t = 0$ or paths in which output or inflation are unbounded. We rule out large root indeterminacy by using the transversality condition. In particular, straightforward algebra shows that if (53) is satisfied, $\beta\alpha\lambda_2 > 1$ so that the transversality condition implies that the system does not have large root indeterminacy, that is that $\omega_2' = 0$. To rule out small root indeterminacy we show that the only unbounded sequences satisfying (99) and (100) have either output going to plus infinity or inflation to minus infinity, so that they violate the boundedness conditions. In particular, with some algebra we can show that $(\lambda_1 - a')/b' > 0$ so that from (101), it follows that, if $\omega_1' > 0$, y_t converges to plus infinity and if $\omega_1' < 0$, π_t converges to minus infinity. Thus, $\omega_1' = \omega_2' = 0$, so that (101) implies that

$y_t = \pi_t = 0$ for all $t \geq 1$. An argument identical to that in the proof of Proposition 5 shows that reversion policies can be designed in period 0 to make best responses controllable, so that the sophisticated policies uniquely implement any competitive equilibrium. *Q.E.D.*