Assortative Learning*

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Abstract

Because of sorting, more skilled workers are more productive in higher type firms. They also learn at different rates about their productivity and therefore have different wage paths across firms. We show that under supermodularity there is always Positive Assortative Matching: differential learning is always dominated by the productivity. Surprisingly, this holds even if learning is faster in the low type firm. The key assumption driving this result is Bayesian updating and that this is a pure learning model. The model provides realistic predictions about wage variance, turnover and the wage distribution. We also derive a new equilibrium condition in this class of continuous time models in addition to the common smooth-pasting and value-matching conditions. This *no-deviation condition* captures sequential rationality and results in a restriction on the second derivative of the value function.

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1 Introduction

High ability workers sort into more productive jobs. Due to complementarities in production, their higher marginal product allows them to command higher wages. The Beckerian model of assortative matching is very well suited to explain those patterns of sorting. Unfortunately, it is mute on the issue of turnover of workers between different jobs. Instead, the Jovanovic (1979) learning model has long been the canonical framework for analyzing turnover in the labor market[1] over the life cycle. Workers and firms learn about match-specific human capital and will tend to stay in a match if learning reveals the match is good. Experimentation tends to occur early on which leads to decreasing turnover over the life cycle. Because in Jovanovic (1979) learning is about the match and not about the worker, there is neither worker heterogeneity nor sorting. In this paper, we offer a unified approach of learning and sorting. At a theoretical level, we establish a solution method for a market equilibrium in a continuous time economy with multiple learning opportunities (multi-armed bandit). We can solve the model to make realistic predictions about wages, sorting and turnover that can be reconciled with the stylized facts.

In the labor market, the learning experiences of workers are most likely to differ across different firms. Starting in a top law firm or a multinational will induce different paths of information revelation than working in a local family business. The worker now faces a trade-off between different experimentation experiences: take a lower wage at a high productivity firm where information may be revealed at a different rate or accept higher wage and learn more slowly. It is intuitive that sorting and learning are intimately connected.

Modeling the labor market as a multi-armed bandit problem and solving it is challenging. Most existing learning models and continuous time games are tractable because they are essentially one-armed bandit problems with a fixed outside option that acts as an absorbing state. One-armed bandit problems typically have attractive properties, including reservation strategies. Instead, multi-armed bandits in general do not have reservation strategies when arms are correlated, even if the learning rate is the same across firms.[2] But our labor market is not exactly identical to the canonical bandit problem. First, there are a continuum of experimenters, and as a result of two-sided heterogeneity, deviations and off-equilibrium path beliefs non-trivially affect equilibrium. Second, because of competitive wage determination à la Jovanovic (1979), the payoffs are endogenous. Finally, because workers learn about general human capital instead of match-specific human capital, the arms are positively correlated.

We find that it is the combination of competitive wage determination (endogenous payoffs) and the incentives needed to avoid a deviation that give rise to a new condition which we call the *no-deviation condition*. This condition must be satisfied in addition to the common equilibrium

[1] Of course, also the search model inherently exhibits turnover, but with observable types turnover is constant over the life cycle. Moscarini (2005) brings together search and learning in the Jovanovic framework.

[2] See for example Chernoff (1968). Only with multiple independent arms are reservation strategies guaranteed.

conditions of value-matching and smooth-pasting. The no-deviation condition can be interpreted as the continuous time version of the one-shot deviation principle. In other words, there is no assumption of commitment on the part of the firm and the equilibrium wage must be self-enforcing. We prove that the no-deviation condition implies that the second derivative at the cut-off belief is the same for the value functions of the high type as well as the low type firms. Recall that value matching requires that at the cut-off the value functions take the same value, the smooth-pasting condition requires that the first derivative is the same, and now the no-deviation requires equal second derivatives as well.

We show that supermodularity of the production technology is a necessary and sufficient condition for positive assortative matching, and that the equilibrium allocation is unique. Those workers with the highest beliefs about their ability will in equilibrium sort into those firms that are most productive. Moreover, we can analytically solve for the equilibrium allocation in terms of the cut-off belief, and we derive in closed form the stationary distribution of beliefs.

While in most of the analysis we consider common noise across firms, it turns out that the sorting result holds for different learning rates (noise) across firms, even if the rate of learning is slower in the high type firm. It is easy to see that with supermodularity and a learning rate no smaller in high types firms there will be positive sorting. The high type firm is both superior in the signal and in the noise. But if high type firms learn at a sufficiently slower rate (the noise is sufficiently high), then the signal-to-noise ratio in the high type firm may well be lower. The reason why this nonetheless does not affect the learning is that the value of learning also depends on the degree of convexity of the value function (from Ito's Lemma), in addition to the signal-to-noise ratio. But by the no-deviation condition, at the cut-off belief, the degree of convexity is the same in both firms and therefore the equilibrium value of learning is the same, no matter the difference in signal-to-noise ratios. Key here is that wages are endogenous and determined competitively. That is why this property does not hold in the canonical multi-armed bandit problem.

The technical contribution of the no-deviation condition allows us to actually solve the model.[3] We can now employ the framework and generate realistic predictions about labor market variables over the life cycle. Like in the existing learning models, turnover decreases with tenure.

Unlike most existing models, we can also explain that the variance of wages of a cohort is maximized asymptotically. This can provide a rationale for the fact that the variance of wages increases over the life cycle. Like existing models, the signal precision for a given type increases – eventually the type becomes exactly known – and as a result, the variance conditional on a type goes to zero. However, because of sorting the variance *between* types is maximized. With a fixed

[3]The idea of sequential rationality is of course not new and has also been employed in continuous time games by Sannikov (2007) who uses the concept of self generation. And Cohen and Solan (2009) use dependence of strategies on a small interval dt to restrict the set of Markovian strategies, in the spirit of our dt-shot deviation. It is precisely the one-shot deviation *in conjunction* with endogenous payoffs leads to the equalization of the second derivative of the value functions.

outside option, the second source of variance is absent and existing models find decreasing variance in wages over the life cycle. There is conclusive evidence that the life cycle variance of wages is increasing and concave (see amongst others Heathcoate, Violante and Perri (2009)).

We can fully characterize the stationary distribution of posterior beliefs and therefore also of the distribution of wages. At the cut-off belief, there is a discontinuity in the wage schedule due to the fact that the worker is indifferent between a low wage and a high option value of learning in the high type firm and a high wage with a low option value.

The motivation of our analysis and the results are obviously closest related to the labor market learning literature (Jovanovic (1979, 1984), Harris and Holmström (1982), Moscarini (2005) and Papageorgiou (2009).[4] Yet, there is a close relation to both the experimentation literature (Bolton and Harris (1999), Cripps, Rady and Keller (2005), Rosenberg et al. (2007)) and the literature on continuous time games (Sannikov (2007, 2008), Faingold and Sannikov (2009)). Most models of learning have a finite set of players and have an absorbing state. Ours has a continuum of agents and there is learning in all states. Moreover, it is essentially a competitive model with equilibrium prices and therefore payoffs from learning are endogenous.

The idea of analyzing a matching model where the current allocation determines the future type is first explored in Anderson and Smith (2009). They analyze a two-sided matching model of reputations with imperfect information about both matched types, and though they cannot fully characterize the equilibrium allocation, they show that under supermodularity Positive Assortative Matching of reputations may fail. Our set up differs substantially, but the main difference is in the information extraction. Their agents infer the type of each of the matched partners from the realization of a *joint* signal.[5] Another key characteristic of our model is that it is a pure Bayesian learning model where beliefs follow a martingale (Anderson and Smith (2009) allow for a general transition function that maps current matched types into future types and that is not necessarily a martingale). In Section 9 we show that our result holds for Bayesian updating processes other than the Brownian motion (we show our result extends for a generalized Lévy process), and we also establish that Positive Assortative Matching can fail if we have an updating process that is not Bayesian (this can be interpreted for example as a technology of human capital accumulation in addition to the information extraction). The latter is consistent with Anderson and Smith (2009).

[4]Papageorgiou (2009) analyzes a learning model with heterogeneity. He estimates the Roy model version of Moscarini's search model. Unlike the Beckerian model of matching, in the Roy model workers have a two-dimensional skill vector (either fishermen or hunters). Upon the realization of this tuple, agents choose to specialize in either of the two professions given market wages. With search frictions, non-competitive wage setting and a different production technology, the no-deviation condition is not imposed in addition to value matching and smooth pasting.

[5]Our model is more closely related to the standard firm-worker model to which they compare their two-sided model in a discussion. There is only a one-sided inference problem as in our model and they show Positive Assortative Matching arises for extreme beliefs $p = 0$ and 1, but not in the interior.

2 The Model Economy

Population of Firms and Workers. The economy is populated by a unit measure of workers and a unit measure of firms. Both firms and workers are *ex ante* heterogeneous. The firm's type $y \in H, L$ represents its productivity. The type y is observable to all agents in the economy. The fraction of H type firms is π and all firms are infinitely lived. The worker ability $x \in \{H, L\}$ is not observable, both to firms and workers, i.e., information is symmetric.[6] Nonetheless, both hold a common belief about the worker type, denoted by $p \in [0, 1]$. Upon entry, a newly born worker is of type H with probability p_0 and of type L with probability $1 - p_0$. Workers die with exogenous probability δ. New workers are born at the same rate.[7]

Preferences and Production. Workers and firms are risk-neutral and discount future payoffs at rate $r > 0$. Utility is perfectly transferable. Output is produced in pairs of one worker and one firm (x, y). Time is continuous. Positive output produced consists of a divisible consumption good and is denoted by μ_{xy}. We assume that more able workers are more productive in any firm, $\mu_{Hy} \geq \mu_{Ly}, \forall y$ and refer to it as worker monotonicity. While it is often useful, we do not in general assume firm monotonicity, which would be $\mu_{xH} \geq \mu_{xL}, \forall x$. Strict supermodularity is defined in the usual way:

$$\mu_{HH} - \mu_{LH} > \mu_{HL} - \mu_{LL}, \tag{1}$$

and with the opposite sign for strict submodularity. In the entire page, we will refer to strict supermodularity when we just mention supermodularity, likewise for submodularity.

Information. Because worker ability is not observable to both the worker and the firm, parties face an information extraction problem. They observe noisy measure of productivity, denoted by X_t. Cumulative output is assumed to be a Brownian motion with drift μ_{xy} and common variance σ^2

$$X_t = \mu_{xy} t + \sigma Z_t \tag{2}$$

where Z_t is a standard Wiener process and as a result, X_t is normally distributed with mean $\mu_{xy} t$ and variance $\sigma^2 t$. By Girsanov's Theorem the probability measures over the paths of two diffusion processes with the same volatility but different bounded drifts are equivalent, that is, they have the same zero-probability events. Since the volatility of a continuous-time diffusion process is effectively observable, the worker's type could be learned directly from the observed volatility if σ depends on

[6] This substantially simplifies the problem at hand. With private signals Cripps, Ely, Mailath and Samuelson (2008) show that with a finite signal space there will be common learning, but not necessarily with an infinite signal space as is the case in our model here.

[7] Without death, we know the posterior belief will converge with probability one to $p = 1$ or $p = 0$. Death here actually acts as a shuffling device to guarantee a non-trivial stationary distribution of posterior beliefs. Another modeling approach is to assume that such p_0 is drawn from some exogenous given distribution f_0. But as we will show later, this makes it quite difficult to solve the ergodic distribution.

workers' types.[8]

Equilibrium. We consider a (stationary) competitive equilibrium in this economy. With two types of firms and a continuum of p's in this market, take a competitive wage schedule $w_y(p)$ as given which specifies wage for every possible type p worker working in firm y.[9] Denote by V_y the stationary discounted present value of the competitive profits for firm y. The flow profit can be written as rV_y.[10] Now we are ready to define the notion of competitive equilibrium:

Definition 1 *In a (stationary) competitive equilibrium, there is a competitive wage schedule* $w_y(p) = \mu_y(p) - rV_y$*, where* $\mu_y(p) = p\mu_{Hy} + (1-p)\mu_{Ly}$ *denotes worker* p*'s expected productivity for firm* $y = H, L$ *and worker* p *chooses the firm* y *with the highest discounted present value. The market clears such that the measure of workers working in the* L *firm is* $1-\pi$ *and the measure of workers working in the* H *firm is* π*.*

We would like to point out several things about this definition. First, the definition of competitive equilibrium implies identical types will obtain the same payoff. A firm y earns the same flow profit for every p. Our notion of competitive equilibrium puts restrictions on the off-equilibrium prices, as does the Beckerian definition of a matching equilibrium. In the current context this is comparable to the notion of subgame perfect equilibrium. Recall that subgame perfect equilibrium requires that agents behave optimally on any possible subgame. Similarly, here we require: Although type p worker is not employed by firm y on equilibrium path, the hypothetical wage is still $w_y(p) = \mu_y(p) - rV_y$ to guarantee the firm cannot make or lose money if the employment suddenly happens. Second, unlike Anderson and Smith (2008), our wage definition concerns a spot market wage. They parse the wage into a static wage plus a dynamic human capital effect. Instead, our spot wage approach captures the idea that firms cannot commit to future actions. That wage setting process therefore corresponds to the Pareto efficient allocation. Here we take the view that parties cannot commit their wage contracts on future actions (see also Hörner and Samuelson 2009 for a model of experimentation in the presence of spot market contracts). Together with sequential rationality, this therefore requires that the wage contract is self-enforcing. Finally, like all price taking economies, the wage schedule essentially transforms our problem into a decision problem for the workers.

[8] However, we can allow σ to be firm-specific. In section 9 we analyze the general case of firm-dependent σ_y.

[9] Bergemann and Välimäki (1996) and Felli and Harris (1996) consider a two-firm, one-worker/buyer model with strategic price setting in a world with independent arms. With ex ante heterogeneous firms and workers and correlated arms, we instead focus on competitive price setting which is closest in spirit to the Beckerian benchmark.

[10] Notice since there is no free entry, V_y need not to be zero. We could model free entry as long as in equilibrium there is a non-degenerate distribution of firm types in the economy. We consider this does not add to the insights of our model.

3 Preliminaries

3.1 Benchmark: No Learning

Workers differ in the common beliefs p of being a high type. We shut down learning so that beliefs are invariant. This can be viewed as a special case of the learning model with the variance σ^2 going to infinity. We assume that there is no birth or death so we essentially have a static problem. Suppose without of generality that p is uniformly distributed on $[0, 1]$. We continue to maintain the assumption that the worker does not know her true type or that she has no private information about it. Denote $\mu_y(p) = p\mu_{Hy} + (1-p)\mu_{Ly}$ for $y = H, L$ and r as the discount rate.

Under the above notion of competitive equilibrium, it is easy to verify the following claim (All of the results in this paper are in the sense of "almost surely" because we allow a zero measure of players to behave differently):

Claim 1 *Under strict supermodularity, PAM is the unique (stationary) competitive equilibrium allocation: H firms match with workers $p \in [1-\pi, 1]$, L firms match with workers $p \in [0, 1-\pi)$. The opposite (NAM) holds under strict submodularity: H firms match with workers in $[0, \pi)$.*

Since there is no learning, essentially this result is the similar to Becker's (1973) result, but with uncertainty. Noteworthy about this Bayesian version of Becker is that even though for PAM there is supermodularity of the ex-post payoffs ($\mu_{HH} + \mu_{LL} > \mu_{HL} + \mu_{LH}$), there need not be monotonicity in expected payoffs, i.e., $\mu_H(1-\pi)$ may be smaller than $\mu_L(1-\pi)$. In fact, that will be reflected in the firm's equilibrium payoffs: $V_H \geq V_L$ if and only if $\mu_H(1-\pi) \geq \mu_L(1-\pi)$.

As in Becker, the equilibrium allocation is unique, but there may be multiple splits of the surplus. In the case of PAM, we only require that $w_H(1-\pi) = w_L(1-\pi)$. There are multiple equilibrium payoffs if the surplus of a match between L and $p = 0$ is positive. Instead, if $\mu_L(0) = 0$,[11] there is a unique equilibrium payoff.

3.2 Belief Updating

In the presence of learning we can now derive the beliefs and subsequently the value functions. A sufficient statistic for output history, which determines the future prospects of a match, thus also the natural state variable of the bargaining game, is the posterior belief p_t that the worker had a high productivity. Now, it is well-known that we can have the following important result: conditional on the output process $(X_t)_{t \geq 0}$, $(p_t)_{t \geq 0}$ is a martingale diffusion process. Moreover, this process could be represented as a Brownian motion.

Based on the framework of our model, denote $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma$, $y = H, L$, $\Sigma_y(p) = \frac{1}{2}p^2(1-p)^2 s_y^2$ and then we can get:

[11] And there is limited liability, i.e., workers and firms cannot receive negative payoffs.

Lemma 1 *(Belief Consistency) Consider any worker who works for firm y between t_0 and t_1. Given a prior $p_{t_0} \in (0,1)$, the posterior belief $(p_t)_{t_0<t\leq t_1}$ is consistent with the output process $(X_{y,t})_{t_0<t\leq t_1}$ if and only if it satisfies*

$$dp_t = p_t(1-p_t)s_y d\bar{Z}_{y,t}$$

where

$$d\bar{Z}_{y,t} = \frac{1}{\sigma}[dX_{y,t} - (p_t\mu_{Hy} + (1-p_t)\mu_{Ly})dt].$$

The proof of this Lemma is in Faingold and Sannikov (2007) or Daley and Green (2008). The basic idea behind the proof is a combination of Bayes' rule and Ito's lemma. Given the period t posterior belief p_t and dX_t, we know the posterior belief at period $t+dt$ is:

$$p_{t+dt} = \frac{p_t \exp\{-\frac{[dX_t-\mu_{Hy}dt]^2}{2\sigma^2 dt}\}}{p_t \exp\{-\frac{[dX_t-\mu_{Hy}dt]^2}{2\sigma^2 dt}\} + (1-p_t)\exp\{-\frac{[dX_t-\mu_{Ly}dt]^2}{2\sigma^2 dt}\}}.$$

Hence,

$$dp_t = p_{t+dt} - p_t = p_t(1-p_t)\frac{\exp\{-\frac{[dX_t-\mu_{Hy}dt]^2}{2\sigma^2 dt}\} - \exp\{-\frac{[dX_t-\mu_{Ly}dt]^2}{2\sigma^2 dt}\}}{p_t \exp\{-\frac{[dX_t-\mu_{Hy}dt]^2}{2\sigma^2 dt}\} + (1-p_t)\exp\{-\frac{[dX_t-\mu_{Ly}dt]^2}{2\sigma^2 dt}\}}.$$

Apply Ito's Lemma and we obtain the above result.

Lemma 1 establishes that dp depends on three elements: $p(1-p)$, which peaks at 1/2; the signal-to-noise ratio of output, $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma$ and $d\bar{Z}_y$, the normalized difference between realized and unconditionally expected flow output, which is a standard Wiener process with respect to the filtration $\{X_{y,t}\}$. Obviously, beliefs move faster the more uncertain about worker's quality (p close to 1/2); the less variation in the output process (smaller σ) and the larger productivity difference (higher $\mu_{Hy} - \mu_{Ly}$).

Now the learning consideration will change the results. Moreover, the supermodularity not only affects the value of the output created as in the standard Beckerian model, but it also changes the speed of learning. For example, under supermodularity ($\mu_{HH} - \mu_{HL} > \mu_{LH} - \mu_{LL}$), the learning speed is faster in high type firm, which is especially significant for p close to 1/2. Intuitively speaking, learning makes it more attractive to match with a high type firm even though it is better for her to match with a low type firm without learning.

3.3 Value Functions

Consider any interval for the posterior belief $p \in [p_1, p_2]$ where the worker accepts the offer from a type y firm, then the value function is given by[12]:

[12]Note that we critically need the assumption that the worker doesn't know any private information about his type. If this assumption is violated, the worker's value functions could not be written like this.

$$rW_y(p) = \mu_y(p) - V_y + \Sigma_y(p)W_y^{''}(p) - \delta W_y(p), \tag{3}$$

from Ito's Lemma. The term $\mu_y(p) - V_y$ is equal to the flow wage payoff and corresponds to the deterministic component of the diffusion $X_{y,t}$, and the term $\Sigma_y(p)W_y^{''}(p)$ is the second-order term from the transformation W of the diffusion process $X_{y,t}$. All higher-order terms vanish as the time interval shrinks to zero. The general solution to this differential equation is:

$$W_y(p) = \frac{\mu_y(p) - V_y}{r+\delta} + k_{y1}p^{1-\alpha_y}(1-p)^{\alpha_y} + k_{y2}p^{\alpha_y}(1-p)^{1-\alpha_y}, \tag{4}$$

where

$$\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{s_y^2}} \geq 1.$$

First notice that the boundedness of the value function implies that if 0 is included in the domain, then $k_{y1} = 0$ and if 1 is included in the domain, then $k_{y2} = 0$. Second, $\Sigma_y(p)W_y''(p)$ is the value of learning and this is an option value in the sense that the worker has the choice to change his job as he learns his type p. It is easy to verify that this value is zero if the worker never changes his job.[13] From the Martingale property of the Brownian motion, at any p the expected value of p in the next time interval is equal to p. There is as much good news as bad news to be expected in the next period. It is the option value of switching to a *more suitable* match that generates the value of learning.

4 Analysis and Results

4.1 Characterization of the Equilibrium Allocation

Now consider any candidate equilibrium where a type p worker is indifferent between matching with either firm y. Then equilibrium requires the equal-value condition (the worker gets the same value at the cutoff) and the smooth-pasting condition (the marginal of both value functions is identical). For example, if for $p \in [p_1, p_2)$, the worker works in the low type firm and for $p \in [p_2, p_3)$, the worker works in the high type firm, then we must have:[14]

$$W_L(p_2) = W_H(p_2) \quad \text{and} \quad W_L{}'(p_2) = W_H{}'(p_2). \tag{5}$$

It is important to point out that both the equal value-condition and the smooth-pasting condition are on-equilibrium path conditions. It has nothing to do with the off-equilibrium path (i.e.,

[13] In that case, p can take both the values 0 and 1. So the boundedness of the value function requires that both k_{y1} and k_{y2} are zero and hence $W_y''(p) = 0$ for every p.

[14] We slightly abuse notation hers since W_L is not defined on p_2. A more precise way of writing the equations is $W_L(p_2+) = W_H(p_2)$ and $W_L{}'(p_2+) = W_H{}'(p_2)$. In what follows, we will continue to use the expression in the text in order to economize on notation.

instead of accepting offers from low type firms, worker with $p \in [p_1, p_2)$ are tempted to accept offers from high high type firms).

In the following lemmas we characterize the value functions establishing convexity and monotonicty:

Lemma 2 *The equilibrium value functions W_y are strictly convex for $p \in (0,1)$.*

Proof. In Appendix. ■

The intuition for this Lemma is the following. Preferences are linear and the option value of learning is strictly positive, hence the value function with the option of learning is convex. To see this, observe that since the measure of both types of firms are strictly positive, market clearing requires that workers with some p's will be employed by high type firms while workers with other p's will be employed by low type firms. This implies that some worker has to change jobs at some point and the option value of learning is non-zero. Hence we have $W_y''(p) > 0$, for all $p \in (0,1)$. On the other hand, when $p = 0$ or 1, the posterior belief will always stay at 0 or 1 by Bayes' rule such that learning never happens. It is easy to verify that $W_y''(p) = 0$ for $p = 0$ or 1.

Given the strict convexity of equilibrium value functions and the smooth pasting condition, we can immediately derive the following Lemma:

Lemma 3 *The equilibrium value functions W_y are strictly increasing.*

Proof. In Appendix. ■

One important implication is that if we define $\mathcal{W}(p)$ as the envelope of all equilibrium value functions $W_y(p)$. Then this envelope function $\mathcal{W}(p)$ is continuous, strictly increasing and strictly convex for $p \in (0,1)$. Suppose workers with $p \in [0, \underline{p})$ are employed by type y firm and workers with $p \in (\bar{p}, 1]$ are employed by type $-y$ firm. Then we should have: $W_y'(0) = \frac{\mu_{Hy} - \mu_{Ly}}{r+\delta} < W_{-y}'(1) = \frac{\mu^{H,-y} - \mu^{L,-y}}{r+\delta}$. This gives us another result:

Lemma 4 *Under supermodularity, in any equilibrium $p = 0$ worker matches with L firm; $p = 1$ worker matches with H firm. The opposite under strict submodularity. Moreover,*

$$\frac{\min(\Delta_H, \Delta_L)}{r+\delta} < W'(p) < \frac{\max(\Delta_H, \Delta_L)}{r+\delta},$$

where $\Delta_H = \mu_{HH} - \mu_{LH}$ and $\Delta_L = \mu_{HL} - \mu_{LL}$.

Intuitively this result is best understood by using the standard sorting argument from Becker (1973). At $p = 0$ and $p = 1$ there is no value of learning. As a result, there the value function can be interpreted as being determined by the no-learning allocation.

The properties derived above are mainly concerned with on-equilibrium path behavior. We also need to specify what happens in the event of deviations and behavior is off-equilibrium path. We consider the equivalence of a one-shot deviation in continuous time because we think of the continuum as an idealization of discrete time. This amounts to a worker playing the deviant action over an interval $[t, t+dt)$ according to the belief p at time t, and considering the limit as $dt \to 0$.[15] This is very important because it allows us to derive the value function for deviation. On the contrary, if the deviation only takes place at a single time point t, then the value function for deviation is essentially the same as the one without deviation because no information will be extracted from just a single time point.

The next Lemma establishes that if we consider off-the-equilibrium path deviations, we actually need one additional condition, which we call the *no-deviation* condition.

Lemma 5 *To deter possible deviations, a necessary condition is:*

$$W_H''(\underline{p}) = W_L''(\underline{p}) \quad \textit{(No-deviation condition)} \tag{6}$$

for any possible cutoff $\underline{p}$*.*

Proof. Without loss of generality, we assume that on equilibrium path, worker with $p > \underline{p}$ accepts offers from high type firms and worker with $p < \underline{p}$ accepts offers from low type firms. Consider one possible one-shot deviation: a $p > \underline{p}$ worker matches with a low type firm for dt and then switch back. Then the new value function is defined as:

$$\tilde{W}_L(p) = w_L(p)dt + e^{-(r+\delta)dt} W_H(p + dp), \quad \text{where} \quad dp = p(1-p)s_L d\bar{Z}. \tag{7}$$

Apply Ito's Lemma and we get:

$$\tilde{W}_L(p) = w_L(p)dt + e^{-(r+\delta)dt}[W_H(p) + \Sigma_L(p) W_H''(p)dt]. \tag{8}$$

This implies:

$$\lim_{dt \to 0} \frac{\tilde{W}_L(p) - W_H(p)}{dt} = w_L(p) - w_H(p) + [\Sigma_L(p) - \Sigma_H(p)]W_H''(p). \tag{9}$$

The RHS of the above equation must be smaller than zero for any $p > \underline{p}$. Let $p \to \underline{p}$ and we have:

$$\begin{aligned} w_L(\underline{p}) - w_H(\underline{p}) + [\Sigma_L(\underline{p}) - \Sigma_H(\underline{p})]W_H''(\underline{p}) &\leq 0 \\ \Rightarrow w_L(\underline{p}) + \Sigma_L(\underline{p})W_L''(\underline{p}) - (w_H(\underline{p}) + \Sigma_H(\underline{p})W_H''(\underline{p})) + (W_H''(\underline{p}) - W_L''(\underline{p}))\Sigma_L(\underline{p}) &\leq 0 \\ \Rightarrow W_H''(\underline{p}) \leq W_L''(\underline{p}). \end{aligned} \tag{10}$$

[15] This notion is also implicitly used in Sannikov (2007, Proposition 2), and in Cohen and Solan (2009) who consider deviations from Markovian strategies in bandit problems.

Similarly, we can consider another possible one-shot deviation: a $p < \underline{p}$ worker matches with a high type firm for dt and then switch back. Then the new value function is defined as:

$$\tilde{W}_H(p) = w_H(p)dt + e^{-(r+\delta)dt} W_L(p + dp), \quad \text{where} \quad dp = p(1-p)s_H d\bar{Z}. \tag{11}$$

Apply Ito's Lemma once again and we get:

$$\tilde{W}_H(p) = w_H(p)dt + e^{-(r+\delta)dt}[W_L(p) + \Sigma_H(p)W_L''(p)dt], \tag{12}$$

taking the limit

$$\lim_{dt\to 0} \frac{\tilde{W}_H(p) - W_L(p)}{dt} = w_H(p) - w_L(p) + [\Sigma_H(p) - \Sigma_L(p)]W_L''(p) < 0$$

for $p < \underline{p}$. Therefore as p goes to $\underline{p}$, we should have:

$$w_H(\underline{p}) - w_L(\underline{p}) + [\Sigma_H(\underline{p}) - \Sigma_L(\underline{p})]W_L''(\underline{p}) \leq 0 \Rightarrow W_H''(\underline{p}) \geq W_L''(\underline{p}). \tag{13}$$

(10) and (13) imply that $W_H''(\underline{p}) = W_L''(\underline{p})$. ■

This no-deviation condition is quite unique for the two-armed bandit problem. This condition is absent in an one-armed bandit problem. Most of the models in the literature on continuous time learning models (Jovanovic (1979) and Moscarini (2005)) and continuous time games (see amongst others, Sannikov (2009)) are essentially investigating a one-armed bandit problem. There, we can directly look at equilibria in cutoff strategies, a theoretical foundation for which is given by Rosenberg et al. (2007). In the one-armed bandit problems, the safe arm essentially is an absorbing state so we only need to worry about the potential deviation from the risky arm to the safe arm.[16] Then the no-deviation condition becomes $W_H''(\underline{p}) \geq W_L''(\underline{p}) = 0$ but this is already implied by the convexity property.[17]

We provide some intuition for the no-deviation condition. By assuming Sequential Rationality, i.e., the equilibrium is robust to a one-shot deviation, we basically impose that the equilibrium wage is self-enforcing. There is no commitment to future realizations of X_t and therefore of future beliefs p. Now we can interpret W'' as the marginal value of learning: W' is the marginal change of W with respect to the posterior p, and learning changes p and is therefore quantified by the change in W' which is W''. The condition states that there is no deviation if the marginal value of learning at $\underline{p}$ is same in both firms.

But in our two-armed bandit problem, we first need to answer the question whether there exist

[16] For example, in our model assume $\mu_{HL} = \mu_{LL}$ and the return in the low type firm is deterministic.

[17] In a model of option pricing by Dumas (1991), there does exist a condition on the second derivative called the "super contact" condition. It arises as the optimal solution to the option pricing problem with proportional cost.

non-cutoff stationary equilibria, i.e., a worker with $p \in [p_1, p_2)$ accepts the offer from a high type firm, with $p \in [p_2, p_3)$ accepts the offer from a low type firm and with $p \in [p_3, p_4)$ accepts the offer from a high type firm again. Surprisingly, Lemmas 2–9 imply that all stationary competitive equilibria must be in cutoff strategies.

Theorem 1 *PAM is the unique stationary competitive equilibrium allocation under strict supermodularity. Likewise for NAM under strict submodularity.*

To prove this theorem, we only need to prove the following Claim:

Claim 2 *Under strict supermodularity, it is impossible to have $p_1 < p_2$ and equilibrium value functions W_H (for $p \in [p_1, p_2]$), W_{L1} (for $p < p_1$), W_{L2} (for $p > p_2$) such that:*

$$W_H(p_1) = W_{L1}(p_1) \quad and \quad W''_H(p_1) = W''_{L1}(p_1)$$

$$W_H(p_2) = W_{L2}(p_2) \quad and \quad W''_H(p_2) = W''_{L2}(p_2)$$

are satisfied simultaneously.

Under strict submodularity, it is impossible to have $p_1 < p_2$ and equilibrium value functions W_L (for $p \in [p_1, p_2]$), W_{H1} (for $p < p_1$), W_{H2} (for $p > p_2$) such that:

$$W_L(p_1) = W_{H1}(p_1) \quad and \quad W''_L(p_1) = W''_{H1}(p_1)$$

$$W_L(p_1) = W_{H2}(p_2) \quad and \quad W''_L(p_2) = W''_{H2}(p_2)$$

are satisfied simultaneously.

Proof. In Appendix. ■

This result states that it is not benefial for a worker of type p to learn in the high type firm H in the middle as long as there there are still types p on both sides who work in the low type firms. Given the above claim, it is easy to prove the theorem:

Proof. Under supermodularity, by Lemma 9, worker with sufficiently low p's will accept low type firm's offer and worker with sufficiently high p's will accept high type firm's offer. But 2 implies it is impossible to have worker first accept low type firm's offer, then accept high type firm's offer and finally accept low type firm's offer again. Hence, we must have some cutoff $\underline{p}$ such that $p < \underline{p}$ will accept low type firm's offer and $p > \underline{p}$ will accept high type firm's offer. This is exactly a PAM allocation. Use the same logic and you can see NAM is the only possible stationary competitive equilibrium allocation under strict submodularity. ■

Before we turn to the equilibrium distribution, we show that the no-deviation condition in Lemma 9 is not just necessary but also sufficient under supermodularity:

Lemma 6 *Under strict supermodularity, $W''_H(\underline{p}) = W''_L(\underline{p})$ implies that no deviation would happen for the PAM equilibrium.*

Proof. In Appendix. ■

4.2 The Equilibrium Distribution

The previous section shows that under strict supermodularity (submodularity), PAM (NAM) is the unique stationary competitive equilibrium allocation. We still need to construct such an equilibrium. To do that, we assume supermodularity and worker and firm monotonicity: ($\mu_{HH} > \mu_{HL}$ and $\mu_{LH} > \mu_{LL}$)[18]. Now consider a strictly positive assortative matching equilibrium such that workers with beliefs less than $\underline{p}$ will choose L firms and workers with beliefs higher than $\underline{p}$ will choose H firms. From equation (4) we hence have $k_{L1} = 0$ and $k_{L2} > 0$ for $y = L$ and $k_{H2} = 0$ and $k_{H1} > 0$ for $y = H$. Let $k_L = k_{L2}$, $k_H = k_{H1}$ and worker's value functions become:

$$W_L(p) = \frac{w_L(p)}{r+\delta} + k_L p^{\alpha_L}(1-p)^{1-\alpha_L} \tag{14}$$

and

$$W_H(p) = \frac{w_H(p)}{r+\delta} + k_H p^{1-\alpha_H}(1-p)^{\alpha_H}, \tag{15}$$

where

$$\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{s_y^2}} \geq 1.$$

To discuss market clearing conditions, we need to consider the ergodic distribution of p's. From the Fokker-Planck (Kolmogorov forward) equation, the stationary and ergodic density f_y should satisfy the following differential equation:

$$0 = \frac{df_y(p)}{dt} = \frac{d^2}{dp^2}[\Sigma_y(p) f_y(p)] - \delta f_y(p). \tag{16}$$

The general solution to this differential equation is (see also Moscarini (2005)):

$$f_y(p) = [f_{y0} p^{\gamma_{y1}}(1-p)^{\gamma_{y2}} + f_{y1}(1-p)^{\gamma_{y1}} p^{\gamma_{y2}}] \tag{17}$$

where

$$\gamma_{y1} = -\frac{3}{2} + \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} > -1$$

[18] Monotonicity is just to help us find one particular way to divide the surplus. The whole construction of equilibrium also goes through if we do not make this assumption.

and

$$\gamma_{y2} = -\frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} < -2.$$

First, the integrability of f_y requires that $f_{y1} = 0$ if 0 is included in the domain and $f_{y0} = 0$ if 1 is included in the domain. Second, the Fokker-Planck (Kolmogorov forward) equation is only valid for $p \neq p_0$. Since there is a flow in of new workers, for $p = p_0$ we should have a kink in the density function. This also raises the issue of relative position between p_0 and $\underline{p}$. We first consider the case where $\underline{p} < p_0$. We then derive in abbreviated format the result when $\underline{p} > p_0$.

Given any $p_0 \in (0, 1)$, if $\underline{p} < p_0$, then the density functions are:

$$f_H(p) = [f_{H0}p^{\gamma_{H1}}(1-p)^{\gamma_{H2}} + f_{H1}(1-p)^{\gamma_{H1}}p^{\gamma_{H2}}]\mathbb{I}(\underline{p} < p \leq p_0) + f_{H2}(1-p)^{\gamma_{H1}}p^{\gamma_{H2}}\mathbb{I}(p > p_0) \quad (18)$$

and

$$f_L(p) = f_{L0}p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}. \quad (19)$$

The density functions are subject to the following boundary conditions. First, once the posterior belief reaches the equilibrium separation point $\underline{p}$, we should have the cutoff condition:

$$\Sigma_H(\underline{p}+)f_H(\underline{p}+) = \Sigma_L(\underline{p}-)f_L(\underline{p}-). \quad (20)$$

This condition guarantees that the flow of agents who cross $\underline{p}$ from below is equal to the flow of agents who cross from above. The implication is that since the speed from above Σ_H is larger than Σ_L, the densities satsify $f_H(\underline{p}+) < f_L(\underline{p}-)$. It is worth comparing this condition to the standard condition when there is an absorbing state (Cox-Miller (1965), Dixit (1993), and Moscarini (2005)). In the case with only one brownian motion and an absorbing state, what is required is $\Sigma(\underline{p}+)f(\underline{p}+) = 0$ because the probability of obsorption in a time interval dt must equal speed of the flow in of the brownian motion which is proportional to $\sqrt{dt}$ (see Cox and Miller (1965, p.220)). Therefore the speed must be zero near the absorbing boundary.

Second, total flows in and out of the high type firms must balance:

$$\Sigma_H(p_0)[f'_H(p_0-) - f'_H(p_0+)] = \delta\pi + \frac{d}{dp}[\Sigma_H(p)f_H(p)]|_{\underline{p}+}.$$

The left-hand side of the above equation is the total inflow into high type firms, which are new workers who enter into this economy. It must be δ by assumption. The right-hand side of the above equation is the total outflows from the high type firms, which include workers who reach $\underline{p}$ and transfer to low type firms and workers who are hit by the death shock. In the appendix, we manage show that this equation will further imply:

$$\frac{d}{dp}[\Sigma_L(p) f_L(p)]|_{\underline{p}-} = \frac{d}{dp}[\Sigma_H(p) f_H(p)]|_{\underline{p}+}$$

Third, the density function has to be continuous at p_0 (see also Bertola and Caballero (1990)):

$$f_H(p_0-) = f_H(p_0+).$$

This condition is arbitrary in the sense that if it is not satisfied, the distribution will still be ergodic. It is customary to impose this condition as it approximates entry from a non-degenerate distribution instead of entry of identical types p_0.

Finally, we have market clearing condition:

$$\int_{\underline{p}}^{1} f_H(p)dp = \pi \quad \text{and} \quad \int_{0}^{\underline{p}} f_L(p)dp = 1 - \pi.$$

In summary, when $\underline{p} < p_0$, the equilibrium is characterized by a system of eight equations with nine unknowns $(V_L, V_H, k_L, k_H, \underline{p}, f_{H0}, f_{H1}, f_{H2}, f_{L0})$:[19]

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Equal value condition)} \tag{21}$$

$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth pasting condition)} \tag{22}$$

$$W_H''(\underline{p}) = W_L''(\underline{p}) \qquad \text{(No deviation condition)} \tag{23}$$

$$\Sigma_H(\underline{p}+) f_H(\underline{p}+) = \Sigma_L(\underline{p}-) f_L(\underline{p}-) \qquad \text{(Boundary condition)} \tag{24}$$

$$\int_{\underline{p}}^{1} f_H(p)dp = \pi \qquad \text{(Market clearing } H\text{)} \tag{25}$$

$$\int_{0}^{\underline{p}} f_L(p)dp = 1 - \pi \qquad \text{(Market clearing } L\text{)} \tag{26}$$

$$\frac{d}{dp}[\Sigma_L(p) f_L(p)]|_{\underline{p}-} = \frac{d}{dp}[\Sigma_H(p) f_H(p)]|_{\underline{p}+} \qquad \text{(Flow equation at } \underline{p}\text{)} \tag{27}$$

$$f_H(p_0-) = f_H(p_0+) \qquad \text{(Continuous density at } p_0\text{)} \tag{28}$$

Fortunately, Equations (24)–(28) can be solved separatly from Equations (21)–(23). In other words, the procedure of solving this system of equation could be: first we solve $\underline{p}$ jointly with $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ from Equations (24)–(28) and then we plug $\underline{p}$ into Equations (21)–(23) to pin down other unknowns.

[19]Observe that with more unknowns than variables, the solution to our system is indeterminate. In fact, there are potentially a continuum of wages that can be supported in equilibrium, though the allocation will be unique. This indeterminacy is as in Becker: the allocation is unique, but there may be multiple ways to split the surplus. In all that follows, when we use the term uniqueness of equilibrium, we refer to the allocation, not to the wages.

Proposition 1 *Equations (24)-(28) imply $\underline{p} < p_0$ if and only if:*

$$\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp} < \frac{\pi}{1-\pi}. \tag{29}$$

Moreover, if such $\underline{p}$ exists, it must be unique.

Proof. In Appendix. ■

The proof of Proposition 1 is quite straightforward. The idea of the proof is the following: since we have 5 equations with five unknowns, we can first express $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ as a function of $\underline{p}$ and then use the last equation to pin down $\underline{p}$.

The existence and uniqueness of the solution to the system require that $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ change monotonically with $\underline{p}$. Fortunately, this is the case as shown in the appendix. First, from Equation (26), we find f_{L0} is decreasing in $\underline{p}$. Second, Equations (24) and (27) imply that f_{H0}, f_{H1} are also decreasing in $\underline{p}$. Finally, Equation (28) tells us that f_{H2} is also decreasing in $\underline{p}$ given f_{H0}, f_{H1} are decreasing in $\underline{p}$. The monotonicity guarantees that if a solution exists, it must be unique. Furthermore, it enables us to only investigate the boundaries when determining a solution exists. This gives us Equation (29) given in the Proposition.

In the second case, $\underline{p} \geq p_0$. Given any $p_0 \in (0,1)$, if $\underline{p} \geq p_0$, then the density functions are:

$$f_L(p) = f_{L0}p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\mathbb{I}(p < p_0) + [f_{L1}p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} + f_{L2}(1-p)^{\gamma_{L1}}p^{\gamma_{L2}}]\mathbb{I}(p_0 \leq p \leq \underline{p}) \tag{30}$$

and

$$f_H(p) = f_{H0}(1-p)^{\gamma_{H1}}p^{\gamma_{H2}}. \tag{31}$$

Then the system of equations to determine the equilibrium is:

$$\begin{aligned}
W_H(\underline{p}) &= W_L(\underline{p}) && \text{(Equal-value)} && (32)\\
W_H'(\underline{p}) &= W_L'(\underline{p}) && \text{(Smooth-pasting)} && (33)\\
W_H''(\underline{p}) &= W_L''(\underline{p}) && \text{(No-deviation)} && (34)\\
\Sigma_H(\underline{p}+)f_H(\underline{p}+) &= \Sigma_L(\underline{p}-)f_L(\underline{p}-) && \text{(Boundary condition)} && (35)\\
\int_{\underline{p}}^{1} f_H(p)dp &= \pi && \text{(Market clearing } H\text{)} && (36)\\
\int_0^{\underline{p}} f_L(p)dp &= 1-\pi && \text{(Market clearing } L\text{)} && (37)\\
\frac{d}{dp}[\Sigma_L(p)f_L(p)]|_{\underline{p}-} &= \frac{d}{dp}[\Sigma_H(p)f_H(p)]|_{\underline{p}+} && \text{(Flow equation at } \underline{p}\text{)} && (38)\\
f_L(p_0-) &= f_L(p_0+) && \text{(Continuous density at } p_0\text{)} && (39)
\end{aligned}$$

We can now prove the following Proposition, the counterpart to Proposition 1:

Proposition 2 *Equations (35)-(39) imply $\underline{p} \geq p_0$ if and only if:*

$$(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp} \geq \frac{\pi}{1-\pi}. \tag{40}$$

Moreover, if such $\underline{p}$ exists, it must be unique.

The idea for the proof of Proposition 2 is exactly the same as that for the proof of Proposition 1 and the proof is also shown in the appendix. Propositions 1 and 2 together provide a very nice existence and uniqueness result:

Theorem 2 *Under strict supermodularity, for any pair $(p_0, \pi) \in (0,1)^2$, there exists a unique PAM cutoff $\underline{p}$. Moreover, $\underline{p} < p_0$ if and only if:*

$$(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp} < \frac{\pi}{1-\pi}. \tag{41}$$

One of the nice properties about Equation (41) is that the whole equation only depends on p_0, π, δ/s_H^2 and δ/s_L^2. This also gives us a feasible way to compute p_0. Given p_0, π, δ/s_H^2 and δ/s_L^2, we first need to decide the sign of

$$(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp} - \frac{\pi}{1-\pi}.$$

If this sign is negative, then we know that $\underline{p}$ is smaller than p_0 and we can use the system of equations in Case 1 to find out $\underline{p}$. On the contrary, if this sign is not negative, then we know that $\underline{p}$ is larger than p_0 and we can use the system of equations in Case 2 to find out $\underline{p}$. Therefore, we can get a very convenient way to determine the equilibrium the equilibrium cutoff numerically.

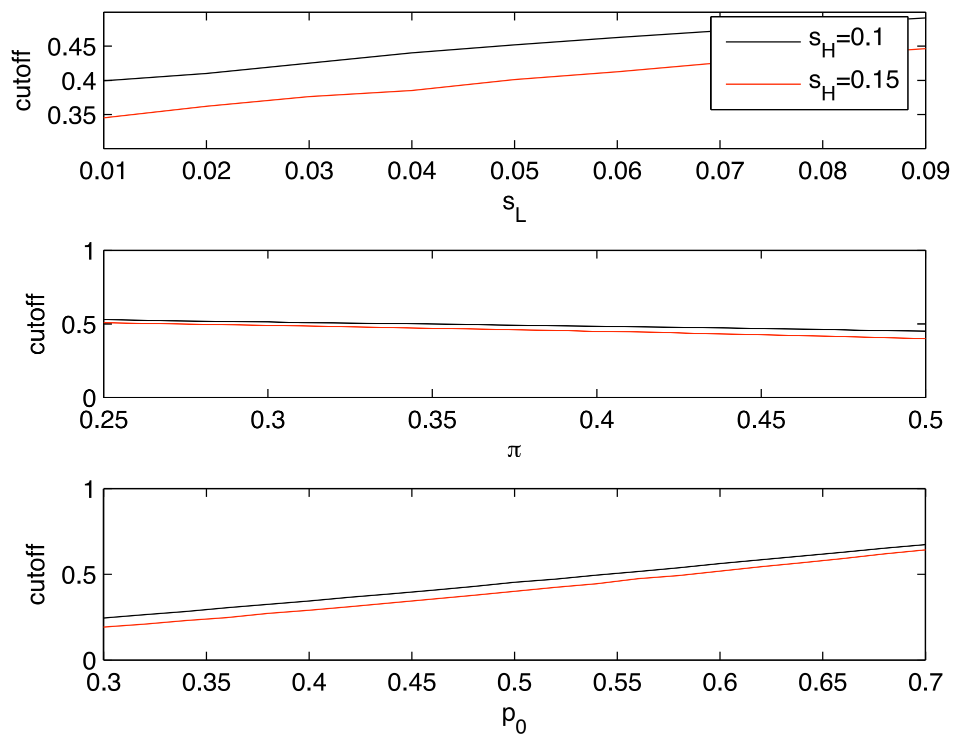
Before presenting the numerical results, we have a simple theoretical comparative static result:

Corollary 1 *$\underline{p}$ is strictly increasing in p_0 and decreasing in π.*

This corollary is proved in the appendix. But the intuition is quite clear: π decreases means there are more low type firms in the economy and hence $\underline{p}$ has to increase to make sure that more workers are matched with low type firms; p_0 increases means the overall quality of the workers is becoming better in the economy and $\underline{p}$ has to increase to make sure that the low type firms can also be matched with better workers.

Mathematically, the relationships between $\underline{p}$ and δ/s_H^2, δ/s_L^2 are not clear. But intuitively speaking, as s_L increases, the degree of supermodularity will decrease while the speed of learning

Figure 1: Equilibrium Cutoff



in low type firms will increase. Both of them make the low type firms more attractive such that $\underline{p}$ should increase in s_L. On the other hand, as s_H increases, both the degree of supermodularity and the speed of learning in high type firms will increase, which will lead to a decrease of $\underline{p}$.

Figure 4.2 plots the value of $\underline{p}$ as a function of s_L, π, p_0, for the case of PAM and with parameter values: $s_H = 0.15, s_L = 0.05, p_0 = 0.5, \pi = 0.5, \delta = 0.01$.

4.3 Equilibrium Analysis: Value Functions

Theorem 2 implies that under supermodularity, the PAM cutoff $\underline{p}$ can be uniquely determined. But given this $\underline{p}$, we still have the following conditions to satisfy:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Equal-value condition)} \tag{42}$$

$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting condition)} \tag{43}$$

$$W_H''(\underline{p}) = W_L''(\underline{p}) \qquad \text{(No-deviation condition)} \tag{44}$$

The last problem is that Equations (51)-(53) are three equations for four unknowns. The equilibrium is indeterminate in the sense that although the allocation $\underline{p}$ is unique, there could be multiple ways to divide the surplus. One way to get rid of this problem is to assume monotonicity and make $\mu_{LL} = 0$. Then non-negative wage requires that $w_L(0)$ has to be zero and hence we have

$V_L = 0$. Equations (51)-(53) then become:

$$\frac{\mu_L(\underline{p})}{r+\delta} + k_L\underline{p}^{\alpha_L}(1-\underline{p})^{1-\alpha_L} = \frac{\mu_H(\underline{p}) - rV_H}{r+\delta} + k_H\underline{p}^{1-\alpha_H}(1-\underline{p})^{\alpha_H}$$

$$\frac{\mu_{HL}-\mu_{LL}}{r+\delta} + k_L\underline{p}^{\alpha_L}(1-\underline{p})^{1-\alpha_L}(\frac{\alpha_L-\underline{p}}{\underline{p}(1-\underline{p})}) = \frac{\mu_{HH}-\mu_{LH}}{r+\delta} + k_H\underline{p}^{1-\alpha_H}(1-\underline{p})^{\alpha_H}(\frac{1-\alpha_H-\underline{p}}{\underline{p}(1-\underline{p})})$$

$$k_L\underline{p}^{\alpha_L-2}(1-\underline{p})^{-1-\alpha_L}\alpha_L(\alpha_L-1) = k_H\underline{p}^{-1-\alpha_H}(1-\underline{p})^{\alpha_H-2}\alpha_H(\alpha_H-1)$$

This system of equations will give us a unique formula for V_H:

$$rV_H = (\mu_{LH}-\mu_{LL}) + \frac{\alpha_H(\alpha_L-1)(\Delta_H-\Delta_L)\underline{p}}{\alpha_H(\alpha_L-1)-(1-\underline{p})(\alpha_L-\alpha_H)}. \tag{45}$$

Here $\Delta_H = \mu_{HH} - \mu_{LH}$ and $\Delta_L = \mu_{HL} - \mu_{LL}$. Furthermore, it is easy to check that both k_H and k_L are strictly larger than zero such that the option value of learning is strictly positive.

Therefore, we finally reach our main result:

Theorem 3 *Under strict supermodularity, the stationary competitive equilibrium is unique in the sense that all equilibria are PAM and the allocation is uniquely determined by Theorem 2. Moreover, assume monotonicity and normalize $V_L = 0$, we can get a unique formula for V_H given by equation (45).*

5 Firm-dependent Volatility: σ_y

A valid criticism of our approach is that we give the H firms too much of an edge under supermodularity (likewise for the L firms under submodularity). Not only are they superior in the production of output, by assuming that the volatility σ is common to both types of firms, effectively the signal-to-noise ratio is higher in H firms:

$$s_H = \frac{\mu_{HH}-\mu_{LH}}{\sigma} > \frac{\mu_{HL}-\mu_{LL}}{\sigma} = s_L,$$

from supermodularity. With firm-dependent volatility, that need not be the case. In particular, for σ_H sufficiently high, it may well be the case that $s_H < s_L$.

Mere observation of the value function in Equation (3), $rW_y(p) = \mu_y(p) - V_y + \Sigma_y(p)W_y''(p) - \delta W_y(p)$, reveals that firm-dependent volatility will play a crucial role here. Since $\Sigma_y = \frac{1}{2}p^2(1-p)^2 s_y^2$, for sufficiently high σ_H and therefore low s_H, it appears intuitive that the value W_H can be smaller than the value of W_L for high p. It turns out that this intuition is wrong. First, in this competitive equilibrium, wages are endogenous and therefore as the value of learning changes, so does $\mu_y(p)-V_y$. Second, the no-deviation condition requires that at the marginal type $\underline{p}$, $W_H'' = W_L''$. It turns out that as a result these two features, in equilibrium the learning effect is the same in both firms, no matter what the volatility σ_y is.

To make this argument formal, when $\sigma_H \neq \sigma_L$, we generally define $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma_y,\ y = H, L$. It is trivial to show that belief updating also satisfies the formula:

$$dp_t = p_t(1-p_t)s_y d\bar{Z}_{y,t}.$$

Furthermore, Lemmas 2–9 still hold because none of these results depend explicitly on σ_y. Then we only need to show that Claim 2 extends. Here we adopt a different approach to prove Claim 2.[20]

Proof. Suppose the situation described by the claim is the case. Then the value-matching conditions imply:

$$\begin{aligned} w_H(p_1) + \Sigma_H(p_1)W''_H(p_1) &= w_L(p_1) + \Sigma_L(p_1)W''_{L1}(p_1) \\ &\text{and} \\ w_H(p_2) + \Sigma_H(p_2)W''_H(p_2) &= w_L(p_2) + \Sigma_L(p_2)W''_{L2}(p_2). \end{aligned}$$

The no-deviation conditions imply:

$$\begin{aligned} \frac{s_H^2 - s_L^2}{s_H^2}(r+\delta)W_H(p_1) &= w_L(p_1) - \frac{s_L^2}{s_H^2}w_H(p_1) \\ &\text{and} \\ \frac{s_H^2 - s_L^2}{s_H^2}(r+\delta)W_H(p_2) &= w_L(p_2) - \frac{s_L^2}{s_H^2}w_H(p_2). \end{aligned}$$

It follows then that:

$$\frac{s_H^2 - s_L^2}{s_H^2}(r+\delta)[W_H(p_2) - W_H(p_1)] = [w_L(p_2) - w_L(p_1)] - \frac{s_L^2}{s_H^2}[w_H(p_2) - w_H(p_1)].$$

W_H is convex and $W'_H(p_1) > \frac{\Delta_L}{r+\delta}$ by Lemma 4. Therefore, we have:

$$\frac{s_H^2 - s_L^2}{s_H^2}\Delta_L(p_2 - p_1) < \Delta_L(p_2 - p_1) - \frac{s_L^2}{s_H^2}\Delta_H(p_2 - p_1),$$

which implies: $\Delta_H < \Delta_L$. Contradiction! ■

With the proof of Claim 2 in hand, the result of Theorem 1 immediately extends: PAM (NAM) is the unique stationary competitive equilibrium allocation under strict supermodularity (submodularity) thus holds for *any* combination of (σ_H, σ_L). Surprisingly, this implies that under strict supermodularity, even if we have an extremely high σ_H such that the learning rate in high type

[20]It is a little bit tricky to prove the sufficiency of the no-deviation condition though because we have to consider both $s_H \geq s_L$ and $s_H < s_L$ cases. In the appendix, we show a proof for a generalized version of Lemma 6.

firms is smaller than that in low type firms, we still have PAM. It is equivalent to assert that the direct productivity consideration dominates the learning in our model. The reason comes from the fact that the equilibrium wage schedules adjust to offset the impact of change in learning rate. The key insight here is the no-deviation condition. At $\underline{p}$, the no-deviation condition requires that the second-order effect on the value function is the same in both firms. This second-order effect W_y'' exactly captures the effect of learning through $\Sigma_y(\underline{p})W_y''(\underline{p})$ where $\Sigma_y = \frac{1}{2}p^2(1-p)^2 s_y^2$. Because equilibrium wages adjust to satisfy the no-deviation condition, the impact of differential learning rates is completely offset by the change of wage schedule, and the equilibrium allocation is solely determined by the productivity consideration.

6 The Planner's Problem

A priori, we might expect the competitive equilibrium not to decentralize the planner's problem. Wage contracts cannot condition on future realizations or actions and are assumed to be self-enforcing. As a result of this lack of commitment, there is a missing market. With incomplete markets, the competitive equilibrium in general does not necessarily decentralize the planner's problem. It turns out however as we show below that this market incompleteness does not preclude the efficiency of the decentralized equilibrium. As will become apparent, this efficiency result is driven by the martingale property present in all models of learning.

We consider a planner's problem under stationarity, i.e., in the presence of an ergodic distribution. The planner chooses an allocation rule and as a consequence of the Kolmogorov forward equation, the ergodic distribution associated with this allocation rule. The objective is to maximize the aggregate flow of output. Given stationarity of the problem, the focus on output maximization yields the same outcome as maximization of aggregate values.

Before we state and prove the efficiency result, we need to derive the stationary distribution under multiple cutoffs. Consider any allocation with multiple cutoffs:

$$0 < \underline{p}_N < \cdots < \underline{p}_1 < 1, \quad N \text{ odd}.$$

Without loss of generality, we assume workers with $p \in (p_1, 1]$ are allocated to the high type firms while workers with $p \in [0, p_N)$ are allocated to the low type firms since for workers with $p = 0$ or 1, there is no need for learning and it is optimal to allocate them according to instantaneous production efficiency (PAM).[21] This also implies that generically N is odd. Denote by Ω_y the set of p's that match with firms of type y.

[21] This property is also established in the one-sided model of Anderson and Smith (2009).

Now the expected belief in high type firms $\mathbb{E}_{\Omega_H} p$ can be written as

$$\int_{\Omega_H} p f_H(p) dp = \sum_i (-1)^i \Upsilon_i^H \quad \text{where} \quad \Upsilon_i^H = \delta^{-1} \left[p \frac{d}{dp} [\Sigma_H(p) f_H(p)] - \Sigma_H(p) f_H(p) \right]_{p \to \underline{p}_i} + p_0.$$

And similarly, the expected posterior belief in low type firms $\mathbb{E}_{\Omega_L} p$ is given by

$$\int_{\Omega_L} p f_L(p) dp = \sum_i (-1)^{i-1} \Upsilon_i^L \quad \text{where} \quad \Upsilon_i^L = \delta^{-1} \left[p \frac{d}{dp} [\Sigma_L(p) f_L(p)] - \Sigma_L(p) f_L(p) \right]_{p \to \underline{p}_i}.$$

The martingale property implies $\mathbb{E}_{\Omega_H} p + \mathbb{E}_{\Omega_L} p = p_0$ or

$$\sum_i (-1)^i \left\{ \left[p \frac{d}{dp} [\Sigma_H(p) f_H(p)] - \Sigma_H(p) f_H(p) \right]_{p \to p_i} - \left[p \frac{d}{dp} [\Sigma_L(p) f_L(p)] - \Sigma_L(p) f_L(p) \right]_{p \to p_i} \right\} = 0.$$

The planner's problem must, as in the case of one cutoff, satisfy the Kolmogorov forward equation, market clearing and the martingale property. We do not know of a known derivation from these constraints of the boundary conditions and flow equations in the presence of multiple cutoffs. Our strategy of proof is therefore to use a variational argument.

The proof heavily uses the martingale property and works as follows. First we consider a candidate allocation with 3 cutoffs and consider an interior interval of p's that are matched to L type firms. We move the bounds of that interval slightly to the left, thus generating a new density in this interval while keeping all other cutoffs and distributions unchanged. We impose market clearing conditions by choosing the new interval. We use the martingale property to show that under supermodularity this experiment lowers aggregate output. This holds until cutoffs coincide such that the interior rang of p's matched with L firms disappears, thus reducing the number of cutoffs to $N = 1$. Observe that while we use market clearing and the martingale property, we do not use the boundary conditions. We establish that output increases in the unconstrained problem (without boundary conditions), therefore it will also increase in the constrained problem. We use a similar argument to establish that output increases when moving from N to $N - 2$ cutoffs. The result then follows by induction. We derive the result under supermodularity. The same logic applies under submodularity.

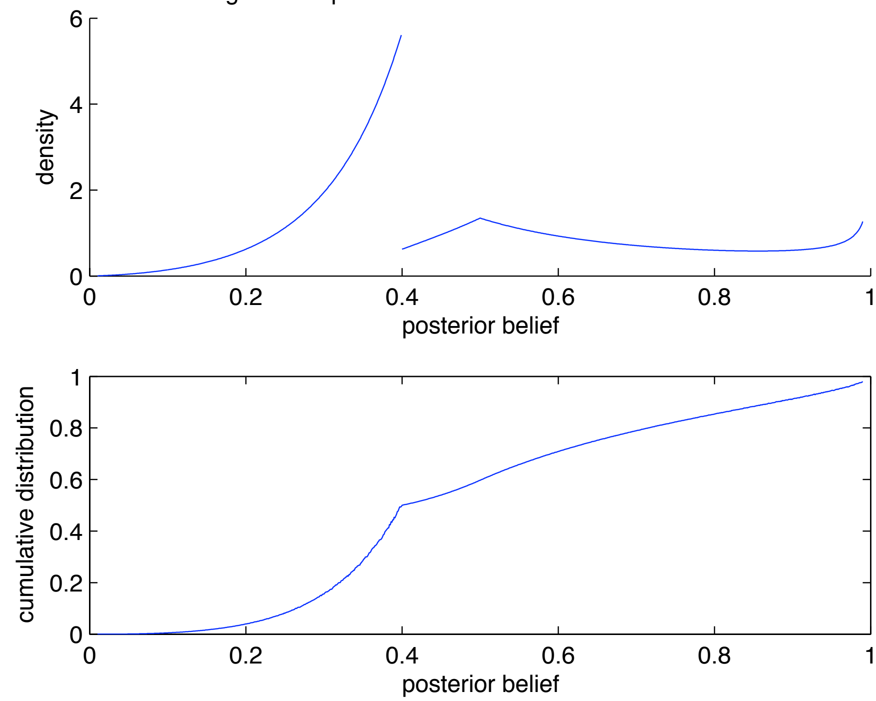
Theorem 4 *The competitive equilibrium decentralizes the planner's stationary solution that maximizes the aggregate flow of output.*

Proof. In Appendix. ■

7 Predictions of the Model

We now turn to some of the predictions of the model.

Figure 2: Equilibrium Distribution of Posterior Beliefs



Wage Gap at $\underline{p}$

We start with an interesting observation:

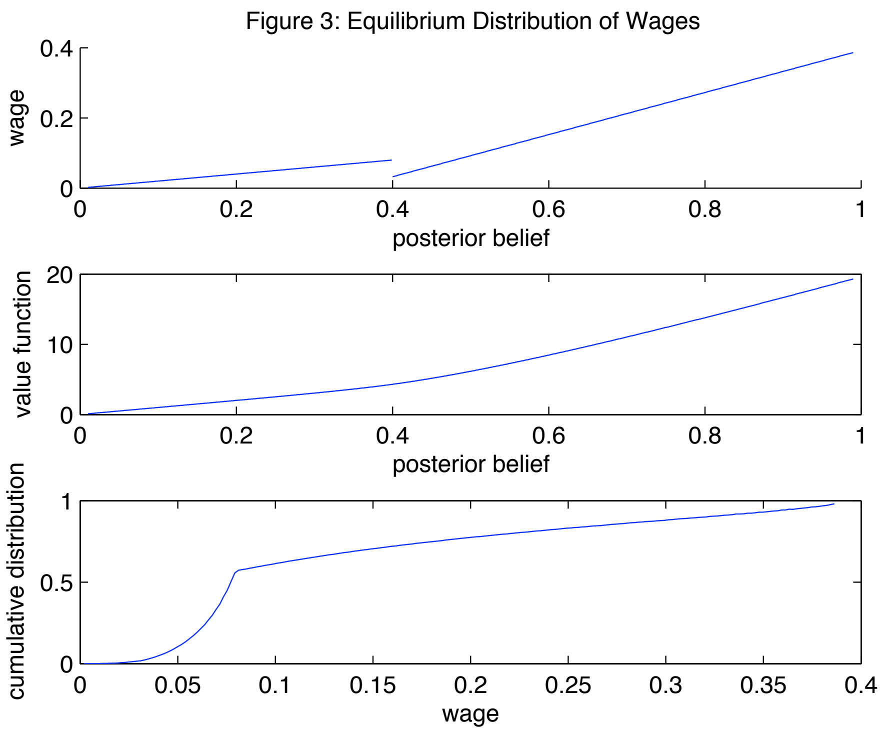
$$\begin{aligned} w_H(\underline{p}) = \mu_H(\underline{p}) - rV_H &= \Delta_H \underline{p} + \mu_{LL} - \frac{\alpha_H(\alpha_L - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H(\alpha_L - 1) - (1 - \underline{p})(\alpha_L - \alpha_H)} \\ &< \Delta_L \underline{p} + \mu_{LL} = \mu_L(\underline{p}). \end{aligned}$$

This implies that the worker with posterior belief slightly higher than $\underline{p}$ will accept high type firm's offer even though the wage provided by the high type firm is lower than the productivity at the low type firm. This obviously comes from the fact that the learning speed in the high type firm is higher and this would compensate the loss in the flow wages.

On the other hand, we can see that the difference in expected productivity at $\underline{p}$ is

$$\mu_H(\underline{p}) - \mu_L(\underline{p}) = (\mu_{HL} - \mu_{LL}) + (\Delta_H - \Delta_L)\underline{p} < rV_H.$$

This implies the high type firm can enjoy a strictly positive rent from a higher learning speed. Because $V_L = 0$, it immediately follows that the wage at $\underline{p}$ in H is lower than in L. This above result does not depend on the assumption $V_L = 0$ and it could be generalized for any possible division of surplus:



Lemma 7 *Under strict supermodularity, for all stationary competitive equilibrium, we have:* $w_H(\underline{p}) < w_L(\underline{p})$ *and* $rV_H - rV_L > \mu_H(\underline{p}) - \mu_L(\underline{p})$.

Ergodic Distribution of Posterior Beliefs

In a steady state, equilibrium sorting transforms the normal output X_t into a piecewise Lévy distribution of posterior beliefs p.

Observe the discontinuity in the density (and the subsequent kink in the cumulative) at $\underline{p}$. This is unlike the learning environments with an absorbing state where the density is continuous. Recall that this is the result of equation (20), which ensures that the flows around $\underline{p}$ are balanced. Given learning is faster in the H firms, the flow from H to L is larger than from L to H, and therefore the measure of types must be smaller to the right of $\underline{p}$. At p_0 the density is continuous but not differentiable, which is due to the fact that there is a measure δ of new entrants in each period.

Note also that in this example, the density is increasing in the neighborhood of $p = 1$. Whether or not the density is increasing depends on the relation between the parameter values δ, s_y and the equilibrium value of p_0, as is shown in Moscarini (2005, Proposition 4).

Wages

The piece-wise Lévy distribution of beliefs maps one-to-one into a wage distribution because of competitive wage determination and the law of one price: for every p there is exactly one wage $w(p)$. As a result, the wage distribution is also piece-wise Lévy. Figure 3 plots $w(p)$ and the density and cumulative distributions.

First, Figure 3A illustrates the wage gap we established earlier, i.e., here is a discontinuity in the wage function $w(p)$ at $\underline{p}$. This follows immediately from value matching $W_L(\underline{P}) = W_H(\underline{p})$ and the fact that learning is faster in H firms. As a result, it must be the case that $w_H(\underline{p}) < w_L(\underline{p})$. Of course, the wage schedule is in H type firms is steeper, $w'_H(p) > w'_L(p)$ due to faster learning.

The wage schedule transforms the belief distribution into the wage distribution. The discontinuity in the wage schedule at $\underline{p}$ happens to coincide with the discontinuity in belief distribution. Not surprisingly, as a result there is exactly one discontinuity in the density of w at $\underline{p}$.

Variance of Beliefs and Wages over the Life Cycle

We investigate the evolution of the posterior belief distribution. To that end, we focus on a group of workers who enter into the market at $t = 0$ and have not died at least at time T. This ensures that we calculate the variance of a surviving cohort without the effect on the variance of the dying workers. The selection of a decreasing population would underestimate the variance. Denote the density function for the posterior beliefs at time $t \leq T$ to be $f_y^T(p,t)$. First notice that from the Martingale property, for any $t \leq T$, the expectation of posterior belies at time $t \leq T$ should stay the same:

$$E(t) = \int_0^{\underline{p}} p f_L^T(p,t)dp + \int_{\underline{p}}^1 p f_H^T(p,t)dp = p_0.$$

Or equivalently, using integration by parts:

$$\begin{aligned}
\frac{dE(t)}{dt} &= \int_0^{\underline{p}} p\frac{d^2}{dp^2}[\Sigma_L(p)f_L^T(p,t)]dp + \int_{\underline{p}}^1 p\frac{d^2}{dp^2}[\Sigma_H(p)f_H^T(p,t)]dp \\
&= \underline{p}\{\frac{d}{dp}[\Sigma_L(p)f_L^T(p,t)]|_{p=\underline{p}}\} - \Sigma_L(\underline{p})f_L^T(\underline{p},t) \\
&\quad -\underline{p}\{\frac{d}{dp}[\Sigma_H(p)f_H^T(p,t)]|_{p=\underline{p}}\} + \Sigma_H(\underline{p})f_H^T(\underline{p},t) = 0.
\end{aligned} \tag{46}$$

Our interest is with the variance of this distribution, which can be written as:

$$Var(t) = \int_0^{\underline{p}} p^2 f_L^T(p,t)dp + \int_{\underline{p}}^1 p^2 f_H^T(p,t)dp - p_0^2.$$

Beginning with any initial distribution, the variance can of course decrease. However, as time goes by, eventually the variance must start to increase. This is established in the following result.

Proposition 3 *The variance of the distribution of beliefs will eventually increase.*

Proof. The Fokker-Planck (Kolmogorov forward) equation implies:

$$\frac{df_y(p)}{dt} = \frac{d^2}{dp^2}[\Sigma_y(p)f_y(p)]. \tag{47}$$

Then integration by parts by parts and using the Fokker-Planck equation yields:

$$\begin{aligned}
\frac{dVar(t)}{dt} &= \int_0^{\underline{p}} p^2 \frac{d^2}{dp^2}[\Sigma_L(p)f_L^T(p,t)]dp + \int_{\underline{p}}^1 p^2 \frac{d^2}{dp^2}[\Sigma_H(p)f_H^T(p,t)]dp \\
&= \underline{p}^2\{\frac{d}{dp}[\Sigma_L(p)f_L^T(p,t)]|_{p=\underline{p}}\} - 2\underline{p}\Sigma_L(\underline{p})f_L^T(\underline{p},t) \\
&\quad -\underline{p}^2\{\frac{d}{dp}[\Sigma_H(p)f_H^T(p,t)]|_{p=\underline{p}}\} + 2\underline{p}\Sigma_H(\underline{p})f_H^T(\underline{p},t) \\
&\quad + \int_0^{\underline{p}} 2\Sigma_L(p)f_L^T(p,t)dp + \int_{\underline{p}}^1 2\Sigma_H(p)f_H^T(p,t)dp.
\end{aligned} \tag{48}$$

From Equation (46), Equation (48) could be further simplified as:

$$\begin{aligned}
\frac{dVar(t)}{dt} &= \underline{p}\Sigma_H(\underline{p})f_H^T(\underline{p},t) - \underline{p}\Sigma_L(\underline{p})f_L^T(\underline{p},t) \\
&\quad + \int_0^{\underline{p}} 2\Sigma_L(p)f_L^T(p,t)dp + \int_{\underline{p}}^1 2\Sigma_H(p)f_H^T(p,t)dp.
\end{aligned} \tag{49}$$

The martingale convergence theorem implies that as t goes to T and as T goes to infinity, we have both $f_H^T(\underline{p},t)$ and $f_L^T(\underline{p},t)$ go to zero for $\underline{p} \in (0,1)$. Meanwhile,

$$\int_0^{\underline{p}} 2\Sigma_L(p)f_L^T(p,t)dp + \int_{\underline{p}}^1 2\Sigma_H(p)f_H^T(p,t)dp \geq 0.$$

Hence, we have $\lim_{T\to\infty} \frac{dVar(T)}{dT} \geq 0$. ■

One feature of the standard learning model is that over he life cycle, the variance of wages decreases. As time goes by, workers are increasingly likely to have found a high productivity match, and eventually the posterior belief converges to one. As a result, the variance decreases and goes to zero. Yet, there is ample evidence that the variance of wages over the life cycle increases and is concave. For some of the most recent evidence, see Heathcoate, Violante and Perri (2009). This is not captured by the standard learning model. As we have shown in Proposition 4, the variance of posterior beliefs must eventually increase. Proposition 4 below establishes that also the distribution of wages will eventually increase.

This is not immediate from Proposition 4 since the wage function, while piece-wise linear, is discontinuous with changing slopes. To that end, we establish in the following Lemma that as t

goes to T and as T goes to infinity, we have both $f_H^T(\underline{p}, t)$ and $f_L^T(\underline{p}, t)$ go to zero for $\underline{p} \in (0, 1)$. At the same time, $\frac{\partial f_y^T(p,t)}{\partial t}$ also goes to zero, which implies that $\frac{d^2}{dp^2}[\Sigma_y(p) f_y^T(p, t)]$ will converge to zero too.

Lemma 8 *As $t \to T$ and as $T \to \infty$, both $f_H^{T\prime}(\underline{p}, t)$ and $f_L^{T\prime}(\underline{p}, t) \to 0$.*

Proof. In Appendix. ■

Proposition 4 *The variance of the distribution of wages will eventually increase.*

Proof. From the above Lemma, as t goes to T and as T goes to infinity, we have:

$$\begin{aligned} \frac{dVar_w(t)}{dt} &= \int_0^{\underline{p}} w_L(p)^2 \frac{d^2}{dp^2}[\Sigma_L(p) f_L^T(p, t)]dp + \int_{\underline{p}}^1 w_H(p)^2 \frac{d^2}{dp^2}[\Sigma_H(p) f_H^T(p, t)]dp \\ &\quad -2Ew_t\{\int_0^{\underline{p}} w_L(p) \frac{d^2}{dp^2}[\Sigma_L(p) f_L^T(p, t)]dp + \int_{\underline{p}}^1 w_H(p) \frac{d^2}{dp^2}[\Sigma_H(p) f_H^T(p, t)]dp\} \end{aligned}$$

will converge to

$$\int_0^{\underline{p}} 2\Delta_L^2 \Sigma_L(p) f_L^T(p, t)dp + \int_{\underline{p}}^1 2\Delta_H^2 \Sigma_H(p) f_H^T(p, t)dp \geq 0.$$

■

8 On-the-job Human Capital Accumulation

On the job, workers and firms not only learn about their unknown innate skills, they also accumulate human capital. In reality, human capital accumulation is an ongoing, continuous process. The longer the tenure of a worker, the higher her productivity. This monotonically increasing relation between tenure and human capital experience is likely also to be concave. For modeling purposes, here we consider a very simple form that captures this relation. With probability λ, a worker becomes transitions from being unexperienced to being experienced.[22] Once a worker is experienced, her productivity increases to $\mu_{xy} + \xi_x$ and the status of experience is complete information.[23] Now there are the same value functions for experienced workers as before W_y^e.

$$rW_y^e(p) = \mu_y(p) + \xi(p) - rV_y + \Sigma_y^e(p) W_y^{e\prime\prime}(p) - \delta W_y^e(p)$$

[22]Having a continuous relation between tenure and human capital renders the system of differential equations into a system of partial differential equations. Typically there is no solution. In the current setup, there is an additional state (experienced versus unexperienced) and the model remains tractable.

[23]Observe that experience is worker dependent, but not firm dependent. While it is likely a realistic feature to have experience dependent on the job type, the reason is that we would have a different level of experience for different histories which makes the problem non-tractible.

where $\xi(p) = p\xi_H + (1-p)\xi_L$ is the expected experience.[24] For the unexperienced worker there is now one additional value function. As before, there are unexperienced workers who are matched with L firms, and who continue to match with an L firms; and there are those who match with H firms both when unexperienced as well as when experienced. We denote those values by W^u_{LL}, W^u_{HH}. There are now also some types p who match with an L firms when unexperienced and who switch to an H firm when they become experienced, the value of which is denoted by W^u_{LH}. This requires that the reservation type of an experienced worker ($\underline{p}^e$) is lower than that of the unexperienced worker ($\underline{p}^u$). We start from this premise and later verify that this is indeed the case. The value functions then are:

$$\begin{aligned}
rW^u_{yy}(p) &= \mu_y(p) - rV_y + \Sigma^u_y(p)W^{u\prime\prime}_{yy}(p) + \lambda W^e_y(p) - (\delta+\lambda)W^u_{yy}(p) \\
rW^u_{LH}(p) &= \mu_L(p) - rV_L + \Sigma^u_L(p)W^{u\prime\prime}_{LH}(p) + \lambda W^e_H(p) - (\delta+\lambda)W^u_{LH}(p)
\end{aligned}$$

Observe that even though experience is completely observable, it does affect the inference from learning in the sense that the signal-to-noise ratio changes to $(\mu_{Hy} + \xi_H - \mu_{Ly} - \xi_L)$. As a result, Σ_y depends on experience u, e.

$$\begin{aligned}
W^u_{yy}(p) &= \frac{\mu_y(p) - rV_y}{r+\delta+\lambda} + k^u_{y1}p^{1-\alpha^u_y}(1-p)^{\alpha^u_y} + k^u_{y2}p^{\alpha^u_y}(1-p)^{1-\alpha^u_y} \\
&+ \frac{\lambda}{(r+\delta)(r+\delta+\lambda)}[\mu_y(p) + \xi(p) - rV_y] \\
&+ \frac{\lambda}{(\lambda+\delta+r) - \frac{(s^u_y)^2}{(s^e_y)^2}(r+\delta)}[k^e_{y1}p^{1-\alpha^e_y}(1-p)^{\alpha^e_y} + k^e_{y2}p^{\alpha^e_y}(1-p)^{1-\alpha^e_y}] \\
W^u_{LH}(p) &= \frac{\mu_L(p) - rV_L}{r+\delta+\lambda} + k^u_{L1}p^{1-\alpha^u_L}(1-p)^{\alpha^u_L} + k^u_{L2}p^{\alpha^u_L}(1-p)^{1-\alpha^u_L} \\
&+ \frac{\lambda}{(r+\delta)(r+\delta+\lambda)}[\mu_H(p) + \xi(p) - V_H] \\
&+ \frac{\lambda}{(\lambda+\delta+r) - \frac{(s^u_L)^2}{(s^e_H)^2}(r+\delta)}[k^e_{H1}p^{1-\alpha^e_H}(1-p)^{\alpha^e_H} + k^e_{H2}p^{\alpha^e_H}(1-p)^{1-\alpha^e_H}] \\
W^e_y(p) &= \frac{\mu_y(p) + \xi(p) - rV_y}{r+\delta} + k^e_{y1}p^{1-\alpha^e_y}(1-p)^{\alpha^e_y} + k^e_{y2}p^{\alpha^e_y}(1-p)^{1-\alpha^e_y}
\end{aligned}$$

where

$$\alpha^u_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta+\lambda)}{(s^u_y)^2}} \geq 1$$

$$\alpha^e_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{(s^e_y)^2}} \geq 1$$

[24]In this section we maintain the earlier assumption that $\sigma_H = \sigma_L = \sigma$.

There are now two cut-offs $\underline{p}^u, \underline{p}^e$. Since we just want to compare $\underline{p}^u$ and $\underline{p}^e$, we can adopt the following thought experiment. First, we assume that $\underline{p}^u = \underline{p}^e = \underline{p}$. Then we can get two systems of equations: one system is the value matching, smooth pasting and no-deviation conditions for the unexperienced workers and the other one is for the experienced workers. Second, we can solve $\Delta V = V_H - V_L$ as what we did previously but now we can get two possible values for ΔV. Denote them to be ΔV^e and ΔV^u. Notice that ΔV^e and ΔV^u are both increasing in the cutoff $\underline{p}$. Finally, we compare ΔV^e and ΔV^u under the assumption that $\underline{p}^u = \underline{p}^e = \underline{p}$. If $\Delta V^e > \Delta V^u$, this means that we should decrease $\underline{p}^e$ or increase $\underline{p}^u$ and hence $\underline{p}^u > \underline{p}^e$; on the contrary, if $\Delta V^e < \Delta V^u$, this means that we should decrease $\underline{p}^u$ or increase $\underline{p}^e$ and hence $\underline{p}^u < \underline{p}^e$. We derive this in the Appendix and can show this to hold when HC accumulation is not too different for H and L types.

Proposition 5 *Assume supermodularity and $\xi_H \simeq \xi_L$. Then $\underline{p}^e < \underline{p}^u$.*

Proof. In Appendix. ■

With human capital accumulation, we can now characterize the entire equilibrium, including wage schedules and the ergodic distribution of types. Even though there are types who gradually learn they are of low productivity, wages need not decrease over the life cycle as they accumulate human capital.

Turnover and Tenure. We express the expected future duration of a match by tenure $\tau_y(p)$. Tenure relates inversely to turnover. $\tau_y(p)$ satisfies the following differential equation (see also Moscarini 2005):

$$\Sigma_y(p)\tau_y''(p) - \delta p = -1,$$

with solutions:

$$\tau_H^u(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^u}\right)^{1/2-\sqrt{1/4+2\delta/(s_H^u)^2}}\left(\frac{1-p}{1-\underline{p}^u}\right)^{1/2-\sqrt{1/4-2\delta/(s_H^u)^2}}\right\};$$

$$\tau_L^u(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^u}\right)^{1/2-\sqrt{1/4-2\delta/(s_L^u)^2}}\left(\frac{1-p}{1-\underline{p}^u}\right)^{1/2-\sqrt{1/4+2\delta/(s_L^u)^2}}\right\};$$

$$\tau_H^e(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^e}\right)^{1/2-\sqrt{1/4+2\delta/(s_H^e)^2}}\left(\frac{1-p}{1-\underline{p}^e}\right)^{1/2-\sqrt{1/4-2\delta/(s_H^e)^2}}\right\};$$

$$\tau_L^e(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^e}\right)^{1/2-\sqrt{1/4-2\delta/(s_L^e)^2}}\left(\frac{1-p}{1-\underline{p}^e}\right)^{1/2-\sqrt{1/4+2\delta/(s_L^e)^2}}\right\}.$$

An immediate implication of the Proposition above is the following:

Proposition 6 *(Tenure) Assume supermodularity and $\xi_H \simeq \xi_L$. Then, $\tau_L^u(p) > \tau_L^e(p)$ for $p < \underline{p}^e$ and $\tau_H^u(p) < \tau_H^e(p)$ for $p > \underline{p}^u$. For $p \in (\underline{p}^e, \underline{p}^u)$, there is a cutoff such that $\tau_L^u(p) < \tau_H^e(p)$ for p higher than this cutoff and $\tau_L^u(p) > \tau_H^e(p)$ for p smaller than this cutoff.*

For the lowest types p, tenure for the unexperienced worker is longer as the experienced workers are more likely to be hired by an H firm given positive information revelation. The opposite is true for the highest p: the unexperienced types face a higher cut-off type and will therefore upon bad information be more likely to switch to an L firm. In the intermediate range, tenure depends on how close p is to either of the cut-offs.

Wages. In our benchmark model without HC accumulation and despite learning, expected wages of a worker are constant, no matter the tenure. This is due to the martingale assumption, i.e., that the expected posterior belief is equal to the prior p. In conjunction with the fact that wages are linear in beliefs, i.e., $w_y(p) = \mu_y(p) - rV_y$, also average wages in the economy are constant irrespective of the distribution. At any point in time, the average belief is p_0 and given linearity, the average wage is $w(p_0)$.

This is not the case when there is HC accumulation. Now the both the expected wage of a given worker and the average wage of a cohort are increasing over time. Since $w_y^e(p) > w_y^u(p)$ and for an individual worker there is a transition rate from unexperienced into experienced of λ, over time expected wages increase.

9 Robustness

9.1 Generalized Lévy Processes

One may suspect that our results are exclusively driven by the specific assumptions of the Brownian motion. In the section, we illustrate that this is not the case by considering a generalized Lévy process, i.e., a compound Poisson process. Let λ_{xy} denote the expected arrival rate of jumps for a type x worker in a type y firm. Following Cohen and Solan (2009), the worker's value function can be written as:

$$W_y(p) = w_y(p)dt + (1 - rdt - \delta dt)\{[p\lambda_{Hy} + (1-p)\lambda_{Ly}]dtW_{y'}(p_h) + (1 - [p\lambda_{Hy} + (1-p)\lambda_{Ly}]dt)W_y(p + dp)$$

where $p_h = \frac{p\lambda_{Hy}}{p\lambda_{Hy}+(1-p)\lambda_{Ly}}$ and y' is the firm type which matches with worker p_h. If no jump occurs, the updating of the posterior belief in firm y follows:

$$dp = -p(1-p)(\lambda_{Hy} - \lambda_{Ly})dt + p(1-p)s_y d\bar{Z}.$$

Then the value function could be further simplified as:

$$(r+\delta+[p\lambda_{Hy}+(1-p)\lambda_{Ly}])W_y(p) = w_y(p)+[p\lambda_{Hy}+(1-p)\lambda_{Ly}]W_{y'}(p_h)-p(1-p)(\lambda_{Hy}-\lambda_{Ly})W_y'(p)+\Sigma_y(p)W_y''(p).$$

The no-deviation condition derived earlier still holds in this situation. The proof is similar and is omitted here.

Lemma 9 *To deter possible deviations, a necessary condition is:*

$$W_H''(\underline{p}) = W_L''(\underline{p}) \quad \textit{(No-deviation condition)} \tag{50}$$

for any possible cutoff $\underline{p}$.

In order to verify whether PAM is indeed an equilibrium, we need to solve this differential equation. To be able to do so, consider the simplifying assumption that $\lambda_{Ly} = 0$ and denote λ_{Hy} by λ_y. Then p_h is always 1 and we have:

$$(r+\delta+p\lambda_y)W_y(p) = w_y(p) + p\lambda_y W_H(1) - p(1-p)\lambda_y W_y'(p) + \Sigma_y(p)W_y''(p).$$

This is a differential equation that we can actually solve explicitly. By guess and verify, we derive that:

$$W_y(p) = A_y + B_y p + k_{y1}p^{\alpha_1}(1-p)^{1-\alpha_1} + k_{y2}p^{\alpha_2}(1-p)^{1-\alpha_2}$$

where $A_y = \frac{\mu_{Ly}-rV_y}{r+\delta}$, $B_y = \frac{\Delta_y+\lambda_y(W_H(1)-A_y)}{r+\delta+\lambda_y}$ and

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}+\frac{\lambda_y}{s_y^2}+\sqrt{(\frac{1}{2}+\frac{\lambda_y}{s_y^2})^2+\frac{2(r+\delta)}{s_y^2}} > 1+2\frac{\lambda_y}{s_y^2} \\
\alpha_2 &= \frac{1}{2}+\frac{\lambda_y}{s_y^2}-\sqrt{(\frac{1}{2}+\frac{\lambda_y}{s_y^2})^2+\frac{2(r+\delta)}{s_y^2}} < 0.
\end{aligned}$$

Suppose there is PAM, then from the value function we know that $k_{L1} > 0$, $k_{L2} = 0$ and $k_{H1} = 0$, $k_{H2} > 0$. Furthermore, we have:

$$\begin{aligned}
W_L'(0) &= B_L = \frac{\Delta_L + \lambda_L(W_H(1)-A_L)}{r+\delta+\lambda_L} \\
W_H'(1) &= B_H = \frac{\Delta_H + \lambda_H(W_H(1)-A_H)}{r+\delta+\lambda_H}.
\end{aligned}$$

Convexity requires that $B_H > B_L$. We can rewrite the value functions as:

$$\begin{aligned}
W_L(p) &= A_L + B_L p + k_L p^{\alpha_L}(1-p)^{1-\alpha_L} \\
W_H(p) &= A_H + B_H p + k_H p^{1-\alpha_H}(1-p)^{\alpha_H}
\end{aligned}$$

where

$$\alpha_H = \frac{1}{2} + \frac{\lambda_H}{s_H^2} + \sqrt{(\frac{1}{2} + \frac{\lambda_H}{s_H^2})^2 + \frac{2(r+\delta)}{s_H^2}} > 1 + 2\frac{\lambda_H}{s_H^2}$$

$$\alpha_L = \frac{1}{2} + \frac{\lambda_L}{s_L^2} + \sqrt{(\frac{1}{2} + \frac{\lambda_L}{s_L^2})^2 + \frac{2(r+\delta)}{s_L^2}} > 1 + 2\frac{\lambda_L}{s_L^2}.$$

Under PAM, at cutoff $\underline{p}$, we must have:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Equal-value condition)} \tag{51}$$

$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting condition)} \tag{52}$$

$$W_H''(\underline{p}) = W_L''(\underline{p}) \qquad \text{(No-deviation condition)} \tag{53}$$

This immediately implies:

$$A_H - A_L + \frac{\alpha_H(\alpha_L - 1)(B_H - B_L)\underline{p}}{\alpha_H(\alpha_L - 1) - (1 - \underline{p})(\alpha_L - \alpha_H)} = 0.$$

Without loss of generality, we assume as before that $\mu_{LL} = rV_L = 0$ and we only have to solve rV_H. Furthermore, total differentiation implies that rV_H is increasing in $\underline{p}$. This enables us to discuss the existence of PAM equilibrium without knowing the value of $\underline{p}$ since monotonicity implies that rV_H must be bounded by the values it takes for $\underline{p}$ at 0 and 1: $rV_H \in [\mu_{LH}, \mu_{HH} - \Delta_L]$.

We can then establish the following result:

Proposition 7 *Given the Lévy process, PAM is a stationary competitive equilibrium allocation under strict supermodularity.*

The proof is immediate if one notes that $B_H > B_L$ for all $\underline{p} \in [0, 1]$, which follows from the fact that $rV_H \in [\mu_{LH}, \mu_{HH} - \Delta_L]$. We need to pin down $\underline{p}$ from the distribution, but no matter $\underline{p}$, we can always find rV_H and $k_L > 0$, $k_H > 0$ such that PAM is an equilibrium.

Notice also that with the Lévy process, beliefs are formed through Bayesian updating. We conjecture that PAM will always be the competitive equilibrium allocation under strict supermodularity for any stochastic process as long as there is Bayesian updating. This is because under Bayesian learning, the belief updating process is always a martingale. Of course, establishing this result for general information processes is impossible because it requires the explicit solution of the differential equations for the value function, which generally does not exist.

9.2 Non-Bayesian Updating

Suppose instead that the belief updating is not a martingale. Then it must be generated by some non-Bayesian learning process. We will now show for an example that the competitive equilibrium can be non-PAM even if there is supermodularity.

Suppose the belief updating process in firm y is given by: $dp = \lambda_y p dt$ for $p < 1$, with λ_y a constant, and once p reaches 1, $dp = 0$. We may think p as kind of human capital. The accumulation of human capital will stop once p reaches 1. The value function of a worker is given by:

$$(r+\delta)W_y(p) = w_y(p) + \lambda_y p W_y'(p)$$

with solution:

$$W_y(p) = C_y p^{\frac{r+\delta}{\lambda_y}} + \frac{\Delta_y}{r+\delta-\lambda_y}p + \frac{\mu_{Ly} - rV_y}{r+\delta}.$$

Suppose there is PAM, then

$$\lim_{p\to 1} W_H(p) = W_H(1) = \frac{\Delta_H}{r+\delta}p + \frac{\mu_{LH} - rV_H}{r+\delta},$$

which implies that:

$$C_H = -\frac{\lambda_H \Delta_H}{(r+\delta)(r+\delta-\lambda_H)}.$$

At the cutoff $\underline{p}$ we have:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Equal-value condition)} \tag{54}$$

$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting condition)} \tag{55}$$

It is easy to get rid of C_L and find an equation for $\underline{p}$:

$$\frac{\Delta_L}{r+\delta}\underline{p} + \frac{\mu_{LL} - rV_L}{r+\delta} = \frac{\lambda_L - \lambda_H}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}(\underline{p})^{\frac{r+\delta}{\lambda_H}} + (1 - \frac{\lambda_L}{r+\delta})\frac{\Delta_H}{r+\delta-\lambda_H}\underline{p} + \frac{\mu_{LH} - rV_H}{r+\delta}$$

or

$$\frac{\Delta_L - \Delta_H}{r+\delta}\underline{p} + \frac{\mu_{LL} - rV_L}{r+\delta} = \frac{\lambda_H - \lambda_L}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - (\underline{p})^{\frac{r+\delta}{\lambda_H}}] + \frac{\mu_{LH} - rV_H}{r+\delta}.$$

Notice that PAM requires that

$$\frac{\mu_{LL} - rV_L}{r+\delta} > \frac{\mu_{LH} - rV_H}{r+\delta}$$

and

$$\frac{\lambda_H - \lambda_L}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - (\underline{p})^{\frac{r+\delta}{\lambda_H}}] < 0$$

if $\lambda_L > \lambda_H$.

Furthermore, fix any $\underline{p} \in (0,1)$ and $\lambda_H > 0$, if we let $r+\delta$ go to zero and λ_L go to infinity, then it is immediate to see that

$$\frac{\lambda_H - \lambda_L}{r+\delta}\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - (\underline{p})^{\frac{r+\delta}{\lambda_H}}] \to -\infty.$$

If we make Δ_L sufficiently close to Δ_H, the equality cannot be held. This implies that PAM cannot be an equilibrium if $r + \delta$ goes to zero, λ_L is sufficiently large and Δ_L is sufficiently close to Δ_H.

10 Conclusion

In this paper, we have proposed a model of the labor market that unifies sorting and a learning-based theory of turnover. In equilibrium under supermodularity, workers with better posteriors about their ability tend to sort into more productive jobs, and over time, their posterior converges to that of the high type. As a result, turnover decreases over the life cycle. Even though wage variation conditional on type decreases, the inequality in the cross section of a cohort increases. This can explain the increase of observed wage variation in the data.

The main technical contribution of this paper is the role of sequential rationality in the presence of competitively determined payoffs. The one-shot deviation principle in conjunction with endogenous payoffs implies that the value function of the worker has a second derivative that is equal at the cut-off type. As a result, we now have a condition at the cutoff in addition to the standard value-matching (zero-th derivative), smooth-pasting (first derivative) which is one additional order higher: the no-deviation condition.

What is possibly most surprising is that the result of positive sorting under supermodularity is not determined by the speed of learning. In the trade-off between the learning speed and efficiency, efficiency always takes the upper hand. As such, the equilibrium allocation does not depend on the signal-to-noise ratio (the ratio of the average payoff gain, which measures the efficiency, over the noise term). This seems to indicate in this competitive environment the sorting aspect dominates the learning.

Appendix

Proof of Lemma 2

Proof. The worker $p \in (0,1)$ always has the choice that stays in one firm y forever. Then the value is $\frac{\mu_y(p)-rV_y}{r+\delta}$. But obviously, this is not an optimal choice (Suppose not, then all of the workers will stay in one type of firms and the market is not cleared). So we have that the equilibrium value function $W_y(p)$ must satisfy: $W_y(p) > \frac{\mu_y(p)-rV_y}{r+\delta}$. This immediately implies:

$$\Sigma_y(p){W_y}''(p) = (r+\delta)W_y(p) - (\mu^i(p) - rV^i) > 0.$$

So the equilibrium value functions W_y convex for $p \in (0,1)$. ■

Proof of Lemma 3

Proof. Suppose workers with $p \in [0, \underline{p})$ are employed by type y firm. This implies that $W_y(p) = \frac{\mu_y(p)-rV_y}{r+\delta} + k_{y2}p^{\alpha_y}(1-p)^{1-\alpha_y}$ since 0 is included in the domain. It is easy to see that ${W_y}'(0) = \frac{\mu_{Hy}-\mu_{Ly}}{r+\delta} > 0$ and since W_y is strictly convex, $W_y'(p) > 0$ for all $p \in [0, \underline{p})$. At $\underline{p}$, worker will transfer to type $-y$ firm but smooth pasting condition implies ${W'}_{-y}(\underline{p}) = W_y'(\underline{p}) > 0$. Strict convexity implies $W_{y'}'(p) > 0$ so on and so forth. Therefore, we must have the equilibrium value functions W_y are strictly increasing. ■

Proof of Claim 2

Proof. Under supermodularity, suppose the situation described by the claim is the case. Then we have:

$$w_H(p_1) + \Sigma_H(p_1)W_H''(p_1) = w_L(p_1) + \Sigma_L(p_1)W_{L1}''(p_1)$$

and

$$w_H(p_2) + \Sigma_H(p_2)W_H''(p_2) = w_L(p_2) + \Sigma_L(p_2)W_{L2}''(p_2)$$

since

$$W_H(p_2) = W_{L2}(p_2) \quad \text{and} \quad W_H(p_1) = W_{L1}(p_1).$$

Using the expression for w_H, w_L and the fact that

$$W_H''(p_2) = W_{L2}''(p_2) \quad \text{and} \quad W_H''(p_1) = W_{L1}''(p_1),$$

we can get:

$$(\Delta_H - \Delta_L)(p_2 - p_1) = \frac{s_H^2 - s_L^2}{s_H^2}[\Sigma_H(p_1)W_H''(p_1) - \Sigma_H(p_2)W_H''(p_2)].$$

Notice that $\frac{s_H^2 - s_L^2}{s_H^2} = \frac{(\Delta_H - \Delta_L)(\Delta_H + \Delta_L)}{(\Delta_H)^2}$. Then the above equation can be rewritten as:

$$\frac{(\Delta_H - \Delta_L)(\Delta_H + \Delta_L)}{(\Delta_H)^2}(r+\delta)[W_H(p_2) - W_H(p_1)] = \frac{\Delta_L}{\Delta_H}(\Delta_H - \Delta_L)(p_2 - p_1).$$

Meanwhile,

$$W_H(p_2) - W_H(p_1) > {W'}_H(p_1)(p_2 - p_1)$$

by strict convexity and

$$W'_H(p_1) > \frac{\Delta_L}{r+\delta}$$

from Lemma 5. These two inequalities imply:

$$(r+\delta)[W_H(p_2) - W_H(p_1)] > \Delta_L(p_2 - p_1)$$

under strict supermodularity. Hence, finally we get: $\frac{\Delta_H+\Delta_L}{\Delta_H} < 1$. Contradiction!

For the strict submodularity case, it suffices to relabel 'H' by 'L' and 'L' by 'H'. The claim is obviously correct given we have already proved the supermodularity result. ■

Derivation of the Boundary Conditions

Here, we just investigate the boundary conditions for case 1: $\underline{p} < p_0$. The derivation is similar for case 2.

From

$$\frac{\partial f_y(p,t)}{\partial t} = \frac{d^2}{dp^2}[\Sigma_y(p) f_y(p,t)] - \delta f_y(p,t),$$

we should have:

$$\int_0^{\underline{p}} \{\frac{d^2}{dp^2}[\Sigma_L(p) f_L(p)] - \delta f_L(p)\} dp = 0$$

and

$$\int_{\underline{p}}^{p_0} \{\frac{d^2}{dp^2}[\Sigma_H(p) f_H(p)] - \delta f_H(p)\} dp + \int_{p_0}^{1} \{\frac{d^2}{dp^2}[\Sigma_H(p) f_H(p)] - \delta f_H(p)\} dp = 0.$$

The above two equations give us:

$$\frac{d}{dp}[\Sigma_L(p) f_L(p)]|_{\underline{p}-} = \delta(1-\pi)$$

and

$$\Sigma_H(p_0)[f'_H(p_0-) - f'_H(p_0+)] = \frac{d}{dp}[\Sigma_H(p) f_H(p)]|_{\underline{p}+} + \delta\pi$$

since the market clearing conditions imply:

$$\int_0^{\underline{p}} f_L(p) dp = 1 - \pi$$

$$\int_{\underline{p}}^{1} f_H(p) dp = \pi$$

and there is continuity at p_0:

$$f_H(p_0-) = f_H(p_0+).$$

Meanwhile, notice that inflow at p_0 must be the same as δ, which implies that $\Sigma_H(p_0)[f'_H(p_0-) - f'_H(p_0+)] = \delta$. This immediately gives us the flow equation at $\underline{p}$:

$$\frac{d}{dp}[\Sigma_L(p) f_L(p)]|_{\underline{p}-} = \frac{d}{dp}[\Sigma_H(p) f_H(p)]|_{\underline{p}+}.$$

Now apply similar logic and we can get:

$$\int_0^{\underline{p}} \{p\frac{d^2}{dp^2}[\Sigma_L(p)f_L(p)] - p\delta f_L(p)\}dp + \int_{\underline{p}}^1 \{p\frac{d^2}{dp^2}[\Sigma_H(p)f_H(p)] - p\delta f_H(p)\}dp = 0.$$

Notice that

$$\int_0^{\underline{p}} p\delta f_L(p)dp + \int_{\underline{p}}^1 p\delta f_H(p)dp = \delta p_0$$

by the martingale property. Meanwhile, we still have: $\Sigma_H(p_0)[f'_H(p_0-)-f'_H(p_0+)] = \delta$. Hence,after some tedious algebra, we can get:

$$\{p\frac{d}{dp}[\Sigma_L(p)f_L(p)] + \Sigma_L(p)f_L(p)\}|_{\underline{p}-} = \{p\frac{d}{dp}[\Sigma_H(p)f_H(p)] + \Sigma_H(p)f_H(p)\}|_{\underline{p}+}$$

which gives us the boundary condition at $\underline{p}$:

$$\Sigma_H(\underline{p}+)f_H(\underline{p}+) = \Sigma_L(\underline{p}-)f_L(\underline{p}-).$$

Proof of Proposition 1

Proof. First, from Equation (26), we have:

$$f_{L0} = \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp}.$$

Second, from Equation (28), we can get:

$$f_{H2} = f_{H0}(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{H2}} + f_{H1}$$

Finally, from Equation (24) and (27), we can express f_{H0} and f_{H1} as functions of f_{L0}:

$$f_{H0} = \frac{\eta_H+\eta_L}{2\eta_H}\frac{s_L^2}{s_H^2}(\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H}f_{L0}$$

and

$$f_{H1} = -\frac{\eta_L-\eta_H}{2\eta_H}\frac{s_L^2}{s_H^2}(\frac{\underline{p}}{1-\underline{p}})^{\eta_L+\eta_H}f_{L0}.$$

Here,

$$\eta_L = \sqrt{\frac{1}{4}+\frac{2\delta}{s_L^2}} > \eta_H = \sqrt{\frac{1}{4}+\frac{2\delta}{s_H^2}} > 1/2.$$

Next, we want to show that both f_{H0} and f_{H1} are decreasing in $\underline{p}$.

We can rewrite f_{H0} as:

$$f_{H0} = \frac{\eta_H+\eta_L}{2\eta_H}\frac{s_L^2}{s_H^2}(\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H}\frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp}.$$

Notice that

$$(\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H} = \int_0^{\underline{p}}[(\frac{p}{1-p})^{\eta_L-\eta_H}]'dp = \int_0^{\underline{p}}(\eta_L-\eta_H)(\frac{p}{1-p})^{\eta_L-\eta_H-1}(\frac{1}{1-p})^2dp.$$

Let $G_1(p) = p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}$ and $G_2(p) = (\frac{p}{1-p})^{\eta_L-\eta_H-1}(\frac{1}{1-p})^2$. We have:

$$\frac{G_1(p)}{G_2(p)} = p^{-\frac{1}{2}+\eta_H}(1-p)^{-\frac{1}{2}-\eta_H}$$

is increasing in p. Therefore, we must have:

$$(\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H}\frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp}$$

is decreasing in $\underline{p}$[25]. So f_{H0} is decreasing in $\underline{p}$.

Similarly, we can rewrite f_{H1} as:

$$f_{H1} = -\frac{\eta_L-\eta_H}{2\eta_H}\frac{s_L^2}{s_H^2}(\frac{\underline{p}}{1-\underline{p}})^{\eta_L+\eta_H}\frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp}.$$

Meanwhile

$$(\frac{\underline{p}}{1-\underline{p}})^{\eta_L+\eta_H} = \int_0^{\underline{p}}(\eta_L+\eta_H)(\frac{p}{1-p})^{\eta_L+\eta_H-1}(\frac{1}{1-p})^2dp.$$

Let $G_3(p) = (\frac{p}{1-p})^{\eta_L+\eta_H-1}(\frac{1}{1-p})^2$ and we have:

$$\frac{G_1(p)}{G_3(p)} = p^{-\frac{1}{2}-\eta_H}(1-p)^{-\frac{1}{2}+\eta_H}$$

is decreasing in p. Therefore, we must have:

$$-(\frac{\underline{p}}{1-\underline{p}})^{\eta_L+\eta_H}\frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}dp}$$

is decreasing in $\underline{p}$ and hence f_{H1} is also decreasing in $\underline{p}$.

Now we can conclude

$$f_{H2} = f_{H0}(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{H2}} + f_{H1}$$

is also decreasing in $\underline{p}$. Therefore, we can expressing f_{H0}, f_{H1} and f_{H2} as $\xi_0(\underline{p})$, $\xi_1(\underline{p})$ and $\xi_2(\underline{p})$ respectively such that ${\xi_0}' < 0$, ${\xi_1}' < 0$ and ${\xi_2}' < 0$.

Hence, the market clearing condition 25 implies:

$$H(\underline{p}) = \int_{\underline{p}}^{p_0}[\xi_0(\underline{p})p^{\gamma_{H1}}(1-p)^{\gamma_{H2}} + \xi_1(\underline{p})p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}]dp + \int_{p_0}^1 \xi_2(\underline{p})p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}dp = \pi.$$

It is easy to see that $H' < 0$ since ${\xi_0}' < 0$, ${\xi_1}' < 0$ and ${\xi_2}' < 0$. Therefore, there exists $\underline{p} \in (0, p_0)$ such that $H(\underline{p}) = \pi$ if and only if $\lim_{x\to 0} H(x) > \pi$ and $\lim_{x\to p_0} H(x) < \pi$.

It is easy to verify that as $\underline{p} \to 0$, $f_{H0} = \xi_0(\underline{p}) \to \infty$ and $f_{H1} = \xi_1(\underline{p}) \to 0$. We thus have:

$$\lim_{x\to 0} H(x) \to \infty > \pi.$$

[25] Actually, we are using the result that if $\frac{G_2(p)}{G_1(p)}$ is decreasing in p, then $\frac{\int_0^{\underline{p}} G_2(p)dp}{\int_0^{\underline{p}} G_1(p)dp}$ will also be decreasing in $\underline{p}$. This is true because by the definition of Riemann integral, $\int_0^{\underline{p}} G_1(p)dp$ and $\int_0^{\underline{p}} G_2(p)dp$ could be written as the limit of Riemann sum. The ratio of two Riemann sums is always decreasing in $\underline{p}$ since $\frac{G_2(p)}{G_1(p)}$ is decreasing in p.

Meanwhile, when $\underline{p} \to p_0$, it is obvious that $H(\underline{p}) \to \int_{p_0}^1 f_{H2} p^{\gamma_{H2}} (1-p)^{\gamma_{H1}} dp$. Notice that

$$f_{H2} = f_{H0}(\frac{p_0}{1-p_0})^{\gamma_{H1}-\gamma_{H2}} + f_{H1} \to \frac{s_L^2}{s_H^2}(\frac{p_0}{1-p_0})^{\eta_L+\eta_H} \frac{1-\pi}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp}$$

as $\underline{p} \to p_0$.

Therefore, $\lim_{x\to p_0} H(x) < \pi$ if and only if:

$$\frac{s_L^2}{s_H^2}(\frac{p_0}{1-p_0})^{\eta_L+\eta_H} \frac{1-\pi}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp} \int_{p_0}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} dp < \pi,$$

which gives us the condition in the proposition. Moreover, since H is strictly decreasing, the solution to $H(x) = \pi$ must be at most one. This completes our proof of Proposition 1. ■

Proof of Corollary 1

Proof. It is also straightforward to prove Corollary 1 from the equation $H(\underline{p}; \pi, p_0) = \pi$. Obviously, H is linear in $(1-\pi)$. So as π increases, $\pi/(1-\pi)$ increases and we have to decrease $\underline{p}$ to keep the equation. On the other hand,

$$\begin{aligned}\frac{\partial H}{\partial p_0} = \xi_0(\underline{p}) p_0^{\gamma_1^H}(1-p_0)^{\gamma_2^H} + \xi_1(\underline{p}) p_0^{\gamma_2^H}(1-p_0)^{\gamma_1^H} - \xi_2(\underline{p}) p_0^{\gamma_2^H}(1-p_0)^{\gamma_1^H} \\ + \int_{p_0}^1 \frac{\partial \xi_2(\underline{p})}{\partial p_0} p^{\gamma_2^H}(1-p)^{\gamma_1^H} dp.\end{aligned}$$

It is easy to verify that the first term is zero while the second term is negative. Hence $H(\underline{p}; \pi, p_0)$ is decreasing in p_0 and we have to increase $\underline{p}$ to keep the equation as p_0 increases.

The proof for the comparative statics for $\underline{p} > p_0$ case is similar and hence is omitted. ■

Proof of Proposition 2

Proof. First, from equation (36), we have:

$$f_{H0} = \frac{\pi}{\int_{\underline{p}}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} dp}.$$

Then Equations (35) and (38) imply:

$$f_{L1} = \frac{\eta_L - \eta_H}{2\eta_L} \frac{s_H^2}{s_L^2} (\frac{\underline{p}}{1-\underline{p}})^{-\eta_L-\eta_H} f_{H0}$$

and

$$f_{L2} = \frac{\eta_L + \eta_H}{2\eta_L} \frac{s_H^2}{s_L^2} (\frac{\underline{p}}{1-\underline{p}})^{\eta_L-\eta_H} f_{H0}.$$

Here,

$$\eta_L = \sqrt{\frac{1}{4} + \frac{2\delta}{s_L^2}} > \eta_H = \sqrt{\frac{1}{4} + \frac{2\delta}{s_H^2}} > 1/2.$$

It is easy to see that f_{H0}, f_{L1}, f_{L2} are increasing in $\underline{p}$ and hence $f_{L0} = f_{L1} + f_{L2}(\frac{p_0}{1-p_0})^{-2\eta_L}$ is also increasing in $\underline{p}$ by Equation (39).

Hence, we can express f_{L0}, f_{L1}, f_{L2} as $\xi_0(\underline{p})$, $\xi_1(\underline{p})$ and $\xi_2(\underline{p})$ respectively such that $\xi_0{}' > 0$, $\xi_1{}' > 0$ and $\xi_2{}' > 0$.

Finally, the market clearing condition (37) implies:

$$H(\underline{p}) = \int_0^{p_0} \xi_0(\underline{p}) p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} dp + \int_{p_0}^{\underline{p}} [\xi_1(\underline{p}) p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} + \xi_2(\underline{p}) p^{\gamma_{L2}}(1-p)^{\gamma_{L1}}] dp = 1 - \pi.$$

Obviously, $H(\cdot)$ is strictly increasing, which guarantees the solution is unique if it exists and $\lim_{x \to p_0} H(x) \leq 1 - \pi$ will give us Equation (40) in Proposition 2. ■

Proof of Lemma 6

Proof. Here we will try to prove a generalized version of Lemma 6. More specifically, we want to show that Lemma 6 is true for any combination of (s_H, s_L).

First of all, we want to show all of the one-shot deviations are ruled out by our no-deviation condition as $dt \to 0$.

Under strict supermodularity, the value functions are given by:

$$W_L(p) = \frac{w_L(p)}{r+\delta} + k_L p^{\alpha_L}(1-p)^{1-\alpha_L}$$

and

$$W_H(p) = \frac{w_H(p)}{r+\delta} + k_H p^{1-\alpha_H}(1-p)^{\alpha_H}.$$

Let

$$\mathcal{G}_L(p) = k_L p^{\alpha_L}(1-p)^{1-\alpha_L}\left(\frac{\alpha_L - p}{p(1-p)}\right) > 0$$

and

$$\mathcal{G}_H(p) = k_H p^{1-\alpha_H}(1-p)^{\alpha_H}\left(\frac{1-\alpha_H - p}{p(1-p)}\right) < 0$$

be the first derivatives for the non-linear parts of the value functions. Smooth pasting at $\underline{p}$ implies:

$$\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p}) = \frac{\Delta_H}{r+\delta} + \mathcal{G}_H(\underline{p}).$$

For $p < \underline{p}$, define:

$$\begin{aligned} Z_L(p) = \lim_{dt \to 0} \frac{\tilde{W}_H(p) - W_L(p)}{dt} &= w_H(p) - w_L(p) + [\Sigma_H(p) - \Sigma_L(p)] W_L''(p) \\ &= w_H(p) - w_L(p) + \frac{s_H^2 - s_L^2}{s_L^2} \Sigma_L(p) W_L''(p) \\ &= w_H(p) - w_L(p) + \frac{s_H^2 - s_L^2}{s_L^2}((r+\delta)W_L(p) - w_L(p)). \end{aligned}$$

Obviously, we have $\lim_{p \nearrow \underline{p}} Z_L(p) = 0$ from Lemma 9. If we can show that $Z_L(p)$ is increasing in p as p increases from 0 to $\underline{p}$, then we are done. Notice that

$$Z_L'(p) = \Delta_H - \frac{s_H^2}{s_L^2}\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)W_L'(p)$$

and $W_L'(p)$ lies between $\frac{\Delta_L}{r+\delta}$ and $\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p})$ for $p \in [0, \underline{p}]$.[26]

If $s_H^2 \geq s_L^2$, then

$$Z_L'(p) \geq \Delta_H - \frac{s_H^2}{s_L^2}\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)\frac{\Delta_L}{r+\delta} = \Delta_H - \Delta_L > 0;$$

if $s_H^2 < s_L^2$, then

$$\begin{aligned} Z_L'(p) &\geq \Delta_H - \frac{s_H^2}{s_L^2}\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)[\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p})] \\ &= \Delta_H - \frac{s_H^2}{s_L^2}\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)[\frac{\Delta_H}{r+\delta} + \mathcal{G}_H(\underline{p})] \\ &= \frac{s_H^2}{s_L^2}(\Delta_H - \Delta_L) + \frac{s_H^2 - s_L^2}{s_L^2}(r+\delta)\mathcal{G}_H(\underline{p}) > 0. \end{aligned}$$

Therefore, we conclude that $Z_L'(p) > 0$ for both $s_H \geq s_L$ and $s_H < s_L$ cases, which implies that $Z_L(p) < 0$ for all $p < \underline{p}$ and hence there is no profitable one-shot deviation as dt is sufficiently small.

For $p > \underline{p}$, similarly define:

$$Z_H(p) = \lim_{dt \to 0} \frac{\tilde{W}_L(p) - W_H(p)}{dt} = w_L(p) - w_H(p) + [\Sigma_L(p) - \Sigma_H(p)]W_H''(p). \tag{56}$$

Under PAM equilibrium, we have $Z_H(\underline{p}) = 0$ from Lemma 9. Secondly, notice that

$$Z_H(p) = w_L(p) - w_H(p) + [\Sigma_L(p) - \Sigma_H(p)]W_H''(p) = w_L(p) - w_H(p) + \frac{s_L^2 - s_H^2}{s_H^2}((r+\delta)W_H(p) - w_H(p)),$$

with $W_H'(p)$ lies between $\frac{\Delta_H}{r+\delta} + \mathcal{G}_H(\underline{p})$ and $\frac{\Delta_H}{r+\delta}$ for $p \in [\underline{p}, 1]$.[27] Hence, if $s_L^2 > s_H^2$

$$Z_H'(p) \leq \Delta_L - \Delta_H < 0;$$

and if $s_L^2 \leq s_H^2$

$$Z_H'(p) \leq \Delta_L - \frac{s_L^2}{s_H^2}\Delta_H + \frac{s_L^2 - s_H^2}{s_H^2}(r+\delta)(\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p})) < 0.$$

Therefore, $Z_H'(p) < 0$ for both $s_H \geq s_L$ and $s_H < s_L$ cases and hence $Z_H(p) < 0$ for all $p > \underline{p}$.

Second, since there is no one-shot deviation for any p, obviously there will be no any other deviation for any p. Consider any deviation starting at p. Then the above result says it is better not to deviate for at least dt time. Suppose after dt, we achieve a new p'. Similarly, there should

[26] This comes from the fact that

$$\mathcal{G}_L(p) = k_L(\frac{p}{1-p})^{\alpha_L - 1}(\frac{\alpha_L - p}{1-p})$$

is increasing in p.

[27] This comes from the fact that

$$\mathcal{G}_H(p) = k_H(\frac{1-p}{p})^{\alpha_H - 1}(\frac{1-\alpha_L - p}{p})$$

is decreasing in p.

be no deviation for at least dt' time. Iterate using the same logic and we can see that any deviation is not profitable. ■

Proof of Theorem 4

Proof. We establish the proof under supermodularity. The same logic goes through for submodularity. Now consider the following three steps: 1. for $N = 3$ we show that the planner can increase output when changing the cutoffs; 2. for $N = 3$ no allocation dominates PAM; 3. For any N, the allocation with $N - 2$ cutoffs dominates that with N cutoffs.

1. For $N = 3$, output increases from changing the cutoffs

Consider any allocation with three cutoffs $0 < \underline{p}_3 < \underline{p}_2 < \underline{p}_1 < 1$ such that workers with $p \in (\underline{p}_1, 1]$ and $p \in (\underline{p}_3, \underline{p}_2)$ are allocated to the high type firms while workers with $p \in [0, \underline{p}_3)$ and $p \in (\underline{p}_2, \underline{p}_1)$ are allocated to the low type firms. Furthermore, denote the ergodic density function for this allocation to be f_y and for p close to 0, let the density function be $f_L(p) = \tilde{f}_{L0} p^{\gamma_L} (1-p)^{1-\gamma_L}$ while the ergodic density function for p close to 1 is denoted by $f_H(p) = \tilde{f}_{H0} p^{1-\gamma_H} (1-p)^{\gamma_H}$ where $\tilde{f}_{L0}$ and $\tilde{f}_{H0}$ are constants. Correspondingly, denote the ergodic density under the PAM allocation to be f_y^* with the unique cutoff $\underline{p}$.

1. Suppose the planner changes the allocation by moving the interval to the left: $(\underline{p}_2, \underline{p}_1) \to (\underline{p}_2', \underline{p}_1')$ where $(\underline{p}_2', \underline{p}_1') = (\underline{p}_2 - \epsilon_2, \underline{p}_1 - \epsilon_1)$. Choose ϵ_1, ϵ_2 such that market clearing is satisfied:
$$\int_{\underline{p}_1'}^{\underline{p}_1} f_H(p)dp = \int_{\underline{p}_2'}^{\underline{p}_2} f_H(p)dp.$$

2. Given the new cutoffs, the Kolmogorov forward equation will pin down a new density $\hat{f}_L$ in the interval $(\underline{p}_2', \underline{p}_1')$. Globally, we need to satisfy market clearing and the martingale condition. The market clearing condition for the H types is satisfied by the construction. For the L type firms it requires that:
$$\int_{\underline{p}_2'}^{\underline{p}_1'} \hat{f}_L(p)dp = \int_{\underline{p}_2}^{\underline{p}_1} f_L(p)dp.$$
The martingale condition requires that $\mathbb{E}_{\Omega_H'} p + \mathbb{E}_{\Omega_L'} p = p_0$ or:[28]
$$\int_0^{p_3} p f_L(p)dp + \int_{p_3}^{p_2'} p f_H(p)dp + \int_{p_2'}^{p_1'} p \hat{f}_L(p)dp + \int_{p_1'}^{1} p f_H(p)dp = p_0.$$

3. Then comparing the original allocation to the new one, we get
$$\mathbb{E}_{\Omega_H'} p - \mathbb{E}_{\Omega_H} p = \int_{\underline{p}_1'}^{\underline{p}_1} p f_H(p)dp - \int_{\underline{p}_2'}^{\underline{p}_2} p f_H(p)dp > 0$$
since by construction
$$\int_{\underline{p}_1'}^{\underline{p}_1} f_H(p)dp = \int_{\underline{p}_2'}^{\underline{p}_2} f_H(p)dp$$

[28] Things are slightly different if we have $p_0 \in (p_2', p_1')$. Then we have four new distribution coefficients but we also have two new equations: $\hat{f}_L(p_0-) = \hat{f}_L(p_0+)$ and $\Sigma_L(p_0)(\hat{f}_L'(p_0-) - \hat{f}_L'(p_0+)) = \delta$.

and the interval $[\underline{p}_2', \underline{p}_1']$ is strictly to the left of $[\underline{p}_2, \underline{p}_1]$. From Lemma 10 below, $\mathbb{E}_{\Omega_H'} p > \mathbb{E}_{\Omega_H} p$ implies the planner prefers allocation Ω' over Ω.

4. Similarly, we can consider another transform which is to move the interval to the right: $(\underline{p}_3, \underline{p}_2) \to (\underline{p}_3', \underline{p}_2')$ where $(\underline{p}_3', \underline{p}_2') = (\underline{p}_3 + \epsilon_2, \underline{p}_2 + \epsilon_1)$. This can also lead to output increases. Keep on doing such transformations and eventually, we can have both the distance and the measure between $\underline{p}_3'$ and $\underline{p}_1'$ are arbitrarily small.

2. For $N = 3$, no allocation dominates PAM

1. We now show by contradiction that an allocation with $\underline{p}_3'$ and $\underline{p}_1'$ arbitrary close is dominated by the PAM allocation. Suppose on the contrary that there exists $\tilde{p}_1$, $\tilde{p}_2$ and $\tilde{p}_3$ which dominates the PAM allocation. Then by Lemma 10, we must have:

$$\int_{\tilde{p}_1}^{1} p f_H(p) dp + \int_{\tilde{p}_3}^{\tilde{p}_2} p f_H(p) dp > \int_{\underline{p}}^{1} p f_H^*(p) dp \tag{57}$$

and

$$\int_{\tilde{p}_2}^{\tilde{p}_1} p f_L(p) dp + \int_{0}^{\tilde{p}_3} p f_L(p) dp < \int_{0}^{\underline{p}} p f_L^*(p) dp. \tag{58}$$

Let $\hat{p}_1$ and $\hat{p}_3$ be defined as:

$$\int_0^{\hat{p}_3} f_L(p) dp = (1 - \pi) \quad \text{and} \int_{\hat{p}_1}^{1} f_H(p) dp = \pi.$$

Then by definition we should have:

$$\int_{\hat{p}_1}^{1} p f_H(p) dp > \int_{\tilde{p}_1}^{1} p f_H(p) dp + \int_{\tilde{p}_3}^{\tilde{p}_2} p f_H(p) dp > \int_{\underline{p}}^{1} p f_H^*(p) dp$$

and

$$\int_0^{\hat{p}_3} p f_L(p) dp < \int_{\tilde{p}_2}^{\tilde{p}_1} p f_L(p) dp + \int_0^{\tilde{p}_3} p f_L(p) dp < \int_0^{\underline{p}} p f_L^*(p) dp.$$

2. The next step of the proof requires Lemma 11 below. The Lemma implies that we should have $\tilde{p}_3 < \hat{p}_3 < \underline{p} < \hat{p}_1 < \tilde{p}_1$ to guarantee that

$$\int_{\hat{p}_1}^{1} p f_H(p) dp > \int_{\underline{p}}^{1} p f_H^*(p) dp \;\text{ and }\; \int_0^{\hat{p}_3} p f_L(p) dp < \int_0^{\underline{p}} p f_L^*(p) dp.$$

Therefore, inequalities (57) and (58) only hold when $\tilde{p}_1 - \tilde{p}_3 > \hat{p}_1 - \hat{p}_3 > 0$ which contradicts that fact that we can make the distance between $\tilde{p}_1$ and $\tilde{p}_3$ arbitrarily small while still keeping the inequalities (57) and (58) (Notice that by making distance between $\tilde{p}_1$ and $\tilde{p}_3$ smaller and smaller, we can increase the aggregate payoff and make the allocation with $N = 3$ cutoffs even better). Hence, no allocation with $N = 3$ cutoffs could be better than the PAM allocation in terms of aggregate surplus.

3. For N cutoffs, the allocation is dominated by any allocation with $N - 2$ cutoffs.

Consider three adjacent cutoffs $\underline{p}_{n-1}, \underline{p}_n$ and $\underline{p}_{n+1}$ such that workers with $p \in (\underline{p}_{n-1}, \underline{p}_{n-2})$ and $p \in (\underline{p}_{n+1}, \underline{p}_n)$ are allocated to high type firms; workers with $p \in (\underline{p}_n, \underline{p}_{n-1})$ and $p \in (\underline{p}_{n+2}, \underline{p}_{n+1})$

are allocated to low type firms. Suppose the density functions are such that the market clears and the expectation of p's is p_0. Then we just need to choose ϵ such that

$$\int_{\underline{p}_{n-1}-\epsilon}^{\underline{p}_{n-1}} f_H(p)dp = \int_{\underline{p}_{n+1}}^{\underline{p}_n} f_H(p)dp.$$

Now $\underline{p}_{n-1}$, $\underline{p}_n$ and $\underline{p}_{n+1}$ converge to $\underline{p}_{n-1} - \epsilon$ but p_{n+2} is kept to be the same. The market clearing condition requires that

$$\int_{\underline{p}_{n+2}}^{\underline{p}_{n-1}-\epsilon} \tilde{f}_L(p)dp = \int_{\underline{p}_n}^{\underline{p}_{n-1}} f_L(p)dp + \int_{\underline{p}_{n+2}}^{\underline{p}_{n+1}} f_L(p)dp.$$

Meanwhile, the martingale condition requires that:

$$\int_{\underline{p}_1}^{1} pf_H(p)dp + \cdots + \int_{\underline{p}_{n-1}-\epsilon}^{\underline{p}_{n-2}} pf_H(p)dp + \int_{\underline{p}_{n+2}}^{\underline{p}_{n-1}-\epsilon} p\tilde{f}_L(p)dp + \cdots + \int_0^{\underline{p}_N} pf_L(p)dp = p_0.$$

We can also solve the two coefficients from these two equations and no other conditions are needed. As before,

$$\mathbb{E}_{\Omega_H} p = \int_{\Omega_H} pf_H(p)dp$$

must become higher and this allocation with $N-2$ cutoffs will generate a higher aggregate payoff.

Finally, by the standard induction argument, we can conclude that the PAM allocation with one cutoff dominates any allocation with $N \geq 3$ cutoffs in aggregate surplus. ■

Lemma 10

Lemma 10 *Consider two possible allocations with ergodic density functions $f_H(p)$, $f_L(p)$ (allocation 1) and $\tilde{f}_H(p)$, $\tilde{f}_L(p)$ (allocation 2) respectively. Then if and only if $\int_{\Omega_H} pf_H(p)dp > \int_{\tilde{\Omega}_H} p\tilde{f}_H(p)dp$ or alternatively, $\int_{\Omega_L} pf_L(p)dp < \int_{\tilde{\Omega}_L} p\tilde{f}_L(p)dp$, we have allocation 1 must be better than the allocation 2. Here, Ω_H ($\tilde{\Omega}_H$) is the set of ps that matches with high type firms for allocation 1 (2) and vice versa.*

Proof. The total expected surplus for allocation 1 could be written as:

$$S = \int_{\Omega_H} (\Delta_H p + \mu_{LH}) f_H(p)dp + \int_{\Omega_L} (\Delta_L p + \mu_{LL}) f_L(p)dp.$$

Notice that we always have: $\int_{\Omega_H} f_H(p)dp = \pi$ and $\int_{\Omega_L} f_L(p)dp = 1-\pi$. Hence, the total expected surplus for allocation 1 could be rewritten as:

$$S_1 = \int_{\Omega_H} \Delta_H p f_H(p)dp + \pi\mu_{LH} + \int_{\Omega_L} \Delta_L p f_L(p)dp + (1-\pi)\mu_{LL}.$$

The martingale property implies that the expectation of p is always p_0, which implies that

$$\int_{\Omega_H} pf_H(p)dp + \int_{\Omega_L} pf_L(p)dp = p_0.$$

Based on this fact, we can furthermore rewrite S_1 as:

$$S_1 = (\Delta_H - \Delta_L) \int_{\Omega_H} p f_H(p) dp + \Delta_L p_0 + \pi \mu_{LH} + (1-\pi)\mu_{LL}.$$

And similarly,

$$S_2 = (\Delta_H - \Delta_L) \int_{\tilde{\Omega}_H} p f_H(p) dp + \Delta_L p_0 + \pi \mu_{LH} + (1-\pi)\mu_{LL}.$$

Therefore, $S_1 > S_2$ if and only if $\int_{\Omega_H} p f_H(p) dp > \int_{\tilde{\Omega}_H} p \tilde{f}_H(p) dp$ or alternatively, $\int_{\Omega_L} p f_H(p) dp < \int_{\tilde{\Omega}_L} p \tilde{f}_L(p) dp$. ■

Lemma 11

Lemma 11 *Let $\hat{p}_1$ be such that $\int_{\hat{p}_1}^1 f_H(p) dp = \pi$, then $\int_{\hat{p}_1}^1 p f_H(p) dp$ is increasing in $\hat{p}_1$. Let $\hat{p}_3$ be such that $\int_0^{\hat{p}_3} f_L(p) dp = (1-\pi)$, then $\int_0^{\hat{p}_3} p f_L(p) dp$ is also increasing in $\hat{p}_3$.*

Proof. We just prove the case that $\hat{p}_1 > p_0$. The other cases are similar. From

$$\int_{\hat{p}_1}^1 p f_H(p) dp = \int_{\hat{p}_1}^1 p (\frac{d^2}{dp^2}[\Sigma_H(p) f_H(p)] - \delta f_H(p)) dp$$

we have:

$$\int_{\hat{p}_1}^1 p f_H(p) dp = \pi \hat{p}_1 + \frac{\pi \hat{p}_1 (1-\hat{p}_1)}{\eta_H + \hat{p}_1 - 1} = \frac{\pi \hat{p}_1 (\eta_H + \frac{1}{2})}{\eta_H + \hat{p}_1 - \frac{1}{2}}$$

is increasing in $\hat{p}_1$ since

$$\eta_H = \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} > \frac{1}{2}.$$

■

Proof of Lemma 8

Proof. First of all, as t goes to T and as T goes to infinity, we have:

$$\frac{d^2}{dp^2}[\Sigma_y(p) f_y^T(p,t)]$$

goes to

$$\Sigma_y(p) f_y^{T\prime\prime}(\underline{p}, t) + s_y^2 \underline{p}(1-\underline{p})(1-2\underline{p}) f_y^{T\prime}(\underline{p}, t).$$

We actually have to consider three cases:

1. $\underline{p} = 1/2$. This implies that $f_y^{T\prime\prime}(\underline{p}, t)$ has to go to zero. And by the Taylor expansion, for any $p < \underline{p}$ and p sufficiently close to $\underline{p}$,

$$f_L^T(p,t) = f_L^T(\underline{p}, t) + f_L^{T\prime}(\underline{p}, t)(p - \underline{p})$$

goes to zero. This implies that $f_L^{T\prime}(\underline{p}, t)$ goes to zero. Similarly, we have $f_H^{T\prime}(\underline{p}, t)$ goes to zero.

2. $\underline{p} < 1/2$. This implies that $f_y^{T\prime}(\underline{p},t)$ and $f_y^{T\prime\prime}(\underline{p},t)$ have opposite signs. Now consider any $p < \underline{p}$ and p sufficiently close to $\underline{p}$, then

$$f_L^T(p,t) = f_L^T(\underline{p},t) + f_L^{T\prime}(\underline{p},t)(p-\underline{p}) + \frac{1}{2}f_L^{T\prime\prime}(\underline{p},t)(p-\underline{p})^2$$

should go to zero. This can only happen if

$$\lim_{t\to T,\ T\to\infty} f_L^{T\prime}(\underline{p},t) = f_L^{T\prime\prime}(\underline{p},t) = 0.$$

From equation (46), it is immediately to see that $f_H^{T\prime}(\underline{p},t)$ should also go to zero.

3. $\underline{p} > 1/2$. The proof is similar to the proof for case 2.

■

On the Job Human Capital Accumulation

Under the assumption of $\underline{p}^u = \underline{p}^e = \underline{p}$, the value functions could be written down as:

$$\begin{aligned}
W_y^u(p) &= \frac{\mu_y(p)-V_y}{r+\delta+\lambda} + k_{y1}^u p^{1-\alpha_y^u}(1-p)^{\alpha_y^u} + k_{y2}^u p^{\alpha_y^u}(1-p)^{1-\alpha_y^u} \\
&+ \frac{\lambda}{(r+\delta)(r+\delta+\lambda)}[\mu_y(p)+\xi(p)-V_y] \\
&+ \frac{\lambda}{(\lambda+\delta+r)-\frac{(s_y^u)^2}{(s_y^e)^2}(r+\delta)}[k_{y1}^e p^{1-\alpha_y^e}(1-p)^{\alpha_y^e} + k_{y2}^e p^{\alpha_y^e}(1-p)^{1-\alpha_y^e}] \\
W_y^e(p) &= \frac{\mu_y(p)+\xi(p)-V_y}{r+\delta} + k_{y1}^e p^{1-\alpha_y^e}(1-p)^{\alpha_y^e} + k_{y2}^e p^{\alpha_y^e}(1-p)^{1-\alpha_y^e}
\end{aligned}$$

where

$$\alpha_y^u = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta+\lambda)}{(s_y^u)^2}} \geq 1$$

$$\alpha_y^e = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{(s_y^e)^2}} \geq 1$$

Notice that $W_y^u(p)$ could be further written as:

$$\begin{aligned}
W_y^u(p) &= \frac{\mu_y(p)-V_y}{r+\delta+\lambda} + k_{y1}^u p^{1-\alpha_y^u}(1-p)^{\alpha_y^u} + k_{y2}^u p^{\alpha_y^u}(1-p)^{1-\alpha_y^u} \\
&- \frac{\lambda\frac{(s_y^u)^2}{(s_y^e)^2}}{(r+\delta+\lambda)[(\lambda+\delta+r)-\frac{(s_y^u)^2}{(s_y^e)^2}(r+\delta)]}[\mu_y(p)+\xi(p)-V_y] \\
&+ \frac{\lambda}{(\lambda+\delta+r)-\frac{(s_y^u)^2}{(s_y^e)^2}(r+\delta)}W_y^e(p)
\end{aligned}$$

Similarly,

$$W_L^e(\underline{p}) = W_H^e(\underline{p}), \quad W_L^{e\prime}(\underline{p}) = W_H^{e\prime}(\underline{p}), \quad W_L^{e\prime\prime}(\underline{p}) = W_H^{e\prime\prime}(\underline{p})$$

would imply:

$$\tilde{V}_H^e = (\mu_{LH} - \mu_{LL}) + \frac{\alpha_H^e(\alpha_L^e - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H^e(\alpha_L^e - 1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)}.$$

And from

$$W_L^u(\underline{p}) = W_H^u(\underline{p}), \quad W_L^{u\prime}(\underline{p}) = W_H^{u\prime}(\underline{p}), \quad W_L^{u\prime\prime}(\underline{p}) = W_H^{u\prime\prime}(\underline{p}),$$

we can get another equilibrium payoff $\tilde{V}_H^u$ as:

$$\begin{aligned}
\tilde{V}_H^u &= (\mu_{LH} - \frac{A_L}{B_L}\frac{B_H}{A_H}\mu_{LL}) - \frac{B_H}{A_H}\frac{\lambda\xi_L}{r+\delta+\lambda}(\frac{1-A_H}{B_H} - \frac{1-A_L}{B_L}) \\
&+ \frac{B_H}{A_H}\frac{\alpha_H^u(\alpha_L^u - 1)(D_H - D_L)\underline{p}}{\alpha_H^u(\alpha_L^u - 1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)},
\end{aligned}$$

where

$$D_H = \frac{A_H}{B_H}\Delta_H - \frac{1-A_H}{B_H}\frac{\lambda\Delta_\xi}{r+\delta+\lambda}$$

$$D_L = \frac{A_L}{B_L}\Delta_L - \frac{1-A_L}{B_L}\frac{\lambda\Delta_\xi}{r+\delta+\lambda}$$

$$A_H = 1 - \frac{(s_H^u)^2}{(s_H^e)^2} \quad B_H = (\lambda+\delta+r) - \frac{(s_H^u)^2}{(s_H^e)^2}(r+\delta)$$

$$A_L = 1 - \frac{(s_L^u)^2}{(s_L^e)^2} \quad B_L = (\lambda+\delta+r) - \frac{(s_L^u)^2}{(s_L^e)^2}(r+\delta).$$

Proof of Proposition 5

Proof. Supermodularity is equivalent to $\Delta_H > \Delta_L$, and $\xi_H \simeq \xi_L$ is equivalent to $\Delta_\xi \to 0$. The proof cand be divided into three parts. We want to show:

1.
$$(\mu_{LH} - \frac{A_L}{B_L}\frac{B_H}{A_H}\mu_{LL}) - \frac{B_H}{A_H}\frac{\lambda\xi_L}{r+\delta+\lambda}(\frac{1-A_H}{B_H} - \frac{1-A_L}{B_L}) < (\mu_{LH} - \mu_{LL})$$

2.
$$\frac{B_H}{A_H}(D_H - D_L) < \Delta_H - \Delta_L$$

and

3.
$$\frac{\alpha_H^u(\alpha_L^u - 1)\underline{p}}{\alpha_H^u(\alpha_L^u - 1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)} < \frac{\alpha_H^e(\alpha_L^e - 1)\underline{p}}{\alpha_H^e(\alpha_L^e - 1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)}.$$

First of all, notice that $\frac{(s_H^u)^2}{(s_H^e)^2} > \frac{(s_L^u)^2}{(s_L^e)^2}$ since $\Delta_H > \Delta_L$. And it is easy to see that $\frac{A_H}{B_H} < \frac{A_L}{B_L}$ and $\frac{1-A_H}{B_H} > \frac{1-A_L}{B_L}$ because of that. Hence we can get the first two inequalities.

For the last one, we can just compare:

$$\alpha_H^u(\alpha_L^u-1)[\alpha_H^e(\alpha_L^e-1)-(1-\underline{p})(\alpha_L^e-\alpha_H^e)]$$

and

$$\alpha_H^e(\alpha_L^e-1)[\alpha_H^u(\alpha_L^u-1)-(1-\underline{p})(\alpha_L^u-\alpha_H^u)].$$

Notice this is equivalent to compare $\alpha_H^u(\alpha_L^u-1)(\alpha_L^e-\alpha_H^e)$ and $\alpha_H^e(\alpha_L^e-1)(\alpha_L^u-\alpha_H^u)$. From the expressions of αs, we have:

$$(\alpha_L^e-\alpha_H^e)(\alpha_L^e+\alpha_H^e-1)=2(r+\delta)[\frac{\sigma^2}{(\Delta_L+\Delta_\xi)^2}-\frac{\sigma^2}{(\Delta_H+\Delta_\xi)^2}]$$

and

$$(\alpha_L^u-\alpha_H^u)(\alpha_L^u+\alpha_H^u-1)=2(r+\delta+\lambda)[\frac{\sigma^2}{\Delta_L^2}-\frac{\sigma^2}{\Delta_H^2}].$$

Hence, when $\Delta_\xi=0$, we only need to compare:

$$(r+\delta)\alpha_H^u(\alpha_L^u-1)(\alpha_L^u+\alpha_H^u-1)$$

and

$$(r+\delta+\lambda)\alpha_H^e(\alpha_L^e-1)(\alpha_L^e+\alpha_H^e-1).$$

Meanwhile, we have:

$$\begin{aligned}(r+\delta)\alpha_H^u(\alpha_L^u-1)\alpha_L^u &= (r+\delta)\alpha_H^u\frac{2(r+\delta+\lambda)}{\Delta_L^2}\\ >(r+\delta+\lambda)\alpha_H^e(\alpha_L^e-1)\alpha_L^e &= (r+\delta+\lambda)\alpha_H^e\frac{2(r+\delta)}{\Delta_L^2}\end{aligned}$$

and

$$\begin{aligned}(r+\delta)\alpha_H^u(\alpha_L^u-1)(\alpha_H^u-1) &= (r+\delta)(\alpha_L^u-1)\frac{2(r+\delta+\lambda)}{\Delta_H^2}\\ >(r+\delta+\lambda)\alpha_H^e(\alpha_L^e-1)(\alpha_H^e-1) &= (r+\delta+\lambda)(\alpha_L^e-1)\frac{2(r+\delta)}{\Delta_H^2}\end{aligned}$$

since $\alpha_y^u>\alpha_y^e$. This implies:

$$\alpha_H^u(\alpha_L^u-1)(\alpha_L^e-\alpha_H^e)>\alpha_H^e(\alpha_L^e-1)(\alpha_L^u-\alpha_H^u)$$

and therefore,

$$\frac{\alpha_H^u(\alpha_L^u-1)\underline{p}}{\alpha_H^u(\alpha_L^u-1)-(1-\underline{p})(\alpha_L^u-\alpha_H^u)}<\frac{\alpha_H^e(\alpha_L^e-1)\underline{p}}{\alpha_H^e(\alpha_L^e-1)-(1-\underline{p})(\alpha_L^e-\alpha_H^e)}.$$

Then $\tilde{V}_H^u<\tilde{V}_H^e$, and as a result $\underline{p}^e<\underline{p}^u$. QED. ■

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