Taxation, redistribution, and debt in incomplete market economies with aggregate shocks*

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Abstract

We study an incomplete market economy with aggregate shocks and agents with heterogeneous skills. To redistribute and to finance exogenous stochastic government expenditures, the government imposes non-linear taxes on labor income. Only relative assets positions determined, so assets of the government or one of the agents can be set to zero always. Optimal labor distortions are history dependent but do not necessarily to drift to zero. Optimal labor distortions and debts in a heterogenous agent economy differ markedly from representative agent models like Aiyagari et al (2002). Differences stem from the restriction that lump sum transfers are non-negative in those representative agent economies, a restriction that we do not impose. In representative agent economies, the binding non-negative lump sum transfers restriction determines dynamics of distorting taxes and government debt. If that restriction were imposed in our heterogenous agent economy, it would not necessarily bind.

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1 Introduction

With complete markets, a government should rely mostly on history-contingent debt to adjust to random fluctuations in government purchases g_t; distorting taxes on labor should be approximately constant and depend only on the Markov state driving g_t. These conclusions emerged from Lucas and Stokey's (1983) and Chari and Kehoe's (1999) studies of environments with a representative agent and linear taxes on labor earnings. They also emerged from Werning's (2007) study of settings with nonlinear taxes and heterogenous agents.

A perception that governments have limited access to state-contingent debt motivated subsequent researchers to study Ramsey debt and tax policies without complete markets. Aiyagari et. al. (2002, a.k.a. AMSS) and Farhi (2010) studied optimal debt and linear taxes with incomplete markets and a representative consumer. In contrast to outcomes under complete markets, they found that the linear tax rate on labor earnings is history dependent and that it acquires a very persistent component rationalizing Barro (1979). Government debt dynamics are an important determinant of the evolution of the optimal linear tax rate. Market incompleteness gives the government a precautionary motive to acquire claims on the public, imparting a downward drift to the distorting tax rate. AMSS constructed a quasi-linear preference example in which the government's precautionary savings dynamics asymptotically drive the distorting tax to zero almost surely.[1]

To incorporate more realistic tax systems, this paper studies Ramsey debt and tax policies when taxes on labor income are allowed to be nonlinear. Figure 1 shows that U.S. taxes on labor earnings are nonlinear, and that while a linear tax is a poor approximation, an affine tax is a pretty good approximation. Motivated by that observation, we study nonlinear tax systems. In the AMSS incomplete markets tradition, we suppose that the government and all types of agent trade only a one-period risk-free bond. Given an exogenous government expenditure process $\{g_t\}_{t=0}^{\infty}$, our Ramsey government chooses transfers and distorting taxes on labor income partly to pay for g_t and partly to redistribute resources across households. We study how distorting taxes, transfers, and debts respond to government expenditure shocks. We study a sequence of nonlinear taxes that assume various functional forms and alternative sets of conditioning information for taxes on labor earnings.

We start with the simplest departure from linear taxes, an affine tax system that consists of a proportional labor income tax or subsidy rate τ together with a constant T that we regard

[1] Mike and Tom: Somewhere put a footnote of the Ali-Mike-Anmol discussion about the consequences of allowing private agents to trade a complete set of Arrow securities, but allowing the government to trade only a risk-free one-period security. Put in the link or lack of link to a Werning economy.

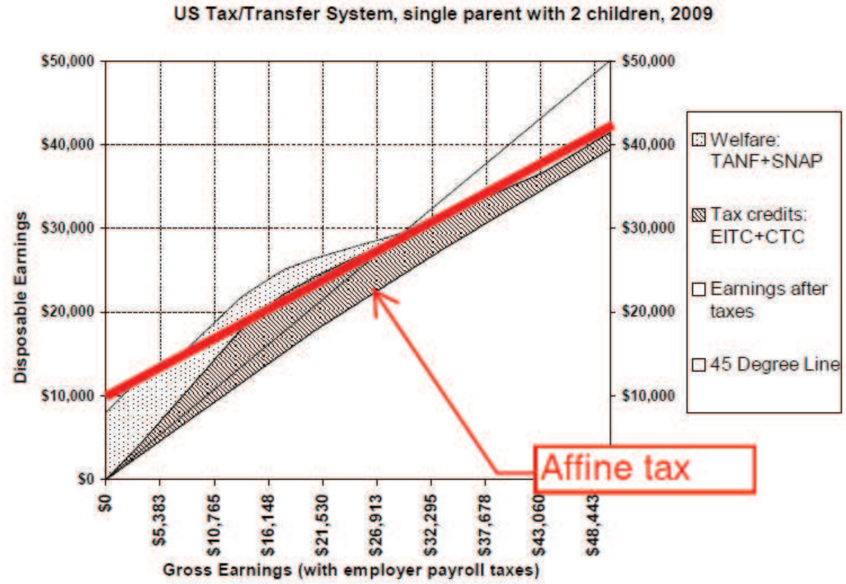

Figure 1: The U.S. tax code is poorly approximated as linear, better approximated as affine.

as a lump-sum tax (or subsidy). The analysis of Ramsey taxes and debt under affine taxes reveals forces that also drive outcomes under a broader class of nonlinear taxes on labor coupled with other restrictions on taxes that prevent the government from effectively completing debt markets by taxing returns on assets.

Both quantitatively and qualitatively, paths of optimal taxes and government debt in the heterogeneous agent economies differ substantially from those previously obtained in representative agent models. We show that while optimal labor distortions are persistent and history dependent, they generally have no downward drift. Optimal government debt is indeterminate. More specifically, in an economy with I agents and a government, debts of one of the $I + 1$ agents can be normalized to be zero always. While agents' trading of risk-free assets facilitates smoothing distortions, the Ramsey planner doesn't care whether private agents smooth those distortions by trading only with each other or by trading also with the government.

The different prescriptions for the optimal taxes come from the presence or absence of non-negativity restrictions on transfers and their distinct ramifications in the two types of economies. In homogeneous agent economies, lump sum *taxes* are the optimal way to finance government expenditures, and to rule them out a constraint that lump sum transfers must be non-negative is exogenously imposed. This restriction gives government debt an important role in helping to reduce the planner's Lagrange multiplier on this constraint. Under the optimal policy, the government accumulates assets to relax future non-negativity constraints on transfers. When interest rates equal the planner's discount rate, as in the benchmark quasi-linear example in AMSS, the government stops accumulating assets only when an undistorted 'first best' tail allocation is attained. In contrast, in heterogeneous agent economies either when the government wants enough redistribution or when the lowest skilled agents are sufficiently poor, even if imposed, a non-negativity constraint on transfers never binds. That completely disarms the precautionary motive that the government in a representative agent model has to accumulate assets for the purpose of decreasing labor tax distortions over time.

After studying affine taxes, we consider optimal taxes when the government has access to richer sets of tax instruments. A planner constrained only by its inability to observe agents' skills directly would choose allocations that depend only on the current realization of the aggregate shock so that distortions are not history dependent. We show that these optimal allocations can be achieved only if the planner has access to either a non-linear tax on asset holdings or a state-dependent tax on asset holdings; that is, ironically, only if the planner can either shut down or complete the incomplete asset markets. When the government can't do that, optimal distortions are history-dependent.

2 Environment

There are I types of infinitely lived agents. There is a mass π_i of a type $i \in I$ agent, with $\sum_{i=1}^{I} \pi_i = 1$. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}$ and labor supply $\{l_{i,t}\}$ are ordered by

$$\mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t U^i\left(c_{i,t}, l_{i,t}\right) \tag{1}$$

where $\beta \in (0,1)$ is a discount factor and, except in some special examples, $U^i : \mathbb{R}_+^2 \to \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. Let $U^i_{x,t}$ or $U^i_{xy,t}$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t. We assume that $l_i \in \left[0, \bar{l}_i\right]$ and that $\lim_{x \to \bar{l}_i} U^i_l\left(c, x\right) = \infty$, $\lim_{x \to 0} U^i_l\left(c, x\right) = 0$ for all c and i.

An agent of type i who supplies l_i units of labor supply produces $\theta_i l_i$ units of output, where $\theta_i \in \Theta$ is a nonnegative scalar. Let g_t denote government purchases. Feasible allocations satisfy

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i \theta_i l_{i,t}. \tag{2}$$

Government expenditures g_t follow an irreducible finite state process Markov. Let $s_t = g_t$ and $s^t = \left(s_0, ..., s_t\right)$. We use two notations to keeps track of histories. Most of the time we use a notation z_t to denote a random variable with a time t conditional distribution that is a function of the history g^t. In some places, we use a more explicit notion $z\left(s^t\right)$ to denote a realization of the stochastic process z_t at a particular history s^t.

The government's preferences over stochastic process for consumption and work are ordered by

$$\mathbb{E}_0 \sum_{i=1}^{I} \pi_i \alpha_i \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, l_{i,t}\right) \tag{3}$$

where $\alpha_i \geq 0$, $\sum_{i=1}^{I} \alpha_i = 1$ is a set of Pareto weights.

2.1 Incomplete markers and affine taxes

Under an affine tax system, agent's i budget constraint is

$$c_{i,t} + b_{i,t} = \left(1 - \tau_t\right) \theta_i l_{i,t} + R_{t-1} b_{i,t-1} + T_t \tag{4}$$

where $b_{i,t}$ denotes asset holdings of agent i in time $t \geq 0$, R_{t-1} is a gross risk-free one-period interest rate from time $t-1$ to time t for $t \geq 1$ and $R_{-1} \equiv 1$. For $t \geq 0$, R_t is measurable with respect to time t information. To rule out Ponzi schemes, we assume that $b_{i,t}$ must be bounded from below.

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^{I} \pi_i \theta_i l_{i,t} - T_t + R_{t-1} B_{t-1}, \tag{5}$$

where B_t denotes government assets at time t. We assume that government debt must be bounded from below. Our assumptions about preferences imply that the government can collect finite revenues in each period, so this restriction rules out Ponzi schemes for the government.

We assume that agents $i \in I$ and the government start with initial assets $\{b_{i,-1}\}_{i=1}^{I}$ and B_{-1}, respectively, and that asset holdings satisfy

$$\sum_{i=1}^{I} \pi_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \tag{6}$$

Since all $b_{i,t}$ and B_t are bounded from below, constraint (6) implies that they must also be bounded from above.

We allow the government to choose a feasible sequence of transfers $\{T_t\}$ and do not restrict their signs at any particular dates or histories. For example, if Pareto weight $\alpha_i > 0$ and type i has $\theta_i = 0$ and no initial wealth, the present value of transfers to agent i must necessarily be nonnegative. All results in the present paper include this example as a special case.

Remark: We choose this structure of taxes and markets to preserve key tensions that pervade economies with incomplete markets. We know that if the government can trade risk-free debts of different maturities (as in Angeletos (2002) or Buero and Nicolini (2004)) or if it can tax bonds in response to the shocks (as in Chari, Christiano, and Kehoe (1994)), it can effectively replicate equilibrium allocations for a complete market economy. To preserve market incompleteness, we rule out these instruments. Similarly, it is well know that a combination of consumption and labor taxes can replicate taxes on debt, so we rule out consumption taxes too. We conjecture that most of our results will continue to hold if we allow those taxes but require that they be measurable functions of information available one period in advance.

We now turn to the definition of equilibrium.

Definition 1 *For given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$, *a competitive equilibrium with affine taxes is a sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *such that* $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ *maximizes (1) subject to (4),* $\{\{b_{i,t},\}_i, B_t\}_t$ *are bounded, and constraints (2), (5) and (6) are satisfied.*

When equilibrium allocations are interior, standard arguments show that necessary conditions for consumer optimality are

$$(1-\tau_t)\,\theta_i U^i_{c,t} = -U^i_{l,t} \tag{7}$$

and

$$U^i_{c,t} = \beta R_t \mathbb{E}_t U^i_{c,t+1}. \tag{8}$$

Unless otherwise stated, we assume that an equilibrium is interior. To characterize an equilibrium, we require

Lemma 1 *Any sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ *is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (7), and (8) and* $b_{i,t}$ *is bounded for all* i *and* t.

Proof. Necessity is obvious. In the technical appendix we use arguments of Magill and Quinzii (1994) and Constantinides and Duffie (1996) to show that any $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (4), (7) and (8) is a solution to consumer i's maximization problem. Equilibrium B_t is then determined by (6) and constraint (5) is then implied by Walras' Law ■

We are now ready to discuss how taxes should optimally respond to aggregate shocks. To find the optimal equilibrium, by Lemma 1 we can choose $\{\{c_{i,t}, l_{i,t}\}_{i,t}, b_{i,t}, R_t, \tau_t, T_t\}$ to maximize (3) subject to (2), (7), and (8). We follow steps similar to ones taken by Lucas and Stokey (1983) and AMSS and apply a first-order approach. Substituting consumers' first-order conditions (7) and (8) into the budget constraints (4) yields implementability constraints of the form

$$c_{i,t} + b_{i,t} = -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + T_t + \frac{U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} U^i_{c,t}} b_{i,t-1} \text{ for all } i, t. \tag{9}$$

Remark: For $I \geq 2$, we can use constraint (9) for one of the agents, e.g. $i = 1$, to eliminate T_t from (9) for all other agents. Define $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$ and represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ = \quad & -\frac{U^i_{l,t}}{\theta_i U^i_{c,t}} (\theta_i l_{i,t} - \theta_1 l_{1,t}) + \frac{U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} U^i_{c,t}} \tilde{b}_{i,t-1} \text{ for all } i > 1, t. \end{aligned} \tag{10}$$

With this representation of the implementability constraints, the maximization problem depends only on the $I - 1$ variables $\tilde{b}_{i,t-1}$. Therefore, any combinations of $\{\{b_{i,t},\}_i, B_t\}_t$ that satisfy (6) and give rise to the same $\left\{\tilde{b}_{i,t}\right\}_{i,t}$ support the same allocations.

The remark leads to

Proposition 1 *(i) We are free to normalize the asset holdings $\{b_{i.t}\}_i$ of one of the agents i or B_t of the government to zero in all t. (ii) For all initial distributions of assets $(\{b_{i,-1}\}_i, B_{-1})$ for which $\left\{\tilde{b}_{i,-1}\right\}_{i=2}^{I}$ is fixed, optimal allocations are identical.*

The first part of proposition 1 asserts that the sequence $(\{b_{i,t},\}_{i,t}, B_t)$ is indeterminate. This is a version of Ricardian equivalence.[2] Given any equilibrium, for example, one in which the government occasionally does not balance its contemporaneous budget, there exists another equilibrium with the same allocation and interest rate sequence, but in which the government runs a balanced budget at all dates and all histories. The economics driving this outcome underlies the classic insight of Barro (1974): for a sequence of government surpluses (deficits), of constant present value, the government could increase (reduce) transfers today and reduce (increase) transfers tomorrow, appropriately adjusting for the costs of servicing its one-period debt, uniformly across across agents. This would leave the present value of distorting taxes for *each agent* unchanged and allow each agent to preserve his consumption and labor supply streams by adjusting only his path of assets.

These Ricardian equivalence statements do *not* say that "debt is irrelevant" for optimal transfers and distorting taxes. Rather, it means that the Ramsey planner is indifferent between, on the one hand, adjusting government debt and, on the other hand, letting agents trade debt

[2] Also see Werning (2007) for a related proof of Ricardian equivalence in a complete market environment.

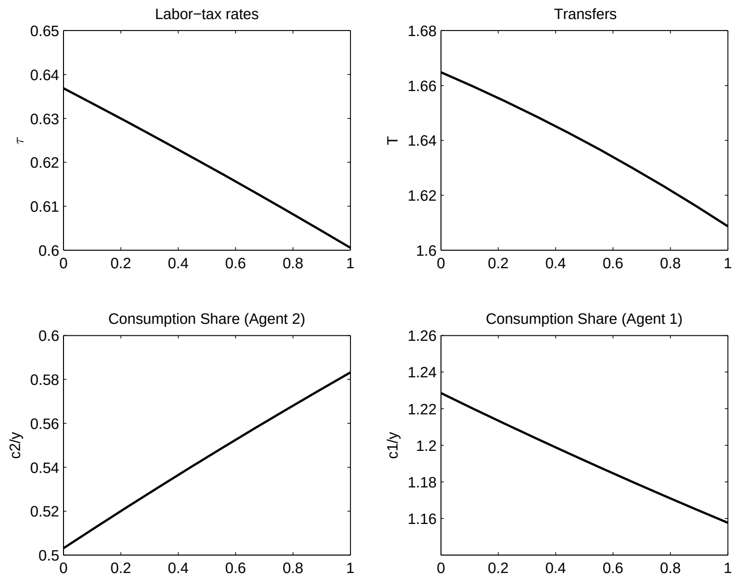

Figure 2: Distorting taxes τ, transfers T, consumption of high θ type, and consumption of low θ type, all as functions of fraction of initial government debt owned by low θ type 2 agent.

privately among themselves to smooth responses to aggregate shocks. As we will highlight in the examples to be discussed below, debt markets are used extensively to smooth fluctuations in taxes.

The second part of the proposition shows that by itself the initial level of government debt B_{-1} is not informative about how distortionary are the taxes levied to service it. To estimate the magnitudes of distortions that a given level of government indebtedness causes, one needs to know the *distribution* of asset holdings, $\{b_{i,-1}\}_i$ across private agents. In a nutshell, what matters is not just how much debt government the government owes, but who owns it.

2.2 Nonstochastic stationary example

Figures 2, 3, 4, and 5 show outcomes for a simple nonstochastic stationary economy. We have set parameters so that government purchases are about 10 percent of what would be GDP without distorting taxes, and we have set productivities and Pareto weights so that after tax wages for the high type are about 5 times what they are for the low type, an approximation to the 90-10 decile ratio in the U.S. There are two types with $\theta_1 = 8$ and $\theta_2 = 1$. The measures of the two types of workers are $\pi_1 = .5$ and $\pi_2 = .5$. Preferences of both types are ordered by (1) with $\beta = .98$ and

$$U^i = \frac{c_{it}^{1-\sigma}}{1-\sigma} - \frac{l_{it}^{1+\gamma}}{1+\gamma}$$

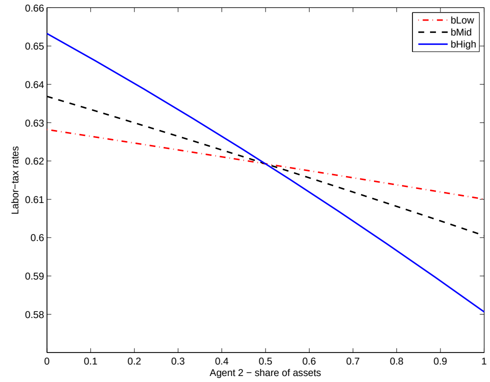

Figure 3: Distorting labor tax for different levels of initial government debt, all as functions of fraction of initial government debt owned by low θ type 2 agent.

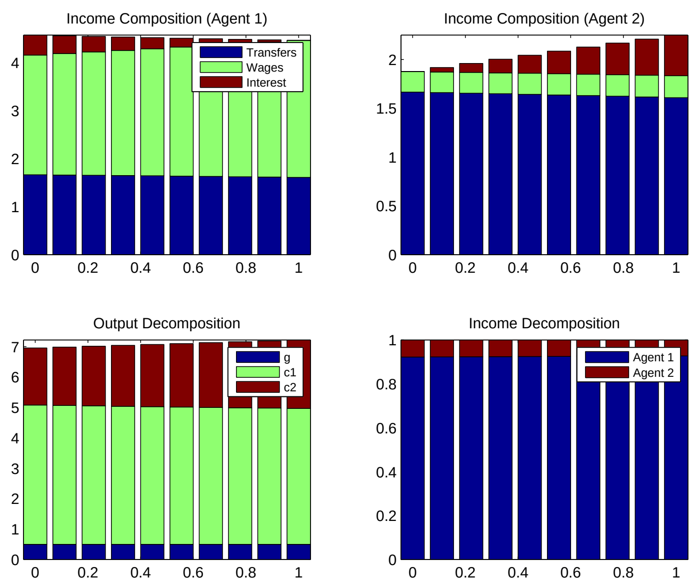

Figure 4: Top panels: sources of income for high θ and low θ types; lower panels, decomposition of output by use and distribution of income across types, all as functions of fraction of initial government debt owned by low θ type 2 agent.

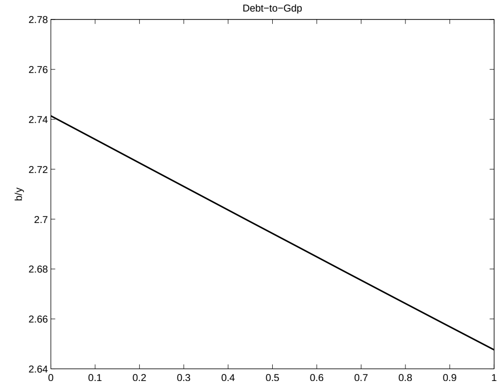

Figure 5: Debt to GDP ratio as function of fraction of initial government debt owned by low θ type 2 agent.

with $\sigma = 1$ and $\gamma = .3$. Government expenditures $g_t = 1$ for all t. Initial government debt is $B = 10$, which is double undistorted full employment GDP. The relatively unskilled type 2 agents initially own a fraction $x \in [0, 1]$ of the initial government debt. Pareto weights are $\alpha_1 = .5, \alpha_2 = .5$. To normalize, we assume that the two private agents do not borrow or lend with each other, only with the government. We take $\tilde{b}_2$ as an initial condition. Because g is nonstochastic and constant, there are complete markets.

Pareto optimal allocations have constant c_1, c_2, τ, T, all of which are functions of $g, \tilde{b}_2$; $R = \beta^{-1}$ Figure 2 shows τ and T as functions of x as well as the consumption levels of the two types as functions of the fraction x of government debt initially in the hands of the low θ type 2 agent. Figure 3 shows τ as a function of x for three levels of initial government debt $-B$. Figure 2 shows the two agents' sources of income as functions of x.

The government sets affine taxes to finance g and to transfer from the high θ type to the low θ type through two types of transfers: T, the constant in the affine tax schedule, and $\pi_1 RxB$, the interest payments on the government debt received by the low θ type. Figures 2, 4, and 5 show outcomes. Figure 2 shows that the government sets a lower distorting tax rate τ and a lower explicit transfer T_t, the higher is x. Higher levels of initial government debt steepen the slopes of the τ on x curve because the larger is B, the more potent interest payments become as a means of subsidizing the low θ type.

Remark: A downward slope of the labor tax as a function of the fraction x of initial government debt in the hands of the low θ type agent prevails so long as the Pareto weight α_1 attached to the high θ agent is sufficiently high (.15 or above with the other parameters set at the values for our figures). For a fixed α_1, the consumption share of the low θ agent 2 rises with his share

x of initial government debt. The downward slope of the distorting tax function requires that his interest earnings can rise enough as his share of initial assets rises. When α_1 is too low, his interest earnings can be too low, meaning that rising consumption of a type 2 agent as a function of x might have to be achieved with higher transfers and hence higher labor taxes.

3 Optimal affine taxes

3.1 Lagrangian formulation

Though economic outcomes differ markedly, the mathematical structure of the Ramsey problem in our heterogeneous agent incomplete markets economy with affine labor taxes resembles that for the representative agent economies with linear taxes studied by AMSS and Fahri (2010). We can follow their steps of analysis. Multiply (10) by $U^i_{c,t}$ and define $\tilde{a}_{i,t} \equiv \tilde{b}_{i,t} U^i_{c,t}$ to obtain

$$U^i_{c,t}\left(c_{i,t} - c_{1,t}\right) + \frac{U^i_{l,t}}{\theta_i}\left(\theta_i l_{i,t} - \theta_1 l_{1,t}\right) + \tilde{a}_{i,t} = \frac{U^i_{c,t}}{\beta \mathbb{E}_{t-1} U^i_{c,t}} \tilde{a}_{i,t-1} \quad \text{for all } i > 1, t. \tag{11}$$

Let $\beta^t \Pr\left(s^t\right) \psi_i\left(s^t\right)$ be a Lagrange multiplier on this constraint in a Lagrangian for the Ramsey planner. First order conditions with respect to $\tilde{a}_i(s^t)$ imply

$$\begin{aligned} \psi_{i,t} &= \left(\mathbb{E}_t\left[U^i_{c,t+1}\right]\right)^{-1} \mathbb{E}_t\left[U^i_{c,t+1}\psi_{i,t+1}\right] \\ &= \mathbb{E}_t \psi_{i,t+1} + \left(\mathbb{E}_t\left[U^i_{c,t+1}\right]\right)^{-1} Cov_t\left(U^i_{c,t+1}, \psi_{i,t+1}\right). \end{aligned} \tag{12}$$

This is a multi-agent counterpart of equation (17) of AMSS (2002) for their economy with a representative agent and linear tax on labor. The Lagrange multiplier $\psi_{i,t}$ measures the distortion of the tax system, since when $\psi_{i,t} = 0$ for all i, the distorting tax τ_t equals zero. Equations like (12) are usually interpreted to imply that distortions follow a "random-walk-like" process, partly confirming Barro's (1979) insight about tax smoothing. In representative agent economies with linear taxes, they also can imply that asymptotically optimal distortions decline (possibly to zero) while government asset holdings grow[3]. We have already seen from Proposition 1 that, with heterogeneous agents and affine taxes, government assets are indeterminate and therefore by themselves play no necessary role in shaping equilibrium allocations and distortions. Nevertheless, the next section reveals a Barro-AMSS-like insight about smoothing distortions that holds with affine taxation despite the fact that salient outcomes about long run levels of government held assets and tax distortions obtained by AMSS and Fahri (2010) under linear taxes evaporate.

[3] See, e.g. Aiaygari et al (2002) and Marcet et al (2011).

3.2 Bellman equation

In the spirit of Kydland and Prescott (1980) and Farhi (2010), we can formulate the planner's problem recursively. Let $\tilde{\mathbf{b}} = \left(\tilde{b}_2, ..., \tilde{b}_I\right)$ and $\mathbf{u} = \left(U_c^1, ..., U_c^I\right)$. The planner's Bellman equation is

$$V(\tilde{\mathbf{b}}, \mathbf{u}, g_-) = \max_{\tilde{\mathbf{b}}', \mathbf{c}, \mathbf{l}} \sum_s \Pr(g|g_-) \left[\sum_i \pi_i \alpha_i U^i(g) + \beta V(\tilde{\mathbf{b}}', \mathbf{U}_c^i(g), g)\right] \tag{13}$$

$$(c_i(g) - c_1(g)) + \tilde{b}_i'(g) + \frac{U_l^i(g)}{\theta_i U_c^i(g)} (\theta_i l_i(g) - \theta_1 l_1(g)) = \frac{u_i}{\beta \mathbb{E}_{s_-} U_c^i(g)} \tilde{b}_i \tag{14}$$

$$\frac{\sum_s \Pr(g|g_-) U_c^i(g)}{u_i} = \frac{\sum_s \Pr(g|g_-) U_c^j(g)}{u_j} \ \forall i, j, g, \tag{15}$$

$$\frac{U_l^i(g)}{\theta_i U_c^i(g)} = \frac{U_l^j(g)}{\theta_j U_c^j(g)} \ \forall i, j, g, \tag{16}$$

$$\sum_i \pi_i c_i(g) + g = \sum_i \pi_i \theta_i l_i(g) \ \forall g. \tag{17}$$

Given $\tilde{b}_{-1}, g_0$, the planner chooses $\mathbf{u}$ to maximize $V(\tilde{\mathbf{b}}_{-1}, \mathbf{u}, g_0)$ subject to (16) and (17) for $t = 0$. We will use this formulation to calculate Ramsey plans numerically. Before doing that, we characterize some special cases that highlight the main economic forces that drive optimal taxes and allocations.

3.3 Quasi-linear preferences

AMSS obtained sharp results when preferences of their representative agent are quasi-linear in consumption, i.e.,

$$U^i(c, l) = c - h_i(l). \tag{18}$$

The AMSS case of a representative agent with affine taxes is a convenient benchmark against which to compare optimal allocations in our heterogeneous agent economy. In AMSS, the government acquires a precautionary motive to accumulate assets. With quasi-linear preferences, eventually the government finances all revenue needs from its earnings on these assets. The labor tax τ_t follows a persistent process that converges to zero.

We temporarily adopt a quasi linear preference specification for each type of agent $i \in I$ in our heterogeneous agent economy with affine taxes. We assume that for all types i there is a common finite lower bound on consumption, $\underline{c} \leq 0$, potentially arbitrarily low[4]

$$c \geq \underline{c}. \tag{19}$$

[4] The reader can think of $\underline{c}$ as being zero. We use a more general bound to show that some of our results will hold even for arbitrarily low $\underline{c}$.

In our heterogeneous agent economy with affine taxes, Proposition 1 implies outcomes for government debt that are very different from AMSS. When preferences are quasi-linear, the covariance term in (12) equals zero, so application of a martingale convergence theorem allows a sharp characterization of the long-run joint distribution of assets and consumption across agents.[5]

To contrast outcomes in our environment with those in AMSS's, we restrict our attention to parameters for which the equilibrium is interior in the sense that constraint (19) does not bind. It is easy to show that there is a large set of parameters $\{\alpha_i, \theta_i, g\}$ that verify interiority. In the next section, we will consider an economy that does not assume interiority.

Proposition 2 *Suppose that preferences are of form (18) for all i and that the equilibrium is interior. Suppose that h_i satisfies*[6]

$$0 \leq h_i''' \leq \left(h_i''\right)^2 / h_i' \text{ for all } i. \tag{20}$$

Then the optimal tax, τ_t^*, satisfies $\tau_t^* = \tau^*$ and an optimum debt pattern $\left\{b_{i,t}^*, B_t^*\right\}_{i,t}$ can be chosen to satisfy $b_{i,t}^* = b_{i,-1}$ for all $i,\ t \geq 0$ and $B_t^* = B_{-1}$ for all $t \geq 0$.

Proof. [**The algebra needs to be double checked**] When equilibrium allocations are interior, the first order condition (7) becomes $(1-\tau_t)\,\theta_i = h_i'(l_{i,t})$. We can invert function $h_i'(\cdot)$ to find labor supply l_i as a function of $(1-\tau)$. Call this function $H_i(1-\tau)$. Note that $H_i' > 0$.

For our purposes it is more convenient to express all labor allocations as a function of $(1-\tau)$ and optimize with respect to τ rather than $\{l_i\}_i$. When the equilibrium allocations are interior, $R = 1/\beta$ and the implementability constraint (9) becomes

$$c_{i,t} + b_{i,t} - (1-\tau_t)\,H_i(1-\tau_t) = T_t + \beta^{-1} b_{i,t-1}. \tag{21}$$

We can find optimal allocations by maximizing

$$\max_{\{c_{i,t}, b_{i,t}, \tau_t, T_t\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^{I} \alpha_i \pi_i \beta^t \left[c_{i,t} - h_i\left(H_i(1-\tau)\right)\right]$$

subject to (21) and

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i \theta_i H_i(1-\tau_t).$$

[5] In addition to the papers discussed above, see also work of Farhi (2010), Battaglini and Coate (2007, 2008) who use this property of quasilinear preferences to characterize the long run distribution.

[6] It can be easily verified that this condition is satisfied for example when preferences exhibit constant elasticity of substitution $c - \frac{1}{\gamma} l^\gamma$ with $\gamma \geq 2$.

Let $\beta^t \pi_i \eta_{i,t}$ and $\beta^t \lambda_t$ be the Lagrange multipliers on the feasibility and the implementability constraints. The first order conditions for $c_{i,t}$ and T_t imply that the Lagrange multipliers do not depend on t, and in particular that $\lambda_t = 1$ for all t, and that $\eta_{i,t} = 1 - \alpha_i$. The first order condition with respect to $(1-\tau)$ is

$$\sum_{i=1}^{I} \alpha_i \pi_i \left[h_i' \left(H_i \left(1-\tau\right)\right) H_{i,t}' \right] + \sum_{i=1}^{I} \pi_i \eta_i \left(H_{i,t} + \left(1-\tau_t\right) H_t' \right) - \lambda \sum_{i=1}^{I} \pi_i \theta_i H_{i,t}' = 0$$

Since $h_i' \left(H_i \left(1-\tau\right)\right) = \left(1-\tau_t\right) \theta_i$, this equation can be written as

$$\sum_{i=1}^{I} \left[\alpha_i \pi_i \theta_i + \pi_i \eta_i \right] \left(1-\tau_t\right) H_{i,t}' + \sum_{i=1}^{I} \pi_i \eta_i H_{i,t} = \lambda \sum_{i=1}^{I} \pi_i \theta_i H_{i,t}' \tag{22}$$

Under the assumptions of the theorem, the left side of equation (22) is increasing in $(1-\tau_t)$ while the right side is increasing. Therefore there exists a unique τ^* that solves equation(22).[7] This τ^* pins down a unique constant equilibrium labor supply l_i^*. Let $b_{i,t}^* = b_{i,-1}$ for all t and define

$$c_{i,t}^* = \frac{1-\beta}{\beta} b_i^* - h_i' \left(l_i^*\right) l_i^* + T_t^*$$

where T_t^* solves

$$\sum_{i=1}^{I} \pi_i \left(\frac{1-\beta}{\beta} b_i^* - h_i' \left(l_i^*\right) l_i^* + T_t^* \right) + g_t = \sum_{i=1}^{I} \pi_i \theta_i l_i^*. \tag{23}$$

By inspection, T_t^* depends only on g_t and not on the history g^{t-1}. ■

In the affine-tax proposition 2 economy, fluctuations in lump sum taxes and transfers do all the work. Furthermore, if the planner wants to redistribute enough towards low skilled types, this lump sum component can be positive at all dates and states. In such an economy, the planner always uses lump sum *transfers* and never uses lump sum *taxes*, so even if we had imposed the AMSS constraint $T_t \geq 0$, that constraint would never bind.

The Lucas and Stokey (1983) and AMSS (2002) representative agent models impose $T_t = 0$. Outcomes would be unaltered if the restriction $T_t = 0$ were to be weakened to $T_t \geq 0$, because

[7] [**I leave it here for now to make it easy to double check my arguments**] To double check this analysis, we have $H(h'(l)) = l$, therefore $H'h'' = 1$ and $H'' \left(h''\right)^2 + H'h''' = 0$. If $h''' \geq 0$ then $H'' \leq 0$ which shows that the right hand side of (22) is decreasing in $(1-\tau)$. On the other hand, we have $(1-\tau) H' = h'/h''$, and, since l is monotonically increasing in $(1-\tau)$, $(1-\tau) H'$ increases in $(1-\tau)$ if and only if h'/h'' increases in l. We have

$$\frac{\partial}{\partial l} \left(h'/h'' \right) = \frac{\left(h''\right)^2 - h'h'''}{\left(h''\right)^2},$$

which is positive if $h''' \leq \left(h''\right)^2 / h'$.

When $h(l) = \frac{1}{\gamma} l^\gamma$, $h' = l^{\gamma-1}, h'' = (\gamma-1) l^{\gamma-2}$ and $h''' = (\gamma-1)(\gamma-2) l^{\gamma-3}$, we have that $h''' \geq 0$ if $\gamma \geq 2$ and

$$\left(h''\right)^2 - h'h''' = (\gamma-1) \left[(\gamma-1) l^{2\gamma-4} - (\gamma-2) l^{2\gamma-4} \right] > 0.$$

a government in those models has no incentive to use distorting taxes to finance positive lump sum *transfers*. Instead, the government would like to impose lump sum *taxes*. Distributional motives make the situation become very different in our model.

3.4 Comparison with representative agent economies

Authors who study optimal taxation and debt in representative agent economies routinely assume that, in addition to setting linear taxes on labor, the government can also provide lump sum transfers, but not impose lump sum taxes.[8] This is equivalent to adding one restriction to the tax system that we studied so far, namely,

$$T_t \geq 0 \text{ for all } t. \tag{24}$$

Relative to that literature, we have taken two departures. We assumed that consumers are heterogeneous and we dropped constraint (24). A natural question is: which of the two departures drive outcomes in Propositions 1 and 2? We answer this question by reporting in the following way. First, we show that if there exists a type who is sufficiently "poor", then constraint (24) does not bind in the optimal tax problems that we studied so far. Second, we study a version of our quasi-linear economy with the added restriction (24) to highlight key differences between heterogeneous and homogeneous agent models.

Constraint (24) need not bind

First, we show some conditions under which constraint (24) does not bind.

Proposition 3 *Suppose that a Ramsey planner chooses optimal taxes in the heterogenous agent economy of section 2 subject to the additional constraint (24). Suppose that* $\underline{c} = 0$ *and that there is some type* j *for whom* $\theta_j = 0$ *and* $b_{j,-1} \leq 0$*. Then constraint (24) does not bind.*

Proof. Let $\left\{c^*_{i,t}, l^*_{i,t}, b^*_{i,t}, R^*_t, \tau^*_t, T^*_t\right\}_{i,t}$ be the optimal competitive equilibrium when T_t is unresticted. Note that since $\theta_j = 0$,

$$c^*_{j,t} = R^*_{t-1} b^*_{j,t-1} + T^*_t - b^*_{j,t} \geq 0.$$

Let $\hat{T}_t = R^*_{t-1} b^*_{j,t-1} + T^*_t - b^*_{j,t}$, $\hat{b}_{j,t} = 0$ and construct $\hat{b}_{i,t}$ for $i \neq j$ as in the proof of Proposition 1. Then $\left\{c^*_{i,t}, l^*_{i,t}, \hat{b}_{i,t}, R^*_t, \tau^*_t, \hat{T}_t\right\}_{i,t}$ is a competitive equilibrium that satisfies the additional constraint (24). ■

[8] See, e.g. AMSS (2002).

This proposition shows that constraint (24) generally does not bind when some agents cannot afford to pay lump sum taxes. We could prove a similar proposition stating that if the planner wants enough redistribution, in the sense that he assigns a sufficiently high Pareto weight to types with low present values of income, then constraint (24) typically does not bind.

When constraint (24) won't bind eventually

It is also instructive to study a version of the quasi-linear economy of section 3.3 with the additional restriction that taxes must satisfy constraint (24). We maintain the assumption that equilibrium allocations are interior. When $I = 1$ this economy is identical to one studied in Section III of AMSS (2002).

When constraint (24) binds, it is no longer the case that Proposition 1 holds and that only the net position of assets $\left\{\tilde{b}_{i,t}\right\}_{i>1,t}$ determines optimal allocations. As a result, we cannot obtain implementability constraints like (11). Instead, we will have an implementability constraint for each type, namely,

$$c_{i,t} + b_{i,t} = h_i'(l_{i,t})\, l_{i,t} + \beta^{-1} b_{i,t-1} + T_t. \tag{25}$$

The altered optimal tax problem is

$$\max_{\{c_{i,t}, b_{i,t}, l_{i,t}, T_t\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^{I} \alpha_i \pi_i \beta^t \left[c_{i,t} - h_i(l_{i,t})\right] \tag{26}$$

subject to (25), (24), (2) and

$$h_i'(l_{i,t}) / \theta_i = h_j'(l_{j,t}) / \theta_j \text{ for all } i, j.$$

Proposition 4 *Suppose h_i satisfies (20). Let $\beta^t \chi_t$ be the Lagrange multiplier on constraint (24) in maximization problem (26). Then $\chi_t \to 0$ a.s.*

Proof. Let $\beta^t \pi_i \eta_{i,t}$ and $\beta^t \lambda_t$ be the Lagrange multipliers on the feasibility and the implementability constraints. First-order conditions for $c_{i,t}$, $b_{i,t}$ and T_t are

$$\alpha_i + \eta_{i,t} = \lambda_t$$

$$\eta_{i,t} = \mathbb{E}_t \eta_{i,t+1}$$

and

$$\sum \pi_i \eta_{i,t} = \chi_t \geq 0. \tag{27}$$

These constraints imply that λ_t is a positive martingale and therefore converges to a constant a.s. This, in turn, implies that $\eta_{i,t}$ and χ_t converge to constants a.s. and, under assumption

(20), that $l_{i,t}$ and $(1-\tau_t)$ converge a.s. to some long run values l_i^* and $(1-\tau^*)$ Constraint (2) implies that some $c_{i,t}$'s must fluctuate to offset fluctuations in g_t. If $\chi_t \to \chi > 0$, then $T_t \to 0$. Summing (25) gives

$$\sum \pi_i \theta_i l_i^* - g_t + B_t = \sum \pi_i h_i' \left(l_i^*\right) l_i^* + \beta^{-1} B_{t-1}.$$

For any bound on B_t we can find a sequence of shocks g_t so that eventually this bound will be violated, leading to a constradiction. This implies that $\chi_t \to 0$. ■

This proposition highlights the key force behind the long run results obtained by AMSS (2002). Since the risk-free interest rate equals the discount rate, a Ramsey planner who faces constraint (24) and who is in a setting in which the constraint threatens to bind in the future always wants to save a bit more to relax future constraints (24). This motive endures until the planner has saved enough to render all future constraints (24) slack. In the representative agent economy of AMSS, constraint (24) binds each period until the government has acquired enough assets that it never again has to use distortionary taxes τ_t. This explains the AMSS (2002) result that the government collects no taxes in the long run. When agents are heterogenous and the government cares about redistribution, things can be very different. As we have discussed earlier in this section, in some settings with heterogenous agents, constraints (24) do not bind, and as a result the government has no reason to accumulate assets or to smooth distortions imperfectly over time.

3.5 Distortion smoothing with risk-aversion

Our quasi-linear example is limiting in at least one important respect. Since all agents are risk-neutral, they are indifferent to fluctuations that leave the ex-ante present value of consumption unaltered. With risk-averse agents, things becomes more complicated because it is more difficult to isolate a pure "labor distortion", since the distortion generated by τ_t will depend partly on agents' diverse accumulations of assets. Their asset accumulations will in turn depend on their aversions to risk and the implied precautionary motives. In general, analysis of such economies requires numerical computations. To highlight the main economic forces, it is possible to construct special economies that still allow us to separate "labor distortion" and "risk aversion" effects. We accomplish this by assuming that some agents' decisions are adversely affected only by a fluctuating distorting labor tax, while others are affected only by their aversion to consumption risk.

Here is a simple example of such an economy. There are only two types of agents. A type 1 agent has quasilinear preferences as in the previous section with $\theta_1 = 1$, while a type 2 agent is risk averse and has $\theta_2 = 0$; his preferences can be represented with a strictly concave, twice differentiable utility function $u(c_{2,t})$ that satisfies Inada conditions. We call this an *AMSS-like economy*. Higher curvature in u makes fluctuations in $c_{2,t}$, and hence in transfers T_t, more costly.

This simple AMSS-like economy highlights key forces governing optimal taxes and transfers with incomplete markets. Optimal allocations and taxes are generally *history-dependent*, in the sense that the optimal allocations at time t depend not only on the current realization of the government expenditure shock g_t but also on the history of shocks. This result contrast sharply both with the complete market economies of Lucas and Stokey (1983) and Werning (2007) and the constrained optima to be discussed in section 6.1, where the optimal allocations in period t depend only on g_t. We also use this economy to highlight different ways that taxes, transfers, and debts can adjust to aggregate shocks. We show these same forces again in numerical examples for more general economies.

Proposition 5 *Suppose that there is unique $\hat{l}$ that solves $h''(\hat{l})\hat{l} + h'(\hat{l}) = 1$. Let $c^*_{1,t}$ be an optimal allocation of consumption of the risk-neutral agent 1 in the AMSS-like economy. Then $c^*_{1,t} = \underline{c}$ infinitely often almost surely.*

Proof. We show this result by contradiction. Suppose that *(19)* does not bind after some period $\bar{\mathcal{T}}$. Then the interest rate that period is β^{-1} and, since u satisfies Euler equation,

$$u'(c_t) = \mathbb{E}_t u'(c_{t+1}).$$

Since the optimal allocations in period t are recursive in $\left(\tilde{b}_{1,t-1}, u_c(c_{t-1})\right)$, the optimal allocations after $\bar{\mathcal{T}}$ can be found by solving the following optimization problem

$$\max_{\{c_1,c_2,l_1,\tilde{b}\}} \mathbb{E}_{\mathcal{T}} \sum_{t=\bar{\mathcal{T}}+1}^{\infty} \beta^{t-\bar{\mathcal{T}}-1} \left[\alpha_1 \left(c_{1,t} - h(l_{1,t})\right) + \alpha_2 u(c_{2,t})\right] \tag{28}$$

subject to the constraints that the sequence $\left\{\tilde{b}_{i,s}\right\}_{i,s>\bar{\mathcal{T}}}$ is bounded and

$$c_{2,t} - c_{1,t} + \tilde{b}_t + h'(l_{1,t})l_{1,t} = \frac{1}{\beta}\tilde{b}_{t-1}, \tag{29}$$

$$c_{1,t} + c_{2,t} + g_t = l_{1,t}, \tag{30}$$

$$u_{c,t} = \mathbb{E}_t u_{c,t+1}, \tag{31}$$

$$u_{c,\mathcal{T}} = \bar{u} \tag{32}$$

Equation (31) implies that $u_{c,t}$ is a supermartingale and therefore converges. It cannot converge to zero, since then $c_{2,t}$ would diverge to infinity and violate (30) (this follows since $c_{1,t}$

is bounded from below and $l_{1,t}$ cannot diverge to infinity). Therefore, $u_{c,t}$ and $c_{2,t}$ both converge to finite values, so $c_{2,t} \to c_2^*$. Consumption of agent 1, $c_{1,t}$, is then determined as a residual from (30) and follows the same Markov process as g_t.

In the appendix, we show that the Lagrange multiplier η_t on constraint (29) must converge to some finite value η^*. Then the first order conditions for $c_{1,t}$ imply that the multiplier on the feasibility constraint (30), λ_t, must also converge to a finite value $\lambda^* = \alpha_1 - \eta^*$. The first order condition for l_1

$$h'(l_{1,t}) \left(\alpha_1 - \eta_t \left(1 + \frac{h''(l_{1,t})l_{1,t}}{h'(l_{1,t})} \right) \right) = \lambda_t$$

implies that l_1 converges to some l^*.

At this stage it is helpful to consider particular histories s^t explicitly. Pick any s^t such that $s^{\bar{\mathcal{T}}} < s^t$. Choose any $s^\infty > s^t$ and substitute repeatedly into (29) for all s^k that satisfy $s^t \le s^k < s^\infty$ to get

$$\sum_{k=0}^{\infty} \beta^k \left[2c_2 \left(s^{t+k} \right) + \left(h'(l_1(s^{t+k})) - 1 \right) l_1(s^{t+k}) \right] + \sum_{k=0}^{\infty} \beta^k g \left(s^{t+k} \right) + \lim_{\mathcal{T} \to \infty} \beta^{\mathcal{T}} \tilde{b} \left(s^{t+\mathcal{T}+1} \right) = \frac{1}{\beta} \tilde{b}(s^t). \tag{33}$$

If we choose t sufficiently large, the first integral is sufficiently close to a constant for almost all possible paths s^k. But different paths of s^k lead to different values of $\sum_{k=0}^{\infty} \beta^k g \left(s^k \right)$, which implies that for some $s^\infty > s^t$, $\lim_{\mathcal{T} \to \infty, s^{t+\mathcal{T}+1} < s^\infty} \beta^{\mathcal{T}} \tilde{b} \left(s^{t+\mathcal{T}+1} \right) \neq 0$. This implies that $\tilde{b} \left(s^{t+\mathcal{T}+1} \right)$ is unbounded along that history, which leads to a contradiction. ■

Proposition 5 reveals key forces. The government wants (a) to smooth labor distortions caused by taxes, and (b) to smooth consumption of the risk-averse agent. To smooth labor distortions, the government should keep the marginal tax on labor constant across all realizations of s_t. To smooth consumption of the risk averse type 2 agent, the government could (i) keep transfers T_t constant by borrowing from or lending to the risk-neutral type 1 agent in response to shocks to g_t; or (ii) let T_t fluctuate and have the risk-averse agent borrow from or lend to the risk-neutral type 1 agent to smooth consumption. With incomplete markets, the government cannot do either of these things perfectly. There is always a long enough sequence of bad shocks so that either in case (i) the government runs into its borrowing limit and must adjust the distorting tax rate to raise more revenues; or in case (ii) the risk-averse type 2 agent runs into his borrowing limit and can no longer smooth his consumption, in which case the government must adjust the distorting tax rate to help the type 2 risk-averse agent smooth consumption.

This example indicates that optimal allocations will generally be *history dependent*, in the sense that allocations in period t will depend not only on the current realization of g_t but also on the entire history of shocks g^{t-1}. In the next section, we show that the same insights continue to hold with flexible nonlinear tax systems, and that this implies that the optimal distortions τ_t are generally history-dependent as in AMSS and Barro (1979).

4 Distortion smoothing more generally

[**Numerical example goes here. Highlight history-dependence, how $\tilde{b}$ behaves over time**]

5 Simple non-linear taxation

The previous section studied affine taxation. Figure 1 indicates that affine taxes are a good first approximation to the U.S. tax system, but more that refined approximations are better. So in this section, we relax the assumption that taxes are affine and allow them to be an arbitrary function of current labor income. We do not allow other taxes. We will show that while such non-linear taxes allow more redistribution and achieve higher welfare, the main lessons from our study of affine taxes prevail.

With simple non-linear taxes, an agent's budget constraint (4) becomes

$$c_{i,t} + b_{i,t} = \theta_i l_{i,t} - T_t(\theta_i l_{i,t}) + R_{t-1} b_{i,t-1} \tag{34}$$

and the government budget constraint becomes

$$g_t + B_t = \sum_{i=1}^{I} \pi_i T_t(\theta_i l_{i,t}) + R_{t-1} B_{t-1}. \tag{35}$$

A competitive equilibrium with simple non-linear taxes can then be defined analogously to Definition 1.

Section 3 emphasized two sets of results. First, the net distribution of initial assets $\left\{\tilde{b}_{i,-1}\right\}_{i>1}$ rather than $\{b_{i,-1}\}_{i=1}^{I}$ determines welfare under the optimal allocation, so that government assets B_t can be set to zero in all states without loss of generality. Second, optimal allocations are generally history dependent. For example, it is easy to show that in the AMSS-like economy of section 3.5, consumption of the risk averse type may fluctuate initially as he faces interest rates β^{-1}, but then it either converges to a constant, or the interest rate diverges from β^{-1} as the consumption of the type 1 agent hits its lower bound $\underline{c}$. Both of these conclusions continue to hold when taxes are non-linear. Consider a tax schedule $T_t(y_t)$ for which $\left\{c_{i,t}^*, l_{i,t}^*, b_{i,t}^*, B_t^*, R_t^*\right\}_t$ is part of a competitive equilibrium. Construct an alternative tax schedule $\hat{T}_t(y)$ that supports the same equilibrium allocations but set government debt to zero. Define the alternative tax schedule $\hat{T}_t$ as

$$\hat{T}_t\left(y\right) = B_t^* + T_t\left(y\right) - R_{t-1}^* B_{t-1}^*.$$

When consumers face taxes and prices $\left\{\hat{T}_t(y), R_t^*\right\}_{t=0}^{\infty}$, exactly the same $\left\{c_{i,t}^*, l_{i,t}^*\right\}_{i,t}$ are available to them as when they face $\{T_t(y), R_t^*\}_{t=0}^{\infty}$ and vice versa. To see that, consider any sequence $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (34) for $\{T_t(y), R_t^*\}_{t=0}^{\infty}$ with $\{b_{i,t}\}_{i,t}$ being bounded. Define $\left\{\hat{b}_{i,t}\right\}_{i,t}$ as

$$\hat{b}_{i,t} - R_{t-1}^* \hat{b}_{i,t-1} - \hat{T}_t\left(\theta_i l_{i,t}\right) = b_{i,t} - R_{t-1}^* b_{i,t-1} - T_t\left(\theta_i l_{i,t}\right).$$

This implies that

$$\begin{aligned}
\hat{b}_{i,t} &= \left(b_{i,t} + B_{i,t}\right) - R_{t-1}^*\left(b_{i,t-1} + B_{i,t-1}\right) + R_{t-1}^* \hat{b}_{i,t-1} \\
&= \left(b_{i,t} + B_{i,t}\right) - R_{-1}^*\left(b_{i,-1} + B_{i,-1}\right).
\end{aligned}$$

Since $\{b_{i,t}\}_{i,t}$ is bounded, so is $\left\{\hat{b}_{i,t}\right\}_{i,t}$. Therefore, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a feasible allocation under $\left\{\hat{T}_t(y), R_t^*\right\}_{t=0}^{\infty}$ and hence the same $\left\{c_{i,t}^*, l_{i,t}^*\right\}_{i,t}$ must be optimal for taxes $\{T_t(y)\}_t$ and $\left\{\hat{T}_t(y)\right\}_t$. Summing across agents i and using the asset market clearing condition (6), we can conclude that the new government debt $\left\{\hat{B}_t\right\}_{t=0}^{\infty}$ satisfies $\hat{B}_t = 0$ for all t. We can also prove the second part of Proposition 1 for this setting.

It is also easy to see that the optimal simple non-linear taxes leads to history-dependent allocations. Consider again the AMSS-like economy of section 3.5. In periods when the allocation is interior, the marginal utility of consumption of agent 2 will follow a martingale (31). This, together with the fact that total output is bounded from above, implies that $c_{1,t}$ will either converge to a constant or that $c_{2,t}$ will hit its lower bound, in which case the interest rate R_t will differ from β^{-1}. Either of these situations implies that for the same realization of government expenditures g_t, allocations for small t will generally differ from the allocations for large t. In the following section, we show that there is a sense in which the same conclusions continue to hold more generally in economies with incomplete markets.

6 Constrained optimum and more general taxes

The history <u>dependence</u> of the optimal allocations under affine and simple non-linear taxes described in sections 3 and 4 contrasts with the history <u>independence</u> of constrained optimal allocations, which can be thought of as optimal allocations with a sufficiently rich set of tax instruments. In this section, we first characterize constrained optimal allocations and show that they are not history dependent, so that allocations in period t depend only on the realization of shock g_t but not on the history g^{t-1} or the time period. Then we identify features of a tax system that are required to implement a constrained optimal allocation as a competitive equilibrium with taxes. We will show that generally such a system effectively completes markets by making returns on assets state-contingent. We conclude by showing that when the government is not able to complete markets through tax policies, optimal allocations are history dependent.

6.1 Constrained optimum

A recent literature on optimal taxation, sometimes referred to as the New Dynamic Public Finance (NDPF), approaches taxes from a different angle than does a Ramsey analysis.[9] The NDPF literature makes explicit assumptions characterizes optimal allocations that respect information gaps between agents and the government. Then it studies taxes and other interventions that decentralize an optimal allocation.

In the spirit of the NDPF, we assume that θ is private information and that the government observes labor θl and c for each agent. Constrained optimal allocations solve the mechanism design problem

$$\max_{\{c_{i,t},y_{i,t}\}} \mathbb{E}_0 \sum_{i=1}^{I} \alpha_i \pi_i \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i} \right) \tag{36}$$

subject to the incentive constraints

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i} \right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i \left(c_{j,t}, \frac{y_{j,t}}{\theta_i} \right) \text{ for all } i, j \tag{37}$$

and the feasibility constraint

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i y_{i,t}. \tag{38}$$

Let $\eta_{i,j}$ be Lagrange multipliers on (37). Let $W^i(c_{i,t}, y_{i,t})$ be defined as

$$W^i(c_{i,t}, y_{i,t}) = \left(\alpha_i \pi_i + \eta_{i,i}\right) U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i} \right) - \sum_{j \neq i} \eta_{j,i} U^j \left(c_{j,t}, \frac{y_{j,t}}{\theta_i} \right).$$

Then we can re-write the optimization problem (36) as

$$\max_{\{c_{i,t},y_{i,t}\}} \min_{\{\eta_{ij}\}} \mathbb{E}_0 \sum_{i=1}^{I} \sum_{t=0}^{\infty} \beta^t W^i(c_{i,t}, y_{i,t})$$

subject to (38). This problem is equivalent to solving a sequence of static problems for each realization of g_t. Therefore, the optimal allocation depends only on the realization of g, but not on the time period t or the history g^{t-1}.

[9] For a survey, see Golosov, Tsyvinski, and Werning (2007) Kocherlakota (2010).

6.2 Decentralization of the optimum

In this section, we show that while constrained optimum allocations do not depend on history, the taxes that decentralize those allocations generally do. Moreover, such taxes complete markets either by making the returns on assets state-contingent or by effectively eradicating all asset trades.

We consider a general non-linear tax $T_t(y_t, X_t)$ where X_t is a vector of additional agent-specific variables, and ask what information this vector should contain. The vector X_t will not be unique, since, for example past labor incomes and asset returns are equivalent ways of tracking labor income and consumption. Still, we will highlight common features a "fully optimal" tax system $T_t(y_t, X_t)$ must have.

As in the previous section, we assume that agents begin with initial debt holdings $\{b_{i,-1}\}_i$ and that each period they are able to trade a one-period bond with non-state-contingent return R. The budget constraint of an agent is the same as (34) except that $T_t(y_t)$ is replaced with $T_t(y_t, X_t)$. We modify the definition of a competitive equilibrium with these more general taxes accordingly.

We have the following

Proposition 6 *(i) Constrained optimal allocations can be decentralized as a competitive equilibrium with tax* $T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)$ *where* $F\left(\{y_s\}_{s=0}^{t-1}\right)$ *is some function of previous labor earnings.*
(ii) The marginal tax on debt $\frac{\partial T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)}{\partial b}$ *must be either a function of* $\left(y_t, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)$ *or a non-linear function of* b_{t-1}*, and we are free to set* $T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right) = y_t + \max\{R_t b_t, 0\}$ *if* $b_t \neq 0$.

Most of the results are easy to see. Let $\left\{c_{i,t}^{sp}, y_{i,t}^{sp}\right\}_{i,t}$ be the constrained optimal allocation. There are in general many tax systems that implement this allocation, for example, one that sets $T(y_t, b_{t-1}, y_{t-1}, ..., y_0) = y_{it}^{sp} - c_{i,t}^{sp}$ if vector $(y_t, b_{t-1}, ..., y_0) = \left(y_{i,t}^{sp}, 0, y_{i,t-1}^{sp}, ..., y_{i,0}^{sp}\right)$ and an arbitrarily high number for all other $(y_t, b_{t-1}, ..., y_0)$. This assures that an agent can choose only among sequences $\left\{c_{i,t}^{sp}, y_{i,t}^{sp}\right\}_{i,t}$ that are compatible by construction.

The optimal allocation $\left\{c_{i,t}^{sp}, y_{i,t}^{sp}\right\}_{i,t}$ generally has the property that agents' marginal rates of substitution are not equalized

$$\frac{\beta \mathbb{E}_t U_c^i\left(c_{i,t+1}^{sp}, y_{i,t+1}^{sp}/\theta_i\right)}{U_c^i\left(c_{i,t}^{sp}, y_{i,t}^{sp}/\theta_i\right)} \neq \frac{\beta \mathbb{E}_t U_c^j\left(c_{j,t+1}^{sp}, y_{j,t+1}^{sp}/\theta_j\right)}{U_c^j\left(c_{j,t}^{sp}, y_{j,t}^{sp}/\theta_j\right)}.$$

As a result, marginal returns on assets $\frac{\partial T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)}{\partial b}$ cannot be linear in asset holdings and must either depend on an agent's income, which typically will make them be state-contingent, or else be non-linear. Debt plays no useful role, since, as part ii of Proposition 6 indicates, the government can implement the optimum by taxing away *all* of an agent's income from assets if his debt differs from zero. Thus, the planner effectively completes asset markets either by making returns state-contingent or by shutting them.

6.3 History-dependent taxes when asset taxes are not available

The optimal tax schedule described in section 6.2 requires that the government can observe and tax assets of the households. When the government can do that, all impediments to optimality coming from preexisting debts and market incompleteness vanish. In period 0, the government can effectively redistribute assets as it wants and then can use asset taxes to assure that the distribution of assets never changes. There is never a distribution of asset holdings that is inefficient or that prevents the government from achieving the constrained optimal allocation.

One objection to this characterization of an optimal tax policy is that in practice asset holdings of the households are not easily observed because many financial transactions are anonymous and hidden from the government. That makes taxation of assets difficult. In this section, we discuss the implications for optimal taxation that follow from a government's inability to tax assets when nevertheless it has access to an arbitrary non-linear tax $T_t\left(y_t, ..., y_0\right)$. Since consumption taxes can replicate asset taxes, we rule them out.

In the appendix, we state an optimal tax problem for general non-linear labor income taxes $T_t\left(y_t, ..., y_0\right)$. We also characterize a particular example that exhibits the same features encountered in Section 5, namely, that distortions and allocations generally depend on the history of past shocks and the initial distribution of net assets $\left\{\tilde{b}_{i,-1}\right\}_{i=2}^{I}$. The example is basically a variant of the AMSS-like economy of Section 3.5 where both agents can now work. We show that in that economy either the consumption of agent 1, $c_{1,t}$, must eventually hit the lower bound, as in the case of affine taxes, or the consumption of agent 2, $c_{2,t}$ must initially fluctuate but eventually converge to a constant. Both of those cases imply that optimal taxes and allocations depend not only on the current g but also on its history.

This example highlights the following general features of the incomplete markets models. When the government's ability to tax assets is limited, the distribution of assets $\left\{\tilde{b}_{i,t}\right\}_{i=2}^{I}$ are key state variables that influence optimal allocations. When an aggregate shock occurs, the distribution $\left\{\tilde{b}_{i,t}\right\}_{i=2}^{I}$ changes endogenously. Expenditures g_t affect agents' decisions about how much to save in period t, $\left\{\tilde{b}_{i,t}\right\}_{i=2}^{I}$. When markets are incomplete, the *ex post* return on these assets in period $t+1$ does not depend on the shock g_{t+1}. As a result, there is history dependence in the optimal allocations and there are distortions not present under the optimal mechanism design allocation of section 6 or in economies with complete markets. This result is very robust to a variety of assumptions about the tax structure so long as the government's ability to tax assets state-contingently or nonlinearity is limited.

6.4 Role of debt markets?

Proposition 6 shows that debt markets play no useful role when taxes are sufficiently flexible, and in particular when they can depend on the entire history of labor earnings. For then the government can effectively shut the debt market by taxing away all income if it observes that the agent uses this market. The government can then provide all insurance through the tax code.

An optimal tax schedule can be less extreme. Various decentralizations popular in the literature use a continuous but non-linear tax on asset income, such as Werning (2012) or XXXX. Despite being continuous, these taxes effectively achieve the same goal as attained in the extreme example in Proposition 6: they make it prohibitively costly for agents to depart from the allocation assigned by the planner. This result is not surprising, for it is known (see, for example, Golosov and Tsyvinski (2007)) that welfare in the mechanism design problems is lower when agents can trade freely or face linear returns on assets.

In the economy with affine taxes considered in section 3.5, debt markets played an important role. They allowed the agents and the government to smooth consumption and distortions in response to the aggregate shocks.

The contrasting roles of debt markets in these two situations induces us to ask, "when are debt markets useful?" In general, unrestricted debt markets play two roles, one that is welfare decreasing, another that is welfare enhancing. Optimal non-linear income taxes are generally not convex. When agents can freely trade assets, they can convexity their budget sets. On the one hand, private asset trading improves the utility of each individual agent but it reduces the government's ability to redistribute and therefore lowers social welfare. This is why in the mechanism design problems it is welfare decreasing for agents to trade behind the planner's back. On the other hand, when the tax system is less than perfect, trading assets usually cannot provide an agent with an optimal amount of consumption smoothing. The affine taxes in section 3.5 illustrate that. If agents could not trade among themselves, consumption of the risk averse agent, $c_{2,t}$, would fluctuate much more, leading to welfare losses.

Allowing taxes on current income to be non-linear, as in Section 5, helps the government provide more insurance to agents through the tax code, although less than perfectly. As a result, whether welfare is higher in the economy in which agents can trade assets freely or in the economy in which agents cannot trade assets depends on which force dominates. It is easy to construct examples in which one or the other of these economies has higher welfare depending on the redistributive objectives of the government or the nature of the aggregate shocks. [**should we actually put an example of that?**] This highlights a general point. The more restricted is the available set of tax instruments, the more important it is for the agents to be able to trade assets freely. We conjecture that this point would be more important in economies with richer shock structures.

7 Appendix

7.1 Proof of Lemma 1

We prove a slight more general version of our result. Consider an infinite horizon, incomplete markets economy in which an agent maximizes utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$ subject to an infinite sequence of budget constraints. We assume that U is concave and differentiable. Let $\mathbf{p}(s^t)$ be a price vector in state s^t with $p_i(s^t)$ denoting the price of good i. We use a normalization $p_1\left(s^t\right) = 1$ for all s^t. There is a risk-free bond.

Let $b(s^t)$ be the agent's bond holdings, and let $\mathbf{e}\left(s^t\right)$ be a stochastic vector of endowments.

Consumer maximization problem

$$\max_{\mathbf{x}_t, b_t} \sum_{t=0}^{\infty} \beta^t \Pr\left(s^t\right) U(\mathbf{x}\left(s^t\right)) \tag{39}$$

subject to

$$\mathbf{p}\left(s^t\right)\mathbf{x}\left(s^t\right) + b\left(s^t\right) = \mathbf{p}\left(s^t\right)\mathbf{e}\left(s^t\right) + R(s^{t-1})b\left(s^{t-1}\right) \tag{40}$$

and $\left\{b\left(s^t\right)\right\}$ is bounded.

The Euler conditions are

$$\begin{aligned} \mathbf{U}_x(s^t) &= U_1(s^t)\mathbf{p}(s^t) \\ \Pr\left(s^t\right) U_1\left(s^t\right)) &= \beta R(s^t) \sum_{s^{t+1} \geq t^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right). \end{aligned} \tag{41}$$

Proposition 7 *Consider an allocation $\{\mathbf{x}_t, b_t\}$ that satisfies (40), (41) and $\{b_t\}_t$ is bounded. Then $\{\mathbf{x}_t, b_t\}$ is a solution to (39).*

Proof. The proof follows closely Constantinides and Duffie (1996). Suppose there is some other budget feasible allocation $\mathbf{x} + \mathbf{h}$ that maximizes (39). Since U is strictly concave,

$$\begin{aligned} & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t + \mathbf{h}_t) - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t) \\ \leq \; & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}_t \end{aligned} \tag{42}$$

To attain $\mathbf{x} + \mathbf{h}$, the agent must deviate by φ_t from his original portfolio b_t such that $\{\varphi_t\}_t$ is bounded, $\varphi_{-1} = 0$ and

$$\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) = R\left(s^{t-1}\right)\varphi(s^{t-1}) - \varphi(s^t)$$

Multiply by $\beta^t \Pr\left(s^t\right) U_1(s^t)$ to get:

$$\begin{aligned}
\beta^t \Pr\left(s^t\right) U_1(s^t)\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) &= U_1(s^t)R\left(s^{t-1}\right)\varphi(s^{t-1}) - U_1(s^t)\varphi(s^t) \\
&= \beta^t \Pr\left(s^t\right) U_1(s^t)R\left(s^{t-1}\right)\varphi(s^{t-1}) - \beta^t R(s^t) \sum_{s^{t+1}\geq t^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right)\varphi(s^t)
\end{aligned}$$

where we used the second part of (41) in the second equality. Sum over the first T periods and use the first part of (41) to eliminate $\mathbf{U}_x(\mathbf{x}_t) = U_1(s^t)\mathbf{p}(s^t)$

$$\sum_{t=0}^{T} \beta^t \Pr\left(s^t\right) \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}\left(s^t\right) = -\sum_{s^T} \Pr\left(s^T\right)\beta^T U_1\left(s^{T+1}\right)\varphi(s^T).$$

Since $\{\varphi_t\}_t$ is bounded there must exist $\bar{\varphi}$ s.t. $|\varphi_t| \leq \bar{\varphi}$. By Theorem 5.2 of Magill and Quinzii (1994),

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \Pr\left(s^t\right) \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}\left(s^t\right) = 0.$$

Substitute this into (42) to show that $\mathbf{h}$ does not improve utility of consumer. ■

7.2 Proof of technical details of Proposition 5

In this appendix we show that $c_{2,t}$ and η_t must converge to finite values.

Equation (31) implies that $u_{c,t}$ is a supermartingale and therefore converges. It cannot converge to zero, since then $c_{2,t}$ would diverge to infinity. Since $c_1 \geq \underline{c}$, constraint (30) can only be satisfied if $l_1\left(s^t\right) \to \infty$, but then (29) would imply that $\tilde{b}_t \to -\infty$ violating boundedness. Therefore, $u_{c,t}$ and $c_{2,t}$ both converge to finite values, so $c_{2,t} \to c_2^*$.

The first-order conditions for $\tilde{b}$ imply that

$$\eta_t = \mathbb{E}_t \eta_{t+1}.$$

Thus, η_t is a martingale. But does not necessarily have to be bounded, so we cannot apply a standard martingale convergence result. We use a different argument to prove our result.

Let $\beta^t \Pr\left(s^t\right)\eta(s^t)$, $\beta^t \Pr\left(s^t\right)\lambda(s^t)$, and $\beta^t \Pr\left(s^t\right)\zeta(s^t)$ be Lagrange multipliers on (29), (30), and (31), respectively. The first-order conditions for $c_{1,t}$ and $l_{1,t}$ are

$$\alpha_1 - \eta(s^t) = \lambda(s^t)$$

and

$$\alpha_1 h'(s^t) - \eta\left(s^t\right)\left[h''(s^t)l(s^t) + h'(s^t)\right] = \lambda(s^t).$$

These conditions imply that if $l(s^t)$ converges, so does $\eta(s^t)$. We shall show that if $\eta(s^t)$ does not converge, then $l(s^t)$ must converge, which will establish a contradiction.

Combine the first-order conditions for $c_{1,t}$ and $c_{2,t}$ to get

$$\alpha_2 u_c(s^t) + \left(\zeta(s^{t-1}) - \zeta(s^t)\right) u_{cc}(s^t) = \alpha_1 - 2\eta(s^t). \tag{43}$$

Since $u_c(s^t)$ and $u_{cc}(s^t)$ converge to finite values, if $\eta(s^t)$ does not converge to a constant, neither does $\zeta(s^{t-1}) - \zeta(s^t)$. Rewrite (43) as

$$\zeta(s^t) = \zeta(s^{t-1}) + \frac{\alpha_2 u_c(s^t) - \alpha_1}{u_{cc}(s^t)} + 2\frac{\eta(s^t)}{u_{cc}(s^t)}. \tag{44}$$

Choose a t sufficiently large that $\frac{\alpha_2 u_c(s^t)-\alpha_1}{u_{cc}(s^t)}$ and $u_{cc}(s^t)$ are close to being constants. Since $\eta(s^t)$ is a martingale that does not converge, we can find an $\varepsilon > 0$ and a history $\hat{s}^\infty$ such that

$$\left| \frac{\alpha_2 u_c(s^k) - \alpha_1}{u_{cc}(s^k)} + 2\frac{\eta(s^k)}{u_{cc}(s^k)} \right| > \varepsilon \text{ for } s^k \in \hat{s}^\infty$$

for infinitely many k. Then from (44) $|\zeta(s^t)| \to \infty$ for any $s^t \in \hat{s}^\infty$.

Rewrite (28) as

$$\max_{\{c_1,c_2,l_1,\tilde{b}\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pr\left(s^t\right) \left[\alpha_1 \left(c\left(s^t\right) - h(l_1\left(s^t\right))\right) + \alpha_2 u(c_2\left(s^t\right)) + \left(\zeta\left(s^{t-1}\right) - \zeta\left(s^t\right)\right) u_c\left(s^{t+1}\right)\right] \tag{45}$$

subject to constraints (29), (30) and a requirement that sequence $\left\{\tilde{b}_t\right\}$ is bounded. This problem is recursive. We can find the optimal allocations $\left\{c_1\left(s^t\right), c_2\left(s^t\right), l_1\left(s^t\right), \tilde{b}\left(s^t\right)\right\}$ for $s^t > s^m$ for some s^m if we take the sequence of multipliers $\left\{\zeta\left(s^t\right)\right\}_{s^t > s^m}$ and $\tilde{b}(s^m)$ as given to solve

$$\max_{\{c_1,c_2,l_1,\tilde{b}\}} \sum_{t=m+1}^{\infty} \sum_{s^t > s^m} \beta^t \Pr\left(s^t\right) \left[\frac{\alpha_1}{\zeta\left(s^{t-1}\right)} \left(c\left(s^t\right) - h(l_1\left(s^t\right))\right) + \frac{\alpha_2}{\zeta\left(s^{t-1}\right)} u(c_2\left(s^t\right)) + \left(1 - \frac{\zeta\left(s^t\right)}{\zeta\left(s^{t-1}\right)}\right) u_c\left(s^{t+1}\right)\right] \tag{46}$$

subject to constraints (29), (30) and a requirement that sequence $\left\{\tilde{b}_t\right\}$ is bounded. Denote by $\beta^t \Pr\left(s^t\right) \hat{\eta}(s^t)$ and $\beta^t \Pr\left(s^t\right) \hat{\lambda}(s^t)$ the Lagrange multipliers of this re-normalized problem. The first-order conditions for this problem with respect to c_1, l_1, and c_2 are

$$\frac{\alpha_1}{\zeta\left(s^{t-1}\right)} - \hat{\eta}(s^t) = \hat{\lambda}(s^t), \tag{47}$$

$$\frac{\alpha_1}{\zeta\left(s^{t-1}\right)} h'\left(s^t\right) - \hat{\eta}(s^t)\left[h''(s^t) l(s^t) + h'(s^t)\right] = \hat{\lambda}(s^t). \tag{48}$$

Combine these equations to get

$$\frac{\alpha_1}{\zeta\left(s^{t-1}\right)} h'\left(s^t\right) - \hat{\eta}(s^t)\left[h''(s^t) l(s^t) + h'(s^t)\right] = \frac{\alpha_1}{\zeta\left(s^{t-1}\right)} - \hat{\eta}(s^t).$$

Consider a history $\hat{s}^{\infty}$ and choose $s^m \in \hat{s}^{\infty}$. We know that $|\zeta\left(s^{t-1}\right)| \to \infty$. Suppose that $h'(s^t)$ remains bounded. Then the above equation implies that $l(s^t)$ converges to a value $\hat{l}$ that satisfies

$$h''(\hat{l})\hat{l} + h'(\hat{l}) = 1.$$

The assumptions of the theorem imply that this value is unique. This establishes that $\eta(s^t)$ converges. Alternatively, suppose that $h'(s^t) \to \infty$. Substitute (30) into (29):

$$2c_2\left(s^t\right) + g(s^t) + \left(h'(l_1(s^t)) - 1\right) l_1(s^t) = \frac{1}{\beta}\tilde{b}(s^{t-1}) - \tilde{b}\left(s^t\right).$$

Since $\tilde{b}(s^t)$ is bounded, the right side of this expression is bounded. Both $g(s^t)$ and $c_2(s^t)$ are finite, so $\left(h'(l_1(s^t)) - 1\right) l_1(s^t)$ must remain bounded, which rules out the possibility that $h'(s^t) \to \infty$. This completes the proof.

7.3 Addition details for Section 6.3

First, we provide a general discussion of the optimal tax problem described in Section 6.3.

The planner chooses I state-contingent bundles of pre-tax labor income $\{y_{i,t}\}_{i,t}$ and after tax labor income $\{x_{i,t}\}_{i,t}$. Each agent chooses his preferred bundle and re-trades at market clearing interest rates R_t.

The problem of agent i who chooses a bundle j is

$$V_i\left(\{x_{j,t}, y_{j,t}, R_t\}_t\right) = \max_{\{c,b\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U_i\left(c^j_{i,t}, \frac{y_{j,t}}{\theta_i}\right)$$

s.t.

$$c^j_{i,t} + b^j_{i,t} = x_{j,t} + R_{t-1} b^j_{i,t-1}.$$

By standard arguments, allocations $c_{i,t}$ are characterized by the budget constraint, Euler condition

$$\frac{\partial U_i\left(c^j_{i,t}, \frac{y_{j,t}}{\theta_i}\right)}{\partial c} = \beta R_t \mathbb{E}_t \frac{\partial U_i\left(c^j_{i,t+1}, \frac{y_{j,t+1}}{\theta_i}\right)}{\partial c}$$

together with a constraint that $\left\{b^j_{i,t}\right\}_t$ are bounded for all i, j.The interest rates $\{R_t\}_t$ clear the asset markets when all agents choose their allocations optimally.

By revelation principle, we can cast this problem as a mechanism design in which all agent choose their allocations truthfully subject to the incentive constraints

$$V_i\left(\{x_{i,t}, y_{i,t}, R_t\}_t\right) \geq V_i\left(\{x_{j,t}, y_{j,t}, R_t\}_t\right) \text{ for all } i, j.$$

The IC constraint is equivalent to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U_i \left(c_{i,t}^i, \frac{y_{i,t}}{\theta_i} \right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U_i \left(c_{i,t}^j, \frac{y_{j,t}}{\theta_i} \right) \text{ for all } i, j$$

$$c_{i,t}^j + b_{i,t}^j = x_{j,t} + R_{t-1} b_{i,t-1}^j$$

$$\frac{\partial U_i \left(c_{i,t}^j, \frac{y_{j,t}}{\theta_i} \right)}{\partial c} = \beta R_t \mathbb{E}_t \frac{\partial U_i \left(c_{i,t+1}^j, \frac{y_{j,t+1}}{\theta_i} \right)}{\partial c}$$

$$\sum_i \pi_i c_{i,t} + g_t = \sum_i \pi_i y_{i,t} \text{ for all } t$$

$$b_{i,t}^j \geq \underline{B}.$$

The last condition is a no Ponzi game condition. The planner chooses the allocations $\left\{ c_{i,t}^j, b_{i,t}^j, y_{i,t}, R_t \right\}_{t,i,j}$ that satisfy these constraints and maximize

$$\max_{\left\{ c_{i,t}^j, b_{i,t}^j, y_{i,t}, R_t \right\}_{t,i,j}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_i \alpha_i \pi_i \beta^t U_i \left(c_{i,t}^i, \frac{y_{i,t}}{\theta_i} \right)$$

We characterize a version of the AMSS-like economy set up in Section 3.5. Unlike that section, we assume that the risk averse agent can work, his productivity is θ_2 and his utility is given by $u(c) - h_2 \left(\frac{y}{\theta_2} \right)$. We assume that the planner assigns a sufficiently high Pareto weight on agent 1 so that it is agent 2 incentive constraint which binds.

Similarly to the discussion of Section 3.5, two cases are possible. The equilibrium can either be interior, in which case $c_{1,t}$ is always above $\underline{c}$, or constraint (19) eventually binds. The latter case automatically implies history-dependence of allocations, so we consider the former case. The optimal tax problem is

$$\max_{\left\{ c_{i,t}^j, b_{i,t}^j, y_{i,t}, R_t \right\}_{t,i,j}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 \left(c_{1,t} - h_1 \left(\frac{y_{1,t}}{\theta_1} \right) \right) + \alpha_2 \left(u(c_{2,t}) - h_2 \left(\frac{y_{2,t}}{\theta_2} \right) \right) \right]$$

subject to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{2,t}) - h_2 \left(\frac{y_{2,t}}{\theta_2} \right) \right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{2,t}^1) - h_2 \left(\frac{y_{1,t}}{\theta_2} \right) \right)$$

$$c_{2,t} + b_{2,t} = x_{2,t} + \frac{1}{\beta} b_{2,t-1}$$

$$u'(c_{2,t+1}) = \mathbb{E}_t u'(c_{2,t+1})$$

$$c_{2,t}^1 + b_{2,t}^1 = x_{1,t} + \frac{1}{\beta} b_{2,t-1}^1$$

$$u'(c_{2,t+1}^1) = \mathbb{E}_t u'(c_{2,t+1}^1)$$

$$c_{1,t} + b_{1,t} = x_{1,t} + \frac{1}{\beta} b_{1,t-1}$$

$$c_{1,t} + c_{2,t} + g_t = y_{1,t} + y_{2,t}$$

$$b^j_{i,t} \geq \underline{B}.$$

There are a few redundant equations there. Without loss of generality we can set $b_{1,t} = 0$ for all t in which case $c_{1,t} = x_{1,t}$. Moreover it is clear that the Lagrange multiplier on the second constraint must be zero. Therefore we have

$$\max_{\left\{c^j_{i,t}, b^j_{i,t}, y_{i,t}\right\}_{t,i,j}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 \left(c_{1,t} - h_1\left(\frac{y_{1,t}}{\theta_1}\right)\right) + \alpha_2 \left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right)\right]$$

s.t.

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c^1_{2,t}) - h_2\left(\frac{y_{1,t}}{\theta_2}\right)\right)$$

$$c^1_{2,t} + b^1_{2,t} = c_{1,t} + \frac{1}{\beta} b^1_{2,t-1}$$

$$u'\left(c_{2,t+1}\right) = \mathbb{E}_t u'\left(c_{2,t+1}\right)$$

$$u'\left(c^1_{2,t+1}\right) = \mathbb{E}_t u'\left(c^1_{2,t+1}\right)$$

$$c_{1,t} + c_{2,t} + g_t = y_{1,t} + y_{2,t}$$

Guess that both Euler constraints do not bind. Take the first order conditions

$$\alpha_1 - \eta^1_{2,t} = \lambda_t \tag{49}$$

$$u'\left(c_{2,t}\right)\left(\alpha_2 + \mu\right) = \lambda_t \tag{50}$$

$$\mu u'\left(c^1_{2,t}\right) = \eta^1_{2,t} \tag{51}$$

$$\eta^1_{2,t} = \mathbb{E}_t \eta^1_{2,t+1} \tag{52}$$

From (52) $\eta^1_{2,t}$ is a martingale, therefore from (49) λ_t is a martingale, and therefore $u'\left(c_{2,t}\right)$ and $u'\left(c^1_{2,t}\right)$ are martingales, which confirms out guess. Moreover, $u'\left(c_{2,t}\right)$, $u'\left(c^1_{2,t}\right)$, λ_t and $\eta^1_{2,t}$ must all converge. Next we discuss what they must converge to.

Note that since λ_t converges to a constant, the first order conditions for $y_{2,t}$ and $y_{1,t}$ imply that they also converge to constants. Since $c_{2,t}$ converges to a constant, $c_{1,t}$ must fluctuate to offset fluctuations in g_t. If $c_{1,t}$ fluctuates, then $u'\left(c^1_{2,t}\right) \to 0$ and therefore $\eta^1_{2,t} \to 0$. This implies that in the long run this economy converges to the constrained optimum allocations discussed in Section 6.1. Intuitively what happens is that as $c_{1,t}$ fluctuates, the agent 2, if he deviates, accumulates infinitely large amount of assets. When he does that, smoothing fluctuations in $c_{1,t}$ have no effect on his welfare, and we get back to the fully constrained optimum allocations.

Note that $c_{2,t}$ cannot be constant in all t. If it were, then $c^1_{2,t}$ would also have to be constant in all t, which is impossible since $c_{1,t}$ must fluctuate.

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