Sources of Entropy in Representative Agent Models

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Questions

- Q1. What does the pricing kernel look like?
  - **Dispersion**: **entropy**
  - **Dynamics**: $n-$period entropy and **horizon dependence**
  - **Disasters**: entropy and **high-order cumulants**
  - Illustration: the Vasicek model

- Q2. How do these pricing kernels compare?
  - Power utility
  - Recursive preferences
  - Habits
  - Jumps and disasters

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  - **Dynamics**: $n-$period entropy and **horizon dependence “small”**
  - **Disasters**: entropy and **high-order cumulants help with both**
  - Illustration: the Vasicek model
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Facts about excess returns (% per month)

Asset	Mean	Standard Deviation	Skewness	Excess Kurtosis
S&P 500	**0.40**	5.56	−0.40	7.90
Fama-French (small, low)	−0.30	11.40	0.28	9.40
Fama-French (small, high)	**0.90**	8.94	1.00	12.80
Pound Sterling	0.35	3.16	−0.50	1.50
5 year bond	0.15	1.90	0.10	4.87

- Also ... the nominal 60-month term spread is about 0.1%/month

Facts: summary

- Facts
    - “Big” excess returns, 1% $\gg$ equity premium
    - “Small” term spreads, $\pm$0.1%
    - Skewness and kurtosis evident
- Each tells us something about the pricing kernel

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Entropy

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- Conditional entropy

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- Applications
  - Entropy, $EL_t(m_{t+1})$
  - Horizon dependence

$$H(n) = \underbrace{n^{-1} EL_t(m_{t,t+n})}_{\text{avg over n periods}} - \underbrace{EL_t(m_{t+1})}_{\text{one period}}$$

Properties of entropy

- Dispersion: entropy bound

$$EL_t(m_{t+1}) \quad \geq \quad E\left(\log r_{t+1} - \log r_t^1\right)$$

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- Disasters: high-order cumulants

$$L_t(m_{t+1}) = \underbrace{\kappa_{2t}(\log m_{t+1})/2!}_{\text{normal term}} + \underbrace{\kappa_3(\log m_{t+1})/3! + \kappa_4(\log m_{t+1})/4! + \cdots}_{\text{high-order cumulants}}$$

What the pricing kernel looks like

- Dispersion
  - Entropy ≥ 0.01 = 1% a month
- Dynamics
  - Horizon dependence ≤ 0.001 = 0.1% a month
- Disasters
  - Something besides the normal distribution

Vasicek model: an example

- Pricing kernel

$$\begin{aligned}
\log m_{t+1} &= \log m + a(B) w_{t+1} \\
&= \log m + \underbrace{a_0 w_{t+1}}_{\text{entropy}} + \underbrace{a_1 w_t + a_2 w_{t-1} + \cdots}_{\text{horizon dependence}} \\
w &\sim \text{NID}(0,1)
\end{aligned}$$

- Interest rate

$$y_t^1 = -\log E_t(e^{\log m_{t+1}}) = -\log m - a_0^2/2 - a_1 w_t - a_2 w_{t-1} - \cdots$$

- ARMA(1,1) for $\log m_t$ is AR(1) [Vasicek] for the interest rate

$$a_{j+1} = \varphi a_j \,, \; j \geq 1$$

Vasicek model: properties

- Partial sums

$$A_n = a_0 + a_1 + a_2 + \cdots + a_n$$

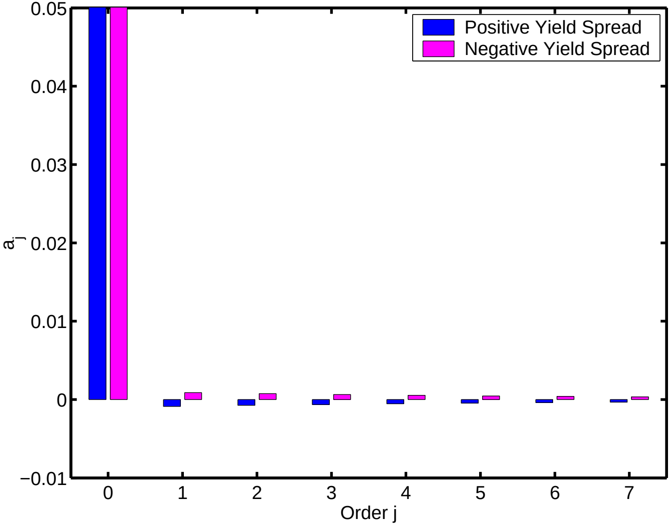
- Entropy

$$EL_t(m_{t+1}) = a_0^2/2 = A_0^2/2 \Rightarrow a_0 \text{ “big”}$$

- Horizon dependence

$$H(n) = n^{-1} \sum_{j=1}^{n} (A_{j-1}^2 - A_0^2)/2 \Rightarrow a_j \text{ “small”}$$

Vasicek model: moving average coefficients



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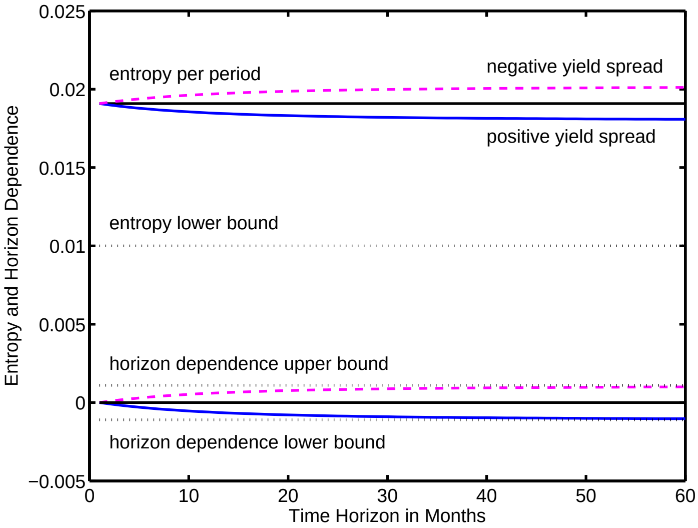
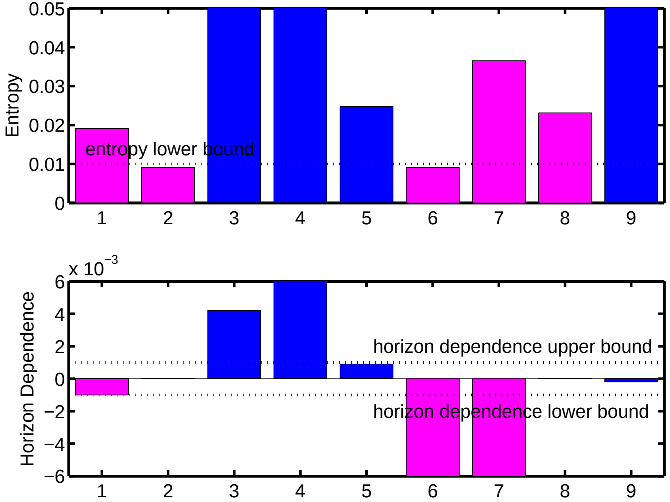


Representative-agent models

- Additive power utility
- Recursive preferences
  - Bansal-Yaron with persistent consumption growth
  - ... and stochastic volatility
- Habits
  - Ratio habits
  - Difference habits
  - Campbell-Cochrane
- Jumps and disasters

Model summary



Recursive preferences

- Preferences

$$
\begin{aligned}
U_t &= \left[(1-\beta)c_t^{\rho} + \beta\mu_t(U_{t+1})^{\rho}\right]^{1/\rho} \\
\mu_t(U_{t+1}) &= \left(E_t U_{t+1}^{\alpha}\right)^{1/\alpha} \\
&\alpha, \rho \leq 1
\end{aligned}
$$

- Interpretation

$$
\begin{aligned}
EIS &= 1/(1-\rho) \\
CRRA &= 1-\alpha \\
\alpha &= \rho \quad \Rightarrow \quad \text{additive power utility}
\end{aligned}
$$

Consumption and pricing kernel

- Consumption growth

$$\begin{aligned} \log g_t &= g + \gamma(B) v^{1/2} w_t \\ \{w_t\} &\sim \text{NID}(0,1) \end{aligned}$$

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- Pricing kernel

$$\begin{aligned} \log m_{t+1} &= \text{constants} \\ &+ \underbrace{[(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)] v^{1/2}}_{a_0} w_{t+1} \\ &+ \underbrace{(\rho-1)\gamma_1 v^{1/2}}_{a_1} w_t + \underbrace{(\rho-1)\gamma_2 v^{1/2}}_{a_2} w_{t-1} + \cdots \end{aligned}$$

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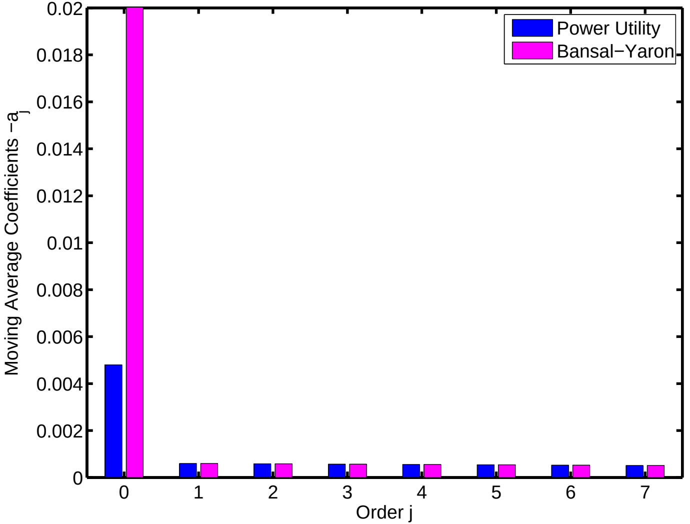
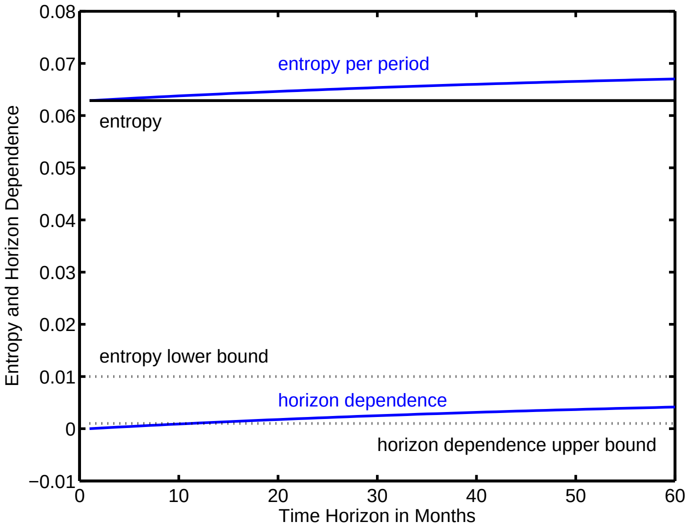
- Critical term: $\gamma(b_1) = \gamma_0 + b_1\gamma_1 + b_1^2\gamma_2 + \cdots$

Power and recursive preferences: moving average coefficients



Recursive preferences: entropy and horizon dependence



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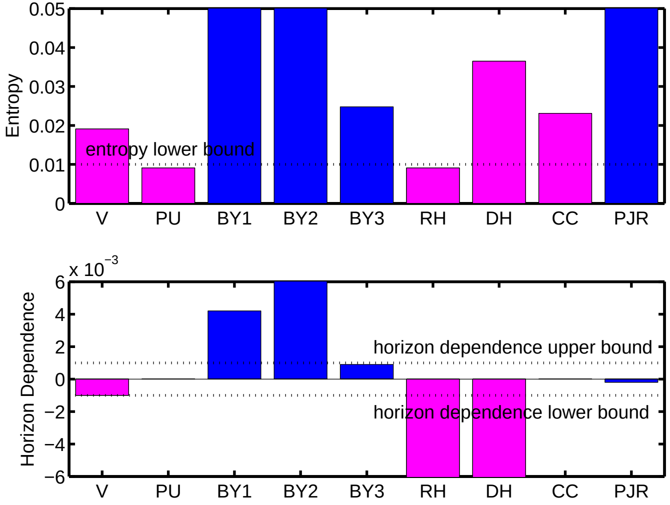

Answers to questions

- Q1. What does the pricing kernel look like?
  - Substantial dispersion: entropy $\geq$ 1% monthly
  - Limited horizon dependence: $\leq$ 0.1% monthly
  - Probably not normal
  - Useful diagnostics for any model
- Q2. How do representative-agent models compare?
  - It is easy to get lots of entropy
  - But it often generates too much horizon dependence
  - All this conditional on parameters
- Q3. What's next?
  - Heterogeneous agents?
  - Business cycle models?

Related work (some of it)

- Bounds
  - Alvarez-Jermann, Bansal-Lehmann, Hansen-Jagannathan
- Recursive preferences
  - Preferences: Epstein-Zin, Kreps-Porteus, Weil
  - Asset pricing: Bansal-Yaron, Campbell, Hansen-Heaton-Li
- Habits
  - Abel, Campbell-Cochrane, Chan-Kogan, Constantinides, Heaton, Sundaresan
- Jumps and disasters
  - Barro, Barro-Nakamura-Steinsson-Ursua, Bekaert-Engstrom, Benzoni-Collin-Dufresne-Goldstein, Branger-Rodrigues-Schlag, Drechsler-Yaron, Eraker-Shaliastovich, Gabaix, Garcia-Luger-Renault, Longstaff-Piazzesi, Wachter

Derivation of the Entropy Bound

- Fundamental Theorem of Asset Pricing

$$\begin{aligned} E_t(m_{t+1}r_{t+1}) &= 1, \\ E_t \log m_{t+1} + E_t \log r_{t+1} \leq \log(1) &= 0, \text{ with equality iff } m_{t+1}r_{t+1} = 1 \end{aligned}$$

- Risk-free rate

$$\log r^1_{t+1} = -\log E_t(m_{t+1}) = -L_t(m_{t+1}) - E_t \log m_{t+1}$$

- Subtract from above:

$$L_t(m_{t+1}) \geq E_t(\log r_{t+1} - \log r^1_{t+1})$$

- Unconditional entropy: $L(m_{t+1}) = EL_t(m_{t+1}) + L(E_t(m_{t+1}))$
- Therefore,

$$L(m_{t+1}) \geq E(\log r_{t+1} - \log r^1_{t+1}) + L(E_t(m_{t+1})) \geq E(\log r_{t+1} - \log r^1_{t+1})$$

the bound is tighter

Back

Entropy and HJ bounds (App A.2)

- Entropy: High-return asset

$$\log r_{t+1} = -\log m_{t+1}$$

- Max excess return over time (iid)

$$L(m_{t,t+n}) = n\left[k^1(1) - \kappa_1\right]$$

- Excess log-return (normal)

$$log r_{t+1} \sim \mathcal{N}(\log r^1_{t+1} + \kappa_{1t}, \kappa_{2t})$$
$$E_t(\log r_{t+1} - \log r^1_{t+1}) = \kappa_{1t}$$

- HJ: High-return asset

$$r_{t+1} = \alpha_t - \frac{m_{t+1}}{Var_t(m_{t+1})^{1/2}}$$

- Max SR over time (iid)

$$\frac{Var(m_{t,t+n})}{E(m_{t,t+n})^2} = e^{n[k^1(2)-2k^1(1)]} - 1$$

- SR (normal)

$$SR_t = \frac{e^{\kappa_{1t}+\kappa_{2t}/2} - 1}{e^{\kappa_{1t}+\kappa_{2t}/2}(e^{\kappa_{2t}} - 1)^{1/2}}$$

Back