Unions and Unemployment*

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PRELIMINARY

Abstract

This paper examines the impact of unions on unemployment and wages in a dynamic equilibrium search model. We model a union as imposing a minimum wage and rationing jobs to ensure that the union's most senior members are employed. This generates rest unemployment, where following a downturn in their labor market, unionized workers are willing to wait for jobs to reappear rather than search for a new labor market. Introducing unions into a dynamic equilibrium model has two implications, which others have argued are features of the data: the hazard of exiting unemployment at long durations is very low when the union-imposed minimum wage is high; and a high union-imposed minimum wage generates a compressed wage distribution and a high turnover rate of jobs.

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1 Introduction

This paper examines the impact of unions on unemployment and wages. We model a union as imposing a minimum wage on employers. The minimum wage binds in at least some states of the world, in which event the union rations jobs to ensure that its most senior members are employed.[1] Our focus is on the implications of such a policy on workers' decision to enter and exit unionized labor markets. We prove that a laid-off union member will never immediately exit her labor market to search elsewhere for a job. Instead, she will endure a spell of rest unemployment, waiting for labor market conditions to improve. We find that the hazard rate of reentering employment generally declines during an unemployment spell, so unionized workers will experience both frequent short spells and infrequent long spells of unemployment.

Our starting point is a simple static model of unions. Suppose workers are risk-neutral and can earn a competitive wage of w^*. A unionized sector offers a higher wage $\hat{w}$. In equilibrium workers must be indifferent between seeking jobs in the two sectors, and so workers face unemployment risk in the unionized sector. This implies $w^* = (1-u)\hat{w}$, where u is the probability that a unionized worker is unemployed and we have normalized an unemployed worker's income to zero. The unemployment rate in the unionized sector is $u = 1 - w^*/\hat{w}$, increasing in the relative wage of unionized jobs.

Our model generalizes this calculation to a dynamic setting where wages are set according to a seniority rule. This has several effects. First, when unions use seniority to allocate jobs, not all workers are equally likely to be unemployed. Loosely speaking, the previous calculation applies for the marginal worker, while the unemployment rate for inframarginal workers will be lower. This reduces the equilibrium unemployment rate. Second, in a dynamic framework, we find that workers who are on the margin of exiting an industry are currently unemployed. If they stay, they expect to be employed at some future date. Because workers are impatient, current unemployment weighs more heavily on them and so this too reduces the equilibrium unemployment rate. Finally, the presence of a union sector may affect the wage of the non-unionized sector in a general equilibrium.

Our modeling strategy closely follows Alvarez and Shimer (2008) and Alvarez and Shimer (2011), which in turn builds on Lucas and Prescott (1974). The economy consists of a large number of labor markets that produce imperfect substitutes. There are many workers and firms in each labor market, so in the absence of unions, wages and output prices are determined competitively within each labor market. Productivity shocks induce workers to

[1] Our model fits into the "monopoly union" approach which stresses that unions may distort labor market outcomes by raising wages and rationing jobs. We do not analyze any potentially beneficial effects of unions, e.g. the "collective voice/insitutional response" stressed by Freeman and Medoff (1984).

move between labor markets. We study two versions of the model, first where workers can move costlessly between markets, and second where labor reallocation across markets is costly because of search frictions.

Both papers distinguish between rest and search unemployment. While in rest unemployment, individuals do not work, enjoying a value of leisure higher than working but lower than being outside the labor force. Moreover, the rest unemployed retain the possibility of returning instantly and at no cost to the labor market where they last worked. Search unemployment enables a worker to locate in any labor market. Our previous paper argued that the existence of rest unemployment may be important for understanding the dynamic behavior of wages. This paper focuses on the possibility that rest unemployment may arise because of unionization. We believe that the two explanations are complementary. Still, it is interesting to note that if there is no leisure advantage to resting rather than working, there is rest unemployment if and only if the minimum wage is binding. In this sense, binding minimum wages create rest unemployment.

Technically, the main difference between the two papers is that in our earlier work, each labor market cleared at each point in time. Whenever a worker was rest unemployed, she weakly preferred rest unemployment to working in her labor market at that instant. In fact, that paper assumed that workers within a market are homogeneous and so all workers were indifferent about working whenever there was rest unemployment in their labor market. In this paper, union-mandated minimum wages and seniority rules make the rest unemployed worse off than the employed. This means we need to keep track of workers' seniority in order to understand their decision to enter and leave labor markets.

We show that if a union has any effect, it generates rest unemployment. This result does not depend on the leisure value of unemployment, nor does it depend on whether there are search frictions. Whenever the minimum wage binds, workers with low seniority who are rationed out of a job decide to stay in the labor market, waiting for the conditions to improve so that they can return to work at the minimum wage. When labor market conditions are bad enough, workers with the lowest seniority among those who are rationed out of employment will leave. The prospects of a labor market are limited by the fact that as conditions improve, new workers will arrive via search. These newcomers will have the lowest seniority, and hence will be most vulnerable to bad shocks, but they will only arrive in a labor market when it is booming. The situation of newcomers depends on how high the minimum wage is. If it is not that high, so that it binds only for bad shocks, they will immediately start working. If the minimum wage is sufficiently high, it always binds. In this case, newcomers arrive when prospects are very good, but are forced to queue until enough good shocks have arrived before they can start work. In such a labor market, there is always a queue of workers waiting either to start or resume employment.

This paper connects with an older literature that examines the impact of unions on labor market outcomes. Medoff (1979) argues that unionized firms lay off workers at a much higher rate than non-unionized firms. Using data at the state and 2-digit-manufacturing level, he concludes that the monthly layoff rate for a non-unionized establishment was 0.5 percent from 1965 to 1969, while the monthly layoff for a comparable establishment that is unionized is 2.3 percent. Similar results at the three digit level from 1958 to 1971 yield a smaller but still substantial difference, 1.0 percent versus 2.2 percent. Medoff (1979) also provides some evidence on the role of seniority in layoffs. 81 percent of union contracts in a Bureau of Labor statistics sample explicitly refer to layoff procedures. Of those, 58 percent state that seniority is the "sole" or "primary" factor determining who is laid off. Medoff concludes that "with additional services comes the right to remain employed until employees with less service have been laid off."

Using their own survey, Abraham and Medoff (1984) confirm that seniority is an important determinant of layoffs in unionized firms. 84 percent of unionized hourly workers who had witnessed a layoff report that a senior employee is never laid off before a more junior one, compared with 42 percent of non-unionized hourly workers and 24 percent of non-unionized salaried workers. There are fewer studies of whether recalls are based on seniority, perhaps because the conclusion is self-evident. Blau and Kahn (1983) find that unions use seniority to allocate fixed-duration layoffs rather than indefinite layoffs, and again give more senior workers a priority in getting recalled from indefinite duration layoff. Tracy (1986) cites one particular 1971 union contract as saying "seniority will apply to layoffs and rehires. The last employee hired shall be the first laid off, and the last laid off shall be the first rehired."

Jacobson, LaLonde, and Sullivan (1993) document that workers displaced from heavily unionized industries suffer unusually large and persistent income declines. This too is consistent with the way we model seniority in unionized industries. In non-unionized industries, workers' welfare is limited by the possibility of new entrants coming to the industry. But in unionized industries, high seniority workers may be significantly better off than new entrants. When they are displaced, the consequences are then disproportionately severe.

Our model also addresses a large literature which argues that unions compress wages. Blau and Kahn (1996) observe that wages in the U.S. are more dispersed than in other OECD countries, particularly towards the bottom of the distribution and argue that this is due to the absence of centralized wage-setting mechanisms. Mourre (2005) confirms this using more recent and detailed data for the European Union. Bertola and Rogerson (1997) show that such wage compression may be important for understanding why other labor market institutions, especially restrictions on turnover, are not particularly correlated with measured job creation and destruction rates. In our model, unions can affect labor market institutions only by compressing wages and so we can confirm that high unemployment rates are associated with substantial wage compression.

Our approach to modeling unemployed union members as rest unemployed builds on Summers (1986), who argues that that union-induced wage rigidities can explain a large portion of unemployment in the U.S.. Unemployed workers who lose their job because of sectoral shocks spend little time searching for jobs, but instead seem to be waiting either for wages to fall or for the shocks to be reversed. Harris and Todaro (1970) propose an extreme version of "wait unemployment" in less developed countries. When rural workers move to the city, they must queue for a job before they can start work. They are willing to do so even though the marginal product of labor is positive in the countryside. Both of these findings are consistent with our model. A spell of rest unemployment ends only if the shocks that caused it are reversed or if the worker becomes so discouraged that she leaves the labor market. In either case, workers can spend a considerable amount of time unemployed. If the minimum wage is sufficiently high relative to the extent of search fricitons, it will bind in all states of the world. Then even new entrants will not be able to go to work immediately. Instead, they must queue until productivity has risen sufficiently for their marginal product to exceed the minimum wage. While they are queueing, increases in productivity raise their seniority—their position in the queue—until they eventually reach the gates of the factory and get employed.

Although it is not our main focus, our paper gives a novel perspective on why unions might choose to raise wages above the market-clearing level. Many authors have recognized that this may be optimal for more senior union members who are protected from the risk of layoff (Freeman and Medoff, 1984). Blanchard and Summers (1986) argue that for an "insider-outsider" theory of European unemployment, where unions run by insiders generate unemployment because wages are set to exclude disenfranchised outsiders. We find that a union that cares equally about insiders and outsiders opts for a minimum wage policy. More precisely, we consider a union that sets the wage, or equivalently the employment level, at each instant in order to maximize the utilitarian welfare of all of its members, insiders and outsiders. We show that the union's policy is characterized by a constant minimum wage, where the minimum wage is a markup over the leisure value of rest unemployment. By setting this minimum wage, the union effectively restricts output so that it never exceeds the monopoly level. When the available number of workers is less the number needed to produce the monopoly output, all the union members are employed and the minimum wage does not bind. At other times the minimum wage binds and there is rest unemployment. Thus the difference between a monopoly producer and a monopoly union is simply an issue of who keeps the monopoly rents. In other words, we find that unions may generate unemployment not because more senior members may have an undue influence on wage setting procedures, but rather because they can only raise the well-being of all their members by constraining output in some states of the world.

Finally, our model is consistent with the finding in Nickell and Layard (1999) that unions raise the unemployment rate only in countries where they cannot effectively coordinate their bargaining. In our model, the equilibrium without unions is Pareto optimal. While any individual union can improve its workers' well-being through a minimum wage, all workers are better off if unions do not exploit their monopoly power. Thus if unions can collude, they be able to avoid generating rest unemployment.

The next section of the paper presents a simplified version of our model without search frictions, where workers can costlessly move between labor markets. This gives a sense of how the dynamics in the model work and how minimum wages affect the unemployment rate. We describe our full model in Section 3 and characterize the equilibrium in Section 4. We first prove that a minimum wage affects the wage distribution if and only if it generates rest unemployment. Then we show how minimum wages affect workers' decision to enter and exit labor markets. Finally, we characterize the search and rest unemployment rates and the hazard rate of exiting unemployment in a labor market with a binding minimum wage. Section 5 explains why a utilitarian union would find it optimal to impose a minimum wage. We finish in Section 6 with a numerical example intended to illustrate the properties of the model.

2 Frictionless Model

We consider a continuous time, infinite-horizon model. We focus for simplicity on an aggregate steady state and assume markets are complete.

2.1 Goods

There is a continuum of goods indexed by $j \in [0, 1]$ and a large number of competitive producers of each good. Each good is produced in a separate labor market with a constant returns to scale technology that uses only labor. In a typical labor market j at time t, there is a measure $l(j,t)$ workers. Of these, $e(j,t)$ are employed, each producing $Ax(j,t)$ units of good j, while the remaining $l(j,t) - e(j,t)$ are rest-unemployed. Competition forces firms to price each good at marginal cost, so the wage in labor market j, $w(j,t)$, is equal to the product of the price of good j, $p(j,t)$, and the productivity of each worker in labor market

j, $Ax(j,t)$.

A is the aggregate component in productivity while $x(j,t)$ is an idiosyncratic shock that follows a geometric random walk,

$$d \log x(j,t) = \mu_x dt + \sigma_x dz(j,t), \tag{1}$$

where μ_x measures the drift of log productivity, $\sigma_x > 0$ measures the standard deviation, and $z(j,t)$ is a standard Wiener process, independent across goods.

To keep a well-behaved distribution of labor productivity, we assume that the market for good j shuts down according to a Poisson process with arrival rate δ, independent across goods and independent of good j's productivity. When this shock hits, all the workers are forced out of the labor market. A new good, also named j, enters with positive initial productivity $x \sim F(x)$, keeping the total measure of goods constant. We assume a law of large numbers, so the share of labor markets experiencing any particular sequence of shocks is deterministic.

2.2 Households

There is a representative household consisting of a measure 1 of members. The large household structure allows for full risk sharing within each household, a standard device for studying complete markets allocations.

At each moment in time t, each member of the representative household engages in one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the intermediate goods (or equivalently labor) markets.
  - $E(t)$ of these workers are employed at the prevailing wage and get leisure 0.
  - $U_r(t) = L(t) - E(t)$ of these workers are rest-unemployed and get leisure b_r.
- The remaining $1 - E(t) - U_r(t)$ household members are inactive, getting leisure b_i.

We assume $b_r < b_i$, so rest unemployment gives less leisure than inactivity. Household members may costlessly move between these three states. However, whenever they enter (or reenter) a market, they start with the lowest level of seniority. In addition to the endogenous decision to leave a market, we allow for two other exogenous reasons why a worker may exit her market: it shuts down at rate δ; and she is hit by an idiosyncratic shock according to a Poisson process with arrival rate q, independent across individuals and independent of their labor market's productivity. We introduce the idiosyncratic "quit" shock q to account for separations that are unrelated to the state of the labor market.

We represent the household's preferences via the utility function

$$\int_0^\infty e^{-\rho t}\left(\log C(t) + b_i\big(1 - E(t) - U_r(t)\big) + b_r U_r(t)\right) dt, \tag{2}$$

where $\rho > 0$ is the discount rate and $C(t)$ is the household's consumption of a composite good

$$C(t) = \left(\int_0^1 c(j,t)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}, \tag{3}$$

and $c(j,t)$ is the consumption of good j at time t. We assume that the elasticity of substitution between goods, θ, is greater than 1. The cost of this consumption is $\int_0^1\int_0^1 p(j,t)c(j,t)djdn$, which we assume the household finances using its labor income.

Standard arguments imply that the demand for good j satisfies

$$c(j,t) = \frac{C(t)P(t)^\theta}{p(j,t)^\theta}, \tag{4}$$

where

$$P(t) = \left(\int_0^1 p(j,t)^{1-\theta} dj\right)^{\frac{1}{1-\theta}} \tag{5}$$

is the price index, which we normalize to equal 1.

To ensure a well-behaved distribution of wages, we impose two restrictions on preferences and technology. First, we require

$$\delta > (\theta - 1)\big(\mu_x + \tfrac{1}{2}(\theta - 1)\sigma_x^2\big), \tag{6}$$

so industries exit sufficiently quickly to offset the drift in the stochastic process for productivity. If this condition failed, workers could attain infinite utility. Second, we require

$$X \equiv \left(\int_0^\infty x^{\theta-1} dF(x)\right)^{\frac{1}{\theta-1}} \in (0,\infty), \tag{7}$$

a restriction on the distribution of productivity in new labor markets. If this condition failed, the wage would be either zero or infinite.

2.3 Unions

Unions constrain the wage in labor market j, introducing a restriction $w(j,t) \geq \hat{w}(j)$. For most of our analysis, we treat the minimum wage $\hat{w}(j)$ as exogenous and consider its consequences. To see whether the minimum wage constraint binds, first note that if all the workers in the labor market were employed, they would produce $Ax(j,t)l(j,t)$ units of good j. Inverting the demand curve equation (4) and eliminating the price level using $P(t) = 1$, the relative price of good j would be

$$p(j,t) = \left(\frac{\bar{C}(t)}{Ax(j,t)l(j,t)}\right)^{\frac{1}{\theta}}.$$

The wage in the labor market would then be $p(j,t)Ax(j,t)$ or

$$w(j,t) = \left(\frac{C(t)\big(Ax(j,t)\big)^{\theta-1}}{l(j,t)}\right)^{\frac{1}{\theta}}. \tag{8}$$

This is increasing in the productivity of the labor market and decreasing in the number of workers. In particular, if there are too many workers in the market, the minimum wage constraint binds. In that case, $w(j,t) = \hat{w}(j)$ and employment is determined at the level that makes the price of good j equal to $\hat{w}(j)/Ax(j,t)$,

$$e(n_j,t) = \frac{C(t)\big(Ax(j,t)\big)^{\theta-1}}{\hat{w}(j)^{\theta}}, \tag{9}$$

increasing in productivity and decreasing in the minimum wage.

We assume that when the minimum wage constraint binds, more senior workers have the first option to work, where seniority is measured by the amount of time spent in the union. Consider a worker with relative seniority $s \in [0,1]$, where we measure relative seniority s as the percentage of workers in the labor market with lower seniority, so $s = 1$ corresponds to the worker with the greatest seniority. Assuming she wants the job, she is guaranteed to be employed if $e(j,t)/l(j,t) \geq 1 - s$ or, from equation (9),

$$s \geq 1 - \frac{C(t)\big(Ax(j,t)\big)^{\theta-1}}{\hat{w}(j)^{\theta}l(j,t)}. \tag{10}$$

A worker with a given seniority is more likely to be employed when productivity is higher, the minimum wage is lower, or the number of workers in the labor market is smaller.

Since workers are typically not indifferent about working, those with more seniority are weakly better off. Thus to analyze a worker's decision to enter or stay in a labor market, we need to examine not only the behavior of wages in the market, but also how the entry and exit of other workers influences each worker's seniority.

2.4 Equilibrium

We look for a competitive equilibrium of this economy, subject to the constraints imposed by minimum wages. At each instant, each household chooses how much of each good to consume and how to allocate its members between employment, rest unemployment, and inactivity, in order to maximize utility subject to the constraints imposed by seniority rules; and each goods producer j maximizes profits by choosing how many workers to hire taking as given the wage in its labor market and the price of its good. Moreover, the demand for labor from goods producers is equal to the supply from households in each market unless the minimum wage constraint binds, in which case labor demand may be less than labor supply; and households' demand for goods is equal to the supply from firms. We focus on parameter values for which the household keeps some of its members inactive, which requires that the leisure value of inactivity b_i is sufficiently large.

We look for a stationary equilibrium where all aggregate quantities and prices are constant, as is the joint distribution of wages, productivity, output, employment, and rest unemployment across labor markets. We suppress the time argument as appropriate in what follows. With identical households and complete markets, consumption is equal to current labor income and hence we also ignore financial markets.

2.5 Characterization

In this section, we prove that the number of workers in labor market j satisfies

$$l(j,t) = \frac{C\big(Ax(j,t)\big)^{\theta-1}}{\bar{w}(j)^{\theta}} \tag{11}$$

for some constant $\bar{w}(j)$, where C is the constant level of consumption. We also characterize $\bar{w}(j)$. In unionized markets with a binding minimum wage $\hat{w}(j)$, we prove that $\bar{w}(j) < \hat{w}(j)$. Equation (10) implies that a worker is employed if and only if

$$s \geq 1 - \left(\frac{\bar{w}(j)}{\hat{w}(j)}\right)^{\theta} \equiv \hat{s}(j) \in (0,1). \tag{12}$$

The unemployment rate in labor market j is equal to $\hat{s}(j)$. In labor markets where the minimum wage $\hat{w}(j)$ is not binding, $\bar{w}(j) = w^*$, a constant that satisfying $w^* \geq \hat{w}(j)$. All workers are employed and have the same expected utility, regardless of their seniority. In what follows, we suppress the name of the labor market j.

To prove this, first consider a non-unionized labor market, for example a labor market with no minimum wage. We claim that, regardless of the sequence of shocks hitting the industry, a worker earns a constant wage w^* and is always employed. To prove this and characterize w^*, we use the assumption that some members of the household are inactive. Since the household can freely move workers between inactivity and a job in a non-unionized labor market, it must be indifferent between the two activities. An inactive worker contributes b_i utils to the household, while a worker employed at w^* contributes w^*/C, since the marginal utility of consumption is $1/C$. Combining these, we find that $w^* = b_i C$. As long as the minimum wage is smaller than this level, $\hat{w} \leq w^*$, it does not bind. As an industry with a non-binding minimum wage is hit by productivity shocks, the number of workers varies according to equation (11), while the wage stays constant at w^*. The workers in such industries move between as necessary while avoiding any unemployment spells.

Now consider the case where $\hat{w} > w^* = b_i C$. The analysis in the previous paragraph is inapplicable because the minimum wage is binding. We conjecture that in equilibrium a worker's value depends only on her relative seniority $v(s)$, where $s \in [0, 1]$ is the fraction of workers with lower seniority. A worker exits an industry when her seniority falls to 0 and the industry is hit by an adverse shock. She works whenever her seniority exceeds the threshold defined in equation (12) for some value of $\bar{w} < \hat{w}$ to be determined.

By taking limits of discrete-time, discrete-state model, we show in Section A.1 that the worker's value function may be expressed as

$$\rho v(s) = R(s) + \lambda \left(\frac{w^*}{\rho C} - v(s) \right) + v'(s)(1-s)(\theta - 1)\big(\mu_x - \tfrac{1}{2}(\theta - 1)\sigma_x^2\big) + \tfrac{1}{2} v''(s)(1-s)^2(\theta - 1)^2 \sigma_x^2 \quad (13)$$

for all $s > 0$. Here $R(s)$ is the return function:

$$R(s) = \begin{cases} b_r & \text{if } s < \hat{s} \\ \hat{w}/C & \text{if } s \geq \hat{s}. \end{cases} \quad (14)$$

The parameter $\lambda \equiv \rho + \delta$ is the exogenous rate that workers exit markets and $w^*/\rho C = b_i/\rho$ is the utility for a worker in a competitive market or in inactivity. For a worker with seniority $s \in (0, 1)$, the drift in seniority is $(1-s)(\theta-1)\big(\mu_x - \frac{1}{2}(\theta-1)\sigma_x^2\big)$ and the instantaneous standard deviation of seniority is $(1-s)(\theta-1)\sigma_x$.

Equation (13) implies that $v(s)$ is twice continuously differentiable at s where $R(s)$ is continuous, although it is only once differentiable at $s = \hat{s}$. To solve the second order differential equation and find the threshold for unemployment $\hat{s}$, we need three terminal conditions. We use two conditions for new entrants to markets. The value matching condition states that workers with zero seniority are indifferent about participating in the market and going to a competitive market,

$$v(0) = \frac{w^*}{\rho C}.$$

The smooth pasting condition states that the marginal value of seniority is zero at low seniority,

$$v'(0) = 0.$$

We establish the latter condition in A.1. Finally, note seniority $s = 1$ is an absorbing state. In this case, equation (13) reduces to

$$\rho v(1) = \frac{\hat{w}}{C} + \lambda \left(\frac{w^*}{\rho C} - v(1) \right),$$

which ensures that the marginal value of seniority is bounded at $s = 1$.

One can verify that the unique solution to this system of equations is

$$v(s) = \begin{cases} \dfrac{b_r}{\rho + \lambda} + \dfrac{\lambda w^*/C}{(\rho+\lambda)\rho} + \displaystyle\sum_{i=1}^{2} \underline{c}_i (1-s)^{-\eta_i} & \text{if } s < \hat{s} \\ \dfrac{\hat{w}/C}{\rho + \lambda} + \dfrac{\lambda w^*/C}{(\rho+\lambda)\rho} + \displaystyle\sum_{i=1}^{2} \hat{c}_i (1-s)^{-\eta_i} & \text{if } s \geq \hat{s}, \end{cases} \tag{15}$$

where the exponents η are the roots of the characteristic equation

$$\rho + \lambda = (\theta - 1)\mu_x \eta + \tfrac{1}{2}(\theta-1)^2 \sigma_x^2 \eta^2,$$

with $\eta_1 < 0$ and $\eta_2 > 1$; the latter condition is ensured by equation (6). The threshold for working $\hat{s}$, and hence the unemployment rate in the market, is given by[2]

$$\hat{s} = 1 - \left(\frac{w^* - b_r C}{\hat{w} - b_r C} \right)^{\frac{1}{\eta_2}}, \tag{16}$$

[2]Combining equations (12) and (16), we obtain an expression for the constant $\bar{w}$:

$$\bar{w} = \hat{w} \left(\frac{w^* - b_r C}{\hat{w} - b_r C} \right)^{\frac{1}{\theta \eta_2}}.$$

Since $\hat{w} > w^*$, $\theta > 1$, and $\eta_2 > 1$, this implies $\bar{w} < \hat{w}$.

and the constants $\underline{c}_i$ and $\hat{c}_i$ satisfy

$$\underline{c}_1 = \left(\frac{w^*/C - b_r}{\rho + \lambda}\right)\frac{\eta_2}{\eta_2 - \eta_1} > 0, \quad \underline{c}_2 = -\left(\frac{w^*/C - b_r}{\rho + \lambda}\right)\frac{\eta_1}{\eta_2 - \eta_1} > 0,$$
$$\hat{c}_1 = -\left(\frac{w^*/C - b_r}{\rho + \lambda}\right)\frac{\eta_2}{\eta_2 - \eta_1}\left(\left(\frac{\hat{w} - b_r C}{w^* - b_r C}\right)^{\frac{\eta_2 - \eta_1}{\eta_2}} - 1\right) < 0, \quad \hat{c}_2 = 0.$$

The general form of the value function in equation (15) is the unique solution to the differential equation (13) at all points $s \in [0, \hat{s}) \cup (\hat{s}, 1]$. The constants $\underline{c}_1$ and $\underline{c}_2$ are pinned down by the value-matching and smooth-pasting conditions. The restriction $\hat{c}_2 = 0$ is required to be sure that the value function stays bounded as seniority converges to 1. Finally, the choice of $\hat{c}_1$ and $\hat{s}$ is determined by the requirement that the value function is everywhere once differentiable, $\underline{v}(\hat{s}) = \hat{v}(\hat{s})$ and $\underline{v}'(\hat{s}) = \hat{v}'(\hat{s})$.

It is straightforward to verify algebraically that the value function is increasing in s. Since $\underline{c}_1 > 0$ and $\eta_1 < 0$, $\underline{c}_1(1-s)^{-\eta_1}$ is convex in s. Similarly, $\underline{c}_2 > 0$ and $\eta_2 > 1$, which ensures that $\underline{c}_2(1-s)^{-\eta_2}$ is convex. Thus $v'(s)$ is increasing for $s < \hat{s}$. Since smooth pasting imposes that $v'(0) = 0$, $v'(s) > 0$ for $s \in (0, \hat{s})$. At values of $s > \hat{s}$, $v'(s)$ is positive because $\hat{c}_1 < 0$ and $\eta_1 < 0$. This confirms that workers exit their labor market voluntarily only when their seniority falls to 0.

2.6 Unemployment

Equation (16) describes the unemployment rate unemployment rate $\hat{s}$ in a labor market with minimum wage $\hat{w} \geq w^*$. It is equal to 0 if $\hat{w} = w^*$ and is then increasing in the minimum wage $\hat{w}$. To understand the magnitude of unemployment, compare this to a hypothetical labor market with a minimum wage but where jobs are allocated randomly, not based on seniority. If a worker enters such a labor market, she is employed at the minimum wage $\hat{w}$ with probability $1-u$ and rest-unemployed otherwise. Since a household must be indifferent between sending a worker to such a labor market and sending the worker to a competitive labor market, we have

$$w^*/C = (1-u)\hat{w}/C + ub_r \Rightarrow u = 1 - \frac{w^* - b_r C}{\hat{w} - b_r C}.$$

Since $\hat{w} > w^* = b_i C > b_r C$, this defines $u \in (0, 1)$. Moreover, since $\eta_2 > 1$, this defines $u > s$.[3] Relative to a case where jobs are assigned randomly, a seniority rule reduces the

[3] $\eta_2 = 1$ and so $u = s$ only in one extreme case. We require $\mu_x + \frac{1}{2}(\theta - 1)\sigma_x^2 = 0$, so there is no drift in the average level of productivity; $\rho \to 0$, so there is no discounting; and $q = 0$ and $\delta \to 0$, so workers never leave markets exogenously.

unemployment rate associated with a given minimum wage by unevenly distributing the union rents. This encourages marginal workers to leave the labor market rather than lingering in rest unemployment.

With a random assignment of jobs to union members, the unemployment rate depends only on the leisure from inactivity and rest unemployment and the real wage $\bar{w}/C$. With seniority rules, other preference and technology parameters also affect a labor market's unemployment rate through their effect on η_2; equation (16) implies that any parameter which raises η_2 reduces the unemployment rate.[4]

A higher discount rate ρ or a higher exogenous exit rate λ raises η_2 and hence reduces the unemployment rate. Since marginal workers are always unemployed, an increase in ρ implies that workers weigh current unemployment more heavily than the future possibility of employment and so less inclined to stay in the labor market. Similarly, an increase in λ reduces the probability of experiencing future employment in this market and so encourages low-seniority workers to leave.

On the other hand, a higher drift in productivity μ_x or standard deviation of productivity σ_x raises the unemployment rate. A higher drift implies that an initial unemployment spell is unlikely to be repeated, while a higher standard deviation raises the option value of waiting to see how productivity evolves. Finally, a greater elasticity of substitution θ raises the unemployment rate because it amplifies the impact of any productivity shock. None of these possibilities are present in the static model.

3 Full Model

We now extend the model by introducing search frictions. While workers can costlessly move between employment and rest unemployment within a labor market, we assume it takes time to move between markets. This changes our results along several dimensions.

First, productivity shocks cause wage fluctuations within labor markets since search frictions prevent costless arbitrage of any wage differences across markets. With wage fluctuations, we interpret unions as imposing a minimum wage $\hat{w}$ and a seniority rule, rather than just a fixed wage. Following a positive sequence of productivity shocks, the minimum wage constraint may be slack and all the union members employed. More generally, in the presence of search frictions some markets may be more attractive than others, even for a worker without seniority.

[4] The discussion in this paragraph and the next two paragraphs is loose because we implicitly assume that a change in parameters does not affect the level of consumption C. In the next section, we extend the model to have many industries and allow these parameters to differ across industries. If we followed a similar approach here, the comparative statics with respect to λ, μ_x, σ_x, and θ would be relevant in the cross-section.

Second, we find that workers need not experience a spell of unemployment when they enter a market. Workers enter markets with a moderate minimum wage at times when the minimum wage constraint does not bind. This allows them to start a job immediately. But when markets are hit by adverse shocks, they will not immediately exit. Instead, we prove that they will always experience a spell of rest unemployment before exiting. In this sense, rest unemployment is associated with declining unionized industries. Still, for a sufficiently high minimum wage relative to the search frictions, the minimum wage will always bind and so the market will always have some unemployment.

Finally, search frictions give us a notion of workers who are attached to a labor market. This allows us to consider the objective function of a union that represents those workers.

We also extend the model along one other dimension. We assume there are many industries that produce relatively poor substitutes. Within each industry, there are many goods that are relatively easily substituted. This facilitates comparative statics like the ones at the end of the previous section, at the cost of somewhat more cumbersome notation.

3.1 Goods

There is a continuum of industries indexed by $n \in [0,1]$. Within each industry, there is a continuum of goods indexed by $j \in [0,1]$ and a large number of competitive producers of each good. Thus n_j is the name of a particular good produced in a particular industry. The model from the previous section applies within each industry, although parameters may differ across goods. In labor market n_j at time t, there is a measure $e(n_j,t)$ employed workers, each of whom produce $Ax(n_j,t)$ units of good n_j. There are also $l(n_j,t) - e(n_j,t)$ rest-unemployed workers. Workers are paid their marginal product, so the wage in market n_j solves $w(n_j,t) = p(n_j,t)Ax(n_j,t)$, where $p(n_j,t)$ is the price of good n_j.

A is the aggregate component in productivity while $x(n_j,t)$ is an idiosyncratic shock that follows a geometric random walk with industry-specific drift $\mu_{n,x}$ and industry-specific standard deviation $\sigma_{n,x}$:

$$d\log x(n_j,t) = \mu_{n,x}dt + \sigma_{n,x}dz(n_j,t). \tag{17}$$

As before, we assume that the market for good n_j shuts down according to a Poisson process with arrival rate δ_n, independent across goods and independent of good n_j's productivity. When this shock hits, all the workers are forced out of the labor market. A new good, also named n_j, enters with positive initial productivity $x \sim F_n(x)$, keeping the total measure of goods in industry n constant. We assume a law of large numbers, so the share of labor markets in each industry experiencing any particular sequence of shocks is deterministic.

3.2 Households

There is a representative household consisting of a measure 1 of members. At each moment in time t, each member of the representative household engages in one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the intermediate goods (or equivalently labor) markets.
  - $E(t)$ of these workers are employed at the prevailing wage and get leisure 0.
  - $U_r(t) = L(t) - E(t)$ of these workers are rest-unemployed and get leisure b_r.
- $U_s(t)$ household members are search-unemployed, looking for a new labor market and getting leisure b_s.
- The remaining $1 - E(t) - U_r(t) - U_s(t)$ household members are inactive, getting leisure b_i.

We assume $b_i > b_s$ but no longer impose $b_i > b_r$. Household members may costlessly switch between employment and rest unemployment and between inactivity and searching; however, they cannot switch intermediate goods markets without going through a spell of search unemployment. Workers exit their intermediate goods market for inactivity or search in three circumstances: first, they may do so endogenously at any time at not cost; second, they must do when their market shuts down, which happens at rate δ_n; and third, they must do so when they are hit by an idiosyncratic shock, according to a Poisson process with arrival rate q_n, independent across individuals and independent of their labor market's productivity. We introduce the idiosyncratic "quit" shock q_n to account for separations that are unrelated to the state of the labor market. Finally, a worker in search unemployment finds a job according to a Poisson process with arrival rate α. When this happens, she may enter the intermediate goods market of her choice.

We represent the household's preferences via the utility function

$$\int_0^\infty e^{-\rho t} \left(\log \bar{C}(t) + b_i\big(1 - E(t) - U_r(t) - U_s(t)\big) + b_r U_r(t) + b_s U_s(t)\right) dt, \tag{18}$$

where $\rho > 0$ is the discount rate and $\bar{C}(t)$ is the household's consumption of an aggregate of all goods produced in all industries,

$$\log \bar{C}(t) = \int_0^1 \log C(n,t) dn, \tag{19}$$

$C(n,t)$ is the household's consumption of an aggregate of the goods in industry n,

$$C(n,t) = \left(\int_0^1 c(n_j,t)^{\frac{\theta_n - 1}{\theta_n}} dj \right)^{\frac{\theta_n}{\theta_n - 1}}, \tag{20}$$

and $c(n_j,t)$ is the consumption of good n_j at time t. We assume that the elasticity of substitution between goods in industry n, θ_n, is greater than 1. The cost of this consumption is $\int_0^1 \int_0^1 p(n_j,t)c(n_j,t)djdn$, which we assume the household finances using its labor income.

Standard arguments imply that the demand for good n_j satisfies

$$c(n_j,t) = \frac{C(n,t)P(n,t)^{\theta_n}}{p(n_j,t)^{\theta_n}}, \tag{21}$$

where

$$P(n,t) = \left(\int_0^1 p(n_j,t)^{1-\theta_n} dj \right)^{\frac{1}{1-\theta_n}} \tag{22}$$

is the price index in industry n. The demand for the consumption aggregator in industry n satisfies

$$C(n,t) = \frac{\bar{C}(t)}{P(n,t)}, \tag{23}$$

where we use the price of the aggregate consumption bundle $\bar{C}$ as numeraire, or equivalently normalize

$$\int_0^1 \log P(n,t)dn = 0. \tag{24}$$

To ensure a well-behaved distribution of wages in each industry, we impose two restrictions on preferences and technology, generalizations of equations (6) and (7):

$$\delta_n > (\theta_n - 1)\big(\mu_{n,x} + (\theta_n - 1)\tfrac{1}{2}(\sigma_{n,x})^2\big) \tag{25}$$

$$X_n \equiv \left(\int_0^\infty x^{\theta_n - 1} dF_n(x) \right)^{\frac{1}{\theta_n - 1}} \in (0,\infty) \tag{26}$$

These ensure that expected utility is finite.

3.3 Unions

Unions constrain the wage in labor market n_j, introducing a restriction $w(n_j,t) \geq \hat{w}(n_j)$. To see whether the minimum wage constraint binds, first note that if all the workers in the industry were employed, they would produce $Ax(n_j,t)l(n_j,t)$ units of good n_j. Inverting the demand curve equation (21) and eliminating the price of industry n using equation (23), the relative price of good n_j would be

$$p(n_j,t) = \frac{\bar{C}(t)}{C(n,t)^{\frac{\theta_n-1}{\theta_n}}\big(Ax(n_j,t)l(n_j,t)\big)^{1/\theta_n}}.$$

The wage in the industry would then be $p(n_j,t)Ax(n_j,t)$ or

$$w(n_j,t) = \frac{\bar{C}(t)\big(Ax(n_j,t)\big)^{\frac{\theta_n-1}{\theta_n}}}{C(n,t)^{\frac{\theta_n-1}{\theta_n}}l(n_j,t)^{\frac{1}{\theta_n}}}. \tag{27}$$

This is increasing in the productivity of the labor market and decreasing in the number of workers. In particular, if there are too many workers in the market, the minimum wage constraint binds. In that case, $w(n_j,t) = \hat{w}(n_j)$ and employment is determined at the level that makes the price of good n_j equal to $\hat{w}(n_j)/Ax(n_j,t)$,

$$e(n_j,t) = \frac{\bar{C}(t)^{\theta_n}\big(Ax(n_j,t)\big)^{\theta_n-1}}{C(n,t)^{\theta_n-1}\hat{w}(n_j)^{\theta_n}}, \tag{28}$$

increasing in productivity and decreasing in the minimum wage. We continue to assume that when the minimum wage constraint binds, more senior workers have the first option to work, where seniority is measured by the amount of time spent in the union. When the minimum wage binds, a worker with seniority s works if and only if

$$s \geq 1 - \frac{\bar{C}(t)^{\theta_n}\big(Ax(n_j,t)\big)^{\theta_n-1}}{C(n,t)^{\theta_n-1}\hat{w}(n_j)^{\theta_n}l(n_j,t)}. \tag{29}$$

3.4 Equilibrium

We look for a competitive equilibrium of this economy, subject to the constraints imposed by minimum wages. At each instant, each household chooses how much of each good to consume and how to allocate its members between employment in each labor market, rest unemployment in each labor market, search unemployment, and inactivity, in order to maximize utility subject to technological constraints on reallocating members across labor markets and the minimum wage constraints, taking as given the stochastic process for wages and seniority in each labor market; and each goods producer n_j maximizes profits by choosing how many workers to hire taking as given the wage in its labor market and the price of its good. Moreover, the demand for labor from goods producers is equal to the supply from households in each market unless the minimum wage constraint binds, in which case labor demand may be less than labor supply; and households' demand for goods is equal to the supply from firms.

We look for a stationary equilibrium where all aggregate and industry-specific quantities and prices are constant, as is the joint distribution of wages, productivity, output, employment, and rest unemployment across labor markets within industries. We suppress the time argument as appropriate in what follows. We continue to ignore financial markets.

4 Characterization of Equilibrium

At any point in time, a typical labor market n_j is characterized by its productivity x and the number of workers l. We look for an equilibrium in which the ratio $x^{\theta_n - 1}/l$ follows a Markov process. Workers enter labor markets when the ratio exceeds a threshold and exit labor markets when it falls below a strictly smaller threshold. Moreover, equation (29) shows that this ratio and a worker's seniority determines whether she has the option to work.

4.1 The Marginal Value of Household Members

We start by computing the marginal value of an additional household member engaged in each of the three activities. These are related by the possibility of reallocating household members between activities.

Consider first a household member who is permanently inactive. It is immediate from equation (18) that he contributes

$$\underline{v} = \frac{b_i}{\rho} \tag{30}$$

to household utility. Since the household may freely shift workers between inactivity and search unemployment, this must also be the incremental value of a searcher, assuming some members are engaged in each activity. A searcher gets flow utility b_s and the possibility of finding a labor market at rate α, giving capital gain $\bar{v} - \underline{v}$, where $\bar{v}$ is the value to the household of having a worker in the best labor market. This implies $\rho \underline{v} = b_s + \alpha(\bar{v} - \underline{v})$ or

$$\bar{v} = \underline{v} + b_i \kappa, \text{ where } \kappa \equiv \frac{b_i - b_s}{b_i \alpha} \tag{31}$$

is a measure of search costs, the percentage loss in current utility from searching rather than inactivity times the expected duration of search unemployment $1/\alpha$. Conversely, a worker may freely exit her labor market, and so the lower bound on the value of a household member in a labor market, either employed or search unemployed, is $\underline{v}$. If the household values a worker at some intermediate amount, it will be willing to keep her in her labor market rather than having her search for a new one.

Finally, consider the margin between employment and resting for a worker in a labor market paying a wage w. A resting worker generates b_r utils while an employed worker generates income valued at $w/\bar{C}$, where $1/\bar{C}$ is the marginal utility of the consumption aggregate. Since switching between employment and resting is costless, all workers prefer to work in any labor market with $w/\bar{C} > b_r$ and prefer to rest in any market with $w/\bar{C} < b_r$. This implies that if $\hat{w}/\bar{C} \leq b_r$, the minimum wage never binds because workers' willingness to enter rest unemployment endogenously keeps the wage above $\hat{w}$. Conversely, if $\hat{w}/\bar{C} > b_r$, the minimum wage may sometimes bind.

4.2 Wage and Labor Force Dynamics

Consider a labor market in industry n with l workers, productivity x, and a minimum wage $\hat{w}$. Let $P(l,x)$ denote the price of its good, $Q(l,x)$ denote the amount of the good produced, $W(l,x)$ denote the wage rate, and $E(l,x)$ denote the number of workers who are employed. Competition ensures that the wage is equal to the marginal product of labor, $W(l,x) = P(l,x)Ax$, while the production function implies $Q(l,x) = E(l,x)Ax$. From equation (27), the wage solves

$$W(l,x) = \bar{C}\max\{e^{\hat{\omega}}, e^{\omega}\} \tag{32}$$

where

$$\omega \equiv \frac{(\theta_n - 1)(\log(Ax) - \log C(n)) - \log l}{\theta_n}, \tag{33}$$

is the logarithm of the "full-employment wage" measured in utils, the wage that would prevail if there were full employment in the labor market and

$$\hat{\omega} \equiv \max\{\log \hat{w} - \log \bar{C}, \log b_r\} \tag{34}$$

is the maximum of the log minimum wage expressed in utils and the utility from rest unemployment. From equation (28), the level of employment is $E(l,x) = le^{\theta_n(\omega-\hat{\omega})}$ if the minimum wage binds, $\omega < \hat{\omega}$, and l otherwise. Hence the amount of the good produced is

$$Q(l,x) = lAx\min\{1, e^{\theta_n(\omega-\hat{\omega})}\}. \tag{35}$$

When $\omega \geq \hat{\omega}$, the wage exceeds the minimum wage and so there is no rest unemployment. Otherwise, enough workers rest to the raise the log wage in utils to $\hat{\omega}$.

Since the wage only depends on ω, we look for an equilibrium in which any labor market with $\omega > \bar{\omega}_n(\hat{\omega})$ immediately attracts new entrants to push the log full employment wage back to $\bar{\omega}_n(\hat{\omega})$ and workers with the least seniority immediately exit any labor market with $\omega < \underline{\omega}_n(\hat{\omega})$ until the log full employment wage increases to $\underline{\omega}_n(\hat{\omega})$. The thresholds $\underline{\omega}_n(\hat{\omega}) \leq$

$\bar{\omega}_n(\hat{\omega})$ are endogenous and depend on both the industry n and the minimum wage $\hat{\omega}$. Workers neither enter nor endogenously exit from labor markets with $\omega \in (\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega}))$, although a fraction of the workers $q_n\, dt$ quit during an interval of time dt. We allow for the possibility that $\underline{\omega}_n(\hat{\omega}) = -\infty$ so workers never exit labor markets. When a positive shock hits a labor market with $\omega = \bar{\omega}_n(\hat{\omega})$, ω stays constant and the labor force l increases. Conversely, negative shocks reduce ω, with l falling as workers exogenously quit the market. At $\underline{\omega}_n(\hat{\omega}) < \omega < \bar{\omega}_n(\hat{\omega})$, both positive and negative shocks affect ω, while l falls deterministically at rate q_n. When $\omega = \underline{\omega}_n(\hat{\omega})$, a negative shock reduces l without affecting ω, while a positive shock raises ω, with l falling due to quits.

If there is an equilibrium with this property, its definition in equation (33) implies ω is a regulated Brownian motion in each market n_j. When $\omega(n_j, t) \in (\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega}))$, only productivity shocks change ω, so

$$d\omega(n_j, t) = \frac{\theta_n - 1}{\theta_n} d \log x(n_j, t) + \frac{q_n}{\theta_n} dt = \mu_n dt + \sigma_n dz(n_j, t), \tag{36}$$

where

$$\mu_n \equiv \frac{\theta_n - 1}{\theta_n} \mu_{n,x} + \frac{q_n}{\theta_n} \quad \text{and} \quad \sigma_n \equiv \frac{\theta_n - 1}{\theta_n} \sigma_{n,x},$$

i.e., in this range $\omega(n_j, t)$ has drift μ_n and instantaneous standard deviation σ_n. When the thresholds $\underline{\omega}_n(\hat{\omega})$ and $\bar{\omega}_n(\hat{\omega})$ are finite, they act as reflecting barriers, since productivity shocks that would move ω outside the boundaries are offset by the entry and exit of workers.

4.3 The Value of a Worker

Now consider a typical worker in a labor market with log minimum wage $\hat{\omega}$ in industry n. The key to our analysis is to recognize that we can analyze the behavior of such a worker in isolation from the rest of the economy. For notational convenience, we suppress the dependence of the value function on industry-specific variables whenever there is no loss of clarity.

We make two assumption on parameter values. The first is that there is non-trivial variability, so that option value is an issue. The second is that discounting is large enough so that the problem is well behaved we assume that discounting being is large enough. Let $\lambda \equiv q + \delta$ and $+ \eta_i$ be the roots of

$$0 = \rho + \lambda - \eta\mu - \eta\sigma^2/2$$

where the values of μ and σ for this industry's ω process are given by equation (36).

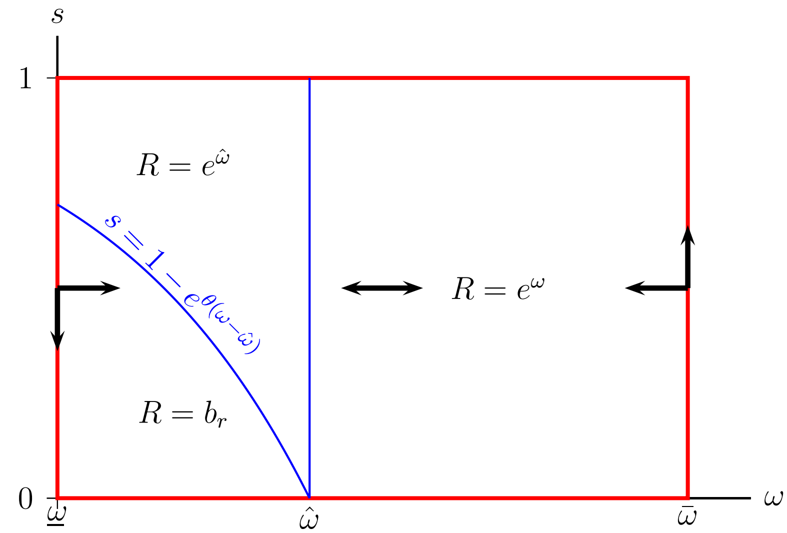

Figure 1: The dynamics of ω and s. All new markets enter at $(\bar{\omega}, 0)$. Markets with $\omega \geq \hat{\omega}$ have no unemployment, while markets with $\omega < \hat{\omega}$ have all workers with $s < 1 - e^{\theta(\omega - \hat{\omega})}$ unemployed.

Assumption 1. *The parameter satisfy: $\sigma > 0$, $\rho + \lambda > 0$ and $\rho + \lambda > \mu + \sigma^2/2$, so that $\eta_1 < 0 < 1 < \eta_2$.*

The worker's state is described by the log full employment wage in her labor market ω and her seniority s, as well as the characteristics of her labor market, including the log minimum wage, the stochastic process for productivity, and the substitutability of goods. But from the worker's perspective, it suffices to know that the log full employment wage is a regulated Brownian motion with endogenous, labor-market specific barriers $\underline{\omega} < \bar{\omega}$. Her seniority in her labor market is her percentile in the tenure distribution in the industry. When a worker arrives, she starts at $s = 0$. Subsequently when workers enter or exit the labor market, the seniority of all workers evolves so as to maintain a uniform distribution of s on $[0, 1]$. Thus s increases only when $\omega = \bar{\omega}$ and falls only when $\omega = \underline{\omega}$; Figure 1 shows the dynamics of ω and s. Each worker exits at the first time $\tau(\underline{\omega}, 0)$ that her state hits $(\underline{\omega}, 0)$, i.e. the first time she is the least senior worker in a market with log full employment wage $\underline{\omega}$. She also exits exogenously at rate $\lambda \equiv q + \delta$, the sum of the quit rate and the rate at which the labor market shuts down.

To compute the value v of a worker in state (ω, s), let

$$R(\omega, s) = \begin{cases} e^{\omega} & \text{if } \omega \geq \hat{\omega} \\ e^{\hat{\omega}} & \text{if } \omega < \hat{\omega} \text{ and } s \geq 1 - e^{\theta(\omega - \hat{\omega})} \\ b_r & \text{if } \omega < \hat{\omega} \text{ and } s < 1 - e^{\theta(\omega - \hat{\omega})} \end{cases} \tag{37}$$

denote the flow payoff of a worker in each state, where we suppress the dependence of the elasticity of substitution θ, and hence the return function R, on the industry n. Figure 1 shows the flow payoff in (ω, s) space. If $\omega \geq \hat{\omega}$, all workers are employed at log wage ω. Otherwise, the most senior workers are employed at $\hat{\omega}$ and the less senior workers are unemployed and get leisure b_r. By construction $b_r \leq e^{\hat{\omega}}$, so employed workers are always weakly better off than unemployed workers. Workers in a particular labor market are indifferent between employment and unemployment only if $b_r = e^{\hat{\omega}}$ and $\omega \leq \hat{\omega}$.

Using this expression, the value of a worker in state (ω_0, s_0) in a market characterized by log minimum wage $\hat{\omega}$ and thresholds $\underline{\omega} < \bar{\omega}$ is

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}\Bigg(\int_0^{\tau(\underline{\omega},0)} e^{-(\rho+\lambda)t}\big(R(\omega(t), s(t)) + \lambda \underline{v}\big) dt \\ + e^{-(\rho+\lambda)\tau(\underline{\omega},0)}\underline{v} \,\Bigg|\, (\omega(0), s(0)) = (\omega_0, s_0) \Bigg), \quad (38)$$

where expectations are taken with respect to the random stopping time τ and the path of the state $(\omega(t), s(t))$ prior to the stopping time. Both the stopping time and the path of the state depends on the thresholds $\underline{\omega}$ and $\bar{\omega}$, while the period return function depends on $\hat{\omega}$. In equilibrium, workers must be willing to exit the labor market in state $(\underline{\omega}, 0)$ and to enter labor markets in state $(\bar{\omega}, 0)$. That is, $\underline{\omega}$ and $\bar{\omega}$ must satisfy

$$v(\underline{\omega}, 0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \underline{v} \quad (39)$$

$$v(\bar{\omega}, 0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \bar{v}, \quad (40)$$

where the values $\underline{v}$ and $\bar{v}$ are common to all labor markets and are determined by the leisure from search and inactivity and by the extent of search frictions; see equations (30)–(31). In addition, workers must be willing to stay in labor markets in all other states, and to stay in labor markets otherwise,

$$v(\omega, s; \hat{\omega}, \underline{\omega}, \bar{\omega}) \geq \underline{v} \text{ for all } (\omega, s) \in [\underline{\omega}, \bar{\omega}] \times [0, 1]. \quad (41)$$

Note that in the presence of a binding minimum wage, workers in some states (ω, s) may attain a value strictly larger than $\bar{v}$. Workers from outside the labor market cannot move directly into such states because they do not have the requisite seniority.

In equilibrium, workers are just indifferent about exiting the labor market at the stopping time $\tau(\underline{\omega}, 0)$. This means that the value of a worker who stays in the labor market until she is hit by the exogenous quit shock is the same as the value of a worker who stays until either she is hit by the quit shock or the first time she reaches state $(\underline{\omega}, 0)$,

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}\left(\int_0^\infty e^{-(\rho+\lambda)t}\big(R(\omega(t), s(t)) + \lambda \underline{v}\big) dt \,\middle|\, (\omega(0), s(0)) = (\omega_0, s_0) \right), \tag{42}$$

when $(\underline{\omega}, \bar{\omega})$ solve equations (39) and (40) and all other workers follow the prescribed policy, exiting the first time they hit state $(\underline{\omega}, 0)$. The equivalence between the value functions in equations (38) and (42) simplifies our exposition.

4.4 Existence of Rest Unemployment

Our first result is that whenever the minimum wage binds, it generates some rest unemployment. As a starting point, consider the case where the minimum wage is zero, or equivalently $\hat{\omega} = -\infty$, the situation we analyzed in Alvarez and Shimer (2008) and Alvarez and Shimer (2011). We proved in Propositions 1 and 2 of Alvarez and Shimer (2008) that, conditional on the other parameters in the model, there exists a threshold $\bar{b}_r > 0$ such that if $b_r < \bar{b}_r$, there is no rest unemployment. Moreover, in this case there exists a unique equilibrium characterized by thresholds $\bar{\omega}^* > \underline{\omega}^* > \log b_r$ where workers enter and exit labor markets so as to regulate wages in $[\underline{\omega}^*, \bar{\omega}^*]$. These thresholds and the associated value function satisfies equations (39))–(42).

Using these definitions, we prove that there will always be some rest unemployment if the minimum wage is higher than $\underline{\omega}^*$.

Proposition 1. *Assume that there is an equilibrium for any value of $\hat{\omega}$. A minimum wage $\hat{\omega} \leq \underline{\omega}^*$ does not bind, so $\bar{\omega} = \bar{\omega}^*$ and $\underline{\omega} = \underline{\omega}^*$. A minimum wage $\hat{\omega} > \underline{\omega}^*$ binds and causes some rest unemployment, $\hat{\omega} > \underline{\omega}$.*

Proof. First consider $\hat{\omega} \leq \underline{\omega}^*$. When $\hat{\omega} \leq \underline{\omega}$, $R(\omega, s) = e^\omega$ for all s and so the value function in equation (42) is independent of $\hat{\omega}$. Thus if $(\underline{\omega}^*, \bar{\omega}^*)$ solve equations (39)–(41) for $\hat{\omega} = -\infty$, they solve the same equations for any $\hat{\omega} \leq \underline{\omega}^*$.

Now suppose $\hat{\omega} > \underline{\omega}^*$. To find a contradiction, suppose $\hat{\omega} \leq \underline{\omega}$. The argument in the previous paragraph implies that $R(\omega, s) = e^\omega$ for all s. But in this case we know from Alvarez and Shimer (2008) that the unique solution to equations (39)–(42) is $(\underline{\omega}^*, \bar{\omega}^*)$, and in particular $\underline{\omega} = \underline{\omega}^*$, a contradiction. □

One might have imagined that a binding minimum wage simply raised the lower threshold for ω so $\underline{\omega} = \hat{\omega}$. This is not the case. Since the standard deviation of productivity per unit of time explodes when the time horizon is short, the option value of entering rest unemployment, at least briefly, always exceeds the option value of immediately exiting the labor market when productivity falls too far.

Proposition 1 assumes that there is an equilibrium, a results that is established below.

4.5 Characterization of the Value Function

We prove in Appendix A.2 that the value function is twice differentiable on the interior of the state space, except at points where $R(\omega, s)$ is discontinuous, i.e. on the locus $s = 1 - e^{\theta(\omega - \hat{\omega})}$, where it is once differentiable. Moreover, taking the limit of a discrete time, discrete state space model, we show in Appendix A.3 that the value function satisfies the following partial differential equations. First, for all (ω, s),

$$\rho v(\omega, s) = R(\omega, s) + \lambda(\underline{v} - v(\omega, s)) + \mu v_\omega(\omega, s) + \tfrac{1}{2}\sigma^2 v_{\omega\omega}(\omega, s). \tag{43}$$

At the highest and lowest wages and for all s,

$$v_\omega(\underline{\omega}, s) = v_s(\underline{\omega}, s)(1 - s)\theta \tag{44}$$

$$v_\omega(\bar{\omega}, s) = v_s(\bar{\omega}, s)(1 - s)\theta. \tag{45}$$

For a worker who is at the exit threshold,

$$v_\omega(\underline{\omega}, 0) = 0. \tag{46}$$

Finally, the highest level of seniority is an absorbing state until the worker exits the labor market, which ensures that

$$v_\omega(\underline{\omega}, 1) = v_\omega(\bar{\omega}, 1) = 0. \tag{47}$$

These act as boundary conditions and are used in our proof that the thresholds uniquely determine the value function. We summarize these results in the following proposition.

PROPOSITION 2. For any $\bar{\omega} > \underline{\omega}$, $v(\omega, s)$ is uniquely determined by equations (43)–(47) and the condition that it is almost everywhere twice differentiable.

In Appendix A.4 we provide a closed form solution for the value function in a typical industry with an arbitrary minimum wage $\hat{\omega}$ and thresholds $\underline{\omega} \leq \bar{\omega}$. We display expressions for three cases depending on whether the minimum wage $\hat{\omega}$ is i) smaller than $\underline{\omega}$, ii) in the interval $(\underline{\omega}, \bar{\omega})$, or iii) equal to $\bar{\omega}$.

In what follows we concentrate in the case for $\lambda = b_r = 0$. The assumption that $\lambda = 0$ simplifies the expressions and it is completely without loss of generality. The assumption that the leisure of value of rest equals the leisure value of work, i.e. $b_r = 0$ simplifies the notation too, but more importantly it emphasizes the role of minimum wages and seniority, as opposed to preference for rest vs work, to create rest unemployment. Using the characterization in Proposition 2 the next proposition establishes that value function is monotone in the log full employment wage and the worker's seniority:

Proposition 3. *Assume that $\lambda = 0$ and $b_r = 0$. The value function $v(\omega, s)$ is strictly increasing in ω and is strictly increasing in s if $\hat{\omega} > \underline{\omega}$ and independent of s otherwise.*

This proposition is proved in the appendix in Appendix A.6. Monotonicity of the value function ensures that equation (41) is satisfied. The fact that the value function is independent of seniority when $\hat{\omega} \leq \underline{\omega}$ is straightforward to verify algebraically. Economically, seniority matters only if the minimum wage sometimes binds, in the sense that unemployed workers are worse off than employed workers within the same market.

Proposition 2 and Proposition 3 establish the existence of an equilibrium for any thresholds and minimum wage $\underline{\omega} \leq \hat{\omega} \leq \bar{\omega}$ if the values of the parameters (b_i, b_s, α) are such that the values of $\underline{v}$ and $\bar{v}$ satisfy equation (39) and equation (40). Since the observable implication for the industry depends only on $(\underline{\omega}, \hat{\omega}, \bar{\omega})$, we can use this result to find the implied values for (b_i, b_s, α) to rationalize such equilibrium. In the next section we turn to the inverse mapping: fixing the parameters that determine $\underline{v}, \bar{b}$ and fixing $\hat{\omega}$ we show that there exist thresholds $\underline{\omega}, \bar{\omega}$ for which there is an equilibrium.

4.6 Existence of equilibrium

In this section we establish the existence of an industry equilibrium given parameters the search and inaction values $\underline{v}, \bar{v}$ and minimum wage ω.

Theorem 1. *Assume that $\lambda = 0$ and $b_r = 0$. Fix the value of inaction and search satisfying $0 < \underline{v} < \bar{v}$. Then, there are two thresholds ${}_*\hat{\omega} < \hat{\omega}^*$ so that for any minimum wage $\hat{\omega} \in ({}_*\hat{\omega}, \hat{\omega}^*)$ for this industry, there exists equilibrium thresholds $\underline{\omega} < \bar{\omega}$, i.e. equations (39) and (40) are satisfied for $(\underline{\omega}, \hat{\omega}, \bar{\omega})$.*

The assumption that $b_r = 0$ makes the point that, even spite the fact that rest gives the same utility as work, there is rest in equilibrium. This is to compared with the model without minimum wages analyzed in Alvarez and Shimer (2011) where we show that there is rest in equilibrium only if $b_r > 0$.

Note that Proposition 2 explains how to compute $\underline{v}$ and $\bar{v}$ given $\underline{\omega}$ and $\bar{\omega}$. Theorem 1 tells us that the mapping is invertible, i.e. that given $\underline{v}$ and $\bar{v}$, we can find $\underline{\omega}$ and $\bar{\omega}$. We conjecture that thresholds are unique, but only have a proof in the case when the minimum wage does not bind, $\hat{\omega} = \log b_r$ (see Alvarez and Shimer, 2008, Proposition 1), and when the minimum wage always binds, $\hat{\omega} \geq \bar{\omega}$.

4.7 Aggregation

Consider a typical industry n and minimum wage rate $\hat{\omega}$. Denote the thresholds for that industry by $\underline{\omega}_n(\hat{\omega})$ and $\bar{\omega}_n(\hat{\omega})$. Given these, we can compute the fraction of workers at each value of $\omega \in [\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega})]$. Note that this is different than the fraction of labor markets at each value of ω, since there are typically more workers in labor markets with a higher log full employment wage.

Proposition 4. *The steady state density of workers' log full employment wage in industry n, minimum wage $\hat{\omega}$ is*

$$f_n(\omega; \hat{\omega}) = \frac{\sum_{i=1}^{2} |\eta_{i,n} + \theta_n| e^{\eta_{i,n}(\omega - \underline{\omega}_n(\hat{\omega}))}}{\sum_{i=1}^{2} |\eta_{i,n} + \theta_n| \frac{e^{\eta_{i,n}(\bar{\omega}_n(\hat{\omega}) - \underline{\omega}_n(\hat{\omega}))} - 1}{\eta_i}}, \tag{48}$$

where $\eta_{1,n} < \eta_{2,n}$ solve the characteristic equation $\delta_n + q_n = -\mu_n \eta_n + \frac{\sigma_n^2}{2} \eta_n^2$ and $\underline{\omega}_n(\hat{\omega}) < \bar{\omega}_n(\hat{\omega})$ are the thresholds for that industry and minimum wage.

The proof of this result is identical to Proposition 3 in Alvarez and Shimer (2008) and hence omitted. That proposition also shows how to close the model to compute the number of workers labor markets and the consumption of each good, results that we do not repeat here. Note that under condition (25), $\eta_{1,n} \leq -\theta$ and $\eta_{2,n} > 0$.

Using this result, we can compute the rest and search unemployment rates for each industry n and minimum wage $\hat{w}$. To reduce the notation, we suppress the dependence of the thresholds on n and $\hat{w}$. If $\hat{\omega} \leq \underline{\omega}$, there is no rest unemployment in any such labor market. Otherwise when $\omega < \hat{\omega}$, all workers with seniority $s < 1 - e^{\theta_n(\omega - \hat{\omega})}$ are rest unemployed. This gives the rest unemployment rate in such a labor market. Integrating across markets using equation (48) gives the industry- and minimum wage-specific rest unemployment rate

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \int_{\underline{\omega}}^{\min\{\hat{\omega}, \bar{\omega}\}} \left(1 - e^{\theta_n(\omega - \hat{\omega})}\right) f_n(\omega; \hat{\omega})\, d\omega.$$

where $U_{r,n}(\hat{\omega})$ is the number of rest unemployed and $L_n(\hat{\omega})$ is the number of (employed or unemployed(workers in such labor markets. This gives

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \frac{\frac{\theta}{\eta_{2,n}}(e^{\eta_{2,n}(\hat{\omega} - \underline{\omega})} - 1) - \frac{\theta}{\eta_{1,n}}(e^{\eta_{1,n}(\hat{\omega} - \underline{\omega})} - 1)}{\sum_{i=1}^{2} |\theta_n + \eta_{i,n}| \frac{e^{\eta_{i,n}(\bar{\omega} - \underline{\omega})} - 1}{\eta_{i,n}}}, \tag{49}$$

when $\hat{\omega} \in (\underline{\omega}, \bar{\omega})$ and

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \frac{\left(1 - e^{\theta_n(\bar{\omega}-\hat{\omega})} + \frac{\theta_n}{\eta_{2,n}}\right)\left(e^{\eta_{2,n}(\hat{\omega}-\underline{\omega})} - 1\right) - \left(1 - e^{\theta_n(\bar{\omega}-\hat{\omega})} + \frac{\theta_n}{\eta_{1,n}}\right)\left(e^{\eta_{1,n}(\hat{\omega}-\underline{\omega})} - 1\right)}{\sum_{i=1}^{2} |\theta_n + \eta_{i,n}| \frac{e^{\eta_{i,n}(\bar{\omega}-\underline{\omega})}-1}{\eta_{i,n}}}, \quad (50)$$

when $\hat{\omega} \geq \bar{\omega}$. Using these equations, we can easily compute how the level of the minimum wage affects the unemployment rate within an industry and how a given minimum wage affects the unemployment rate in different industries.

Now we turn to the search unemployed connected to a particular industry n and minimum wage $\hat{w}$. Let $N_{s,n}(\hat{\omega})$ be the number of workers that leave their labor market per unit of time, either because conditions are sufficiently bad or because they exogenously quit or because their labor market has exogenously shut down. We prove in Alvarez and Shimer (2008) that this satisfies

$$N_{s,n}(\hat{\omega}) = \left(\frac{\theta_n \sigma_n^2}{2} f_n(\underline{\omega}; \hat{\omega}) + \delta_n + q_n\right) L_n(\hat{\omega}). \quad (51)$$

The first term gives the fraction of workers who leave their labor market to keep ω above $\underline{\omega}$. The second term is the fraction of workers who exogenously leave their market. In steady state, the fraction of workers who leave labor markets must balance the fraction of workers who arrive in labor markets. The latter is given by the fraction of workers engaged in searching for this industry and minimum wage, $U_{s,n}(\hat{\omega})$, times the rate at which they arrive to the labor market α, so $\alpha U_{s,n}(\hat{\omega}) = N_{s,n}(\hat{\omega})$. Solve equation (51) using equation (48) to obtain an expression for the ratio of search unemployment to workers in labor markets:

$$\frac{U_{s,n}(\hat{\omega})}{L_n(\hat{\omega})} = \frac{1}{\alpha}\left(\frac{\theta_n \sigma_n^2}{2} \frac{\eta_{2,n} - \eta_{1,n}}{\sum_{i=1}^{2} |\theta_n + \eta_{i,n}| \frac{e^{\eta_{i,n}(\bar{\omega}-\underline{\omega})}-1}{\eta_{i,n}}} + \delta_n + q_n\right) \quad (52)$$

To compute the aggregate rest and search unemployment rates, simply aggregate across minimum wages and industries.

In some special cases, the formulae for search and rest unemployment rates simplify further. Consider an industry with $\mu_{n,x} = -(\theta_n - 1)(\sigma_{n,x})^2/2$, or equivalently $\mu_n = q_n/\theta_n - \theta_n \sigma_n^2/2$. Then one can show that the search and rest unemployment rates are well behaved even if markets never shut down, $\delta_n \to 0$. Although the variance of the productivity distribution explodes, the roots of the characteristic equation in Proposition 4 converge to $\eta_{1,n} = -\theta_n$ and $\eta_{2,n} = 2q_n/\theta_n \sigma_n^2$. Substituting into equation (48), we find

$$f(\omega) = \frac{\eta_{2,n} e^{\eta_{2,n}(\omega-\underline{\omega})}}{e^{\eta_{2,n}(\bar{\omega}-\underline{\omega})} - 1}.$$

If $q_n = 0$ as well, this simplifies further to $f(\omega) = 1/(\bar{\omega} - \underline{\omega})$, i.e. f is uniform on its support, while for positive q_n the density is increasing in ω.[5] Using this, we can compute the search and rest unemployment rates. When $\delta \to 0$, these converge to

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \frac{e^{\eta_{2,n}(\min\{\bar{\omega},\hat{\omega}\}-\underline{\omega})}\left(1 - \frac{\eta_{2,n}}{\theta_n+\eta_{2,n}} e^{\theta_n \min\{\bar{\omega}-\hat{\omega},0\}}\right) - 1 + \frac{\eta_{2,n}}{\theta_n+\eta_{2,n}} e^{-\theta_n(\hat{\omega}-\underline{\omega})}}{e^{\eta_{2,n}(\bar{\omega}-\underline{\omega})} - 1}, \tag{53}$$

$$\frac{U_{s,n}(\hat{w})}{L_n(\hat{w})} = \frac{q_n}{\alpha\left(1 - e^{-\eta_{2,n}(\bar{\omega}-\underline{\omega})}\right)}. \tag{54}$$

These expressions simplify further when there are no quits, $q_n = 0$ and so $\eta_{2,n} \to 0$.

4.8 Hazard Rate of Exiting Unemployment

When there is no rest unemployment, the hazard of exiting unemployment is simply α. This section characterizes the hazard of exiting unemployment when there is rest unemployment, $\hat{\omega} > \underline{\omega}$, but not in the best markets, $\bar{\omega} > \hat{\omega}$. A worker who just switched between employment and rest unemployment is at the margin between the two states. A small shock will move her back. But the longer a worker remains unemployed, the more likely her labor market has suffered a series of adverse shocks, reducing the hazard of finding a job. The low hazard rate of exiting long-term unemployment may be important for understanding the coexistence of many workers who move easily between jobs and a relatively small number of workers who suffer extended unemployment spells (Juhn, Murphy, and Topel, 1991).

To see how this hazard is determined, consider a worker with seniority s who is rest unemployed whenever $s < 1 - e^{\theta(\omega-\hat{\omega})}$. Using the definition of ω in equation (33) and suppressing the dependence of these variables on the industry and minimum wage, we can write this as a condition relating the number of more senior workers in the market, $l(1-s)$, to the current productivity of the market x_0,

$$l(1-s) > \left(\frac{Ax_0}{C}\right)^{\theta-1} e^{-\theta\hat{\omega}}.$$

The worker exits rest unemployment and returns to this market the next time this inequality is violated, i.e. when productivity reaches $\hat{x}$ solving

$$l(1-s) = \left(\frac{A\hat{x}}{C}\right)^{\theta-1} e^{-\theta\hat{\omega}}.$$

[5]This result does not depend on the order in which δ and q converge to 0.

Conversely, she exits rest unemployment and leaves the market when she first reaches state $(\underline{\omega}, 0)$, which occurs at the productivity level $\underline{x}$ satisfying

$$l(1-s) = \left(\frac{A\underline{x}}{C}\right)^{\theta-1} e^{-\theta\underline{\omega}},$$

so the log full employment wage is $\underline{\omega}$ if there are $l(1-s)$ workers left in the market. She also exits the market exogenously if she quits or the market breaks down, at rate $\lambda = q + \delta$. Thus the hazard of ending a spell of rest unemployment depends on competing hazards of productivity rising to $\hat{x}$ or falling to $\underline{x}$. The key observation is that the ratio of these two thresholds is monotone in the distance between $\hat{\omega}$ and $\underline{\omega}$,

$$\hat{\omega} - \underline{\omega} = \frac{\theta - 1}{\theta}\big(\log \hat{x} - \log \underline{x}\big),$$

and so is the same for all workers in an industry, regardless of their seniority.

We let $h(t)$ denote the hazard of ending a (rest or search) unemployment spell of duration t. We will show that this solves

$$h(t) = \hat{h}_r(t)\frac{u_r(t)}{u_r(t) + u_s(t)} + \alpha\frac{u_s(t)}{u_r(t) + u_s(t)},$$

where $\frac{u_r(t)}{u_r(t)+u_s(t)}$ is the probability that a worker with unemployment duration t is rest-unemployed. For a search-unemployed worker, spells end at rate α, independent of the duration of the spell.[6] For a rest-unemployed worker, her spell ends when local labor market conditions improve enough for her to reenter employment. We let $\hat{h}_r(t)$ denote that hazard rate of this event. It is also useful to let $\underline{h}_r(t)$ denote the hazard of endogenously exiting rest unemployment for search unemployment. The previous logic suggests that these hazards depend only on $\hat{\omega} - \underline{\omega}$. Using existing results for the hitting times of a regulated BM with two barriers, in Appendix A.7 we prove that

$$\begin{aligned}
\hat{h}_r(t) &= \frac{\sum_{m=1}^{\infty} m^2 e^{-t\psi_m}}{\sum_{m=1}^{\infty} \frac{m^2}{\psi_m} e^{-t\psi_m}\left(1 - (-1)^m e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)} \\
\underline{h}_r(t) &= \frac{-\sum_{m=1}^{\infty} m^2 e^{-t\psi_m}(-1)^m e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}{\sum_{m=1}^{\infty} \frac{m^2}{\psi_m} e^{-t\psi_m}\left(1 - (-1)^m e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)},
\end{aligned} \tag{55}$$

[6]Here we use the assumption that $\hat{\omega} < \bar{\omega}$, so when a search-unemployed worker finds a market, he goes to work immediately.

where

$$\psi_m \equiv \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{m^2\pi^2\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\right).$$

These sums are easily calculated numerically.

We then compute the duration-contingent unemployment rates by solving a system of two ordinary differential equations with time-varying coefficients:

$$\dot{u}_r(t) = -u_r(t)(\delta + q + \underline{h}_r(t) + \hat{h}_r(t)) \text{ and } \dot{u}_s(t) = -u_s(t)\alpha + u_r(t)(\delta + q + \underline{h}_r(t)) \tag{56}$$

for all $t > 0$. The number of workers in rest unemployment falls as markets shut down and workers exogenously quit, as they exit the market for search unemployment, and as they reenter employment. In the first three events, they become search unemployed, while search unemployment falls at rate α as these workers find jobs. To solve these differential equations, we require two boundary conditions; however, to compute the share of rest unemployed in the unemployed population with duration t, $\frac{u_r(t)}{u_r(t)+u_s(t)}$, we need only a single boundary condition,

$$\frac{\int_0^\infty u_r(t)dt}{\int_0^\infty u_s(t)dt} = \frac{U_r}{U_s}, \tag{57}$$

where U_r and U_s are given in equations (49) and (52).

The hazard rate is particularly easy to characterize both at short and long durations. From the expressions in equation (55) it can be seen that $\lim_{t\to 0} h(t)t = 1/2$. Alternatively, when t is small, we find that $\hat{h}_r(t) \approx \frac{1}{2t}$. Intuitively, consider a worker on the threshold of rest unemployment, $s = 1 - e^{\theta(\omega-\hat{\omega})}$. After a short time interval—short enough that the variance of the Brownian motion dominates the drift—there is a $\frac{1}{2}$ probability that ω has increased, so the worker is reemployed, and a $\frac{1}{2}$ chance it has fallen. But a one-half probability over any horizon t implies a hazard rate $1/2t$. Thus our model predicts that unionized workers will experience many short spells of unemployment, which perhaps can be interpreted as temporary layoffs.

On the other hand, when t is large, the first term of the partial sum in equation (55) dominates,

$$\lim_{t\to\infty} \hat{h}_r(t) = \frac{\psi_1}{1+e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}} \text{ and } \lim_{t\to\infty} \underline{h}_r(t) = \frac{\psi_1 e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}{1+e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}.$$

In addition, if $\alpha > \delta + q + \psi_1$,

$$\lim_{t\to\infty} \frac{u_r(t)}{u_s(t)} = \frac{(\alpha - \psi_1 - \delta - q)\left(1 + e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)}{\delta + q + (\delta + q + \psi_1)e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}},$$

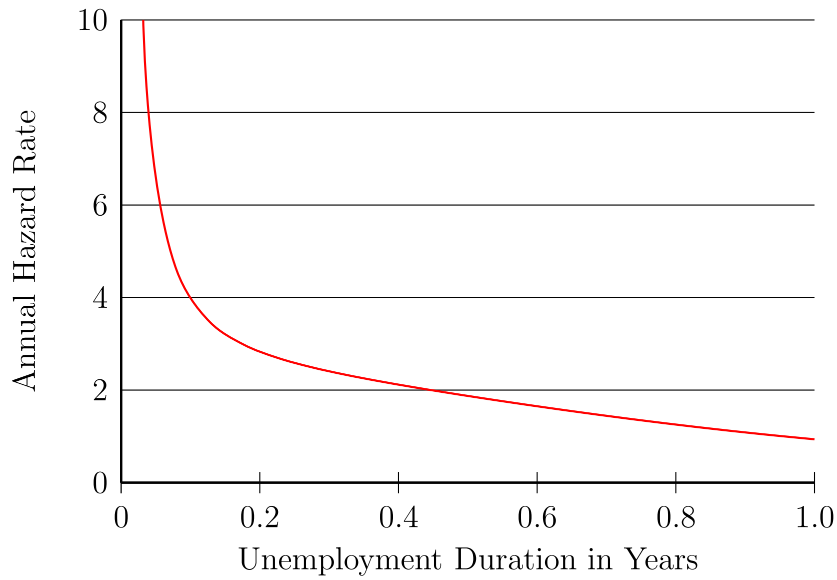

Figure 2: Hazard rate of finding a job as a function of unemployment duration. The parameter values are in the text.

while otherwise the limiting ratio is zero. Together this implies $\lim_{t\to\infty} h(t) = \min\{\alpha, \psi_1 + \delta + q\}$, a function only of the slower exit rate. Since ψ_1 is decreasing in $\hat{\omega} - \underline{\omega}$, the asymptotic exit rate from rest unemployment may be extremely low at long unemployment durations, so unionized workers will sometimes remain unemployed for years with little chance of reemployment.

Figure 7 shows the annual hazard rate of finding a job in our baseline calibration, including a 4.2 percent rest unemployment rate and 1.3 percent search unemployment rate. The overall hazard rate roughly mimics the behavior of $\hat{h}_r(t)$, especially at short unemployment durations, when most unemployed workers are in rest unemployment. Since the rest unemployed find jobs so quickly at the start of an unemployment spell, the share of searchers among the unemployed grows rapidly (Figure 3), peaking at about 54 percent of unemployment after two months duration. After this point, however, the hazard of exiting rest unemployment falls below the hazard of exiting search unemployment and so the share of searchers starts to decline, asymptoting to just 4 percent of unemployment at very long durations.

Our finding of a constant hazard rate for workers in search unemployment and a decreasing hazard rate for workers in rest unemployment is qualitatively consistent with Katz and Meyer (1990) and Starr-McCluer (1993). Katz and Meyer (1990) show that the empirical decline in the job finding hazard rate is concentrated among workers on temporary layoff. Moreover, they find that workers who expect to be recalled to a past employer and are not—in the parlance of our model, workers who end a spell of rest unemployment by searching for a

Figure 3: Fraction of searchers among the unemployed by duration, $\frac{u_s(t)}{u_r(t)+u_s(t)}$. The parameter values are in the text.

new labor market, at hazard $\underline{h}_r(t) + \delta + q$—experience longer unemployment duration than observationally equivalent workers who immediately entered search unemployment. In our model, this last group would correspond to workers experiencing a δ or q shock. Starr-McCluer (1993) finds that the hazard of exiting unemployment is decreasing for workers who move to a job that is similar to their previous one (rest unemployed) while it is actually increasing for workers who move to a different type of job (search unemployed).

5 Union Objective Function

Consider a monopoly union representing the $l(n_j, t)$ workers in labor market n_j at time t. The union's objective is to maximize the total flow utility of those workers,

$$e(n_j,t)w(n_j,t)\frac{1}{\bar{C}} + \big(l(n_j,t) - e(n_j,t)\big)b_r,$$

where $e(n_j, t)$ is the measure of workers who are employed, $w(n_j, t)$ is the wage, and $1/\bar{C}$ is the marginal utility of consumption. For example, we can think of the union setting the wage and then letting competitive firms determine how many workers to hire. From the analysis in Section 3.3, we know that employment is

$$e(n_j,t) = \min\left\{ l(n_j,t), \frac{\bar{C}(t)^{\theta_n}\big(Ax(n_j,t)\big)^{\theta_n - 1}}{C(n,t)^{\theta_n - 1} w^{\theta_n}} \right\}.$$

The solution to the union's problem is to set $w(n_j, t) = \bar{C}e^{\hat{\omega}}$ where

$$\hat{\omega} = \log b_r + \log(\theta_n/(\theta_n - 1)) \tag{58}$$

if this leaves some workers unemployed and otherwise to set a higher level of wages consistent with full employment, $w(n_j, t) = \bar{C}e^{\omega}$, where ω is the log full employment wage defined in equation (33). In other words, the union sets a constant minimum wage which leaves a gap between the utility of the members who work and those who are rest unemployed. The minimum wage is time-invariant, although it will vary across industries depending on the elasticity of substitution θ_n. This is exactly the type of policy that we have analyzed in this paper; the analysis here simply provides a link between the minimum wage and the preference parameters b_r and θ_n.

According to this model, the economy would be perfectly competitive in the absence of unions. By monopolizing a labor market, a union can extract the monopoly rent. It does this by raising wages in order to restrict employment and output and hence raise the price of the good produced by the industry. It achieves exactly the same outcome as would be attained by a monopoly producer facing a competitive industry.[7] The model predicts that unions will be more successful at raising wages in industries producing goods that have poor substitution possibilities, θ_n close to 1.

Some observers have noted that, while unionization raises unemployment rates, the effects are mitigated if unions coordinate their activities (Nickell and Layard, 1999). Our model suggests that this may because coordinated unions are able to internalize the impact of exploiting their monopoly power on other workers. The Pareto optimal allocation is achieved by dropping the minimum wage constraints, so a worker can work whenever $\omega = \log b_r$ (see Alvarez and Shimer, 2008, Appendix B.2). Perhaps coordinated unions are able to avoid the incentive to restrict output in individual labor markets.

6 Example

We set parameters broadly in line with those in our previous paper. Consider an industry with an elasticity of substitution $\theta = 2$. Let the discount rate be $\rho = 0.05$, the leisure value of inactivity be $b_i = 1$, so $\underline{v} = 20$, and the search cost be $\kappa = 2$ so $\bar{v} = 22$. Set the leisure from rest unemployment to $b_r = 0.7$. Fix the standard deviation of wages at $\sigma = 0.12$ and

[7] We do not analyze the interaction between a monopoly producer a monopoly union. In this case, setting a wage and allowing the firm to determine employment is generally inefficient. The two monopolists should agree on both a wage and a level of employment. Still, it seems likely that the equilibrium outcome will be a wage floor.

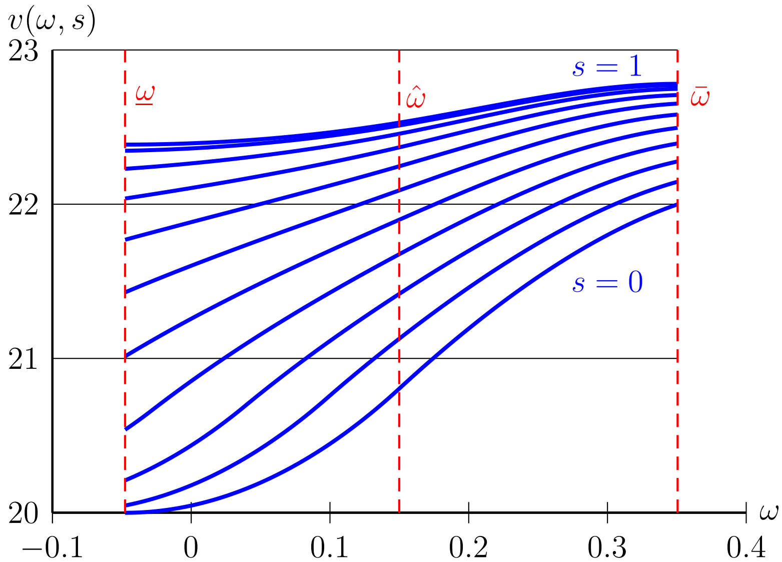

Figure 4: Value functions for $s = 0, 0.1, \ldots, 1$. Parameters are in the text, with $\hat{\omega} = 0.15$.

the quit rate at 0.04. Then let $\mu = q/\theta - \theta\sigma^2/2 \approx 0.0056$ so that we can focus on the limit as $\delta \to 0$. Finally, set the job finding rate for searchers to $\alpha = 3.2$. Since our exploration of parameters is cursory, the results that follow should be considered preliminary.

In the absence of a minimum wage, we find that $\underline{\omega} = -0.258$, higher than $\log b_r = -0.357$. Therefore any minimum wage below this level has no effect. Figure 4 shows the value function $v(\omega, s)$ for different seniorities when $\hat{\omega} = 0.15$. More senior workers are always better off than less senior workers and all workers are better off when the log full employment wage ω is higher, although more senior workers' value function is less sensitive to ω.

Figure 5 shows how the thresholds change as functions of the minimum wage. The lower bound $\underline{\omega}$ increases in $\hat{\omega}$, with a slope less than 1. Put differently, when the minimum wage is higher, the maximum number of workers willing to stay in the industry is smaller for any value of productivity. On the other hand, the upper bound initially falls with $\hat{\omega}$, indicating that a modest degree of monopolization attracts workers to the industry for a given level of productivity. This is true even though the last entrant to the union is the first worker laid off. A monopoly union sets $\hat{\omega} = 0.34$, consistent with a $\bar{\omega} > \hat{\omega} > \underline{\omega}$.

Note that unions in our model generate not only unemployment, but also wage compression (Blau and Kahn, 1996; Bertola and Rogerson, 1997). The range of log wages is given by the distance between the dashed 45 degree line and $\bar{\omega}$. This is declining in the minimum wage, eventually disappearing once all workers are paid $\hat{\omega}$.

Using the computed thresholds, it is straightforward to find out how the rest and search unemployment rates in this industry vary with $\hat{\omega}$ (Figure 6). Initially there is no rest unemployment, although search unemployment is necessary to sustain the industry. As the minimum wage rises, the rest unemployment rate starts to increase while the search un-

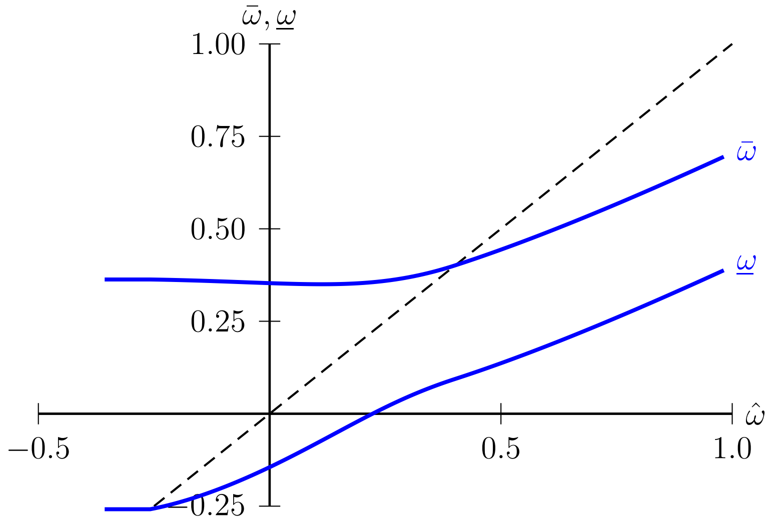

Figure 5: Thresholds as functions of the minimum wage $\hat{\omega}$. Parameters are in the text.

employment rate is approximately unchanged. We conclude from this exercise that union-mandated minimum wages provide a powerful mechanism for generating rest unemployment.

Figure 7 shows the annual hazard rate for two markets with different minimum wages but the other parameters fixed at the benchmark level. In one labor market, the minimum wage is set at $\hat{\omega} = 0$ while in the other it is at the monopoly level, $\hat{\omega} \approx 0.34$. The overall hazard rates are similar at short durations, roughly $1/2t$, because in both cases most unemployed workers are in rest unemployment. With a low minimum wage, few workers get trapped in long-term unemployment because the gap between the minimum wage $\hat{\omega}$ and the exit threshold $\underline{\omega}$ is not that large. That is, most workers either quickly find a job or exit the industry, so mean unemployment duration is 0.33 years and the median duration is 0.21 years. Asymptotically, the exit rate from unemployment converges to $\alpha = 3.2$. But with the monopoly union wage, more workers get stuck in long-term unemployment. In this case, the mean unemployment duration is 0.83 years, the median duration is 0.48 years, and the exit hazard converges to 1.01. In a labor market with such a high minimum wage, the efficiency of search affects the hazard of exiting long-term unemployment only indirectly, through its influence on the distance between the rest unemployment boundaries $\hat{\omega} - \bar{\omega}$.

Finally, it is straightforward to perform simple comparative statics. Take, for example, an industry producing a good that is easy to substitute, $\theta_n = 3$, but with all other parameters unchanged. We find this has little effect on the curves in Figure 5 and Figure 6. Leaving $\hat{\omega}$ fixed at its monopoly value in the industry with $\theta_n = 2$, we find that U_r/L falls from 0.164 to 0.159. But if the minimum wage falls to its new monopoly value, $\hat{\omega} = 0.05$, the rest unemployment rate falls substantially to $U_r/L = 0.031$, while the search unemployment rate

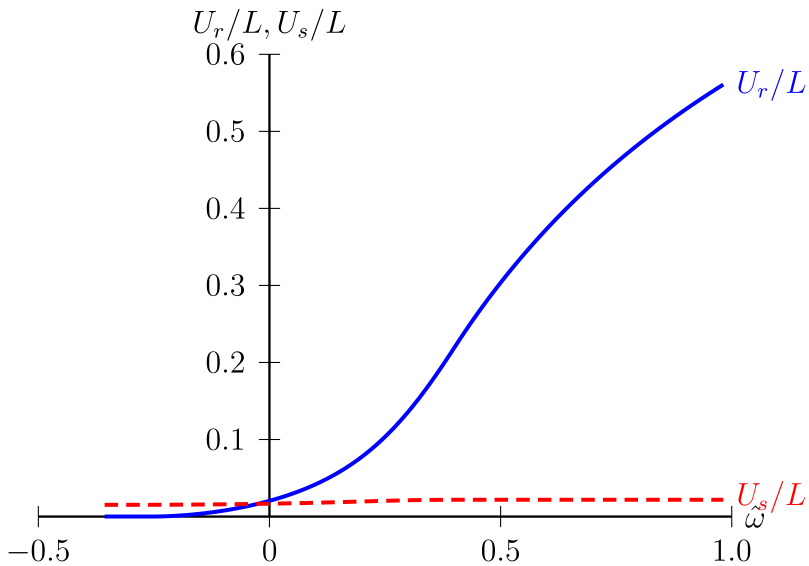

Figure 6: Unemployment as functions of the minimum wage $\hat{\omega}$. Parameters are in the text.

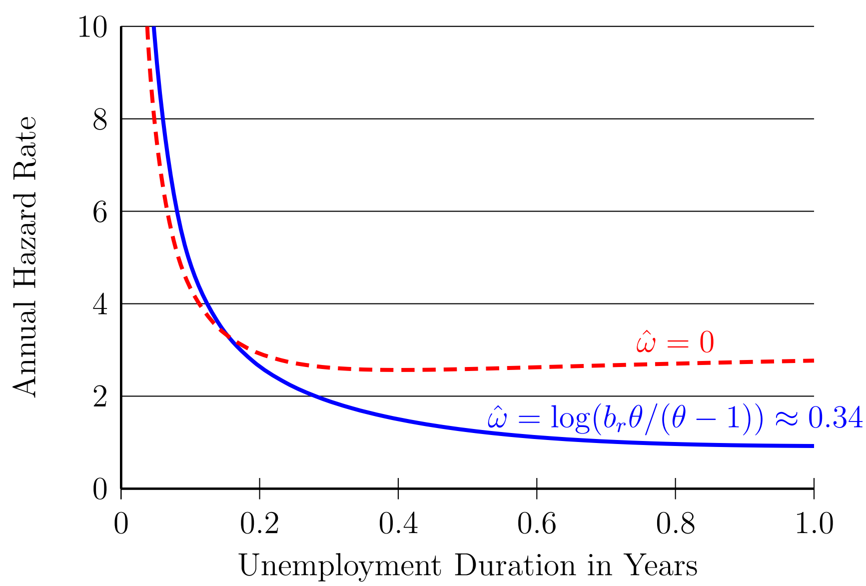

Figure 7: Hazard rate of finding a job as a function of unemployment duration. The parameter values are in the text. The blue solid line uses the monopoly union minimum wage, while in the red dashed line, the minimum wage is $\hat{\omega} = 0$.

is virtually unchanged at $U_s/L = 0.022$.

A Appendix

A.1 Value Function in the Frictionless Model

We consider a discrete time, discrete state space model. The length of the time period is Δt, the discount factor is $1 - \rho\Delta t$, and the exogenous exit probability is $\lambda\Delta t$. We imagine that $\log x$ lies on the countable grid $\{\ldots, -\Delta x, 0, \Delta x, \ldots\}$ while $s \in [0, 1]$. Each period $\log x$ increases by Δx with probability $\frac{1}{2}(1 + \Delta p)$ and otherwise decreases by Δx. Following a decrease in productivity, all workers with seniority below $1 - e^{-(\theta-1)\Delta x}$ exit the market, so as to ensure that equation (11) continues to hold. The seniority of the surviving workers adjusts to

$$s_- = \frac{s - 1 + e^{-(\theta-1)\Delta x}}{e^{-(\theta-1)\Delta x}}, \tag{59}$$

so as to ensure seniority remains uniformly distributed on $[0, 1]$. Conversely, following a positive shock, new workers enter the industry, raising the seniority of a worker from s to

$$s_+ = \frac{s - 1 + e^{(\theta-1)\Delta x}}{e^{(\theta-1)\Delta x}}. \tag{60}$$

Finally, assume $\Delta t = (\Delta x/\sigma_x)^2$ and $\Delta p = \mu_x \Delta x/\sigma_x^2$. This implies that $\log x'$ is a random walk with drift μ_x per unit of time:

$$\mathbb{E}(\log x' - \log x) = \tfrac{1}{2}(1 + \Delta p)\Delta x - \tfrac{1}{2}(1 - \Delta p)\Delta x = \Delta p \Delta x = \mu_x \Delta t.$$

It also has variance σ_x^2 per unit of time, at least in the limit as $\Delta t \to 0$:

$$\begin{aligned}\mathbb{E}(\log x' - \log x - \mu_x\Delta t)^2 &= \tfrac{1}{2}(1 + \Delta p)(\Delta x - \mu_x\Delta t)^2 + \tfrac{1}{2}(1 - \Delta p)(-\Delta x - \mu_x\Delta t)^2 \\ &= (\Delta x)^2 - \Delta p \Delta x \mu_x \Delta t + \mu_x^2(\Delta t)^2 = \sigma_x^2 \Delta t - \mu_x^2(\Delta t)^2\end{aligned}$$

We thus focus on the limiting behavior as Δt converges to 0 holding fixed μ_x and σ_x, which corresponds to the stochastic process that we study in the body of the paper.

Now consider a worker in the discrete time, discrete state model with seniority $s \geq 1 - e^{-(\theta-1)\Delta x}$, so she will remain in the market following the next shock. Her value function satisfies

$$v(s) = R(s)\Delta t + (1 - \rho\Delta t)\Big(\lambda\Delta t b_i/\rho + (1 - \lambda\Delta t)\big(\tfrac{1}{2}(1 + \Delta p)v(s_+) + \tfrac{1}{2}(1 - \Delta p)v(s_-)\big)\Big).$$

Consider a second order Taylor expansion of $v(s_+)$ and $v(s_-)$ around $v(s)$:

$$\begin{aligned} v(s) = R(s)\Delta t + (1-\rho\Delta t)\Big(\lambda\Delta t b_i/\rho + (1-\lambda\Delta t)\Big(v(s) \\ + \tfrac{1}{2}(1+\Delta p)\left(v'(s)(s-1)(e^{-(\theta-1)\Delta x}-1) + \tfrac{1}{2}v''(s)(s-1)^2(e^{-(\theta-1)\Delta x}-1)^2\right) \\ + \tfrac{1}{2}(1-\Delta p)\left(v'(s)(s-1)(e^{(\theta-1)\Delta x}-1) + \tfrac{1}{2}v''(s)(s-1)^2(e^{(\theta-1)\Delta x}-1)^2\right)\Big)\Big). \end{aligned}$$

Next approximate $e^{-(\theta-1)\Delta x} - 1 = -(\theta-1)\Delta x + \frac{1}{2}((\theta-1)\Delta x)^2$ and $e^{-(\theta-1)\Delta x} - 1 = (\theta-1)\Delta x + \frac{1}{2}((\theta-1)\Delta x)^2$, accurate to order $(\Delta x)^2$. Grouping terms and dropping any of higher order than $(\Delta x)^2$, the previous equation becomes

$$\begin{aligned} v(s) = R(s)\Delta t + (1-\rho\Delta t)\Big(\lambda\Delta t b_i/\rho + (1-\lambda\Delta t)\Big(v(s) \\ + v'(s)(1-s)\big((\theta-1)\Delta p\Delta x - \tfrac{1}{2}(\theta-1)^2(\Delta x)^2\big) + \tfrac{1}{2}v''(s)(1-s)^2(\theta-1)^2(\Delta x)^2\Big)\Big). \end{aligned}$$

Now subtract $(1-\rho\Delta t)(1-\lambda\Delta t)v(s)$ from both sides of the equation and replace $\Delta p = \mu_x \Delta x/\sigma_x^2$ and $(\Delta x) = \sigma_x^2 \Delta t$. Divide through by Δt to get

$$\begin{aligned} (\rho+\lambda-\rho\lambda\Delta t)v(s) = R(s) + (1-\rho\Delta t)\Big(\lambda b_i/\rho \\ + (1-\lambda\Delta t)\Big(v'(s)(1-s)\big((\theta-1)\mu_x - \tfrac{1}{2}(\theta-1)^2\sigma_x^2\big) + \tfrac{1}{2}v''(s)(1-s)^2(\theta-1)^2\sigma_x^2\Big)\Big). \end{aligned}$$

In particular, in the limit as $\Delta t \to 0$, we obtain equation (13) and so this holds for all $s > 0$.

Now consider a worker with the lowest seniority, $s = 0$. Her value function satisfies

$$v(0) = R(0)\Delta t + (1-\rho\Delta t)\Big(\lambda\Delta t b_i/\rho + (1-\lambda\Delta t)\big(\tfrac{1}{2}(1+\Delta p)v(s_+) + \tfrac{1}{2}(1-\Delta p)b_i/\rho\big)\Big),$$

since the worker will exit the market following the next shock. It is now sufficient to consider a first order expansion of $v(s_+)$ around $v(0)$ and to also expand $e^{-(\theta-1)\Delta x} - 1 = -(\theta-1)\Delta x$. Using the value matching condition $v(0) = b_i/\rho$, we obtain

$$\rho\Delta t v(0) = R(0)\Delta t + (1-\rho\Delta t)(1-\lambda\Delta t)\tfrac{1}{2}(1+\Delta p)v'(0)(\theta-1)\Delta x.$$

This equation holds in the limit as $\Delta t \to 0$ only if $v'(0) = 0$ as well, the smooth pasting condition.

A.2 Differentiability of the Value Function

Note that for $\omega \in (\underline{\omega}, \bar{\omega})$, a worker's seniority is constant; although some workers exit the market exogenously, they are drawn uniformly from the population of workers. Now let $\tau(\underline{\omega})$ and $\tau(\bar{\omega})$ denote the (stochastic) time when ω first hits $\underline{\omega}$ and $\bar{\omega}$, infinite if it hits the other boundary first. Then we can rewrite equation (42) as

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E}\Bigg(\int_0^{\{\min\{\tau(\underline{\omega}),\tau(\bar{\omega})\}} e^{-(\rho+\lambda)t}\big(R(\omega(t), s_0) + \lambda \underline{v}\big)dt \\ + v(\underline{\omega}, s_0)e^{-(\rho+\lambda)\tau(\underline{\omega})} + v(\bar{\omega}, s_0)e^{-(\rho+\lambda)\tau(\bar{\omega})} \Bigg| \omega(0) = \omega_0 \Bigg).$$

Now let $\pi(\omega|\omega_0)$ be the discounted local time of a Brownian motion prior to the first time it hits the boundary. Rewrite the value function as

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \int_{\underline{\omega}}^{\bar{\omega}} \big(R(\omega(t), s_0) + \lambda \underline{v}\big)\pi(\omega|\omega_0)dt \\ + \mathbb{E}\big(v(\underline{\omega}, s_0)e^{-(\rho+\lambda)\tau(\underline{\omega})} + v(\bar{\omega}, s_0)e^{-(\rho+\lambda)\tau(\bar{\omega})} \,\big|\, \omega(0) = \omega_0\big).$$

Since π is everywhere continuous and is continuously differentiable except at ω_0, differentiating the previous expression gives

$$\frac{\partial v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega})}{\partial \omega_0} = \int_{\underline{\omega}}^{\bar{\omega}} \big(R(\omega(t), s_0) + \lambda \underline{v}\big)\frac{\partial \pi(\omega|\omega_0)}{\partial \omega_0}dt \\ + \big(\lim_{\omega\uparrow\omega_0} R(\omega, s_0) - \lim_{\omega\downarrow\omega_0} R(\omega, s_0)\big)\pi(\omega_0|\omega_0) \\ + v(\underline{\omega}, s_0)\frac{\partial \mathbb{E}e^{-(\rho+\lambda)\tau(\underline{\omega})}}{\partial \omega_0} + v(\bar{\omega}, s_0)\frac{\partial \mathbb{E}e^{-(\rho+\lambda)\tau(\bar{\omega})}}{\partial \omega_0} \,\big|\, \omega(0) = \omega_0.$$

This in turn is continuously differentiable if $R(\omega, s_0)$ is continuous at $\omega = \omega_0$, i.e. if $s_0 \neq 1 - e^{\theta(\omega-\hat{\omega})}$.

A.3 Discrete Time, Discrete State Space Full Model

We consider a discrete time, discrete state space model. The length of the time period is Δt, the discount factor is $1 - \rho\Delta t$, and the exogenous exit probability is $\lambda\Delta t$. We imagine that ω lies on the grid $\{\underline{\omega}, \underline{\omega} + \Delta\omega, \underline{\omega} + 2\Delta\omega, \ldots, \bar{\omega}\}$ while $s \in [0, 1]$. If at the start of the period ω lies on the interior of the grid, it increases by $\Delta\omega$ with probability $\frac{1}{2}(1 + \Delta p)$ and otherwise decreases by $\Delta\omega$, while s stays constant. If $\omega = \underline{\omega}$, it increases to $\underline{\omega} + \Delta\omega$ with probability $\frac{1}{2}(1+\Delta p)$. Otherwise, there is a negative shock. ω is unchanged and all workers with seniority $s < 1 - e^{-\theta\Delta\omega}$ exit the labor market. This ensures that log employment falls by $\theta\Delta\omega$, which according to equation (33) is enough to leave the log full employment wage constant. The seniority of all other workers changes as well, falling from s to

$$s' = \frac{s - 1 + e^{-\theta\Delta\omega}}{e^{-\theta\Delta\omega}}. \tag{61}$$

Conversely, if $\omega = \bar{\omega}$, a negative shock reduces it to $\bar{\omega} - \Delta\omega$ with probability $\frac{1}{2}(1-\Delta p)$. Otherwise there is a positive shock. ω is unchanged, but log employment rises by $\theta\Delta\omega$ in order to leave the log full employment wage constant. The seniority of all workers rises from s to

$$s' = \frac{s - 1 + e^{\theta\Delta\omega}}{e^{\theta\Delta\omega}}. \tag{62}$$

Finally, assume $\Delta t = (\Delta\omega/\sigma)^2$ and $\Delta p = \mu\Delta\omega/\sigma^2$. We focus on the limiting behavior as $\Delta\omega$ converges to 0 holding fixed μ and σ, which corresponds to the stochastic process that we study in the body of the paper.

First take ω on the interior of the grid and an arbitrary s. The Bellman equation implies

$$\begin{aligned} v(\omega, s) = R(\omega, s)\Delta t + (1-\rho\Delta t)\Big(&\lambda\Delta t\underline{v} \\ &+ (1-\lambda\Delta t)\big(\tfrac{1}{2}(1+\Delta p)v(\omega+\Delta\omega, s) + \tfrac{1}{2}(1-\Delta p)v(\omega-\Delta\omega, s)\big)\Big). \end{aligned}$$

Take a second order Taylor expansion to $v(\omega \pm \Delta\omega, s)$ around $v(\omega, s)$ and simplify:

$$\begin{aligned} v(\omega, s) = R(\omega, s)\Delta t + (1-\rho\Delta t)\Big(&\lambda\Delta t\underline{v} \\ &+ (1-\lambda\Delta t)\big(v(\omega, s) + v_\omega(\omega, s)\Delta p\Delta\omega + \tfrac{1}{2}v_{\omega\omega}(\omega, s)\Delta\omega^2\big)\Big). \end{aligned}$$

Subtract $(1-\rho\Delta t)(1-\lambda\Delta t)v(\omega, s)$ from both sides of this equation. Since $\Delta p\Delta\omega = \mu\Delta t$ and $\Delta\omega^2 = \sigma^2\Delta t$, we can divide through by any $\Delta t > 0$ to obtain

$$(\rho + \lambda - \rho\lambda\Delta t)v(\omega, s) = R(\omega, s) + (1-\rho\Delta t)\Big(\lambda\underline{v} + (1-\lambda\Delta t)\big(\mu v_\omega(\omega, s) + \tfrac{1}{2}\sigma^2 v_{\omega\omega}(\omega, s)\big)\Big).$$

Taking the limit as $\Delta t \to 0$ gives equation (43).

Now consider $\omega = \underline{\omega}$. For $s > 1 - e^{-\theta\Delta\omega}$, the Bellman equation solves

$$\begin{aligned} v(\underline{\omega}, s) = R(\underline{\omega}, s)\Delta t + (1-\rho\Delta t)\Big(&\lambda\Delta t\underline{v} \\ &+ (1-\lambda\Delta t)\big(\tfrac{1}{2}(1+\Delta p)v(\underline{\omega}+\Delta\omega, s) + \tfrac{1}{2}(1-\Delta p)v(\underline{\omega}, s')\big)\Big). \end{aligned}$$

where s' solves equation (61). Now take a first order Taylor expansion of v around $(\underline{\omega}, s)$; higher order terms would disappear from the expression. Also approximate $e^{\theta\Delta\omega} - 1 = \theta\Delta\omega$.

$$v(\underline{\omega}, s) = R(\underline{\omega}, s)\Delta t + (1-\rho\Delta t)\Big(\lambda\Delta t\underline{v} \\ + (1-\lambda\Delta t)\big(v(\underline{\omega}, s) + \tfrac{1}{2}(1+\Delta p)v_\omega(\underline{\omega}, s)\Delta\omega - \tfrac{1}{2}(1-\Delta p)v_s(\underline{\omega}, s)(1-s)\theta\Delta\omega\big)\Big).$$

Subtract $(1-\rho\Delta t)(1-\lambda\Delta t)v(\omega, s)$ from both sides of this equation. Recall that $\Delta t = \Delta\omega^2/\sigma^2$ and $\Delta p = \mu\Delta\omega/\sigma^2$ and divide through by $\Delta\omega > 0$:

$$\left(\rho + \lambda - \rho\lambda\frac{\Delta\omega}{\sigma^2}\right)\frac{\Delta\omega}{\sigma^2}v(\underline{\omega}, s) = R(\underline{\omega}, s)\frac{\Delta\omega^2}{\sigma^2} + \left(1 - \rho\frac{\Delta\omega^2}{\sigma^2}\right)\left(\lambda\frac{\Delta\omega}{\sigma^2}\underline{v} \\ + \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\frac{1}{2}\left(\left(1 + \frac{\mu\Delta\omega}{\sigma^2}\right)v_\omega(\underline{\omega}, s) - \left(1 - \frac{\mu\Delta\omega}{\sigma^2}\right)v_s(\underline{\omega}, s)(1-s)\theta\right)\right).$$

Taking the limit as $\Delta\omega \to 0$ gives equation (44). The derivation of equation (45) is almost identical and hence omitted.

Now take $\omega = \underline{\omega}$ and $s \leq 1 - e^{-\theta\Delta\omega}$. In this case, the Bellman equation solves

$$v(\underline{\omega}, s) = R(\underline{\omega}, s)\Delta t + (1-\rho\Delta t)\Big(\lambda\Delta t\underline{v} + (1-\lambda\Delta t)\big(\tfrac{1}{2}(1+\Delta p)v(\underline{\omega} + \Delta\omega, s) + \tfrac{1}{2}(1-\Delta p)\underline{v}\big)\Big).$$

Taking a first order Taylor expansion of v around $(\underline{\omega}, 0)$ and using $v(\underline{\omega}, 0) = \underline{v}$ gives

$$\underline{v} + v_s(\underline{\omega}, 0)s = R(\underline{\omega}, s)\Delta t + (1-\rho\Delta t)\Big(\lambda\Delta t\underline{v} \\ + (1-\lambda\Delta t)\Big(\underline{v} + \tfrac{1}{2}(1+\Delta p)\big(v_\omega(\underline{\omega}, 0)\Delta\omega + v_s(\underline{\omega}, 0)s\big)\Big)\Big).$$

Subtract $(1-\rho\Delta t)(1-\lambda\Delta t)\underline{v}$ from both sides of this equation. Recall that $\Delta t = \Delta\omega^2/\sigma^2$ and $\Delta p = \mu\Delta\omega/\sigma^2$:

$$\left(\rho + \lambda - \rho\lambda\frac{\Delta\omega^2}{\sigma^2}\right)\frac{\Delta\omega^2}{\sigma^2}\underline{v} + v_s(\underline{\omega}, 0)s = R(\underline{\omega}, s)\frac{\Delta\omega^2}{\sigma^2} + \left(1 - \rho\frac{\Delta\omega^2}{\sigma^2}\right)\left(\lambda\frac{\Delta\omega^2}{\sigma^2}\underline{v} \\ + \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\frac{1}{2}\left(1 + \frac{\mu\Delta\omega}{\sigma^2}\right)\big(v_\omega(\underline{\omega}, 0)\Delta\omega + v_s(\underline{\omega}, 0)s\big)\right).$$

Taking the limit as $\Delta\omega \to 0$ gives $v_s(\underline{\omega}, 0) = 0$ for all $s \leq 0$. Combining with equation (44) gives equation (46).

Finally we handle the case of $s = 1$. Since the seniority of such a worker never changes (see equations 61 and 62), we have

$$v(\underline{\omega},1) = R(\underline{\omega},1)\Delta t + (1-\rho\Delta t)\Big(\lambda\Delta t\underline{v} + (1-\lambda\Delta t)\big(\tfrac{1}{2}(1+\Delta p)v(\underline{\omega}+\Delta\omega,1) + \tfrac{1}{2}(1-\Delta p)v(\underline{\omega},1)\big)\Big).$$

Take a first order Taylor expansion of v around $(\underline{\omega},1)$:

$$v(\underline{\omega},1) = R(\underline{\omega},1)\Delta t + (1-\rho\Delta t)\Big(\lambda\Delta t\underline{v} + (1-\lambda\Delta t)\big(v(\underline{\omega},1) + \tfrac{1}{2}(1+\Delta p)v_\omega(\underline{\omega},1)\Delta\omega\big)\Big).$$

Subtract $(1-\rho\Delta t)(1-\lambda\Delta t)v(\underline{\omega},1)$ from both sides of this equation. Recall that $\Delta t = \Delta\omega^2/\sigma^2$ and $\Delta p = \mu\Delta\omega/\sigma^2$ and divide through by $\Delta\omega$:

$$\left(\rho+\lambda-\rho\lambda\frac{\Delta\omega^2}{\sigma^2}\right)\frac{\Delta\omega}{\sigma^2}v(\underline{\omega},1) = R(\underline{\omega},1)\frac{\Delta\omega}{\sigma^2} + \left(1-\rho\frac{\Delta\omega^2}{\sigma^2}\right)\left(\lambda\frac{\Delta\omega}{\sigma^2}\underline{v} + \left(1-\lambda\frac{\Delta\omega^2}{\sigma^2}\right)\frac{1}{2}\left(1+\frac{\mu\Delta\omega}{\sigma^2}\right)v_\omega(\underline{\omega},1)\right).$$

Taking the limit as $\Delta\omega\to 0$ gives $v_\omega(\underline{\omega},1)=0$. A similar logic at $(\bar{\omega},1)$ gives $v_\omega(\bar{\omega},1)=0$, establishing equation (47).

A.4 Expression for the Value Function given Thresholds

A.4.1 High Minimum Wage

We first tackle the case where $\hat{\omega}\geq\bar{\omega}$. We claim that the value function satisfies

$$v(\omega,s) = \begin{cases} \dfrac{e^{\hat{\omega}}+\lambda\underline{v}}{\rho+\lambda} + \hat{c}_1(s)e^{\eta_1(\omega-\hat{\omega})} + \hat{c}_2(s)e^{\eta_2(\omega-\hat{\omega})} & \text{if } s\geq 1-e^{\theta(\omega-\hat{\omega})} \\ \dfrac{b_r+\lambda\underline{v}}{\rho+\lambda} + \underline{c}_1(s)e^{\eta_1(\omega-\hat{\omega})} + \underline{c}_2(s)e^{\eta_2(\omega-\hat{\omega})} & \text{if } s< 1-e^{\theta(\omega-\hat{\omega})}, \end{cases} \tag{63}$$

where $\eta_1<0<\eta_2$ are the roots of the characteristic equation

$$\rho+\lambda-\eta\mu-\eta^2\frac{\sigma^2}{2}=0 \tag{64}$$

and the univariate functions of integration satisfy

$$\hat{c}_1(s) = -\left(\frac{\eta_2}{\eta_2 - \eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right)(1 - e^{-(\eta_2-\eta_1)(\hat{\omega}-\underline{\omega})})(1-s)^{-\eta_1/\theta} \tag{65}$$

$$\hat{c}_2(s) = 0 \tag{66}$$

$$\underline{c}_1(s) = \hat{c}_1(s) + \left(\frac{\eta_2}{\eta_2 - \eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right)(1-s)^{-\eta_1/\theta} \tag{67}$$

$$\underline{c}_2(s) = \hat{c}_2(s) + \left(\frac{-\eta_1}{\eta_2 - \eta_1}\right)\left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda}\right)(1-s)^{-\eta_2/\theta}, \tag{68}$$

where we leave the expressions in a convenient form.

To prove this, note first that equation (63) is the general solution to equation (43). All that remains is to characterize the four functions of integration. The condition that the value function is continuously differentiable even at the boundary between work and rest unemployment, i.e. points of the form $(\omega, 1 - e^{\theta(\omega-\hat{\omega})})$, yields equations (67) and (68). Equations (44) and (45) then reduce to $\hat{c}_i(s)\eta_i = \hat{c}_i'(s)(1-\theta)s$ for $i = 1, 2$, or equivalently $\hat{c}_i(s) = c_i(1-s)^{-\eta_i/\theta}$.

To pin down the two constants c_1 and c_2, we use two more boundary conditions. Differentiate equation (63) to get

$$v_\omega(\omega, s) = c_1(1-s)^{-\eta_1/\theta}\eta_1 e^{\eta_1(\omega-\hat{\omega})} + c_2(1-s)^{-\eta_2/\theta}\eta_2 e^{\eta_2(\omega-\hat{\omega})}$$

when $s \geq 1 - e^{\theta(\omega-\hat{\omega})}$. In particular, equation (47) implies that this should converge to 0 as $s \to 1$ at $\omega = \underline{\omega}$ or $\omega = \bar{\omega}$. Since $\eta_2 > 0 > \eta_1$, this implies $c_2 = 0$, which delivers equation (66). Note that this implies $v_\omega(\omega, 1) = 0$ for all ω, since a worker at $s = 1$ will earn $\hat{\omega}$ regardless of the subsequent sequence of shocks. Finally, use equation (46) to pin down the last constant c_1, yielding equation (65).

A.4.2 Moderate Minimum Wage

Next we turn to the case where $\bar{\omega} > \hat{\omega} \geq \underline{\omega}$. The basic approach is similar. We claim that the value function satisfies

$$v(\omega, s) = \begin{cases} \dfrac{e^\omega}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \dfrac{\lambda \underline{v}}{\rho + \lambda} + \bar{c}_1(s)e^{\eta_1(\omega-\hat{\omega})} + \bar{c}_2(s)e^{\eta_2(\omega-\hat{\omega})} & \text{if } \omega \geq \hat{\omega} \\ \dfrac{e^{\hat{\omega}} + \lambda \underline{v}}{\rho + \lambda} + \hat{c}_1(s)e^{\eta_1(\omega-\hat{\omega})} + \hat{c}_2(s)e^{\eta_2(\omega-\hat{\omega})} & \text{if } \omega < \hat{\omega} \text{ and } s \geq 1 - e^{\theta(\omega-\hat{\omega})} \\ \dfrac{b_r + \lambda \underline{v}}{\rho + \lambda} + \underline{c}_1(s)e^{\eta_1(\omega-\hat{\omega})} + \underline{c}_2(s)e^{\eta_2(\omega-\hat{\omega})} & \text{if } \omega < \hat{\omega} \text{ and } s < 1 - e^{\theta(\omega-\hat{\omega})}, \end{cases} \tag{69}$$

where $\eta_1 < \eta_2$ solve equation (64) and

$$\hat{c}_1(s) = e^{-\eta_1\hat{\omega}} c_1(1-s)^{-\eta_1/\theta} - e^{-\eta_1\hat{\omega}} \left(\frac{e^{\hat{\omega}}}{\rho+\lambda}\right) \left(\frac{\rho+\lambda-(\mu+\frac{1}{2}\sigma^2)\eta_2}{(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)(\eta_2-\eta_1)}\right) \tag{70}$$

$$+ e^{-\eta_1\hat{\omega}} \frac{e^{\hat{\omega}}\left(\frac{e^{-(\hat{\omega}-\underline{\omega})(\eta_2-\eta_1)}(1-e^{-(\bar{\omega}-\hat{\omega})(\eta_2-1)})}{-\eta_1} + \frac{(1-e^{-(\hat{\omega}-\underline{\omega})(\eta_2-\eta_1)})(\rho+\lambda-(\mu+\frac{1}{2}\sigma^2)\eta_2)}{(\rho+\lambda)(\eta_2-\eta_1)}\right)}{(1-e^{-(\bar{\omega}-\underline{\omega})(\eta_2-\eta_1)})(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)} \tag{71}$$

$$\tag{72}$$

$$\hat{c}_2(s) = e^{-\eta_2\hat{\omega}} c_2(1-s)^{-\eta_2/\theta} + e^{-\eta_2\hat{\omega}} \left(\frac{e^{\hat{\omega}}}{\rho+\lambda}\right) \left(\frac{\rho+\lambda-(\mu+\frac{1}{2}\sigma^2)\eta_1}{(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)(\eta_2-\eta_1)}\right) \tag{73}$$

$$- e^{-\eta_2\hat{\omega}} \frac{e^{\hat{\omega}}\left(\frac{e^{(\hat{\omega}-\underline{\omega})(\eta_2-\eta_1)}(1-e^{(\bar{\omega}-\hat{\omega})(1-\eta_1)})}{\eta_2} + \frac{(1-e^{(\hat{\omega}-\underline{\omega})(\eta_2-\eta_1)})(\rho+\lambda-(\mu+\frac{1}{2}\sigma^2)\eta_1)}{(\rho+\lambda)(\eta_2-\eta_1)}\right)}{(1-e^{(\bar{\omega}-\underline{\omega})(\eta_2-\eta_1)})(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)} \text{ where} \tag{74}$$

$$c_1 = -\left(\frac{\eta_2}{\eta_2-\eta_1}\right)\left(\frac{e^{\hat{\omega}}}{\rho+\lambda}\right)\left(1-e^{-(\hat{\omega}-\underline{\omega})(\eta_2-\eta_1)}\right) \quad \text{and } c_2 = 0 \; .$$

For $\omega \geq \hat{\omega}$:

$$a_1 = e^{-\eta_1\hat{\omega}} \left(\frac{e^{\hat{\omega}}}{\rho+\lambda}\right) \left(\frac{\rho+\lambda-(\mu+\frac{1}{2}\sigma^2)\eta_2}{(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)(\eta_2-\eta_1)}\right) \tag{75}$$

$$a_2 = -e^{-\eta_2\hat{\omega}} \left(\frac{e^{\hat{\omega}}}{\rho+\lambda}\right) \left(\frac{\rho+\lambda-(\mu+\frac{1}{2}\sigma^2)\eta_1}{(\rho+\lambda-\mu-\frac{1}{2}\sigma^2)(\eta_2-\eta_1)}\right) \tag{76}$$

If $\omega < \hat{\omega}$ and $s < 1 - e^{\theta(\omega-\hat{\omega})}$:

$$h_1(s) = e^{-\eta_1\hat{\omega}} \left(\frac{\eta_2}{\eta_2-\eta_1}\right) \left(\frac{e^{\hat{\omega}}}{\rho+\lambda}\right) (1-s)^{-\eta_1/\theta} \tag{77}$$

$$h_2(s) = -e^{-\eta_2\hat{\omega}} \left(\frac{\eta_1}{\eta_2-\eta_1}\right) \left(\frac{e^{\hat{\omega}}}{\rho+\lambda}\right) (1-s)^{-\eta_2/\theta}. \tag{78}$$

Once again, equation (69) is the general solution to equation (43). Continuity and differentiability of the value function again allow us to relate $\hat{c}_i(s)$ to $\bar{c}_i(s)$ and $\underline{c}_i(s)$ to $\hat{c}_i(s)$. Then equations (44) and (45) yield a pair of differential equations which can be solved for $\bar{c}_1(s)$ and $\bar{c}_2(s)$ as a function of parameters and the two constants c_1 and c_2. This gives equations (**??**)–(77). Equation (47), evaluated either at $\underline{\omega}$ or $\bar{\omega}$, again pins down $c_2 = 0$ so that $\bar{c}_2(s)$ remains finite as $s \to 1$. Finally, use equation (46) to pin down c_1.

A.4.3 Low Minimum Wage

Finally we study $\hat{\omega} < \underline{\omega}$. This is equivalent to the case when $\hat{\omega} = \underline{\omega}$, since in both situations the minimum wage never binds. Applying the analysis with a moderate minimum wage gives

$$v(\omega, s) = \frac{e^{\omega}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \frac{\lambda \underline{v}}{\rho + \lambda} + c_1 e^{\eta_1(\omega - \underline{\omega})} + c_2 e^{\eta_2(\omega - \underline{\omega})} \tag{79}$$

where $\eta_1 < \eta_2$ solve equation (64) and

$$c_1 = \frac{e^{\underline{\omega}}\left(1 - e^{-(\bar{\omega}-\underline{\omega})(\eta_2 - 1)}\right)}{-\eta_1(1 - e^{-(\bar{\omega}-\underline{\omega})(\eta_2-\eta_1)})(\rho + \lambda - \mu - \frac{1}{2}\sigma^2)} \tag{80}$$

$$c_2 = -\frac{e^{\underline{\omega}}\left(1 - e^{(\bar{\omega}-\underline{\omega})(1-\eta_1)}\right))}{\eta_2(1 - e^{(\bar{\omega}-\underline{\omega})(\eta_2-\eta_1)})(\rho + \lambda - \mu - \frac{1}{2}\sigma^2)}. \tag{81}$$

Note that the value function does not depend on the worker's seniority.

A.5 Alternative representation and monotonicity

We begin with an alternative closed form solution of the value function, which we use to show some of its properties below. The value function takes a different form in different regions of the state space that we label as:

$$\begin{aligned}
\text{Region 1} &\equiv \{\omega, s : \hat{\omega} \le \omega \le \bar{\omega}\ ,\ 0 \le s \le 1\},\\
\text{Region 2} &\equiv \{\omega, s : \underline{\omega} \le \omega \le \hat{\omega}\ ,\ 1 \ge s \ge -\exp\left(-\theta(\hat{\omega} - \omega)\right)\},\\
\text{Region 3} &\equiv \{\omega, s : \underline{\omega} \le \omega \le \hat{\omega}\ ,\ 0 \le s \le -\exp\left(-\theta(\hat{\omega} - \omega)\right)\}.
\end{aligned}$$

Proposition 5. Assume that $\lambda = b_r = 0$. Let $\bar{\omega} \ge \hat{\omega} \ge \underline{\omega}$ with at least one of the inequalities strict. Then there is a unique function $v(\omega, s)$ that satisfies: i) the BJH equation

$$\rho v(\omega, s) = R(\omega, s) + \mu v_\omega(\omega, s) + v_{\omega\omega}(\omega, s)\frac{\sigma^2}{2}$$

at all (ω, s) except those on $\{(\omega, s) : \underline{\omega} \le \omega \le \hat{\omega},\ s = -\exp\left(-\theta\left(\hat{\omega} - \omega\right)\right)\}$, ii) The boundary conditions $v_\omega(\omega, s) = \theta(1 - s)\, v_\omega(\omega, s)$ hold at $\omega \in \{\underline{\omega}, \bar{\omega}\}$ and $s \in [0, 1)$, iii) The smooth pasting condition $v_\omega(\omega, s) = 0$ hold at $(\omega, s) \in \{(\underline{\omega}, 0), (\underline{\omega}, 1), (\bar{\omega}, 1)\}$, iv) The function v is C^2 on $[\underline{\omega}, \bar{\omega}] \times [0, 1)$, except at

$$(\omega, s) \in \{(\omega, s) : \underline{\omega} \le \omega \le \hat{\omega},\ s = -\exp\left(-\theta\left(\hat{\omega} - \omega\right)\right)\}$$

where it is C^1. v) The function $v(\cdot,1)$ is C^2 on $[\underline{\omega},\bar{\omega}]$. Moreover, $v(\omega,s)$ is given by

$$v(\omega,s)=\frac{e^{\omega}}{\rho-\mu-\sigma^2/2}+\sum_{i=1,2}\left[\hat{c}_i(s)+a_i\right]\ e^{\eta_i\omega}$$

in Region 1, i.e. for $\omega\in[\hat{\omega},\bar{\omega}]$,

$$v(\omega,s)=\sum_{i=1,2}\left[\hat{c}_i(s)+h_i(z(s))\right]e^{\eta_i\omega}$$

in Region 3, i.e. for $\omega\in[\underline{\omega},\ \hat{\omega}]$ and $s\le 1-\exp\left(-\theta\left(\hat{\omega}-\underline{\omega}\right)\right)$ and

$$v(\omega,s)=\frac{e^{\hat{\omega}}}{\rho}+\sum_{i=1,2}\hat{c}_i(s)\ e^{\eta_i\omega}$$

if Region 2, i.e. for $\omega\in[\underline{\omega},\ \hat{\omega}]$ and $s\ge 1-\exp\left(\theta\left(\hat{\omega}-\underline{\omega}\right)\right)$. The functions $\hat{c}_i(s):[0,1]\to\mathbb{R}$, and $h_i:[0,\ \theta(\hat{\omega}-\underline{\omega})]\to\mathbb{R}$, and the numbers a_i are as follows. Let the matrices M and N be

$$M\equiv\begin{bmatrix} e^{\eta_1\hat{\omega}} & e^{\eta_2\hat{\omega}} \\ \eta_1 e^{\eta_1\hat{\omega}} & \eta_2 e^{\eta_2\hat{\omega}}\end{bmatrix},$$

$$N\equiv\begin{bmatrix} e^{\eta_1\bar{\omega}}\eta_1 & e^{\eta_2\bar{\omega}}\eta_2 \\ e^{\eta_1\underline{\omega}}\eta_1 & e^{\eta_2\underline{\omega}}\eta_2\end{bmatrix}$$

Then the constants a_i are given by

$$a=M^{-1}\ e^{\hat{\omega}}\ \begin{bmatrix}\frac{1}{\rho}-\frac{1}{\rho-\mu-\sigma^2/2}\\ -\frac{1}{\rho-\mu-\sigma^2/2}\end{bmatrix}.$$

Let the function $z:[0,1]\to\mathbb{R}$ be

$$z(s)\equiv-\log(1-s)\ .$$

Then the functions $h_i(\cdot)$ are given by

$$h_i(z(s))=h_i(0)\,e^{\frac{\eta_i}{\theta}z(s)}$$

for all $s\in[0,1]$ with

$$h(0)=M^{-1}\begin{bmatrix}\frac{e^{\hat{\omega}}}{\rho}\\ 0\end{bmatrix},$$

The functions $c_i(\cdot)$ are given by

$$\hat{c}_2(s) = \hat{c}_2(1)$$

for $i = 2$ for all $s \in [0,1]$. For $i = 1$

$$\hat{c}_1(s) = [\hat{c}_1(0) - \hat{c}_1(1)]\, e^{\eta_1 z(s)} + \hat{c}_1(1)$$

for all $s \in [0,1]$. Finally, $\hat{c}_1(0)$ given by

$$\hat{c}_1(0) = \frac{-e^{\eta_2 \underline{\omega}} \eta_2 \; \hat{c}_2(1) - \sum_{i=1,2} h_i(0)\, e^{\eta_i \underline{\omega}} \eta_i}{e^{\eta_1 \underline{\omega}} \eta_1}$$

and the vector $\hat{c}(s)$ evaluated at $s = 1$ is given by:

$$\hat{c}(1) = \; N^{-1} \begin{bmatrix} -g \\ 0 \end{bmatrix}$$

where the number g is given by

$$g \equiv \frac{e^{\bar{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} a_i e^{\eta_i \bar{\omega}} \eta_i \; .$$

Proof of Proposition 5. Let $\sigma_i(z(s)) \equiv c_i(s)$ or $\sigma_i(\cdot) \equiv c_i(z^{-1}(\cdot))$. Notice that

$$(1-s)\, c_i'(s) = \sigma_i'(z(s))$$

where c_i stands for $\bar{c}_i, \hat{c}_i$ or $\underline{c}_i$ and likewise for σ_i. We use $\bar{c}(s) = \hat{c}(s) + a_i$, $\underline{c}_i(s) = \hat{c}_i(s) + h_i(s)$

Given that $\eta_1 < 0 < \eta_2$ and $\bar{\omega} > \underline{\omega}$ the matrices M and N are invertible since

$$\det M = e^{(\eta_1+\eta_2)\hat{\omega}}(\eta_2 - \eta_1) \neq 0$$

$$\det N = (\eta_1 \eta_2)\left(e^{\eta_1 \bar{\omega} + \eta_2 \underline{\omega}} - e^{\eta_2 \bar{\omega} + \eta_1 \underline{\omega}}\right) \neq 0$$

Step I. Solve for $\bar{\sigma}_i(z)$ as a function of $\hat{\sigma}_i(z)$ for all z.

Write the equations for value matching and smooth pasting between regions 1 and 2 for all $z \in (0, \infty)$ as:

$$\begin{aligned}
\frac{e^{\hat{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} \bar{\sigma}_i(z)\, e^{\eta_i\ \hat{\omega}} &= \frac{e^{\hat{\omega}}}{\rho} + \sum_{i=1,2} \hat{\sigma}_i(z)\, e^{\eta_i\ \hat{\omega}} \\
\frac{e^{\hat{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} \bar{\sigma}_i(z)\, \eta_i e^{\eta_i\ \hat{\omega}} &= \sum_{i=1,2} \hat{\sigma}_i(z)\ \ \eta_i\ e^{\eta_i\ \hat{\omega}}
\end{aligned}$$

Using these two equations we can solve for $\bar{\sigma}(z)$ as a function of $\hat{\sigma}(z)$. The solution gives:

$$\bar{\sigma}_i(z) = \hat{\sigma}_i(z) + a_i$$

for two constants a_1, and a_2. This can be seen by replacing into the equations above and obtaining:

$$\begin{aligned}
\frac{e^{\hat{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} a_i\ e^{\eta_i\ \hat{\omega}} &= \frac{e^{\hat{\omega}}}{\rho} \\
\frac{e^{\hat{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} a_i\ \eta_i e^{\eta_i\ \hat{\omega}} &= 0
\end{aligned}$$

or

$$a = e^{\hat{\omega}}\ M^{-1} \begin{bmatrix} \frac{1}{\rho} - \frac{1}{\rho-\mu-\sigma^2/2} \\ -\frac{1}{\rho-\mu-\sigma^2/2} \end{bmatrix}$$

Step II. Solving $\hat{\sigma}(\infty) = \hat{c}_i(1)$ and $\bar{\sigma}_i(\infty) = \bar{c}_i(1)$.

We have that at $s = 1,\ v_\omega(\omega, 1) = 0$ at $\omega \in \{\underline{\omega}, \bar{\omega}\}$ so that

$$\begin{aligned}
\sum_{i=1,2} \bar{\sigma}_i(\infty)\, e^{\eta_i \bar{\omega}} \eta_i &= -\frac{e^{\bar{\omega}}}{\rho - \mu - \sigma^2/2} \\
\sum_{i=1,2} \hat{\sigma}_i(\infty)\, e^{\eta_i \underline{\omega}} \eta_i &= 0
\end{aligned}$$

Together with $\bar{\sigma}_i(z) = \hat{\sigma}_i(z) + a_i$ this implies:

$$\begin{aligned}
\sum_{i=1,2} \hat{\sigma}_i(\infty)\, e^{\eta_i \bar{\omega}} \eta_i &= -g(\bar{\omega}, \hat{\omega}) \\
\sum_{i=1,2} \hat{\sigma}_i(\infty)\, e^{\eta_i \underline{\omega}} \eta_i &= 0
\end{aligned}$$

where

$$g \equiv \frac{e^{\bar{\omega}}}{\rho-\mu-\sigma^2/2}+\sum_{i=1,2} a_i\left(\hat{\omega}\right) e^{\eta_i \bar{\omega}} \eta_i$$

The solution of this system gives $\hat{\sigma}_i\left(\infty\right)$ or

$$\hat{c}\left(1\right) \equiv \hat{\sigma}\left(\infty\right)= \ N^{-1}\left[\begin{array}{c} -g \\ 0 \end{array}\right]$$

Step III. Solve ode's for $\hat{\sigma}_i\left(z\right)$ for $z \geq \theta\left(\hat{\omega}-\underline{\omega}\right)$

Consider the ode's for $\hat{\sigma}$ and $\bar{\sigma}$ for $s \geq \underline{s}=1-\exp\left(-\theta\left(\hat{\omega}-\underline{\omega}\right)\right)$ or

$$z \geq \theta\left(\hat{\omega}-\underline{\omega}\right)$$

They are given by $v_\omega=\left(1-s\right)\theta v_s$ evaluated at $\omega \in \{\underline{\omega}, \bar{\omega}\}$ or

$$\begin{aligned}
\frac{e^{\bar{\omega}}}{\rho-\mu-\sigma^2/2}+\sum_{i=1,2} \bar{\sigma}_i\left(z\right)\eta_i e^{\eta_i\ \bar{\omega}} &= \theta\sum_{i=1,2} \bar{\sigma}_i'\left(z\right)\ e^{\eta_i\ \bar{\omega}} \\
\sum_{i=1,2} \hat{\sigma}_i\left(z\right)\eta_i e^{\eta_i\ \underline{\omega}} &= \theta\sum_{i=1,2} \hat{\sigma}_i\left(z\right)\ e^{\eta_i\ \underline{\omega}}
\end{aligned}$$

Using the result for $\bar{\sigma}$ in terms of $\hat{\sigma}$:

$$\bar{\sigma}_i'\left(z\right)=\hat{\sigma}_i'\left(z\right)$$

or

$$\begin{aligned}
\frac{e^{\bar{\omega}}}{\rho-\mu-\sigma^2/2}+\sum_{i=1,2} a_i\eta_i e^{\eta_i\ \bar{\omega}}+\sum_{i=1,2} \hat{\sigma}_i\left(z\right)\eta_i e^{\eta_i\ \bar{\omega}} &= \theta\sum_{i=1,2} \hat{\sigma}_i'\left(z\right)\ e^{\eta_i\ \bar{\omega}} \\
\sum_{i=1,2} \hat{\sigma}_i\left(z\right)\eta_i e^{\eta_i\ \underline{\omega}} &= \theta\sum_{i=1,2} \hat{\sigma}_i'\left(z\right)\ e^{\eta_i\ \underline{\omega}}
\end{aligned}$$

The solution of this system is given by

$$\hat{\sigma}_i\left(z\right)=e^{\frac{\eta_i}{\theta}z}\ f_i+d_i$$

where d_i solve

$$\begin{aligned} g + \sum_{i=1,2} d_i \ \eta_i e^{\eta_i \ \bar{\omega}} &= 0 \\ \sum_{i=1,2} d_i \ \eta_i e^{\eta_i \ \underline{\omega}} &= 0 \end{aligned}$$

for g defined as above, thus

$$d = N^{-1} \begin{bmatrix} -g \\ 0 \end{bmatrix}$$

or, by comparing both systems

$$d = \hat{c}(1) \ .$$

Step IV Solve for $\underline{\sigma}(z)$ as functions of $\hat{\sigma}(z)$ for $z \leq \theta(\hat{\omega} - \underline{\omega})$

For $z \in (0, \bar{z})$ we have two other value matching and smooth pasting between regions 2 and 3:

$$\begin{aligned} \sum_{i=1,2} \underline{\sigma}_i(z) \, e^{\eta_i \ \omega(z)} &= \frac{e^{\hat{\omega}}}{\rho} + \sum_{i=1,2} \hat{\sigma}_i(z) \, e^{\eta_i \ \omega(z)} \\ \sum_{i=1,2} \underline{\sigma}_i(z) \, \eta_i e^{\eta_i \ \omega(z)} &= \sum_{i=1,2} \hat{\sigma}_i(z) \ \ \eta_i \ e^{\eta_i \ \omega(z)} \end{aligned}$$

where $\omega(z)$ is given by

$$\omega(z) = \hat{\omega} - \frac{1}{\theta} z \iff s = 1 - \exp(-\theta(\hat{\omega} - \omega))$$

so we have

$$\begin{aligned} \sum_{i=1,2} \underline{\sigma}_i(z) \, e^{-\frac{\eta_i}{\theta} z} e^{\eta_i \ \hat{\omega}} &= \frac{e^{\hat{\omega}}}{\rho} + \sum_{i=1,2} \hat{\sigma}_i(z) \, e^{-\frac{\eta_i}{\theta} z} e^{\eta_i \ \hat{\omega}} \\ \sum_{i=1,2} \underline{\sigma}_i(z) \, \eta_i e^{-\frac{\eta_i}{\theta} z} e^{\eta_i \ \hat{\omega}} &= \sum_{i=1,2} \hat{\sigma}_i(z) \ \ \eta_i e^{-\frac{\eta_i}{\theta} z} \ e^{\eta_i \ \hat{\omega}} \end{aligned}$$

These equations imply that

$$\underline{\sigma}_i(z) = \hat{\sigma}_i(z) + h_i(z)$$

where $h_i(z)$ satisfy

$$\begin{aligned}
\sum_{i=1,2} h_i(z)\, e^{-\frac{\eta_i}{\theta} z} e^{\eta_i \ \hat{\omega}} &= \frac{e^{\hat{\omega}}}{\rho} \\
\sum_{i=1,2} h_i(z)\, \eta_i e^{-\frac{\eta_i}{\theta} z} e^{\eta_i \ \hat{\omega}} &= 0
\end{aligned}$$

for all $z \in (0, \bar{z})$ .We try the solution

$$\begin{aligned}
h_i(z) &= h_i(0; \hat{\omega})\, e^{\frac{\eta_i}{\theta} z} \\
h(0) &= M^{-1} \begin{bmatrix} \frac{e^{\hat{\omega}}}{\rho} \\ 0 \end{bmatrix}
\end{aligned}$$

For future reference notice that $h_i'(z) \ = (\eta_i/\theta)\, h_i(z)$.

Step V. Solve ode's for $\hat{\sigma}$ in $z \in (0, \theta(\bar{\omega} - \underline{\omega}))$

In this region we also have two boundary conditions, corresponding to $v_\omega = v_s \theta (1 - s)$ evaluated at $\omega \in \{\underline{\omega},\ \bar{\omega}\}$. This gives the ode's:

$$\frac{e^{\bar{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} \bar{\sigma}_i(z)\, e^{\eta_i \bar{\omega}} \eta_i = \theta \sum_{i=1,2} \bar{\sigma}_i'(z)\, e^{\eta_i \bar{\omega}}$$

and

$$\sum_{i=1,2} \underline{\sigma}_i(z)\, e^{\eta_i \underline{\omega}} \eta_i = \theta \sum_{i=1,2} \underline{\sigma}_i'(z)\, e^{\eta_i \underline{\omega}}$$

replacing that

$$\begin{aligned}
\bar{\sigma}_i(z) &= \hat{\sigma}_i(z) + a_i \\
\underline{\sigma}_i(z) &= \hat{\sigma}_i(z) + h_i(z)
\end{aligned}$$

we have:

$$\begin{aligned}
& \frac{e^{\bar{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} a_i e^{\eta_i \bar{\omega}} \eta_i + \sum_{i=1,2} \hat{\sigma}_i(z)\, e^{\eta_i \bar{\omega}} \eta_i \\
= \ & \theta \sum_{i=1,2} \hat{\sigma}_i'(z)\, e^{\eta_i \bar{\omega}}
\end{aligned}$$

and

$$\begin{aligned}
& -\theta \sum_{i=1,2i} h_i'(z)\, e^{\eta_i \underline{\omega}} + \sum_{i=1,2} h_i(z)\, e^{\eta_i \underline{\omega}} \eta_i + \sum_{i=1,2} \hat{\sigma}_i(z)\, e^{\eta_i \underline{\omega}} \eta_i \\
= \;& \theta \sum_{i=1,2} \hat{\sigma}_i'(z)\, e^{\eta_i \underline{\omega}}
\end{aligned}$$

or using that $h_i'(z) = (\eta_i/\theta)\, h_i(z)$

$$\sum_{i=1,2} \hat{\sigma}_i(z)\, e^{\eta_i \underline{\omega}} \eta_i = \theta \sum_{i=1,2} \hat{\sigma}_i'(z)\, e^{\eta_i \underline{\omega}}$$

for $z \in (0, \bar{z})$. We can also write this system as:

$$\begin{aligned}
g + \sum_{i=1,2} \hat{\sigma}_i(z)\, e^{\eta_i \bar{\omega}} \eta_i &= \theta \sum_{i=1,2} \hat{\sigma}_i'(z)\, e^{\eta_i \bar{\omega}} \\
\sum_{i=1,2} \hat{\sigma}_i(z)\, e^{\eta_i \underline{\omega}} \eta_i &= \theta \sum_{i=1,2} \hat{\sigma}_i'(z)\, e^{\eta_i \underline{\omega}}
\end{aligned}$$

where g is defined as above.

This system of linear odes is the same as the one that applies for $z \geq \theta(\hat{\omega} - \underline{\omega})$. Since we require $\hat{\sigma}_i(z)$ to be continuously differentiable at $z = \bar{z}$, the solution must also be given $\hat{\sigma}_i(z) = e^{\frac{\eta_i}{\theta} z} f_i + d_i$ for the same constants $f,\ d$ in the entire domain $z \geq 0$. Since we have shown that $d = \hat{c}(1)$, we can write:

$$\begin{aligned}
\hat{\sigma}_i(z) &= e^{\frac{\eta_i}{\theta} z} f_i + \hat{c}_i(1) \\
&= e^{\frac{\eta_i}{\theta} z} (\hat{\sigma}_i(0) - \hat{c}_i(1)) + \hat{c}_i(1)
\end{aligned}$$

for $z \geq 0$.

Step VI. Using the unstable root to determine one initial condition for $\hat{\sigma}_i$.

From the previous step we have that

$$\hat{\sigma}_i(z) = e^{\frac{\eta_i}{\theta} z} f_i + \hat{c}_i(1)$$

Recall that $\eta_1 < 0 < \eta_2$, and that at $s = 1$, or equivalently $z \to \infty$

$$\begin{aligned}
\hat{c}_i(1) &\equiv \lim_{z \to \infty} \hat{\sigma}_i(z) \\
\bar{c}_i(1) &\equiv \lim_{z \to \infty} \bar{\sigma}_i(z)
\end{aligned}$$

we have finite values. Thus the coefficient f_2 associated with the unstable root $\eta_2 > 0$ must be zero so that for all $z \geq 0$:

$$\hat{\sigma}_2(z) = \hat{c}_2(1), \ \bar{\sigma}_2(z) = \hat{c}_2(1) + a_2$$

and thus

$$\underline{\sigma}_2(z) = \hat{c}_2(1) + h_2(0)\, e^{\frac{\eta_2}{\theta} z}$$

for $z \in [0, \bar{z}]$.

Step VII. Using $v_\omega(\underline{\omega}, 0) = 0$ to determine an initial condition for $\hat{\sigma}_i(0)$

Finally at $(\omega, s) = (\underline{\omega}, 0)$ we have $v_\omega = 0$ or

$$\sum_i \underline{\sigma}_i(0)\, e^{\eta_i \underline{\omega}} \eta_i = 0$$

or using $\underline{\sigma}_i(z) = \hat{\sigma}_i(z) + h_i(z)$ at $z = 0$:

$$\sum_{i=1,2} \hat{\sigma}_i(0)\, e^{\eta_i \underline{\omega}} \eta_i = - \sum_{i=1,2} h_i(0)\, e^{\eta_i \underline{\omega}} \eta_i$$

This gives one equation for $\{\hat{\sigma}_i(0)\}_{i=1,2}$.

Step VIII. Solve for $\hat{\sigma}(0)$.

From step VI we have

$$\hat{\sigma}_2(0) = \hat{c}_2(1),$$

and using step VII

$$\sum_{i=1,2} \hat{\sigma}_i(0)\, e^{\eta_i \underline{\omega}} \eta_i = - \sum_{i=1,2} h_i(0)\, e^{\eta_i \underline{\omega}} \eta_i$$

or

$$\hat{\sigma}_1(0) = \frac{-e^{\eta_2 \underline{\omega}} \eta_2 \ \hat{c}_2(1) - \sum_{i=1,2} h_i(0)\, e^{\eta_i \underline{\omega}} \eta_i}{e^{\eta_1 \underline{\omega}} \eta_1}$$

□

We rewrite the statements of Proposition 3 on the monotonicity of v, given thresholds. Let $v(\omega, s)$ be defined above for given triple $\underline{\omega} \leq \hat{\omega} \leq \underline{\omega}$. Then:

1. The derivative $v_s(\omega, s) \ > 0$ for all $s < 1$, except at $(\omega, s) = (\underline{\omega}, 0)$.

2. The derivative $v_\omega(\omega, s) > 0$ everywhere expect at $(\omega, s) \in \{(\underline{\omega}, 0), \ (\underline{\omega}, 1), \ (\bar{\omega}, 1)\}$.

Proof of Proposition 3.

i) Monotonicity of $v(\omega, s)$ w.r.t. to ω.

First recall that in Alvarez and Shimer (2011) we show that $v(\omega, 1)$ is strictly increasing in $\omega \in (\underline{\omega}, \bar{\omega})$. Since this function satisfies: $v_\omega(\underline{\omega}, 1) = 0$, it implies that $v_{\omega\omega}(\underline{\omega}, 1) > 0$, which in turns implies that:

$$\hat{c}_1(1) > 0 \text{ and } \hat{c}_2(2) > 0.$$

This follows because $0 < \eta_1 < \eta_2$

$$\begin{aligned} v_\omega(\underline{\omega}, 1) &= \sum_{i=1,2} \hat{c}_i(1) e^{\eta_i \underline{\omega}} \eta_i = 0, \text{ and} \\ v_{\omega\omega}(\underline{\omega}, 1) &= \sum_{i=1,2} \hat{c}_i(1) e^{\eta_i \underline{\omega}} (\eta_i)^2 > 0. \end{aligned}$$

ii) Monotonicity of $v(\omega, s)$ w.r.t. to s.

For future reference we notice that in region 3 (i.e. $\omega \leq \hat{\omega}$ and $s \leq 1 - \exp(-\theta(\hat{\omega} - \underline{\omega}))$) we have:

$$\begin{aligned} v_s(\omega, s) &= \frac{1}{1-s}\left[\underline{\sigma}_1'(z(s)) e^{\eta_1 \omega} + \underline{\sigma}_2'(z(s)) e^{\eta_2 \omega}\right] \\ &= \frac{1}{1-s}\left[(\hat{\sigma}_1'(z) + h_1'(z(s))) e^{\eta_1 \omega} + h_2'(z(s)) e^{\eta_2 \omega}\right] \\ &= \frac{1}{1-s}\left[\frac{\eta_1}{\theta}\left[\hat{\sigma}_1(z(s)) - \hat{c}_1(1) + h_1(z)\right] e^{\eta_1 \omega} + \frac{\eta_2}{\theta} h_2(z(s)) e^{\eta_2 \omega}\right] \\ &= \frac{1}{1-s}\left[\frac{\eta_1}{\theta}\left[\hat{\sigma}_1(0) - \hat{c}_1(1) + h_1(0)\right] e^{\frac{\eta_1}{\theta} z(s)} e^{\eta_1 \omega} + \frac{\eta_2}{\theta} h_2(0) e^{\frac{\eta_2}{\theta} z(s)} e^{\eta_2 \omega}\right] \end{aligned}$$

Since by construction we have $v_s(\underline{\omega}, 0) = 0$, this implies:

$$-\eta_1\left[\hat{\sigma}_1(0) - \hat{c}_1(1)\right] = \eta_1 h_1(0) e^{\eta_1 \underline{\omega}} + \eta_2 h_2(0) e^{\eta_2 \underline{\omega}}$$

From the two equations that define $h(0)$, using that $\eta_1 < 0 < \eta_2$,we have

$$h_1(0) > 0 \text{ and } h_2(0) > 0.$$

Moreover, let $H(\omega)$ be

$$H(\omega) \equiv \eta_1 h_1(0) e^{\eta_1 \omega} + \eta_2 h_2(0) e^{\eta_2 \omega}$$

so that

$$H_\omega(\omega) = (\eta_1)^2 h_1(0) e^{\eta_1 \omega} + (\eta_2)^2 h_2(0) e^{\eta_2 \omega} > 0$$

By definition of $h(0)$ we have $H(\hat{\omega}) = 0$, then we can write the previous equation as

$$(-\eta_1)\left[\hat{\sigma}_1(0) - \hat{c}_1(1)\right] = H(\underline{\omega}) < H(\hat{\omega}) = 0.$$

Now we use this inequality to show that in region 2 (i.e. $\omega \leq \hat{\omega}$ and $z \geq \theta(\hat{\omega} - \underline{\omega})$) the function $v(\omega, s)$ is increasing in s :

$$v_s(\omega, s) = \frac{1}{1-s}\hat{\sigma}_1'(z(s))\,e^{\eta_1\omega} = \frac{1}{1-s}\frac{\eta_1}{\theta}e^{\frac{\eta_1}{\theta}z(s)}\left(\hat{\sigma}_1(0) - \hat{c}_1(0)\right)e^{\eta_1\omega} \geq 0$$

Moreover in region 1 (i.e., $\omega \geq \hat{\omega}$) we have

$$v_s(\omega, s) = \frac{1}{1-s}\bar{\sigma}_1'(z(s))\,e^{\eta_1\omega} = \frac{1}{1-s}\hat{\sigma}_1'(z(s))\,e^{\eta_1\omega}$$

so that v is also increasing in s. For region 3 notice that we can write the previous expression as

$$(1-s)\,v_s(\omega, s) = J(\omega, z)$$

where

$$J(\omega, z) \equiv \frac{\eta_1}{\theta}\left[\hat{\sigma}_1(0) - \hat{c}_1(1) + h_1(0)\right]e^{\frac{\eta_1}{\theta}z}e^{\eta_1\omega} + \frac{\eta_2}{\theta}h_2(0)\,e^{\frac{\eta_2}{\theta}z}e^{\eta_2\omega}$$

Since $\hat{c}_2(1) > 0$, $h_2(0) > 0$, $\eta_1 < 0 < \eta_2$, and

$$0 = v_\omega(\underline{\omega}, 0) = \eta_1\left[\hat{\sigma}_1(0) - \hat{c}_1(1) + h_1(0)\right]e^{\eta_1\underline{\omega}} + \eta_2\left[\hat{c}_2(1) + h_2(0)\right]e^{\eta_2\underline{\omega}}$$

we have $\underline{\sigma}_1(0) = \hat{\sigma}_1(0) - \hat{c}_1(1) + h_1(0) > 0$. Moreover using these inequalities we obtain:

$$\begin{aligned}
J_\omega(\omega, z) &= \frac{(\eta_1)^2}{\theta}\left[\hat{\sigma}_1(0) - \hat{c}_1(1) + h_1(0)\right]e^{\frac{\eta_1}{\theta}z}e^{\eta_1\omega} + \frac{(\eta_2)^2}{\theta}h_2(0)\,e^{\frac{\eta_2}{\theta}z}e^{\eta_2\omega} > 0 \\
J_z(\omega, z) &= (\eta_1/\theta)^2\left[\hat{\sigma}_1(0) - \hat{c}_1(1) + h_1(0)\right]e^{\frac{\eta_1}{\theta}z}e^{\eta_1\omega} + (\eta_2/\theta)^2\,h_2(0)\,e^{\frac{\eta_2}{\theta}z}e^{\eta_2\omega} > 0
\end{aligned}$$

Thus for $\omega > \underline{\omega}$ and $z > 0$ we have $J(\omega, z) > 0$ since $J(\underline{\omega}, 0) = 0$, and hence $v_s(\omega, s) > 0$ in region 3.

Monotonicity w.r.t. ω.

In region 1 (i.e. $\omega \geq \hat{\omega}$) we have the following:

$$v_\omega(\omega, s) = \frac{e^\omega}{\rho - \mu - \sigma^2/2} + \sum_{i=1}\bar{\sigma}_i(z(s))\,\eta_i e^{\eta_i\omega}$$

and in particular for $s = 1$

$$v_\omega(\omega, 1) = \frac{e^\omega}{\rho - \mu - \sigma^2/2} + \sum_{i=1}\bar{c}_i(1)\,\eta_i e^{\eta_i\omega} > 0$$

for all $\omega \in (\hat{\omega}, \bar{\omega})$, from the result in our previous paper. We have that $\bar{\sigma}_1(z) = \hat{\sigma}_1(z) + a_1$ and

$\bar{\sigma}_2(z) = \hat{c}_2(1) + a_2$ so

$$v_{\omega s}(\omega, s) = \frac{1}{1-s}\eta_1 \hat{\sigma}_1'(z(s))\, e^{\eta_1 \omega} = \frac{1}{1-s}(\eta_1)^2 [\hat{\sigma}_1(0) - \hat{c}_1(1)]\, e^{\frac{\eta_1}{\theta} z(s)} e^{\eta_1 \omega} < 0$$

since we have shown that $\hat{\sigma}_1(0) - \hat{c}_1(1) < 0$. Thus $v_\omega(w, s) > v_\omega(w, 1) > 0$ for all $0 \le s < 1$ and $\omega \ge \hat{\omega}$.

In region 2 we have, from Alvarez and Shimer (2011),

$$v_\omega(\omega, 1) = \sum_{i=1} \hat{c}_i(1)\, \eta_i e^{\eta_i \omega} > 0$$

and as in region 1:

$$v_{\omega s}(\omega, s) = \frac{1}{1-s}\eta_1 \hat{\sigma}_1'(z(s))\, e^{\eta_1 \omega} = \frac{1}{1-s}(\eta_1)^2 [\sigma_1(0) - \hat{c}_1(1)]\, e^{\frac{\eta_1}{\theta} z(s)} e^{\eta_1 \omega} < 0$$

so $v_\omega(w, s) > v_\omega(w, 1) > 0$ for $\omega \le \hat{\omega}$ and $z \ge \theta(\hat{\omega} - \underline{\omega})$.

In region 3, $\omega \le \hat{\omega}$, and $z \le \theta(\hat{\omega} - \underline{\omega})$ we have

$$v_\omega(\underline{\omega}, s) = \theta(1-s)\, v_s(\underline{\omega}, s) > 0$$

for all $s > 0$, where the equality follows by construction, and the strict inequality has been shown above.

We have shown above that $\underline{\sigma}_i(0) = [\hat{\sigma}_i(0) + h_i(0)] > 0$ for $i = 1, 2$. By construction we have $v_\omega(\underline{\omega}, 0) = 0$, and

$$v_{\omega\omega}(\omega, 0) = \sum_{i=1,2} \underline{\sigma}_i(0)\ (\eta_i)^2\ e^{\eta_i \omega}\ > 0$$

Thus for all $\underline{\omega} < \omega \le \hat{\omega}$,

$$v_\omega(\omega, 0) > v_\omega(\underline{\omega}, 0) = 0$$

Also we have

$$\begin{aligned}
& (1-s)\, v_{\omega s}(\omega, s) \\
= \ & \eta_1 [\hat{\sigma}(z(s)) + h_1'(z(s))]\ e^{\eta_1 \omega} + \eta_2 h_2'(z(s))\ e^{\eta_2 \omega} \\
= \ & \frac{(\eta_1)^2}{\theta} e^{\frac{\eta_1}{\theta} z(s)} [\hat{\sigma}_1(0) - c_1(0) + h_1(0)]\ e^{\eta_1 \omega} \\
& + \frac{(\eta_2)^2}{\theta} h_2(0)\, e^{\frac{\eta_2}{\theta} z(s)}\ e^{\eta_2 \omega} \\
> \ & 0
\end{aligned}$$

and thus

$$v_\omega(\omega, s) > v_\omega(\omega, 0) > 0$$

in the interior of region 3.

□

A.6 Existence of Equilibrium for the $\lambda = b_r = 0$ case

We let $v(\omega, s; \underline{\omega}, \bar{\omega}, \hat{\omega})$ be the value function of an agent with state (ω, s) when the economy is characterized by thresholds $(\underline{\omega}, \bar{\omega})$ and minimum wage $\hat{\omega}$. The next proposition characterize the lower bound $\underline{\omega}$ as a function of $\hat{\omega}, \bar{\omega}$ and $\underline{v}$ so that agents with no seniority are indifferent from leaving the industry and becomes searchers again.

Proposition 6. Equilibrium lower bound. Assume that $\lambda = b_r = 0$. Fix any $\hat{\omega}, \bar{\omega}$ and $\underline{v}$ satisfying: $\hat{\omega} \leq \bar{\omega}$ and $0 \leq \underline{v} \leq {}_*\underline{v}$ where ${}_*\underline{v} = v(\hat{\omega}, 0; \hat{\omega}, \bar{\omega}, \hat{\omega})$. Then there is a unique $\underline{\omega} = \underline{\Gamma}(\underline{v}; \bar{\omega}, \hat{\omega}) \leq \hat{\omega}$ that solves

$$v(\underline{\omega}, 0; \underline{\omega}, \bar{\omega}, \hat{\omega}) = \underline{v},$$

where v is given in Proposition 5. The function $\underline{\Gamma}$ is increasing in $\underline{v}$ and decreasing in $\bar{\omega}$. If $\bar{\omega} = \hat{\omega}$ then

$$\underline{\Gamma}(\underline{v}; \hat{\omega}, \hat{\omega}) \to (1/\eta_2) \log \left(\frac{\underline{v}}{h_2(0) \left[1 - \frac{\eta_2}{\eta_1}\right]} \right) = \hat{\omega}(1 - \eta_2) + \frac{1}{\eta_2} \log(\rho\, \underline{v})$$

If $\bar{\omega}$ becomes arbitrarily large

$$\lim_{\bar{\omega} \to \infty} \underline{\Gamma}(\underline{v}; \bar{\omega}, \hat{\omega}) \to (1/\eta_2) \log \left(\frac{\underline{v}}{[h_2(0) - a_2] \left[1 - \frac{\eta_2}{\eta_1}\right]} \right)$$

which is finite, and where $h_2(0)$ and a_2 are given in Proposition 5.

Proof of Proposition 6. To conserve notation let ξ defined as

$$v(\underline{\omega}, 0; \underline{\omega}, \bar{\omega}, \hat{\omega}) \equiv \xi(\underline{\omega}) = [\hat{c}_2(1) + h_2(0)]\, e^{\eta_2 \underline{\omega}} \left[1 - \frac{\eta_2}{\eta_1}\right]$$

where we notice that $\hat{c}_2(1)$ depends on $\underline{\omega}$. We want to show that

$$\begin{aligned}\frac{\partial}{\partial \underline{\omega}}\xi(\underline{\omega}) &= e^{\eta_2 \underline{\omega}}\left[1-\frac{\eta_2}{\eta_1}\right]\left(\left[\hat{c}_2(1)+h_2(0)\right]\eta_2+\frac{\partial}{\partial \underline{\omega}}\hat{c}_2(1)\right)\\ &> e^{\eta_2 \underline{\omega}}\left[1-\frac{\eta_2}{\eta_1}\right]\left[\hat{c}_2(1)+h_2(0)\right]\eta_2>0\end{aligned}$$

We have established that $\hat{c}_2(1)+h_2(0)>0$ and since $\eta_1<0<\eta_2$ it will suffice to show that $\hat{c}_2(1)$ is an increasing function of $\underline{\omega}$. That $\hat{c}_2$ is increasing in $\underline{\omega}$, and that $\hat{c}_2(1)+h_2(0)>0$, implies that as $\underline{\omega}\to-\infty$, then $\xi\to 0$. The vector $\hat{c}(1;\underline{\omega})$ solves:

$$\begin{aligned}\sum_{i=1,2}\hat{c}_i(1)\,e^{\eta_i\bar{\omega}}\eta_i &= -g\\ \sum_{i=1,2}\hat{c}_i(1)\,e^{\eta_i\underline{\omega}}\eta_i &= 0\end{aligned}$$

where g is not a function of $\underline{\omega}$. This system gives

$$\hat{c}_2(1)=\frac{e^{-\eta_1\bar{\omega}}g}{\eta_2\left[e^{(\eta_2-\eta_1)\underline{\omega}}-e^{(\eta_2-\eta_1)\bar{\omega}}\right]}.$$

If $\bar{\omega}=\hat{\omega}$, then $g=0$ and $\hat{c}_2(1)=0$ for all $\underline{\omega}$. This follows because in this case g is given by:

$$g\equiv\frac{e^{\hat{\omega}}}{\rho-\mu-\sigma^2/2}+\sum_{i=1,2}a_i e^{\eta_i\hat{\omega}}\eta_i$$

and a_i satisfies

$$\frac{e^{\hat{\omega}}}{\rho-\mu-\sigma^2/2}+\sum_{i=1,2}a_i\ e^{\eta_i\ \hat{\omega}}\eta_i=0.$$

Moreover, we have that $h(0)$ solves:

$$\begin{aligned}\sum_{i=1,2}h_i(0)\,e^{\eta_i\ \hat{\omega}} &= \frac{e^{\hat{\omega}}}{\rho}\\ \sum_{i=1,2}h_i(0)\,e^{\eta_i\ \hat{\omega}}\eta_i &= 0\end{aligned}$$

or

$$\begin{aligned}e^{\eta_1\hat{\omega}}h_1(0) &= \frac{e^{\hat{\omega}}}{\rho}-e^{\eta_2\hat{\omega}}h_2(0)\\ e^{\eta_1\hat{\omega}}h_1(0) &= -e^{\eta_2\hat{\omega}}h_2(0)\,\frac{\eta_2}{\eta_1}\end{aligned}$$

thus

$$\begin{aligned} h_2(0) &= \frac{e^{\hat{\omega}}/\rho}{e^{\eta_2\hat{\omega}}\left(1-\frac{\eta_2}{\eta_1}\right)} > 0, \\ h_1(0) &= -\frac{\eta_2}{\eta_1}\frac{e^{\hat{\omega}}/\rho}{e^{\eta_1\hat{\omega}}\left(1-\frac{\eta_2}{\eta_1}\right)} > 0 \end{aligned}$$

Replacing $h_2(0)$ into the expression for $\underline{\omega}$:

$$\underline{\omega} = \frac{1}{\eta_2}\log\left(\frac{\underline{v}}{h_2(0)\left[1-\frac{\eta_2}{\eta_1}\right]}\right) = \hat{\omega}(1-\eta_2) + \frac{1}{\eta_2}\log\left(\rho\ \underline{v}\right)$$

When $\bar{\omega} \to \infty$ we have

$$\hat{c}_2(1) = \frac{e^{-\eta_2\bar{\omega}}g}{\eta_2\left[e^{(\eta_2-\eta_1)(\underline{\omega}-\bar{\omega})}-1\right]}$$

$$\begin{aligned} e^{-\eta_2\bar{\omega}}g &= \frac{e^{(1-\eta_2)\bar{\omega}}}{\rho-\mu-\sigma^2/2} + \sum_{i=1,2} a_i e^{(\eta_i-\eta_2)\bar{\omega}}\eta_i \\ &= \frac{e^{(1-\eta_2)\bar{\omega}}}{\rho-\mu-\sigma^2/2} + a_1 e^{(\eta_1-\eta_2)\bar{\omega}}\eta_1 + a_2\eta_2 \end{aligned}$$

and

$$\begin{aligned} \eta_2 &= \frac{-\mu+\sqrt{\mu^2+2\sigma^2\rho}}{\sigma^2} > \frac{-\mu+\sqrt{\mu^2+2\sigma^2\left(\mu+\sigma^2/2\right)}}{\sigma^2} \\ &= \frac{-\mu+\sqrt{\mu^2+2\sigma^2\mu+\sigma^2}}{\sigma^2} = \frac{-\mu+\mu+\sigma^2}{\sigma^2} = 1 \end{aligned}$$

so that

$$1-\eta_2 < 0$$

and thus

$$\lim_{\bar{\omega}\to\infty}\hat{c}_2(1) = \frac{\lim_{\omega\to\infty} e^{-\eta_2\bar{\omega}}g}{-\eta_2} = \frac{\eta_2 a_2}{-\eta_2} = -a_2$$

Also $h_2(0) - a_2 > 0$ since by defintion $h(0) - a$ satisfy

$$[h(0)-a] = M^{-1}\ e^{\hat{\omega}}\begin{bmatrix} \frac{1}{\rho-\mu-\sigma^2/2} \\ \frac{1}{\rho-\mu-\sigma^2/2} \end{bmatrix}$$

and it is easy to verify that $h_2(0) - a_2 < 0$ leads to a contradiction (from the first equation if

$h_2(0) - a_2(0) < 0$, then it must be that $h_1(0) - a_1 > 0$. But then arrive to a contradiction from the second equation, since $\eta_1 < 0 < \eta_2$).

$$\frac{\partial}{\partial \underline{\omega}} \hat{c}_2(1) = \hat{c}_2(1) \frac{(\eta_2 - \eta_1)}{\left[e^{(\eta_2-\eta_1)\underline{\omega}} - e^{(\eta_2-\eta_1)\bar{\omega}}\right]} \geq 0$$

Thus,

$$\frac{\partial}{\partial \underline{\omega}} \xi(\underline{\omega}) = e^{\eta_2 \underline{\omega}} \left[1 - \frac{\eta_2}{\eta_1}\right] \left([\hat{c}_2(1) + h_2(0)] \eta_2 + \hat{c}_2(1; \underline{\omega}) \frac{(\eta_2 - \eta_1)}{\left[e^{(\eta_2-\eta_1)\bar{\omega}} - e^{(\eta_2-\eta_1)\underline{\omega}}\right]} \right) > 0$$

Notice that

$$\begin{aligned} \lim_{\underline{\omega} \to \infty} \xi(\underline{\omega}) &= 0 \\ \xi(\hat{\omega}) &= \underline{{}_* v}(\hat{\omega}, \bar{\omega}) \end{aligned}$$

To see why notice that when $\hat{\omega} = \underline{\omega}$ the minimum wage does not bind and indeed we have:

$$v(\omega, s; \hat{\omega}, \bar{\omega}, \hat{\omega}) = v(\omega, 0; \hat{\omega}, \bar{\omega}, \hat{\omega})$$

for all s. To see why notice that

$$\begin{aligned} v(\hat{\omega}, 1; \hat{\omega}, \bar{\omega}, \hat{\omega}) &= \frac{e^{\hat{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} [\hat{c}_i(1) + a_i] e^{\eta_i \hat{\omega}} \\ v(\hat{\omega}, 0; \hat{\omega}, \bar{\omega}, \hat{\omega}) &= \frac{e^{\hat{\omega}}}{\rho} + \sum_{i=1,2} [\hat{c}_i(1) + h_i(0)] e^{\eta_i \hat{\omega}} \end{aligned}$$

We also have that $v(\omega, 1; \hat{\omega}, \bar{\omega}, \hat{\omega})$ can be written as

$$v(\omega, 1; \hat{\omega}, \bar{\omega}, \hat{\omega}) = \frac{e^{\omega}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} [\hat{c}_i(1) + a_i] e^{\eta_i \omega}$$

where $v_\omega(\bar{\omega}, 1; \hat{\omega}, \bar{\omega}, \hat{\omega}) = v_\omega(\hat{\omega}, 1; \hat{\omega}, \bar{\omega}, \hat{\omega}) = 0$, or $c_i(1) + a_i$ solve:

$$\begin{aligned} \sum_{i=1,2} [\hat{c}_i(1) + a_i] e^{\eta_i \bar{\omega}} \eta_i &= -\frac{e^{\bar{\omega}}}{\rho - \mu - \sigma^2/2} \\ \sum_{i=1,2} [\hat{c}_i(1) + a_i] e^{\eta_i \hat{\omega}} \eta_i &= -\frac{e^{\hat{\omega}}}{\rho - \mu - \sigma^2/2} \end{aligned}$$

Clearly these are implied by our system when $\hat{\omega} = \underline{\omega}$:

$$\begin{aligned}
\frac{e^{\hat{\omega}}}{\rho-\mu-\sigma^2/2} + \sum_{i=1,2} a_i \ e^{\eta_i \ \hat{\omega}} &= \frac{e^{\hat{\omega}}}{\rho} \\
\frac{e^{\hat{\omega}}}{\rho-\mu-\sigma^2/2} + \sum_{i=1,2} a_i \ \eta_i e^{\eta_i \ \hat{\omega}} &= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1,2} \hat{c}_i (1) e^{\eta_i \bar{\omega}} \eta_i &= -\frac{e^{\bar{\omega}}}{\rho-\mu-\sigma^2/2} - \sum_{i=1,2} a_i e^{\eta_i \bar{\omega}} \eta_i \\
\sum_{i=1,2} \hat{c}_i (1) e^{\eta_i \hat{\omega}} \eta_i &= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1,2} h_i (0) e^{\eta_i \ \hat{\omega}} &= \frac{e^{\hat{\omega}}}{\rho} \\
\sum_{i=1,2} h_i (0) \eta_i e^{\eta_i \ \hat{\omega}} &= 0
\end{aligned}$$

Given that ξ is continuous and ranges from 0 to $\underline{{}_*v} (\hat{\omega}, \bar{\omega})$ the existence of $\underline{\Gamma}$ follows from the intermediate value theorem, and its uniqueness from the fact that ξ is strictly increasing.

Finally we show $\underline{\Gamma}$ is decreasing in $\bar{\omega}$. Recognizing the dependence of $\hat{c}_i (1)$ on $\bar{\omega}$, we first show that $\hat{c}_2 (1)$ is increasing in $\bar{\omega}$. We do so using some results from our previous papers. We have that for $s = 1$ and $\omega \geq \hat{\omega}$

$$v (\omega, 1; \underline{\omega}, \bar{\omega}, \hat{\omega}) = \frac{e^{\omega}}{\rho-\mu-\sigma^2/2} + \sum_{i=1,2} [\hat{c}_i (1) + a_i] e^{\eta_i \omega}$$

and for $\omega < \hat{\omega}$:

$$v (\omega, 1; \underline{\omega}, \bar{\omega}, \hat{\omega}) = \frac{e^{\hat{\omega}}}{\rho} + \sum_{i=1,2} \hat{c}_i (1) e^{\eta_i \omega}$$

We have that $v_\omega (\bar{\omega}, 1; \underline{\omega}, \bar{\omega}, \hat{\omega}) = v_\omega (\underline{\omega}, 1; \underline{\omega}, \bar{\omega}, \hat{\omega}) = 0$. Then

$$v_{\bar{\omega}} (\bar{\omega}, 1; \underline{\omega}, \bar{\omega}, \hat{\omega}) = \sum_{i=1,2} \frac{\partial}{\partial \bar{\omega}} \hat{c}_i (1) e^{\eta_i \bar{\omega}} > 0$$

and likewise:

$$v_{\underline{\omega}} (\underline{\omega}, 1; \underline{\omega}, \bar{\omega}, \hat{\omega}) = \sum_{i=1,2} \frac{\partial}{\partial \underline{\omega}} \hat{c}_i (1) e^{\eta_i \underline{\omega}} > 0$$

where the inequalities follow using the result in our previous paper. This implies that at least one of the derivatives $\frac{\partial}{\partial \underline{\omega}}\hat{c}_i(1)$ must be positive. Suppose, by way of contradiction, that $\frac{\partial}{\partial \underline{\omega}}\hat{c}_2(1) < 0$ and thus $\frac{\partial}{\partial \underline{\omega}}\hat{c}_1(1) > 0$. From the formula in our previous paper we also have that

$$v_{\bar{\omega}}(\bar{\omega}, 1; \underline{\omega}, \bar{\omega}, \hat{\omega}) > v_{\underline{\omega}}(\underline{\omega}, 1; \underline{\omega}, \bar{\omega}, \hat{\omega})$$

Thus:

$$\frac{\partial}{\partial \bar{\omega}}\hat{c}_1(1)\left[e^{\eta_1 \bar{\omega}} - e^{\eta_1 \underline{\omega}}\right] + \frac{\partial}{\partial \bar{\omega}}\hat{c}_2(1)\left[e^{\eta_2 \bar{\omega}} - e^{\eta_2 \underline{\omega}}\right] > 0$$

a contradiction since $\eta_1 < 0 < \eta_2$ and $\bar{\omega} > \underline{\omega}$.

Thus $\hat{c}_2(1)$ is increasing in $\bar{\omega}$, $\xi(\underline{\omega})$ shifts up with as $\bar{\omega}$ increases, and hence $\underline{\Gamma}(\underline{v}; \bar{\omega}, \hat{\omega})$ is decreasing in $\bar{\omega}$.

□

The next proposition develop some critical values for the minimum wage $\hat{\omega}$ given $\underline{v}$ and $\bar{v}$ so that the equilibrium will have Regions 1, 2 and 3, i.e. so that the minimum wage will be "moderate".

PROPOSITION 7. Assume that $\lambda = b_r = 0$. Fix the values of inaction and search satisfying $0 < \underline{v} < \bar{v}$. (i) Let $\hat{\omega}_*$ be the value of $\hat{\omega}$ such that there is a $\underline{\omega} = \underline{\omega}_*$ such that

$$\begin{aligned} v(\underline{\omega}_*, 0; \underline{\omega}_*, \hat{\omega}_*, \hat{\omega}_*) &= \underline{v}, \\ v(\hat{\omega}_*, 0; \underline{\omega}_*, \hat{\omega}_*, \hat{\omega}_*) &= \bar{v}. \end{aligned}$$

Hence at $\hat{\omega} = \hat{\omega}_*$ the minimum wage is so high that region 1 is about to disappear, i.e. $\bar{\omega} = \hat{\omega}_*$. There exists a unique value of $\hat{\omega}_*$ that satisfy these equations and is given by a decreasing function $\hat{\gamma}_*$:

$$\hat{\omega}_* = \log(\underline{v}) + \hat{\gamma}_*\left(\frac{\bar{v}}{\underline{v}}\right).$$

(ii) Let ${}_*\hat{\omega}$ be the value of $\hat{\omega}$ such that there is $\bar{\omega} = {}_*\bar{\omega}$ for which

$$\begin{aligned} v({}_*\hat{\omega}, 0; {}_*\hat{\omega}, {}_*\hat{\omega}, {}_*\bar{\omega}) &= \underline{v}, \\ v({}_*\bar{\omega}, 0; {}_*\hat{\omega}, {}_*\hat{\omega}, {}_*\bar{\omega}) &= \bar{v}. \end{aligned}$$

Hence at $\hat{\omega} = {}_*\hat{\omega}$ the minimum wage is just high enough so that it binds, i.e. $\hat{\omega} = \underline{\omega} = {}_*\underline{\omega}$. (iii) The threshold values satisfy ${}_*\hat{\omega} < \hat{\omega}_*$.

Proof of Proposition 7 (i) When $\bar{\omega} = \hat{\omega}$ we have that $\hat{\omega}_*$ must solve

$$\begin{aligned}\bar{v} &= v\left(\hat{\omega}_*, 0; \underline{\omega}_*, \hat{\omega}_*, \hat{\omega}_*\right) = \frac{e^{\hat{\omega}}}{\rho} + \hat{c}_2\left(1\right) e^{\eta_2 \hat{\omega}}\left[1 - e^{(\eta_1 - \eta_2)(\hat{\omega} - \underline{\omega})}\frac{\eta_2}{\eta_1}\right] \\ &\quad - \frac{e^{\eta_1(\hat{\omega} - \underline{\omega})}}{\eta_1}\left[\sum_{i=1,2} h_i\left(0; \hat{\omega}\right) e^{\eta_i \underline{\omega}} \eta_i\right]\end{aligned}$$

since

$$\frac{e^{\hat{\omega}}}{\rho - \mu - \sigma^2/2} + \sum_{i=1,2} a_i \, e^{\eta_i \ \hat{\omega}} = \frac{e^{\hat{\omega}}}{\rho}.$$

Moreover, with $\hat{\omega} = \bar{\omega}$ we have $\hat{c}_2\left(1\right) = \hat{c}_1\left(1\right) = 0$, since $g\left(\hat{\omega}, \hat{\omega}\right) = 0$, and thus:

$$\begin{aligned}\bar{v} &= \frac{e^{\hat{\omega}}}{\rho} - \frac{e^{\eta_1(\hat{\omega} - \underline{\omega})}}{\eta_1}\left[\sum_{i=1,2} h_i\left(0; \hat{\omega}\right) e^{\eta_i \underline{\omega}} \eta_i\right] \\ &= \frac{e^{\hat{\omega}}}{\rho} - e^{\eta_1 \hat{\omega}}\left[h_1\left(0\right) + h_2\left(0\right) e^{(\eta_2 - \eta_1)\underline{\omega}}\frac{\eta_2}{\eta_1}\right] \\ &= \frac{e^{\hat{\omega}}}{\rho} - e^{\eta_1 \hat{\omega}}\left[h_1\left(0\right) + e^{(\eta_2 - \eta_1)\underline{\omega}} h_2\left(0\right)\frac{\eta_2}{\eta_1}\right]\end{aligned}$$

Since $h\left(0\right)$ solves:

$$\begin{aligned}\sum_{i=1,2} h_i\left(0\right) e^{\eta_i \ \hat{\omega}} &= \frac{e^{\hat{\omega}}}{\rho} \\ \sum_{i=1,2} h_i\left(0\right) e^{\eta_i \ \hat{\omega}} \eta_i &= 0\end{aligned}$$

we have

$$\begin{aligned}h_2\left(0\right) &= \frac{e^{\hat{\omega}}/\rho}{e^{\eta_2 \hat{\omega}}\left(1 - \frac{\eta_2}{\eta_1}\right)}, \\ h_1\left(0\right) &= -\frac{\eta_2}{\eta_1}\frac{e^{\hat{\omega}}/\rho}{e^{\eta_1 \hat{\omega}}\left(1 - \frac{\eta_2}{\eta_1}\right)}\end{aligned}$$

Thus replacing into the expression for $\bar{v}$:

$$\begin{aligned}
\bar{v} &= \frac{e^{\hat{\omega}}}{\rho} - e^{\eta_1 \hat{\omega}} \left[h_1(0) + e^{(\eta_2 - \eta_1)\underline{\omega}} h_2(0) \frac{\eta_2}{\eta_1} \right] \\
&= \frac{e^{\hat{\omega}}}{\rho} - \frac{\frac{\eta_2}{\eta_1}}{\left(1 - \frac{\eta_2}{\eta_1}\right)} \frac{e^{\hat{\omega}}}{\rho} e^{\eta_1 \hat{\omega}} \left[-\frac{1}{e^{\eta_1 \hat{\omega}}} + e^{(\eta_2 - \eta_1)\underline{\omega}} \frac{1}{e^{\eta_2 \hat{\omega}}} \right] \\
&= \frac{e^{\hat{\omega}}}{\rho} \left\{ 1 - \frac{\frac{\eta_2}{\eta_1}}{\left(1 - \frac{\eta_2}{\eta_1}\right)} \left[e^{(\eta_1 - \eta_2)(\hat{\omega} - \underline{\omega})} - 1 \right] \right\}
\end{aligned}$$

Using the expression for $\underline{\omega} = \underline{\Gamma}(v, \hat{\omega}, \hat{\omega})$ from our previous Proposition $\underline{v} = \frac{e^{\hat{\omega}}}{\rho} e^{\eta_2 \ (\underline{\omega} - \hat{\omega})}$ or

$$\frac{\rho \underline{v}}{e^{\hat{\omega}}} = e^{\eta_2 \ (\underline{\omega} - \hat{\omega})}$$

or

$$\left(\frac{e^{\hat{\omega}}}{\rho \ \underline{v}} \right)^{\frac{\eta_1 - \eta_2}{\eta_2}} = e^{(\eta_1 - \eta_2)(\hat{\omega} - \underline{\omega})}$$

we obtain

$$\begin{aligned}
\bar{v} &= \frac{e^{\hat{\omega}}}{\rho} \left\{ 1 - \frac{\frac{\eta_2}{\eta_1}}{\left(1 - \frac{\eta_2}{\eta_1}\right)} \left[e^{(\eta_1 - \eta_2)(\hat{\omega} - \underline{\omega})} - 1 \right] \right\} \\
&= \frac{e^{\hat{\omega}}}{\rho} \left\{ 1 - \frac{\frac{\eta_2}{\eta_1}}{\left(1 - \frac{\eta_2}{\eta_1}\right)} \left[\left(\frac{e^{\hat{\omega}}}{\rho \ \underline{v}} \right)^{\frac{\eta_1 - \eta_2}{\eta_2}} - 1 \right] \right\}
\end{aligned}$$

or

$$\left(\frac{\rho \bar{v}}{e^{\hat{\omega}}} \right) = 1 - \frac{\frac{\eta_2}{\eta_1}}{\left(1 - \frac{\eta_2}{\eta_1}\right)} \left[\left(\frac{e^{\hat{\omega}}}{\rho \underline{v}} \right)^{\frac{\eta_1 - \eta_2}{\eta_2}} - 1 \right]$$

or

$$\left(\frac{\rho \bar{v}}{e^{\hat{\omega}}} \right) = 1 + \left(\frac{1}{1 - \frac{\eta_1}{\eta_2}} \right) \left[\left(\frac{\rho \underline{v}}{e^{\hat{\omega}}} \right)^{\left(1 - \frac{\eta_1}{\eta_2}\right)} - 1 \right]$$

Letting $x \equiv \exp(-\hat{\omega})$

$$x \ (\rho \bar{v}) = 1 + \left(\frac{1}{1 - \frac{\eta_1}{\eta_2}} \right) \left[(x \ (\rho \underline{v}))^{\left(1 - \frac{\eta_1}{\eta_2}\right)} - 1 \right]$$

The RHS and LHS are increasing functions of $x \equiv \exp(-\hat{\omega})$. The RHS is strictly convex, and for $x = 0$ the LHS starts at zero, but the RHS starts at a positive value. We are looking for the solution where $x\rho\bar{v} < 1$, since $\bar{v} < \exp(\hat{\omega})/\rho$. In this range there exists a unique solution. The unique solution in this range, $x(\bar{v}, \underline{v})$ is decreasing in $\bar{v}$ and homogenous of degree -1 in $x(\bar{v}, \underline{v})$. Thus $\exp(\hat{\omega})$ is increasing in $\bar{v}$ and homogenous of degree one in $(\underline{v}, \hat{v})$. We let its solution be:

$$\hat{\omega}_* = \log(\underline{v}) + \hat{\gamma}_* \left(\frac{\bar{v}}{\underline{v}}\right)$$

with $\hat{\gamma}_*$ increasing.

(ii) The existence and uniqueness of ${}_*\hat{\omega}$ follows from the results in Proposition 1 in Alvarez and Shimer (2011), where we show the existence and uniqueness of an equilibrium without minimum wages where wages equal full employment wages which vary in an interval $[\underline{\omega}, \bar{\omega}]$ and where seniority plays no role. Setting $\hat{\omega} = \underline{\omega}$ gives the required value.

(iii) Consider the case at $s = 1$ when $\hat{\omega}_* = \bar{\omega} > \underline{\omega}$

$$v(\bar{\omega}, 1) = \frac{e^{\hat{\omega}_*}}{\rho} > v(\hat{\omega}, 0) = \bar{v} > \underline{v}$$

Consider the case where $\bar{\omega} > \underline{\omega} = \hat{\omega}_*$

$$v(\underline{\omega}, 1) = \underline{v} > \frac{e^{\underline{\omega}}}{\rho} = \frac{e^{*\hat{\omega}}}{\rho}$$

Then

$$\frac{e^{\hat{\omega}_*}}{\rho} > \frac{e^{*\hat{\omega}}}{\rho}.$$

□

We are now ready to prove existence of an equilibrium.

Proof of Theorem 1 . Define the function:

$$v(\bar{\omega}, 0; \underline{\Gamma}(\underline{v}; \bar{\omega}, \hat{\omega}), \bar{\omega}, \hat{\omega}) \equiv \psi(\bar{\omega}).$$

We will show that:

i) This is a continuous function.

ii) We argue that for large enough $\bar{\omega}$, we have $\psi(\bar{\omega}) > \bar{v}$.

iii) Finally we argue that at $\bar{\omega} = \hat{\omega}$, we have $\psi(\bar{\omega}) < \bar{v}$.

i) follows from analyzing the function $v(\bar{\omega}, 1)$ defined in Proposition 5

For ii) we can write for $\omega > \hat{\omega}$

$$\begin{aligned} v\left(\omega,0;\underline{\omega},\bar{\omega},\hat{\omega}\right) &= \frac{e^{\omega}}{\rho-\mu-\sigma^2/2}+\sum_{i=1,2}\left[\hat{\sigma}_i\left(0;\underline{\omega},\bar{\omega},\hat{\omega}\right)+a_i\left(\hat{\omega}\right)\right]e^{\eta_i\omega} \\ &= \frac{e^{\omega}}{\rho-\mu-\sigma^2/2}+\left[\hat{c}_2\left(1;\underline{\omega},\bar{\omega},\hat{\omega}\right)+a_2\left(\hat{\omega}\right)\right]e^{\eta_2\omega} \\ & +\hat{\sigma}_1\left(0;\underline{\omega},\bar{\omega},\hat{\omega}\right)e^{\eta_1\omega}+a_1\left(\hat{\omega}\right)e^{\eta_1\omega} \end{aligned}$$

where we include all the arguments of the functions. Using the expression for $\hat{\sigma}_1\left(0;\underline{\omega},\bar{\omega},\hat{\omega}\right)$:

$$\hat{\sigma}_1\left(0;\underline{\omega},\bar{\omega},\hat{\omega}\right)=\frac{-e^{\eta_2\underline{\omega}}\eta_2\ \hat{c}_2\left(1;\underline{\omega},\bar{\omega},\hat{\omega}\right)-\sum_{i=1,2}h_i\left(0,\hat{\omega}\right)e^{\eta_i\underline{\omega}}\eta_i}{e^{\eta_1\underline{\omega}}\eta_1}$$

then

$$\begin{aligned} v\left(\omega,0;\underline{\omega},\bar{\omega},\hat{\omega}\right) &= \frac{e^{\omega}}{\rho-\mu-\sigma^2/2}+\left[\hat{c}_2\left(1;\underline{\omega},\bar{\omega},\hat{\omega}\right)+a_2\left(\hat{\omega}\right)\right]e^{\eta_2\omega} \\ & -e^{\eta_1\omega}\ e^{(\eta_2-\eta_1)\underline{\omega}}\frac{\eta_2}{\eta_1}\ \hat{c}_2\left(1;\underline{\omega},\bar{\omega},\hat{\omega}\right) \\ & -\frac{\sum_{i=1,2}h_i\left(0;\hat{\omega}\right)e^{\eta_i\underline{\omega}}\eta_i}{e^{\eta_1\underline{\omega}}\eta_1}e^{\eta_1\omega}+a_1\left(\hat{\omega}\right)e^{\eta_1\omega} \end{aligned}$$

Recall that our previous proposition establishes that as $\bar{\omega}\to\infty$ $(1/\eta_2)\log\left(\frac{\underline{v}}{\left[h_2(0)-a_2\right]\left[1-\frac{\eta_2}{\eta_1}\right]}\right)$

$$\underline{\omega}_{\infty}\equiv\lim_{\bar{\omega}\to\infty}\underline{\Gamma}\left(\underline{v};\bar{\omega},\hat{\omega}\right)=(1/\eta_2)\log\left(\frac{\underline{v}}{\left[h_2\left(0\right)-a_2\right]\left[1-\frac{\eta_2}{\eta_1}\right]}\right)$$

and recall that we have shown that $h_2\left(0\right)-a_2>0$, and depends only on $\hat{\omega}$. Notice that

$$\begin{aligned} \lim_{\bar{\omega}\to\infty}\left[\hat{c}_2\left(1;\underline{\omega}_{\infty},\bar{\omega},\hat{\omega}\right)+a_2\left(\hat{\omega}\right)\right] &= 0, \\ \lim_{\bar{\omega}\to\infty}a_1\left(\hat{\omega}\right)e^{\eta_1\bar{\omega}} &= 0, \\ \lim_{\bar{\omega}\to\infty}\ \hat{c}_2\left(1;\underline{\omega}_{\infty},\bar{\omega},\hat{\omega}\right) &< \infty \end{aligned}$$

so that

$$\begin{aligned} & v\left(\omega,0;\underline{\omega},\infty,\hat{\omega}\right) \\ = & \frac{e^{\omega}}{\rho-\mu-\sigma^2/2} \\ & -e^{\eta_1\omega}\ \left[e^{(\eta_2-\eta_1)\underline{\omega}}\frac{\eta_2}{\eta_1}\ \hat{c}_2\left(1;\underline{\omega}_{\infty},\infty,\hat{\omega}\right)+\frac{\sum_{i=1,2}h_i\left(0;\hat{\omega}\right)e^{\eta_i\underline{\omega}}\eta_i}{e^{\eta_1\underline{\omega}}\eta_1}\right] \end{aligned}$$

Then

$$\lim_{\omega\to\infty} v\left(\omega, 0; \underline{\omega}, \infty, \hat{\omega}\right) = \lim_{\omega\to\infty} \frac{e^{\omega}}{\rho - \mu - \sigma^2/2} = \infty$$

since $\eta_1 < 0$.

iii) From our previous proposition we know that if $\hat{\omega}_* = \bar{\omega}$ we have

$$\hat{\omega}_* = \log\left(\underline{v}\right) + \hat{\gamma}_*\left(\frac{\bar{v}}{\underline{v}}\right)$$

for $\hat{\gamma}_*\left(\frac{\bar{v}}{\underline{v}}\right)$. Let $\bar{V}$ be the solution:

$$\hat{\omega} = \log\left(\underline{v}\right) + \hat{\gamma}_*\left(\frac{\bar{V}}{\underline{v}}\right)$$

Since we have, by assumption that $\hat{\omega} < \hat{\omega}_*$, and the function $\hat{\gamma}_*$ is increasing, then $\bar{V} < \bar{v}$. But by definition of $\hat{\gamma}_*$ we have

$$\bar{v} > \bar{V} = \psi\left(\hat{\omega}\right).$$

Given i)-iii) the existence of $\bar{\omega}$ follows from the intermediate value theorem.

□

A.7 Hazard Rates

Consider a Brownian motion with initial $\omega \in (\underline{\omega}, \hat{\omega})$. Let $\hat{G}(t;\cdot;\cdot)$ and $\underline{G}(t;\cdot;\cdot)$ denote the cumulative distribution function for the times until each of the barriers is hit, conditional on the initial value of ω:

$$\begin{aligned}
\hat{G}(t;\hat{\omega},\underline{\omega};\omega) &= \Pr\{t \leq T_{\hat{\omega}}, T_{\hat{\omega}} < T_{\underline{\omega}} \,|\omega(0) = \omega\} \\
\underline{G}(t;\hat{\omega},\underline{\omega};\omega) &= \Pr\{t \leq T_{\underline{\omega}}, T_{\underline{\omega}} < T_{\hat{\omega}} \,|\omega(0) = \omega\},
\end{aligned}$$

with associated densities $\hat{g}$ and $\underline{g}$. Kolkiewicz (2002, pp. 17–18) proves

$$\begin{aligned}
\hat{g}(t;\hat{\omega},\underline{\omega};\omega) &= \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left(\frac{\pi n(\omega-\underline{\omega})}{\hat{\omega}-\underline{\omega}}\right) e^{\frac{\mu(2(\hat{\omega}-\omega)-\mu t)}{2\sigma^2}-\frac{\pi^2 n^2\sigma^2 t}{2(\hat{\omega}-\underline{\omega})^2}} \\
\underline{g}(t;\hat{\omega},\underline{\omega};\omega) &= \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left(\frac{\pi n(\hat{\omega}-\omega)}{\hat{\omega}-\underline{\omega}}\right) e^{\frac{-\mu(2(\omega-\underline{\omega})-\mu t)}{2\sigma^2}-\frac{\pi^2 n^2\sigma^2 t}{2(\hat{\omega}-\underline{\omega})^2}}.
\end{aligned}$$

The hazard rate of the first hitting time, conditional on a rest unemployment spell starting at time 0, i.e conditional on $\omega = \hat{\omega}$, is

$$\hat{h}_r(t) \equiv \lim_{\omega \uparrow \hat{\omega}} \frac{\hat{g}(t; \hat{\omega}, \underline{\omega}, \omega)}{1 - \hat{G}(t; \hat{\omega}, \underline{\omega}, \omega) - \underline{G}(t; \hat{\omega}, \underline{\omega}, \omega)} \text{ and } \underline{h}_r(t) \equiv \lim_{\omega \uparrow \hat{\omega}} \frac{\underline{g}(t; \hat{\omega}, \underline{\omega}, \omega)}{1 - \hat{G}(t; \hat{\omega}, \underline{\omega}, \omega) - \underline{G}(t; \hat{\omega}, \underline{\omega}, \omega)}.$$

Equation (55) follows using L'Hopital's rule.

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