Financial innovation and the transactions demand for cash*

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Abstract

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1 Introduction

In this paper we extend the Baumol-Tobin inventory model for cash balances to a dynamic environment which allows for the possibility of withdrawing cash at random times at low or zero cost. We argue that this modification captures the developments in the withdrawal technology such as the increase in the number bank branches and the availability of ATM cards as well as the increase in the number of ATM terminals. The model captures this feature of the withdrawal technology with a single parameter p, the expected number of opportunities for a withdrawal at zero (or low) cost per unit of time. This random specifications captures the idea the the main cost of withdrawals is the opportunity cost of time, so that the events that occur with probability p represent a match of an agent with an ATM (or a bank desk) at a moment of low opportunity cost of the agent's time.

For positive values of p the proposed specification changes some of the predictions of the Baumol-Tobin model in ways that, we argue, are consistent with stylized facts for the cash management behavior of households. For instance, several studies report interest rate elasticities for money demand below one half, the value predicted by the Baumol-Tobin model. It is also widely documented that money holdings decrease as new payment technologies are introduced. In our model both the level, as well as the interest rate elasticity of the money demand are decreasing in the parameter p. Additionally we document other patterns that differ from the predictions of the Baumol-Tobin model, as well as from the Miller and Orr model, using a micro data for the cash-management of Italian households. One is that the interest rate elasticity of the average number of withdrawals is also smaller than 1/2. Another is that the ratio of withdrawal to average cash holdings tends to be below 2, the value predicted by the Baumol-Tobin model. Finally, households report to have substantial amount of cash at the time of withdrawals, as opposed to withdraw when they cash reaches zero, as predicted by the Baumol-Tobin model. We show that as p increases the theoretical model predict patterns of cash-management that are consistent with these observations.

In Section 2 we discuss some patterns of cash management behavior based on a panel data of Italian households. Section 3 analyzes some of the effects of financial diffusion using a deterministic steady-state model that allows a close comparison with the well known results of Baumol and Tobin. The core of this section is a simple model where, as opposed to the case in Baumol and Tobin, agents have a deterministic number of free withdrawals per period. We show that both the level of money demand and the interest rate elasticity decrease as the number of free withdrawals increases. Section 4 introduces our benchmark stochastic dynamic inventory model. In this model agents have random meetings with a financial intermediary in which they can withdraw money at no cost. This is a dynamic version of the model of Section 3. The implications of this model concerning the distribution of currency holdings, aggregate money demand, the average number of withdrawals, the average size of withdrawals, and the average cash balances at the time of a withdrawal are presented in Section 5. We show that, qualitatively, the model reproduces the features of the data that we highlight in Section 2. Section 6 generalizes the model of Section 4 to a more realistic set up, where there is also a small fixed cost at the time of a convenient random meeting. In section 7 we estimate the model using the panel data for Italian household. We discuss the identification of the parameters, the goodness of the fit of the model, and we use the estimated parameters to evaluate the benefits of owning an ATM card, as well as the impact of the technological changes in withdrawal technology.

In the paper we abstract from the intensive as well as extensive margin for the cash/credit decision. That is, we abstract from the decision of whether to have a credit card or not, and for those that have a credit card, whether a particular purchase is done using cash or credit. In particular, the model in our paper, as well as its empirical implementation, takes as given the expenditure that has to be financed using cash. The problem solve in the model is to minimize the cost of financing these purchases using cash. We are able to study this problem for Italian households because we have measures of the consumption purchases using cash. We view our paper as an input on the study cash/credit decision, an important topic that we plan to address in the future.

There is a large literature on both theoretical inventory models of money demand, as well as estimation of their key features. Here we briefly discuss two related models in the literature. These models provide a rationale for an interest rate elasticity smaller than 1/2, the value obtained in the Baumol and Tobin model. The explanation that we propose is complementary to the ones in those papers because it focuses on the level and interest rate elasticity of individual households demand for money.

Miller and Orr (1966) study the optimal inventory policy of cash for an agent subject to stochastic cash inflows and outflows, and obtain an interest rate elasticity of 1/3. Their model is more suitable for the problem faced by firms, given the nature of stochastic cash inflows and outflows. Instead, our paper focuses on a problem that better describes individual consumers problem, since we study the optimal inventory policy of cash for an agent that faces deterministic cash outflows (consumption expenditure) and no cash inflows. To be consistent with our model, when we analyze micro-level household data, we exclude households headed by self-employed.

Mulligan and Sala-i-Martin (2000) also study a model where the aggregate money demand can feature interest rate elasticity smaller than 1/2. In their model agents must pay a fixed cost to have a deposit account. Agents who face a low value for the total benefit of investing their wealth (either because wealth is low or because its return is low) will not pay the fixed cost and hence locally they will show a zero elasticity to changes in interest rates. Their model offers an explanation for a low interest elasticity of *aggregate* money demand. Instead, we concentrate on the interest rate elasticity of individual demands, by using micro-level household data and conditioning on the agents who do possess an interest bearing deposit account.

2 Cash Holdings Patterns of Italian Households

This section presents summary statistics of the cash holdings patters of Italian households from the *Survey of Household Income and Wealth.*[1] We focus on the surveys conducted from 1993 to 2004 because they include a section dedicated to the household cash management. Table 1 reports cross section mean and medians of some key money holdings statistics, normalized by daily cash expenditures.

Three statistics of Table 1 are at odds with the simplest versions of two classic money demand models: the one by Baumol and Tobin (BT henceforth) and the one by Miller and Orr (MO henceforth). First, households withdraw much before their cash balances reach zero, as they report that the holdings upon a withdrawal are about one third of their average cash balances. Second, the average ratio between the withdrawal amount and the currency holdings is smaller than 2, in some cases about 1. For comparison, this ratio is 2 in the BT model and 3/4 in the MO model. Third, the difference between the number of withdrawals and the corresponding number that is implied by the BT model, given by $c/W = c/2M$ is large.

The inconsistency between the Baumol-Tobin model and the Italian data are illustrated in Figure 1, which plots the theoretical prediction (black dashed line) versus the data both for households with ATM (the filled blue dots) and for house-

[1] This is a periodic survey of the Bank of Italy that collects information on several social and economic characteristics of household members, such as age, gender, education, employment, income, real and financial wealth, consumption and saving behavior. Each survey is conducted on a sample of about 8,000 households. Cash consumption is only available since 1993.

Table 1: Households' currency management

Variable		1993	1995	1998	2000	2002	2004
Average currency[a]							
Household w/o account	(*mean*)	17.1	20.0	20.0	21.8	26.8	25.5
	(*median*)	15	16	16.7	18.7	21.4	30
Household w. account							
w/o ATM	(*mean*)	15.4	17.2	19.3	17.7	16.9	17.8
	(*median*)	12.5	15	15	15	13.1	13.6
w. ATM	(*mean*)	10.4	11.2	12.9	12.4	12.9	13.6
	(*median*)	8.2	8	9	8.3	9	9.4
Average withdrawal[a]							
Household w/o ATM	(*mean*)	23.8	20.4	24.5	21.5	21.2	21.7
	(*median*)	18	15	17.5	15	17.1	17.1
Household w. ATM	(*mean*)	10.9	9.3	12.6	11.6	11.3	12.3
	(*median*)	8.5	7.9	9	8.4	8.9	10
Withdrawal to Currency Ratio							
Household w/o ATM	(*mean*)	2.3	1.7	1.9	2.0	2.0	1.9
	(*median*)	1.5	1.0	1	1.1	1.3	1.2
Household w. ATM	(*mean*)	1.5	1.2	1.3	1.4	1.3	1.4
	(*median*)	1	1	1	1	1	1
Cash at withdrawals[a,b]							
Household w/o ATM	(*mean*)	5.3	4.1	7.8	6.6	6.2	na
	(*median*)	3	2	3.7	3.3	3.7	na
Household w. ATM	(*mean*)	3.7	2.8	4.0	4.5	4.5	na
	(*median*)	2.2	1.7	2	2	2.5	na
Number of withdrawals[c]							
Household w/o ATM	(*mean*)	16.4	17.3	25.2	23.9	22.6	23.0
	(*median*)	6	12	12	12	12	12
Household w. ATM	(*mean*)	50.2	51.3	59.1	64.1	57.9	63.1
	(*median*)	38	48	48	48	48	48
Non durable consumption and services[c,d]							
Household w/o ATM	(mean)	12,187	13,345	12,750	13,561	14,420	15,924
Household w. ATM	(mean)	17,373	19,416	19,069	20,948	21,876	23,378
Share of cash expenditures[e]							
Household w/o ATM	(mean)	0.70	0.69	0.66	0.68	0.67	0.65
Household w. ATM	(mean)	0.66	0.64	0.61	0.57	0.55	0.50
N. of observations[f]		6,938	6,970	6,089	7,005	7,112	7,159

Entries are sample mean and medians computed using sample weights and excluding households with a self-employed head.
Notes: [a]Ratio to daily expenditures done in cash. - [b]Reported level of currency at the time of withdrawal. - [c]Per year. - [d]In euros, in year 2000 prices. - [e]Ratio of cash expenditure to consumption of nondurables and services. - [f]Number of respondents for whom the currency and the cash consumption data are available in each survey. Data on withdrawals are supplied by a smaller number of respondents. Source: Bank of Italy - *Survey of Household Income and Wealth.*

holds without ATM (empty red dots). Each dot represents the mean of the values for e.g. currency holdings and number of withdrawals (panel 1,1) observed for the households in a given province-year (the size of the dot is proportional to the number of observations).

Figure 1: Baumol-Tobin and the data

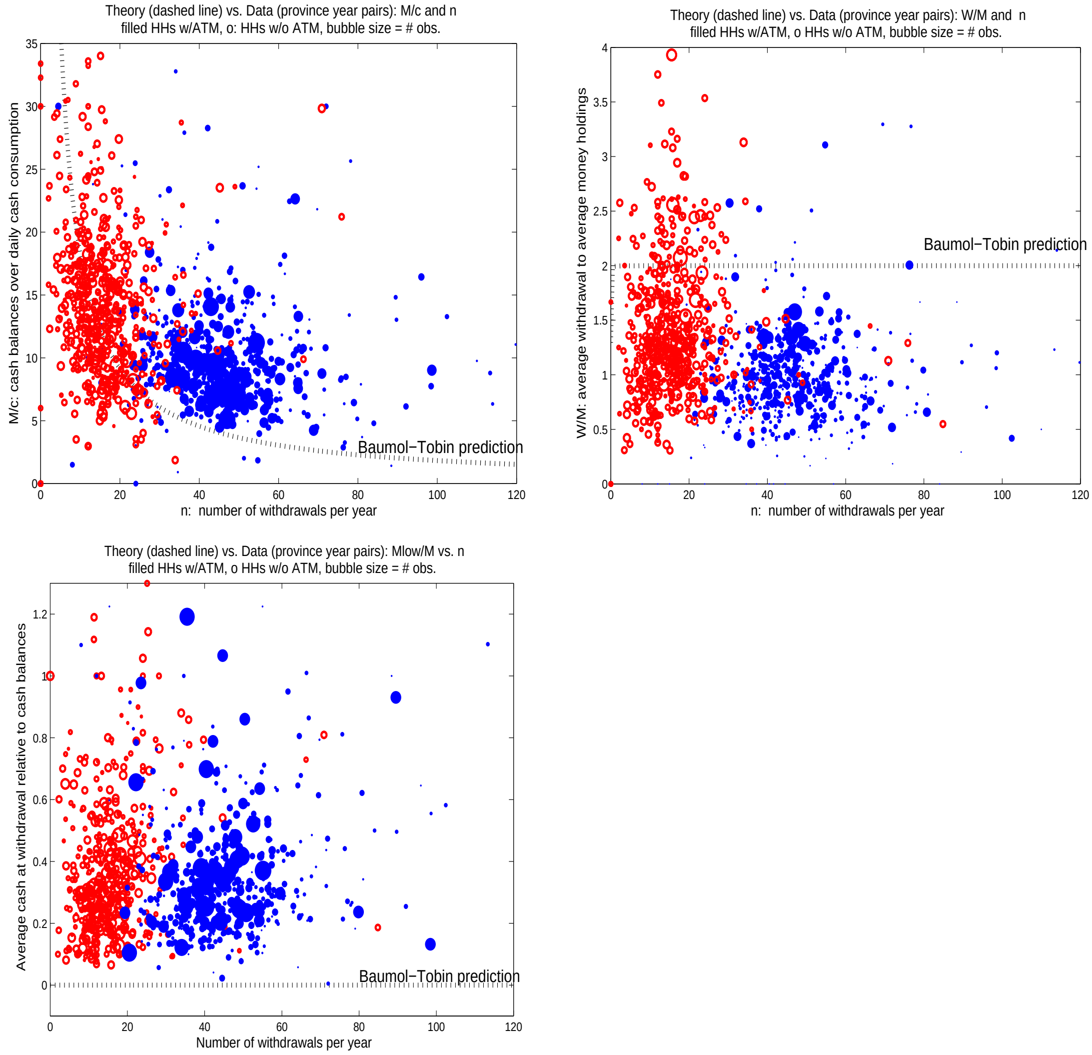


Table 2 reports summary statistics on the supply of bank services, such as the diffusion of bank branches and ATMs, and on the interest rate paid on deposits.[2]

[2]These data are drawn from the Supervisory Reports to the Bank of Italy and the Italian Central

Differences in nominal interest rates across provinces (witnessed by the standard deviations reported in parenthesis) are the result of segmentation in banking markets.[3] Until the early nineties commercial banks faced restrictions to open new bank branches in other provinces. A gradual process of liberalization has occurred since then, which has led to a sharp increase in the number of bank branches and a reduction of the interest rate differentials (see Casolaro, Gambacorta and Guiso (2006) for a review of the main developments in the banking industry during the past two decades).

Table 2: Financial development and interest rates

Variable	1993	1995	1998	2000	2002	2004
Bank branches^{a,b}	0.38	0.42	0.47	0.50	0.53	0.55
	(0.13)	(0.14)	(0.16)	(0.17)	(0.18)	(0.18)
ATM^{a,c}	0.31	0.39	0.50	0.57	0.65	0.65
	(0.18)	(0.19)	(0.22)	(0.22)	(0.23)	(0.22)
Interest rate^{c,d}	6.10	5.23	2.15	1.16	0.77	0.32
	(0.42)	(0.32)	(0.23)	(0.22)	(0.15)	(0.11)

Notes: Cross-section mean (standard deviation in parenthesis). ^{a} Per thousand residents. - ^{b} Elementary data available at the city / year level. - ^{c} Elementary data available at the province / year level. ^{d} Net nominal interest rates expressed in percentages (Source: Central credit register).

Table 3 presents least square regressions of the currency to consumption ratio for households with a deposit account and an ATM card. We think that the identification of the effects of interest rate on cash holdings is complicated by the fact that, as Table 2 shows, both variables display a time trend during the short time period covered by this data set. The regressions thus use year and province dummies in an

Credit Register. Elementary data on ATMs and interest rates are available at the province/year level (the sample covers about 100 provinces; the size of a province is broadly comparable to that of a U.S. county). Elementary data for bank branches are available at the city/year level (the sample covers about 400 cities).

[3]They do not reflect differences in the services or features of the underlying checking account (these statistics are built with the main objective of ensuring comparability and thus focus on a highly homogenous type of service).

attempt to remove unobserved time and regional effects affecting money demand, e.g. differences in the incidence of small crime, a factor which likely reduces currency holdings.

Table 3: The Demand for Currency and tech. development

	Bank account holders without ATM card		
Estimation method	*Ordinary Least Squares* (1)	*Instrumental Variables*[a] (2)	*Household Fixed Effects* (3)
log(cash expenditure)	0.486 (0.018)	0.487 (0.018)	0.393 (0.025)
log(interest rate)	-0.180 (0.092)	-0.236 (0.135)	-0.334 (0.088)
log(interest rate) · Number of bank branches in the city	0.109 (0.036)	0.103 (0.062)	0.098 (0.046)
Number of bank branches in the city	-0.113 (0.058)	-0.159 (0.127)	-0.399 (0.135)
Province dummies	Yes	Yes	No
Year dummies	Yes	Yes	Yes
R^2	0.23	0.23	0.08
Sample size	17,371	17,371	17,371

Note: Robust standard errors in parenthesis. [a]The instruments used for the deposit interest rate and the number of bank branches at the city level are the interest rate lagged value and the number of firms and employees per resident at the city level.

The estimates display a systematic negative correlation between the (log) level of currency holdings and the diffusion of bank branches. The correlation of cash holdings with the interest rate is smaller than one half. The point estimate is about nil for agents who hold an ATM card and around -0.1 for agents without ATM card. The interaction term between bank branches and the interest rate suggests that the interest rate elasticity is decreasing (in absolute value) in the number of bank branches. Thus these regressions imply that more financial development imply less response of currency holdings to interest rate variations.

Finally we compare the estimated interest rate elasticities with the predictions of the BT and MO model. In the BT model, the interest rate elasticity of currency holdings is -1/2 and the elasticity of the average number of withdrawals is 1/2, which are larger in absolute value relative to the ones reported in the regression in Tables 3-6. In the MO model the interest rate elasticity of currency holdings is -1/3, a value close to one estimated in the regressions in Table 4. Nevertheless, the MO model predicts an interest rate elasticity of the average of number of withdrawals of 2/3, which is even larger than the one predicted by the BT model, and larger than the one estimated in the regressions above.

3 A deterministic steady state problem

In this section we model a form of technological progress on the withdrawal technology and discuss its implications for money demand. We conduct the analysis by focusing on steady state calculations. We minimize the steady state cost of attaining a given constant flow of consumption, as opposed to minimizing the expected discounted cost. We do this to increase the comparability with the standard derivation of the Baumol-Tobin money demand and to simplify the exposition of the effect of progress on technology for withdrawals from banks. In particular, this calculation helps understand why the level of the money demand and its interest rate elasticity are smaller for better withdrawal technologies.

Consider the following steady state problem. We let M be the average money balances, and $T(M, c)$ be the average (steady state) number of costly withdrawals from the bank per unit of time required to finance a consumption flow c when the average money balances are M. The function T depends on the withdrawal technology available to agents. We assume that T is decreasing in M, so that fewer withdrawals require higher average balances, and that T, is convex, so the minimization problem is well behaved. We let R be the net nominal interest rate (the marginal cost of forgone interest due to an extra unit of money holdings) and

b the cost of each withdrawal. The average money demand solves the minimization problem

$$\min_M \; R\, M + b\, T\,(M, c) \tag{1}$$

The optimal choice of M must balance the impact on the cost due to forgone interest, $R\ M$ with the effect on the cost of withdrawals, $T\,(M, c)$. The formulation of this problem, as in the traditional BT model, uses three simplifying assumptions:

(i) average (steady state) money balances times the interest rate is used to measure the cost, instead of the discounted interest rate cost, and

(ii) the average (steady state) number of withdrawals from the bank is used as opposed to the discounted (expected) cost of withdrawals,

(iii) R is not an argument of the function T.

The assumptions behind this formulation make the comparative statics analysis of the optimal M simple and intuitive. In particular the combination of iii) and the Fisher equation (say that $R = r + \pi$ for a fixed interest rate r), implies that the inflation rate π is not an argument of T. This is not satisfactory because if c and M denote real variables then the inflation rate should appear as an argument of T, as inflation erodes the real value of money holdings.[4] We will remove these simplifying assumptions in the analysis of Section 4. The first order condition for problem (1) is:

$$1 + \frac{b}{R} T'\,(M, c) = 1 + \frac{b}{R\ c} T' \left(\frac{M}{c}, 1 \right) = 0$$

where the first equality follows by assuming, as it seems natural, that the technology $T\,(M, c)$is homogenous of degree zero (Appendix A presents a thorough discussion of the consequences of the homogeneity assumption for a general cost function

[4]Alternatively, one might take c and M to denote nominal quantities, which is an unsatisfactory characterazion of the consumption behavior. Yet another (not so satisfactory) interpretation is that the inflation rate does not change as R changes, which means that the model comparative statics concern changes in the real rate r.

$T(M, c))$.

In this first order condition $T'(M) \equiv \partial T(M, c)/\partial M$ is the decrease in the cost of withdrawals due to an extra unit of money holdings. We will refer to T' as the marginal cost of a withdrawal and to $-R$ as the marginal benefit of an increase in M. Notice that R , b and c enter the foc as a ratio, hence the money demand per unit of cash consumption, M/c, is conveniently defined as a function (only) of the relative cost $\beta \equiv b/(c\ R)$.

3.1 A technology with (exactly) p free withdrawals

Now we use this setup results to analyze the effect on money demand of a simple form of technological progress in T. We consider

$$T_p(M/c) = \max\{\frac{(c/2)}{M} - p, 0\} \ . \tag{2}$$

The parameter p index the level of technology T, in particular it has the interpretation of the average number of free withdrawals per unit of time. The following is a concrete set-up that gives rise to the assumption of p free withdrawals. Assume that the cost b represents the opportunity cost of the time of a trip to a bank branch or an ATM. Think of an agent who, on her way to the ball game, passes by a bank branch or an ATM, say once a week. In this case we can represent the technology T_p as saying that she has one free withdrawal a week, or $p = 1$ per week. Now imagine that an ATM is installed on the way of her job, and assume that she works 6 days a week. This technological improvement can be represented by an increase in p, so that she gets 7 free withdrawals a week, or $p = 7$ per week.

Setting $p = 0$ in (2) all the trips are costly, and we obtain as a baseline case the classical Baumol-Tobin,

$$T_0(M/c) = \frac{(c/2)}{M}$$

An agent with consumption flow c, withdraws 2 M, which last $2M/c$ periods, and hence has average balances M and makes $(c/2M)$ trips to the bank. Notice that T_0 has a marginal cost function T_0' has a constant elasticity equal to 2, which implies the well known result that the money demand elasticity is 1/2. The interpretation of the case of $p > 0$ is that the agent has p free withdrawals, so that if the total number of withdrawals is $(c/2)/M$, then she pays only for the excess of $(c/2)/M$ over p, which gives the expression (2).

Throughout the analysis in this section we allow T to take any real value. However, the specification of the technology in (2) essentially puts a lower bound of p on T. This is similar to the seminal analysis of Tobin (1956) where the integer constraint on the number of transactions is carefully taken into account. Of course the integer constraint puts a lower bound equal to zero on the number of transactions. Our specification of T_p can be thought of as allowing the lower bound on the transactions to be a parameter that indexes technological change.

The money demand for a technology with $p \geq 0$ is given by

$$M_p\left(R\right)/c = (1/p)\ \sqrt{\frac{1}{4\max\left\{\frac{R}{2\ \hat{b}},\ 1\right\}}} \tag{3}$$

where

$$\hat{b} \equiv bp^2/c\ . \tag{4}$$

Consider the case where $p = 0$, so that it is the BT set-up. In this case, for low R the forgone interest cost is small, so that agents decide to economize in costly withdrawals, and hence choose a large value of M. Now consider the case of $p > 0$. In this case there is no reason to have less than p withdrawals per unit of time, since these are, by assumption, free. Hence, for all $R < 2\hat{b}$ agents will choose the same level money holdings, namely, $M_p(R) = M_p\left(2\hat{b}\right)$, since they are not paying for any withdrawal but they are subject to positive forgone interest rate costs (hence the interest elasticity is zero for $R < 2\hat{b}$). Since for $p > 0$ the money demand is constant for $R < 2\hat{b}$, it is both lower in its level and it has a lower interest rate elasticity than the money demand from the BT model. Indeed, (3)-(4) imply that the range of interest rate R for which the money demand is lower and has lower interest rate elasticity is increasing in p. For future reference, notice that keeping $\hat{b}$ fixed, as increasing p decreases the level of the money demand but it does not change its interest rate elasticity. Hence the departure of the behavior relative to the BT model is captured by the parameter $\hat{b}$.

4 Money demand: a stochastic dynamic problem

This section extends the analysis along two dimensions. First, it takes an explicit account of the dynamic nature of the cash inventory problem, as opposed to the steady state analysis of Section 3. In doing so it also relaxes the steady state assumptions in (A1). Second, it introduces a variation on the withdrawal technology considered in Section 3.1. In particular, the technology considered here is one where agents have a Poisson arrival of free opportunities to withdraw cash , as opposed to the assumption of Section 3.1 of having a deterministic number of free withdrawals per period. We think that, relative to the deterministic number of free withdrawals, this assumption is a more realistic depiction of reality. Our maintained assumption is that the main component of the cost for a withdrawal is the opportunity cost of the households. We imagine that, for a given density of ATMs and bank desk, an agent bumps into them at certain rate per unit of time – denoted by p in the model. These are chance meetings with an intermediary that involves zero cost of withdrawal[5]. We argue that random meetings with a financial intermediary is a more realistic depiction of the opportunities faced by households. Our hypothesis is that, as the density of bank branches and ATMs increases, then households get more of these free opportunities to withdraw.

This model has several advantages, besides realism in the modeling of the search technology, over the one with a fixed deterministic number of withdrawals per period. First, a piece of evidence in favor of the random meeting model is that households withdraw much before their cash balances reach zero (see the statistics on the cash at withdrawals in Table 1). A related feature, is that the model with random meetings implies, as shown in the data of Table 1 –and contrary to the implication of the basic BT model and of the model with exactly p free withdrawals– that the ratio of the average withdrawal to the average cash balances is below 2. Second, the model with random meetings smooths out some of the stark features of the model with exactly p free withdrawals. For instance, it turns out that its interest elasticity is lower than 1/2 for the whole range of interest rates, as opposed to be either 1/2 or 0.

We emphasize that the model solves the problem of minimizing the cost of financing a given cash consumption. We think that the explicit dynamic nature of the model will allow us to use it in future work as a building block of a more complete model of cash management, where the decision of paying with cash is formally introduced. Finally, it turns out that the cost of introducing random meetings in an explicitly dynamic model is small, in the sense that the agent decision problem turns out to be very tractable, with an almost close form solution, a feature that we plan to use in a structural estimation of the model.

We turn now to the description of the agent problem. She faces a cash-in-advance constraint and can withdraw or deposit from an interest bearing account. The

[5] In section 6 we extend the model by assuming that withdrawals that occur upon these chance meetings, rather than being free, are subject to a small fixed cost.

sequence problem is to choose an increasing sequence of stopping times $\{\tau_j\}$ at which to withdraw (or deposit) money in an interest bearing account, and the amounts to withdraw at each time, so as to minimize the expected discounted cost of financing a given constant real consumption flow c, denoted by TC_0:

$$TC_0(\tau, m) = E_0\left[\sum_{j=0}^{\infty} e^{-r\ \tau_j}\left\{b\ I_{\tau_j} + \left(m\left(\tau_j^+\right) - m\left(\tau_j^-\right)\right)\right\}\right] \tag{5}$$

where we use $m(t)$ to denote the real value of the stock of currency. The stock of currency jumps discontinuously up at the time of a withdrawal, so use $m(t^+)$ and $m(t^-)$ to denote the right and left limits of m. Thus the amount of a withdrawal at τ_j is $m\left(\tau_j^+\right) - m\left(\tau_j^-\right)$. The law of motion of the real value of the stock of money between withdrawals is given by

$$\frac{dm(t)}{dt} = -c - m(t)\,\pi \tag{6}$$

where π is the inflation rate and c the real consumption flow . We assume that the agent contacts a financial institution with an exogenous probability p per unit of time. More precisely, contacts with the financial intermediary follow a Poisson process with arrival rate p. In the case of a contact the agent can withdraw (or deposit) money in an interest bearing account without incurring a cost. If the agent wants to withdraw (or deposit) in the financial institution in any other time, it must pay a real cost b. The indicator I_{τ_j} takes the value of zero if the withdrawal (or deposit) takes place at the time $t = \tau_j$ of a contact with a financial intermediary, and takes the value of one otherwise. The agent chooses stopping times and withdrawals as function of the history of contacts with the intermediary. We use r for the real rate at which cash flows are discounted. The initial conditions for the problem are the real cash balances, $m(0) = m_0$ and whether at time $t = 0$ the agent is matched with a financial intermediary or not.

We define the shadow cost of a policy $\{\tau_j, m\}$ as the expected discounted cost of the withdrawals plus the expected discounted opportunity cost of the cash balances held by the agent. We denote the shadow cost as SC_0, which is given by:

$$SC_0(\tau, m) = E_0 \left[\sum_{j=0}^{\infty} e^{-r\ \tau_j} \left\{ b\ I_{\tau_j} + \int_0^{\tau_{j+1}-\tau_j} R\ m(\tau_j + t)\ e^{-rt}\ dt \right\} \right] \tag{7}$$

where R is the nominal interest rate and m follows the law of motion (6). The shadow costs is defined in terms of the opportunity cost R and the parameters used to define the total cost, (r, p, π, b). In the next Proposition we show that, provided the Fisher equation $R = r + \pi$ holds, then the total cost can be written as the shadow cost plus the present value of c.

Proposition 1. *Assume that $R = r + \pi$. For any policy $\{\tau, m\}$ the total cost equals the shadow cost plus the present value of c, or*

$$TC_0 = \frac{c}{r} + SC_0\ .$$

Proof. See appendix C.

Proposition 1 implies that minimizing the shadow cost is equivalent to minimizing the total cost only when $R = r + \pi$. Nevertheless, below we consider the shadow cost problem for the general case of arbitrary values for R, r and π. We keep this general case for two reasons. One is to accommodate other costs and benefits of holding cash (such as the costs of petty crime). The second relates to the literature, such as the classic papers by Baumol and Tobin, that does not impose the Fisher equation as discussed above.

We use $V_s(m)$ for the value function corresponding to the minimization of the shadow cost:

$$V_s(m_0) = \min_{\tau, m} SC_0(\tau, m) \tag{8}$$

subject to $m(0) = m_0$ and where $s = f$ denotes that the agent is matched to a financial intermediary and $s = u$ that she is not. The next section solves for V.

Finally, Proposition 1 also helps linking the dynamic model with the steady state analysis done in Section 3. Each of the terms

$$b\ I_{\tau_j} + R \int_0^{\tau_{j+1}-\tau_j} m(\tau_j + t)\ e^{-rt}\ dt$$

in the summation of the shadow cost is similar to cost $b\,T(M, c) + RM$ in the steady state formulation of Section 3. The difference is that here $\int_0^{\tau_{j+1}-\tau_j} m(\tau_j + t)\ e^{-rt}\ dt$ are real balances, as opposed to the nominal M, and that the consumption flow c in (6) that is to be maintained constant is also real, as opposed to nominal, as discussed above.

4.1 Bellman equation for V and optimal policies

We now describe the Bellman equation for $V_s(\cdot)$, find an analytical solution for it and the associated optimal policy. We first write down the Bellman equation for an agent unmatched with a financial intermediary and holding a real value of cash m. The only decision that this agent must make is whether to remain unmatched, or to pay the fixed cost b and be matched with a financial intermediary. If the agent chooses not to contact the intermediary then, as standard, the Bellman equation states that the return on the value function $rV_u(m)$ must equal the flow cost, given by the opportunity cost Rm, plus the expected change per unit of time. There are two sources of expected changes per unit of time: The first is that she finds a financial intermediary with probability p, upon which she incurs in a change in value $V_f(m) - V_u(m)$. The second is that in the next instant of time the real value of cash balances decreases by the amount $c+m\pi$ due, respectively, to her consumption and the effect of inflation. Thus, denoting by $V_u'(m)$ the derivative of $V_u(m)$ with respect to m, the Bellman equation satisfies:

$$rV_u(m) = Rm + p\left(V_f(m) - V_u(m)\right) + V_u'(m)\left(-c - m\pi\right) \tag{9}$$

On the other hand, if the agent chooses to contact the intermediary, the Bellman equation satisfies

$$V_u(m) = b + V_f(m) \tag{10}$$

Notice that an agent can end up being matched with a financial intermediary either because it exogenously "bumps" into it with probability p, or because she pays the cost b. Regardless of how she is matched, an agent matched with a financial intermediary chooses the optimal withdrawal, which we denote by w, as follows

$$V_f(m) = \min_{w} V_u(m+w) \tag{11}$$

subject to

$$w + m \geq 0 \tag{12}$$

where the constraint stipulates that after the withdrawal, or deposit, the cash balances are non-negative. Inspection of (11) reveals that $V_f(\cdot)$ does not depend on m, so from now we denote this value as V^*.

We now turn to the characterization of the Bellman equations and its optimal policy. We will guess and later verify that the optimal policy is described by two parameters, $0 < m^* < m^{**}$. The threshold m^* is the value of cash that agents choose at a financial intermediary; we refer to it as the cash replenishment level. The threshold m^{**} is a value of cash beyond which agents will pay the cost b, contact the intermediary, and make a deposit so as to leave her cash balances at m^*.

Given our guesses, m^*, m^{**}, $V_u(m)$ and V^* we will assume, and later verify, that

$$V_u(m) < V^* + b \;\text{ for } m \in (0, m^{**})$$

so that for $m \in (0, m^{**})$ is not optimal to pay the cost and contact the intermediary. We have that

$$V_u(0) = V^* + b$$

This equality follows since at $m = 0$ the agent must withdraw, since if she does not in the next instance either $m(t)$ becomes negative or she will not be able to finance her consumption. Similarly,

$$V_u(m) = V^* + b \;\text{ for } m \geq m^{**}$$

which follows from the assumption that agents contact the intermediary for $m \geq m^{**}$. Inserting these guesses into (9), (10), (11) implies that a solution of (8) is given by numbers V^*, m^*, m^{**} and the function $V_u(m)$, satisfying:

$$V^* = V_u(m^*) = \min_z V_u(z) \tag{13}$$

$$V_u(m) = \begin{cases} V^* + b & \text{if } m = 0 \\ \dfrac{Rm + pV^* - V_u'(m)(c + m\pi)}{r + p} & \text{if } m \in (0, m^{**}) \\ V^* + b & \text{if } m \geq m^{**} \end{cases} \tag{14}$$

In Appendix B we display the Bellman equations for the a discrete time version of the model. The appendix provides an alternative derivation of the continuous time Bellman equations (13) and (14) by taking limits of the discrete time case as the length of the time interval goes to zero.

The next proposition gives one non-linear equation whose unique solution determines the cash replenishment value m^* as a function of the parameters of the model: $R,\ \pi,\ r,\ p,\ c$ and b.

Proposition 2. *Assume that $r+\pi+p>0$. The optimal return point m^* is given by the unique positive solution to*

$$\left(\frac{m^*}{c}\pi+1\right)^{1+\frac{r+p}{\pi}}=\frac{m^*}{c}\left(r+p+\pi\right)+1\ +\left(r+p\right)\left(r+p+\pi\right)\frac{b}{cR} \tag{15}$$

for $\pi\neq 0$. See appendix C for a proof, and appendix F for the $\pi=0$ case.

Note that, keeping r and π fixed, the solution for m^*/c is a function of (b/cR), as it is in the steady state derivation of money demand of Section 3. The next proposition gives a closed form solution for the function $V_u(\cdot)$, and the scalar V^* in terms of m^*.

Proposition 3. *Assume that $r+\pi+p>0$. Let m^* be the solution of (15).*
(i) The value for the agents not matched with a financial institution, for $m\in(0,m^{**})$, is given by the convex function:

$$V_u(m)=\left[\frac{pV^*-Rc/\left(r+p+\pi\right)}{r+p}\right]\ +\left[\frac{R}{r+p+\pi}\right]m+\left(\frac{c}{r+p}\right)^2 A\ \left[1+\frac{\pi}{c}m\right]^{-\frac{r+p}{\pi}}$$

where the constant A is given by:

$$A=\frac{r+p}{c^2}\left(R\ m^*+\left(r+p\right)b+\frac{Rc}{r+p+\pi}\right)>0.$$

For $m\geq m^{**}$

$$V_u(m)=V^*+b$$

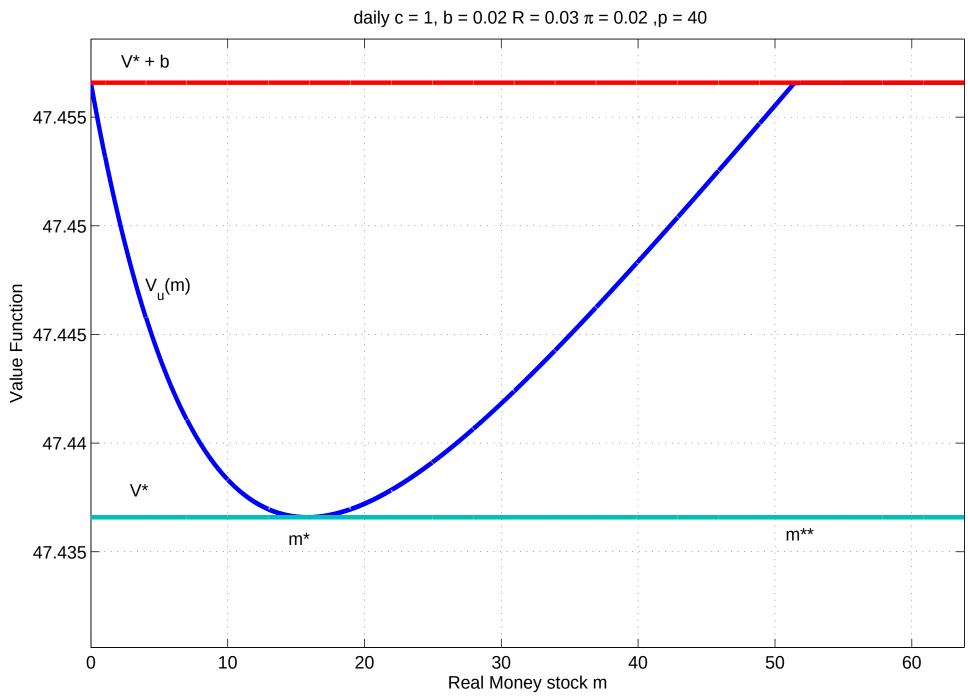
(ii) The value for the agents matched with a financial institution, V^*, is given by:

$$V^* = \frac{R}{r} m^*$$

See appendix C for a proof and appendix F for the $\pi = 0$ case.

The close relationship between the value function at zero cash and the optimal return point $V_u(0) = (R/r) m^* + b$ derived in these proposition will be useful to measure the gains of different financial arrangements. The following picture displays an example value function:

Figure 2: An example Value function



The next proposition uses the characterization of the solution for m^* to conduct some comparative statics.

Proposition 4. *The optimal return point* $m^*(R, r, \pi, c, b, p)$ *has the following properties:*

1. m^* *is homogenous of degree one in* (c, b)*.*

2. The elasticity of m^* *with respect to* b

$$0 \leq \frac{b}{m^*}\frac{dm^*}{db} \leq \frac{1}{2}$$

is decreasing in p*, moreover* $m^* \to 0$ *as* $b \to 0$

3. m^* *is increasing in* c*, and*

$$\frac{c}{m^*}\frac{dm^*}{dc} = 1 - \frac{b}{m^*}\frac{dm^*}{db}.$$

4. The interest rate elasticity satisfies

$$0 \leq -\frac{R}{m^*}\frac{dm^*}{dR} = \frac{b}{m^*}\frac{dm^*}{db} \leq \frac{1}{2}$$

and hence it is decreasing in p*.*

5. For small b/c*, we can approximate* m^* *by the the solution in BT model, or*

$$m^*/c = \sqrt{2\,\frac{b}{cR}} + o\left(\sqrt{\frac{b}{c}}\right)$$

where $o\left(\sqrt{\frac{b}{c}}\right)/\sqrt{\frac{b}{c}} \to 0$ *as* $\sqrt{\frac{b}{c}} \to 0$.

6. Assuming that the Fisher equation holds, in that $\pi = R - r$*, the elasticity of* m^* *evaluated at zero inflation, i.e. at* $R = r$*, satisfies*

$$0 \leq -\frac{p}{m^*}\frac{dm^*}{dp}|_{R=r} \leq \frac{p}{p+r}\ .$$

7. Assuming that the Fisher equation holds, in that $\pi = R - r$*, the elasticity of*

m^* evaluated at zero inflation, i.e. at $R = r$, satisfies

$$-\frac{R}{m^*}\frac{dm^*}{dR}|_{R=r} \leq \frac{1}{2} \ .$$

with strict inequality iff $r + p > 0$.

Proof. *See appendix C.*

Properties 1-4 are the same as in the steady state money demand derived in Section 3. Property 5 says that when b is small relative to c, the resulting money demand is well approximated by the one for the BT model. Property 6. has its analog in the model with p free trips of Section 3.1. The elasticities in 6. and 7. are computed imposing the Fisher equation $R = r + \pi$, in particular we replace inflation using $\pi = R - r$. Instead in the elasticity computed in property 4, as R changes, the inflation rate π and the real rate r are kept constant. The fact that the interest rate elasticity is smaller than 1/2 and decreasing (in absolute value) on p is is one the main results of the model.

5 Distribution of cash balances and average number and size of cash withdrawals

This section derives the distribution of real cash holdings when the policy characterized by the parameters (m^*, p, c) is followed and the inflation rate is π. The policy is to replenish cash holdings up to the return value m^*, either when a match with a financial intermediary occurs, which happens at a rate p per unit of time, or when the agent runs out of money (i.e. real balances hit zero). In the previous section we showed that this is the nature of the optimal policy and we characterized how m^* depends on the fundamental parameters (R, r, π, p, c, b).

Our first result is to compute the expected number of withdrawals per unit of time, denoted by n. This includes both the withdrawals that occur upon an exogenous contact with the financial intermediary and the ones initiated by the agent when her cash balances reach zero.[6]

Proposition 5. *The number of cash withdrawals per unit of time when $\pi \neq 0$ is*

$$n\left(m^{*};c,\ \pi,p\right)=\frac{p}{1-\left(1+m^{*}\pi/c\right)^{-\frac{p}{\pi}}} \tag{16}$$

See appendix C for a proof and appendix F for the $\pi = 0$ case.

For future reference we notice that n is homogenous of degree zero in (m^{*}, c) .As can be seen from the expression the ratio $n/p \geq 1$ since in addition to the p free withdrawals it includes the costly withdrawals that agents do when they exhaust their cash. Notice that n/p is decreasing in m^{*}, indicating that a greater value for the return point allows the agent to finance consumption over a longer time-span. The reciprocal of n gives the expected time between withdrawals. We can see that $1/n$ is a concave and increasing function of $m^{*}\pi/c$. A second order approximation of this function gives:

$$\frac{1}{n\left(m^{*};c\ \pi,p\right)}=\frac{m^{*}}{c}-\frac{1}{2}\left(\pi+p\right)\left(\frac{m^{*}}{c}\right)^{2} \tag{17}$$

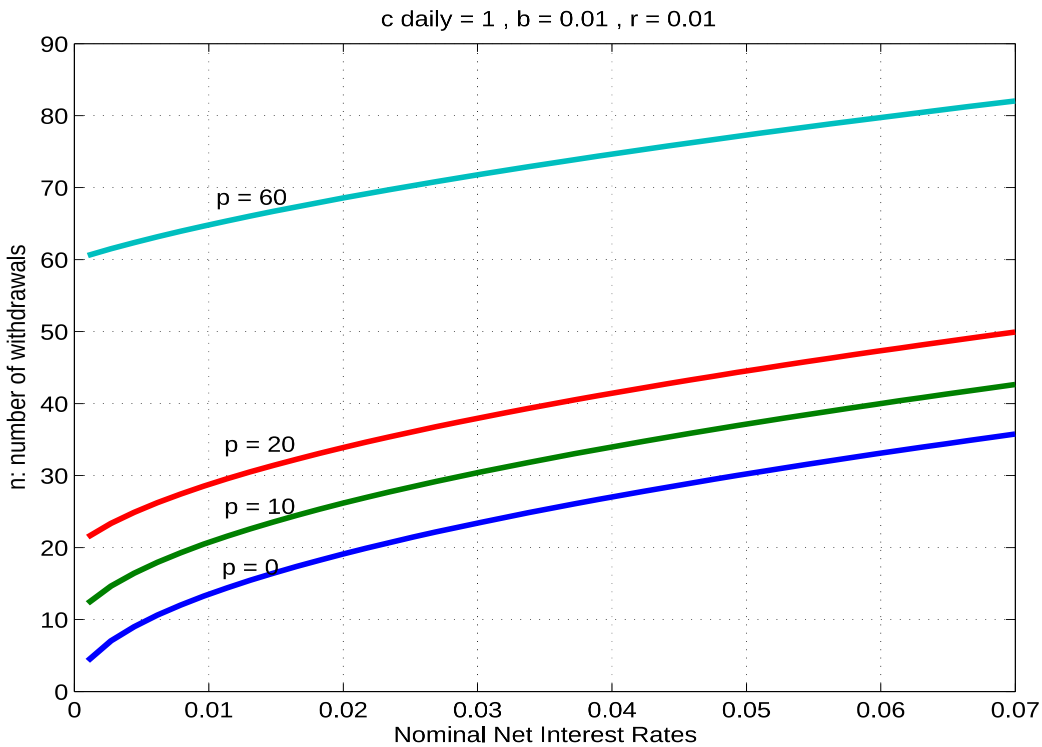
Note how this formula yields exactly the expression in the BT model when $p = \pi = 0$. The formula shows, moreover, that the expected time between withdrawals is decreasing in π and in p.

The next figures displays the average number of withdrawals against the level of interest rates R for different values of the parameter p. All the flow variables are

[6] For instance if $n = 52$ when all the parameters are measured per annum or equivalently if $n_{day} = 52/365 = 1/7$ if measured per day, then the agent withdraws 52 times in a year or, equivalently, she withdraws every 7 days ($1/n_{day} = 7$).

expressed annually, except consumption which is expressed daily, so n is the average number of withdrawals per year.

Figure 3: Number of withdrawals



We use $b = 0.03$ as implying a cost of about 3 percent of daily cash consumption, which is equivalent to less than 2 percent of consumption of non-durables an services (see Table 2), or about 1 percent of daily income. Comparing the numbers in this plot with the ones of Table 1, it seems that a value of p of about 40 is reasonable for those households with an ATM card and one p about 10 may be appropriate for those without an ATM card.

The next Proposition derives the density of the distribution of real cash balances as a function of p, π, c and m^*.

Proposition 6. *(i) The density for the real balances m when $\pi \neq 0$ is*

$$h(m) = \left(\frac{p}{c}\right)\left(\frac{\left[1+\frac{\pi}{c}m\right]^{\frac{p}{\pi}-1}}{\left[1+\frac{\pi}{c}m^*\right]^{\frac{p}{\pi}}-1}\right) \tag{18}$$

(ii) Let $H(m, m_1^*)$ be the cumulative distribution of m for a given m^*. Let $m_1^* < m_2^*$, then $H(m, m_2^*) \leq H(m, m_1^*)$, i.e. $H(\cdot, m_2^*)$ first order stochastically dominates $H(\cdot, m_1^*)$.

See appendix C for a proof and appendix F for the $\pi = 0$ case.

In the proof of Proposition 6 we show that the density of m solves the following ODE:

$$\frac{\partial h(m)}{\partial m} = \frac{(p-\pi)}{(\pi m + c)} h(m)$$

for any $m \in (0, m^*)$. There are two forces determining how the mass is spread out, i.e. determining the shape of this density. One force is that agents meet a financial intermediary at a rate p, where they replenish their cash balances. The other is that inflation eats away the real value of their nominal balances. Notice that if $p = \pi$ these two effects cancel and the density is uniformly constant. If $p < \pi$, the density is downward sloping, with more agents at low values of real balances due to the greater pull of the inflation effect. If $p > \pi$, the density is upward sloping due the the greater effect of the replenishing of cash balances. To see this notice that, as shown in the proof of Proposition 2 (in appendix C), $\pi m^* + c > 0$, thus $\pi m + c > 0$ for all m in the invariant support $(0, m^*)$ and the sign of $\partial h(m)/\partial m$ is given by the sign of $(p - \pi)$.

We can now define the aggregate money demand as

$$M = \int_0^{m^*} m h(m)\, dm.$$

The next proposition gives a formula for M as a function of p, π, c, and m^*.

Proposition 7. *(i) For a given m^* , the aggregate money demand is given by :*

$$M = \mu\left(m^*; c, \pi, p\right) \equiv c\ \frac{\left(1+\frac{\pi}{c}m^*\right)^{\frac{p}{\pi}}\left[\frac{m^*}{c} - \frac{\left(1+\frac{\pi}{c}m^*\right)}{p+\pi}\right] + \frac{1}{p+\pi}}{\left[1+\frac{\pi}{c}m^*\right]^{\frac{p}{\pi}} - 1} \tag{19}$$

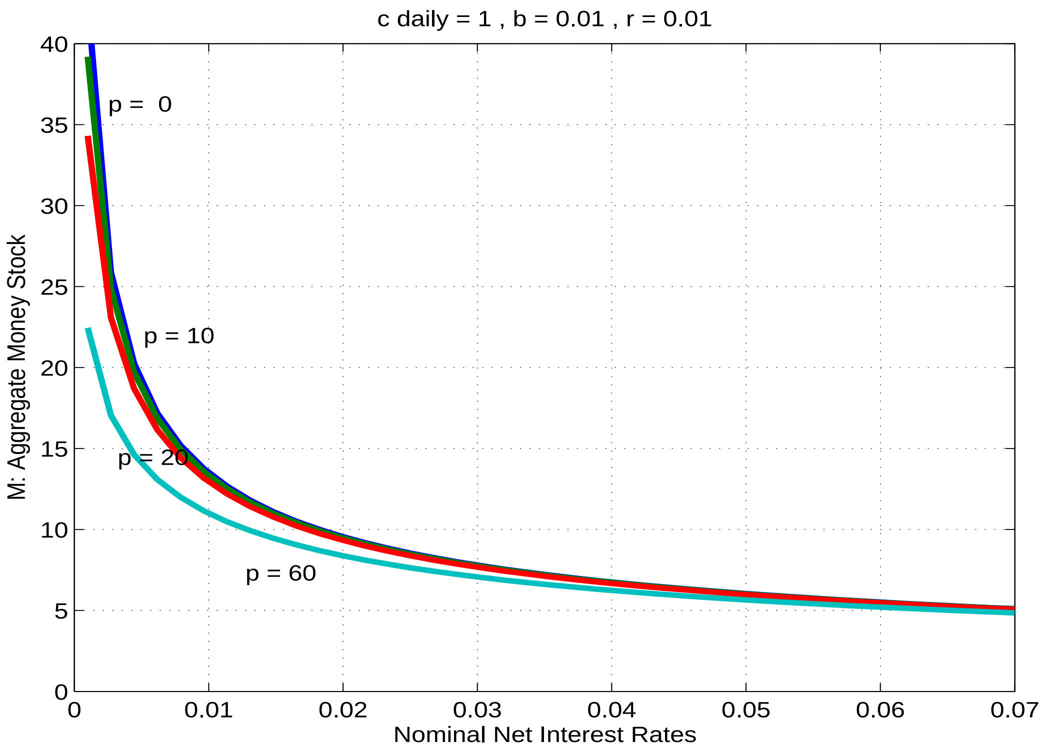
for $\pi \neq 0$.

(ii) M is increasing in m^*.

See appendix C for a proof and appendix F for the $\pi = 0$ case.

The next figure displays a plot of the aggregate money demand M as a function of the nominal interest rate R at various levels of financial diffusion p.

Figure 4: Aggregate money demand



Very roughly, the numbers in this figure suggest that $p = 40$ produces cash balances of similar magnitudes that those of ATM card holders in our Italian data set. On the other hand, matching the cash balances of those households without an

ATM, requires a much lower values of p, closer to 10.

The next proposition compares the interest rate elasticity of the aggregate money demand, with the one for the average number of withdrawals. As a benchmark, recall that in the Baumol Tobin model these two elasticities are $-1/2$ and $1/2$ respectively. In our model, for $p > 0$, the elasticity of the money demand is higher in absolute value than the elasticity of the average number of withdrawals. The intuition for this result is that the average money demand depends on both the target level for cash replenishment m^* and the average number of withdrawals, n. Indeed we have that if the replenishment policy described above is followed then:

$$\frac{M}{c} = \frac{1}{p+\pi}\left[n \ \ (m^*/c) - 1\right], \tag{20}$$

which can be verified by inserting the expression for n given by (16) into the formula for M in (19). But, since for $p > 0$ some withdrawals entail no cost, the households always makes p withdrawals on average. Notice that this is different from the deterministic steady state model with p free withdrawals (Section 3.1), where the two interest rate elasticities were the same. This is also different from the evidence in Tables 4 and 6 for Italian households, where we find similar interest rate elasticities, in absolute values, for M/c and n.

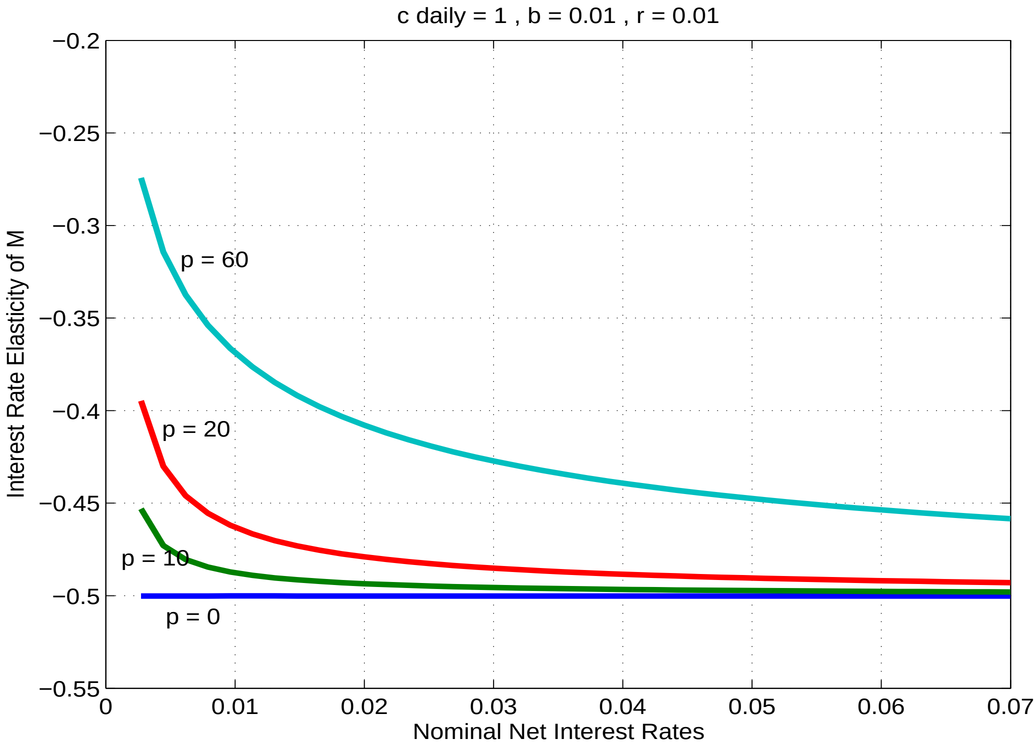
Proposition 8. *The interest rate elasticity of the average cash balances is larger in absolute value than the interest rate elasticity of the average number of withdrawals, evaluated at $\pi = 0$.*

$$-\frac{M\left(R,r,\pi,p\right)}{R}\frac{\partial n\left(R,r,\pi,p\right)}{\partial R}|_{\pi=0} \geq \frac{n\left(R,r,\pi,p\right)}{R}\frac{\partial n\left(R,r,\pi,p\right)}{\partial R}|_{\pi=0}$$

See the appendix for the proof.

The next figure displays the interest rate elasticity of the average money balances for different values of the parameter p.

Figure 5: Interest rate elasticity of M



The figure shows that to obtain an interest rate elasticity of the money demand as low as $1/3$ in absolute value the model requires a relatively large value of p. For instance, with $p = 80$ the interest rate elasticity is $1/3$ when evaluated at an interest of 3 percent.

For future reference, the next proposition studies the relationship between M and m^* :

Proposition 9. *The ratio M/m^* is increasing in p with*

$$\begin{aligned} M/m^* &= \frac{1}{2} \text{ for } p = 0 \\ M/m^* &\rightarrow 1 \text{ as } p \rightarrow \infty. \end{aligned}$$

Proof. To be done.

For the case of $\pi = 0$, the ratio M/m^* has the following a simple expression:

$$M/m^* = \frac{1}{1 - \exp(-pm^*/c)} - \frac{1}{p(m^*/c)}$$

Since the right hand side is a decreasing function of $(m^*/c)\,p$, and since we had shown that the elasticity of m^*/c is (in absolute value) smaller than $p/(p+r)$, then it implies that M^*/c is decreasing in p.

Now we turn to the analysis of the average number of withdrawals, which we denote by W.

Proposition 10. *The average withdrawal is given by:*

$$W = m^* \left[1 - \frac{p}{n}\right] + \left[\frac{p}{n}\right] \int_0^{m^*} (m^* - m)\, h(m)\, dm \tag{21}$$

where

$$\int_0^{m^*} (m^* - m)\, h(m)\, dm = \frac{\frac{\left(1+\frac{\pi}{c}m^*\right)^{\frac{p}{\pi}+1} - 1}{(p+\pi)/c} - m^*}{\left(1 + \frac{\pi}{c}m^*\right)^{\frac{p}{\pi}} - 1}$$

Proof. Follows from Proposition 19 below setting $f = 0$.

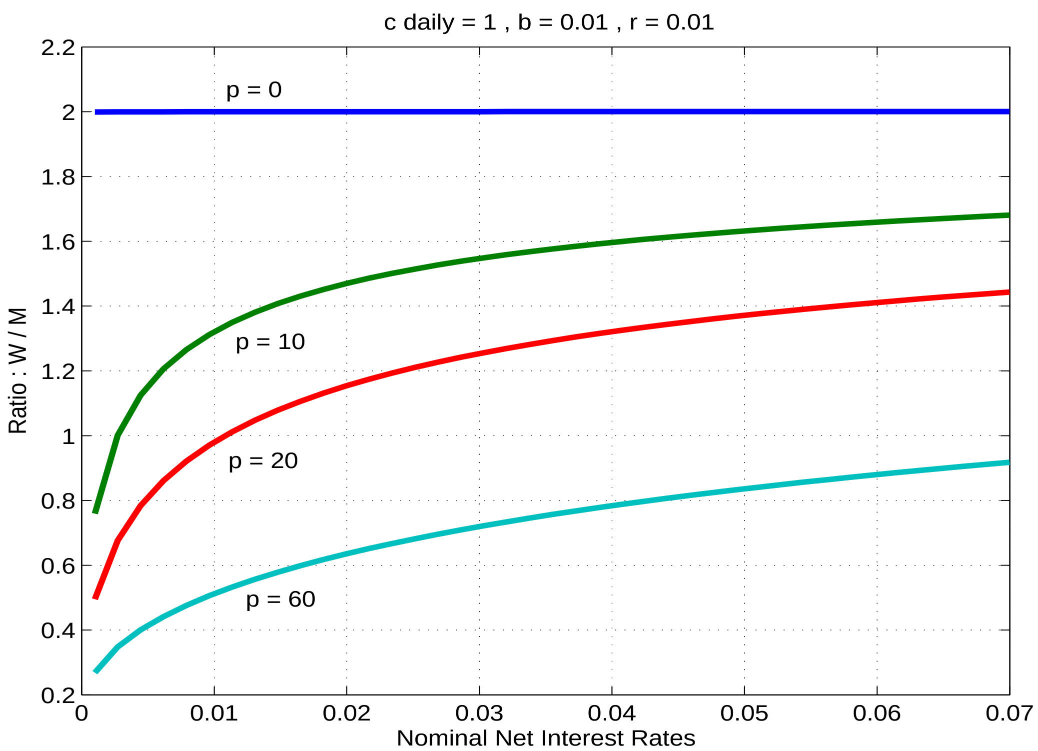
To understand the expression for W notice that $n-p$ is the number of withdrawals in a unit of time that occur because agents reach zero balances, so if we divide it by the total number of withdrawals per unit of time (n) we obtain the fraction of withdrawals that occur the agent reaches zero balances. Each of these withdrawals is of size m^*. The complementary fraction gives the withdrawals that occur due to a chance meeting with the intermediary. A withdrawal of size $m^* - m$ happens with frequency $h(m)$.

Combining the previous results we can see that for $p > 0$, the ratio of withdrawals to average cash holdings is less than 2. To see this, using the definition of W we can write

$$\frac{W}{M} = \frac{m^*}{M} - \frac{p}{n}. \tag{22}$$

Since $M/m^* \geq 1/2$, then it follows that $W/M \leq 2$. Indeed notice that for p large enough this ratio can be smaller than one. We mention this property because for the Baumol - Tobin model the ratio W/M is exactly two, while in the data of Table 1 the average ratio is below 1.4 for those households without an ATM card and about 1.2 for those with an ATM card. The intuition for this result in our model is clear: agents take advantage of the free random withdrawals regardless of their cash balances, hence the withdrawals are distributed on $[0, m^*]$, as opposed to be concentrated on m^*, as in the BT model.

Figure 6: Withdrawal to currency ratio (W/M)



We let $\underline{M}$ be the average amount of money that an agent has at the time of withdrawal. A fraction $[1 - p/n]$ of the withdrawals happens when $m = 0$. For the remaining fraction, p/n, an agent has money holdings at the time of the withdrawal distributed with density h, so that:

$$\underline{M} = 0 \left[1 - \frac{p}{n}\right] + \left[\frac{p}{n}\right] \int_0^{m^*} m \, h(m) \, dm$$

Simple algebra shows that:

$$\underline{M} = m^* - W \tag{23}$$

or inserting the definition of $\underline{M}$ into the expression for M we obtain:

$$\underline{M} = \frac{p}{n} M \tag{24}$$

Notice that in the BT model $\underline{M} = 0$, since agents withdraw only when their cash balances reach zero.

The theoretical model that we have proposed adds one parameter, namely p, to a dynamic version of the BT model, which allows a larger class of cash-management policies. In the BT model average withdrawal to currency $W/M = 2$, average cash at withdrawal $\underline{M}/M = 0$, and interest rate elasticities of M/c and n equal $1/2$. The model with $p > 0$, admits a wider range of values, including $W/M < 2$, $\underline{M}/M > 0$, and lower interest rate elasticities of M/c and n. Indeed it is possible to obtain the polar opposite of the BT model: as $p \to \infty$, then $W/M \to 0$, $\underline{M}/M \to 1$, and the interest rate elasticities of M/c and n tend to zero. Indeed the next proposition finds a one dimensional index, $\hat{b}$ that determines how close is the cash-management behavior of households is to the one prescribed by the BT model. The index $\hat{b}$ is defined as

$$\hat{b} \equiv (p + r)^2 \, b/c$$

For low values of $\hat{b}/R$, the cash-management behavior is as in the BT model, and for high values it is its polar opposite. The next proposition specializes the analysis for $\pi = 0$ and, for most of it, for $r \to 0$. The cost of consider this special set up has been discussed before when we compare our set-up to the one in BT. The benefits of consider this case is that the resulting expressions are very simple. For the next proposition, we consider the case of $\pi = 0$ and define the normalized optimal return point x as

$$x \equiv m^* \left(r + p\right) / c \ .$$

Proposition 11. *The normalized return point x is given by a strictly increasing function $\gamma\left(\hat{b}/R\right)$ and has an interest rate elasticity of $-\left(R/x\right)\left(\partial x/\partial R\right)$ that is strictly decreasing in $\hat{b}/R$:*

$$-\frac{R}{\gamma}\frac{\partial \gamma}{\partial R} \to 1/2 \ \textit{as} \ \left(\hat{b}/R\right) \to 0, \ \textit{and} \ -\frac{R}{\gamma}\frac{\partial \gamma}{\partial R} \to 0 \ \textit{as} \ \left(\hat{b}/R\right) \to \infty.$$

ii) There are functions $\phi\left(\cdot\right)$, $\omega\left(\cdot\right)$, $\alpha\left(\cdot\right)$, such that:

$$\begin{aligned} \left(M/c\right) p &= \phi\left(x \left[\frac{p}{p+r}\right]\right) \ , \\ W/M &= \alpha\left(\omega\left(x\right)\right) \quad \textit{and} \quad \frac{\underline{M}}{M} = 1/\omega\left(x\right) \end{aligned}$$

so that, as $r \to 0$, the ratios: $\left(M/c\right)p$, W/M and $\underline{M}/M$ are functions of $x = \gamma\left(\hat{b}/R\right)$ only, and satisfy:

$$\begin{aligned} W/M &\to 0 \ \textit{as} \ x \to \infty \ \textit{and} \ W/M \to 2 \ \textit{as} \ x \to 0, \\ \overline{M}/M &\to 1 \ \textit{as} \ x \to \infty \ \textit{and} \ \overline{M}/M \to 0 \ \textit{as} \ x \to 0, \\ \frac{-R}{M/c}\frac{\partial M/c}{\partial R} &\to 0 \ \textit{as} \ x \to \infty \ \textit{and} \ \frac{-R}{M/c}\frac{\partial M/c}{\partial R} \to \frac{1}{2} \ \textit{as} \ x \to 0 \end{aligned}$$

moreover, $\partial \log\left(M/c\right)/\partial \log R \ \leq 1.25/2$ for all x, and decreases in x, for $x \geq 20$.

Proof. See appendix C.

This proposition makes clear the sense in which both b and p determine whether cash-management behaves as in BT model or not. In particular, consider first the case where financial development has the effect of decreasing b and increasing p, in such a way as to keep $\hat{b}$ constant. In this case the level of money demand M/c will be converging to zero at the rate $1/p$, but its interest rate elasticity as well as the ratios W/M and $\underline{M}/M$ will stay constant. If instead the increase in p dominates the decrease in b, so that $\hat{b} \to \infty$, then $-R/\left(M/c\right)\ \left(\partial M/c/\partial R\right) \to 0$, $W/M \to 0$ and $\underline{M}/M \to 1$, so the cash-management behavior is the polar opposite of the one in the BT model. Otherwise, if $\hat{b} \to 0$, even as money holdings disappear, the cash-management behavior becomes as in BT: since $-R/\left(M/c\right)\ \left(\partial M/c/\partial R\right) \to 1/2$, $W/M \to 2$ and $\underline{M}/M \to 0$.

Notice that the role of $\hat{b}$ in Proposition 11 generalizes the effect of this parameter in the deterministic model with exactly p free withdrawals analyzed in Section 3.1. The effects on p and $\hat{b}$ on the level and interest rate elasticity of the money demand are a smooth version of the results for the model with exactly p free withdrawals. The model of this section generalizes the effects of $\hat{b}$ to the ratios W/M and $\underline{M}/M$.

Finally, we consider a model where we introduce petty crime as an additional cost of holding money. This is relevant for the empirical implementation of the model. Assume that with probability q per unit of time an agent is robbed, so that her cash balances go to zero, and she must withdraw. As before, we assume that with probability p the agent can make a withdrawal (or deposit) at no cost. The Bellman equation becomes:

$$r\tilde{V}_u\left(m\right) = Rm + p\left(\tilde{V}^* - \tilde{V}_u\left(m\right)\right) + q\left(\tilde{V}^* + b - \tilde{V}_u\left(m\right) + m\right) + \tilde{V}_u'\left(m\right)\left(-c - \pi m\right) \tag{25}$$

or

$$(r+\tilde{p})\,\tilde{V}_u\,(m) = \tilde{R}m + \tilde{p}V^* - \tilde{V}_u'\,(m)\,(c+\pi m) + qb \tag{26}$$

with

$$\tilde{V}^* \equiv \min_z \tilde{V}_u\,(z) \equiv \tilde{V}_u\,(m^*) \tag{27}$$

thus $\tilde{V}_u'\,(m^*) = 0$ where:

$$\tilde{R} = R + q \text{ and } \tilde{p} = p + q \tag{28}$$

Notice that, after redefining the interest rate and the probability of a free withdrawal this Bellman equation differs from the original one only by the constant qb in the flow term. Hence its solution should differ only by a constant, and the optimal policy should be the same, as the next proposition states.

Proposition. If V_u solves

$$(r+\tilde{p})\,V_u\,(m) = \tilde{R}m + \tilde{p}V^* - V_u'\,(m)\,(c+\pi m) \tag{29}$$

for all m and

$$V^* = V_u\,(m^*)\,, \quad V_u'\,(m^*) = 0 \tag{30}$$

then

$$\begin{aligned} \tilde{V}_u\,(m) &= V_u\,(m) + \frac{qb}{r} \\ \tilde{V}^* &= V^* + \frac{qb}{r} \\ \tilde{m}^* &= m^* \end{aligned} \tag{31}$$

solves (26) and (27).

Thus, to find the optimal return point m^* in the model with crime, we simply use the same function as in the case of the model without crime, and replace R by

$\tilde{R} = R + q$ and p by $\tilde{p} = p + q$. Likewise, for a given return point m^*,the expressions for M/c, W/M, and $\underline{M}/M$ are the same as in the model without crime, once we replace the value of p by $\tilde{p} = p + q$. We use this version of the model to reinterpret the relative cost of holding cash, $\tilde{R}/\left(b/c\right)$ as well as to find empirical proxies for $\tilde{p}$.

6 Costly random withdrawals

The dynamic model discussed above has the unrealistic feature that agents withdraw every time a match with a financial intermediary occurs, thus making as many withdrawals as contact with the financial intermediary, many of which of a very small size. In this section we extend the model to the case where the withdrawals (deposits) done upon the random contacts with the financial intermediary are subject to a fixed cost f. We assume that $0 < f < b$.

As mentioned above, this model has a more realistic depiction of the distribution of withdrawals, by limiting the minimum withdrawal size. In particular, we show that the minimum withdrawal size is determined by the fixed cost relative to the interest cost, i.e., f/R. Importantly, the minium withdrawal size is independent of p. On the other hand, if f is large relative to b, the prediction of the model gets closer to the ones of the Baumol-Tobin model. Indeed, if as f goes to b, then then there is no advantage of a chance meeting with the financial intermediary, and hence the model is identical to the one of the previous section, but with $p = 0$.

Agents face a cash-in-advance constraint, and they can withdraw or deposit from an interest bearing account. The sequence problem is to choose an increasing sequence of stopping times $\{\tau_j\}$ at which to withdraw (or deposit) money in an interest bearing account, and the amounts to withdraw at each time, so as to minimize the expected discounted cost of financing a given constant real consumption flow c. The expected discounted total cost, denoted by TC_0 is:

$$TC_0(\tau, m) = E_0 \left[\sum_{j=0}^{\infty} e^{-r\ \tau_j} \left\{ b\ I_{\tau_j} + f\ \hat{I}_{\tau_j} + \left(m\left(\tau_j^+\right) - m\left(\tau_j^-\right) \right) \right\} \right] \tag{32}$$

where we use $m(t)$ to denote the real value of the stock of currency. As before, the stock of currency jumps discontinuously up at the time of a withdrawal (so the amount of a withdrawal at τ_j is $m\left(\tau_j^+\right) - m\left(\tau_j^-\right)$) and the law of motion of the real value of the stock of money between withdrawals is given by equation (6).

As before we assume that contacts with the financial intermediary follow a Poisson process with arrival rate p. In the case of a contact the agent can withdraw (or deposit) money in an interest bearing account at a real cost f . If the agent wants to withdraw (or deposit) in the financial institution in any other time, it must pay a real cost b. The indicator I_{τ_j} takes the value of zero if the withdrawal (or deposit) takes place at the time $t = \tau_j$ of a contact with a financial intermediary, and takes the value of one otherwise. The indicator $\hat{I}_{\tau_j}$ takes the value of one if the withdrawal (or deposit) takes place at the time $t = \tau_j$ of a contact with a financial intermediary, and takes the value of one otherwise. The agent chooses stopping times and withdrawals as function of the history of contacts with the intermediary.

As before, we define the shadow cost of a policy $\{\tau_j, m\}$ as the expected discounted cost of the withdrawals plus the expected discounted opportunity cost of the cash balances held by the agent. We denote the shadow cost as SC_0, which is given by:

$$SC_0(\tau, m) = E_0 \left[\sum_{j=0}^{\infty} e^{-r\ \tau_j} \left\{ b\ I_{\tau_j} + f\ \hat{I}_{\tau_j} + \int_0^{\tau_{j+1}-\tau_j} R\ m\left(\tau_j + t\right)\ e^{-rt}\ dt \right\} \right] \tag{33}$$

The next Proposition is the analogous of Proposition 1.

Proposition 12. *Assume that $R = r + \pi$. For any policy $\{\tau, m\}$ the total cost equals* *the shadow cost plus the present value of* c, *or*

$$TC_0 = \frac{c}{r} + SC_0 \ .$$

Proof. The proof is completely analogous to the one for Proposition 1.

We use $V_s(m)$ for the value function corresponding to the minimization of the shadow cost:

$$V_s(m_0) = \min_{\tau, m} SC_0(\tau, m)$$

subject to $m(0) = m_0$ and where $s = f$ denotes that the agent is matched to a financial intermediary and $s = u$ that she is not. Let V^* be the minimum attained by the value function, i.e. $V^* \equiv V(m^*) = \min_z V(z)$, which is the value attained at the optimal return point m^* and is independent of the state s.

Using notation that is analogous to the one that was used above, the Bellman equation for this problem when the agent is not matched is given by:

$$rV_u(m) = Rm + p\min\{V^* + f - V_u(m), 0\} + V_u'(m)(-c - m\pi) \tag{34}$$

where $\min\{V^* + f - V_u(m)\ ,\ 0\}$ takes into account that it may not be optimal to withdraw/deposit for all contacts with a financial intermediary. Indeed, whether the agent chooses to do so will depend on her level of cash balances.

We will guess, and later verify, a shape for $V_u(\cdot)$ that implies a simple threshold rule for the optimal policy. Our guess is that $V_u(\cdot)$ is strictly decreasing at $m = 0$ and single peaked attaining a minimum at a finite value of m. Then we guess that there will be two thresholds, $\underline{m}$ and $\bar{m}$, that satisfy:

$$V^* + f = V_u(\underline{m}) = V_u(\bar{m}) \tag{35}$$

Under these assumptions the minimized cost takes the form:

$$\min\left\{V^*+f-V_u(m),0\right\}=\begin{cases}V^*+f-V_u(m)<0 & \text{if } m<\underline{m}\\ 0 & \text{if } m\in(\underline{m},\bar{m})\\ V^*+f-V_u(m)<0 & \text{if } m>\bar{m}\end{cases}$$

Thus solving the Bellman equation is equivalent to finding 5 numbers $m^*, m^{**}, \underline{m}, \bar{m},\ V^*$ and a function $V_u(\cdot)$ such that:

$$V^*=V_u(m^*)=\min_z V_u(z)$$

which, given the convexity of V_u, we can write as the following two equations:

$$V^* = V_u(m^*) \tag{36}$$

$$0 = V_u'(m^*) \tag{37}$$

and

$$V_u(m)=\begin{cases}\dfrac{Rm+p\left(V^*+f\right)-V_u'(m)\left(c+m\pi\right)}{r+p} & \text{if } m\in(0,\ \underline{m})\\ \dfrac{Rm-V_u'(m)\left(c+m\pi\right)}{r} & \text{if } m\in(\underline{m},\bar{m})\\ \dfrac{Rm+p\left(V^*+f\right)-V_u'(m)\left(c+m\pi\right)}{r+p} & \text{if } m\in(\bar{m},\ m^{**})\end{cases} \tag{38}$$

and the conditions:

$$V_u(0) = V^*+b \tag{39}$$

$$V_u(m) = V^*+b \text{ for } m>m^{**} \tag{40}$$

Hence the optimal policy in this model is to pay the fixed cost f and withdraw cash when the agent contact the financial intermediary, if her cash balance are in $(0, \underline{m})$ or to deposit if the cash balances are larger than $\bar{m}$. In either case the withdrawal or deposits is such that the post transfer cash balances are set equal to m^*. If the agent contacts a financial intermediary when her cash balances are in $(\underline{m},\ \bar{m})$ then, no action is taken. If the agent cash balances get to zero, then the fixed cost b is paid, and after the withdraw the cash balances are set to m^*. Notice that $m^* \in (\underline{m},\ \bar{m})$. Hence, in this version the withdrawals will have minimum size, namely $m^* - \underline{m}$. This is a more realistic depiction of actual management of cash.

Now we turn to the characterization and solution of the Bellman equation. The solution of the model is similar to the one in the body of the paper, in Propositions 2 and 3. By using the analogous of lemma 1 we obtain the following:

Proposition 13. *For a given $V^*, \underline{m}, \bar{m}, m^{**}$ satisfying $0 < \underline{m} < \bar{m} < m^{**}$:*
The solution of (38) for $m \in (\underline{m}, \bar{m})$ is given by:

$$\begin{aligned} V_u(m) &= \varphi(m, A_\varphi) \equiv \\ &\equiv \frac{-Rc/(r+\pi)}{r} + \left(\frac{R}{r+\pi}\right) m + \left(\frac{c}{r}\right)^2 A_\varphi \left[1 + \frac{\pi}{c} m\right]^{-\frac{r}{\pi}} \end{aligned} \tag{41}$$

for an arbitrary constant A_φ

Likewise, the solution of (38) for $m \in (0,\ \underline{m})$ or $m \in (\bar{m},\ m^{**})$ is given by:

$$\begin{aligned} V_u(m) &= \eta(m, V^*, A_\eta) \equiv \\ &\equiv \frac{p(V^* + f) - Rc/(r+p+\pi)}{r+p} + \left(\frac{R}{r+p+\pi}\right) m + \left(\frac{c}{r+p}\right)^2 A_\eta \left[1 + \frac{\pi}{c} m\right]^{-\frac{r+p}{\pi}} \end{aligned} \tag{42}$$

for an arbitrary constant A_η.

Proof. *See appendix D.*

Next we are going to list a system of 5 equations in 5 unknowns that describes a C^1 solution of $V_u(m)$ on the range $[0, m^*]$. The unknowns in the system are $V^*, A_\eta, A_\varphi, \underline{m}, m^*$. Using proposition 13, and the boundary conditions (35),(36),(37) and (39), the system is given by:

$$\begin{aligned} \varphi_m(m^*, A_\varphi) &= 0 & (43)\\ \varphi(m^*, A_\varphi) &= V^* & (44)\\ \eta(\underline{m}, V^*, A_\eta) &= V^* + f & (45)\\ \eta(0, V^*, A_\eta) &= V^* + b & (46)\\ \varphi(\underline{m}, A_\varphi) &= V^* + f & (47)\end{aligned}$$

In the proof of proposition 14 we show that the solution of this system can be found by solving one non-linear equation in one unknown, namely $\underline{m}$. Once the system is solved it is straightforward to extend the solution to the range: (m^*, ∞).

Proposition 14. *There is a unique solution for the system (43)-(47). The solution characterizes a C^1 function that is strictly decreasing on $(0, m^*)$, convex on $(0, \bar{m})$ and strictly increasing on (m^*, m^{**}). This function solves the Bellman equations described above. The value function satisfies*

$$V_u(0) = \frac{R}{r}\, m^* + b$$

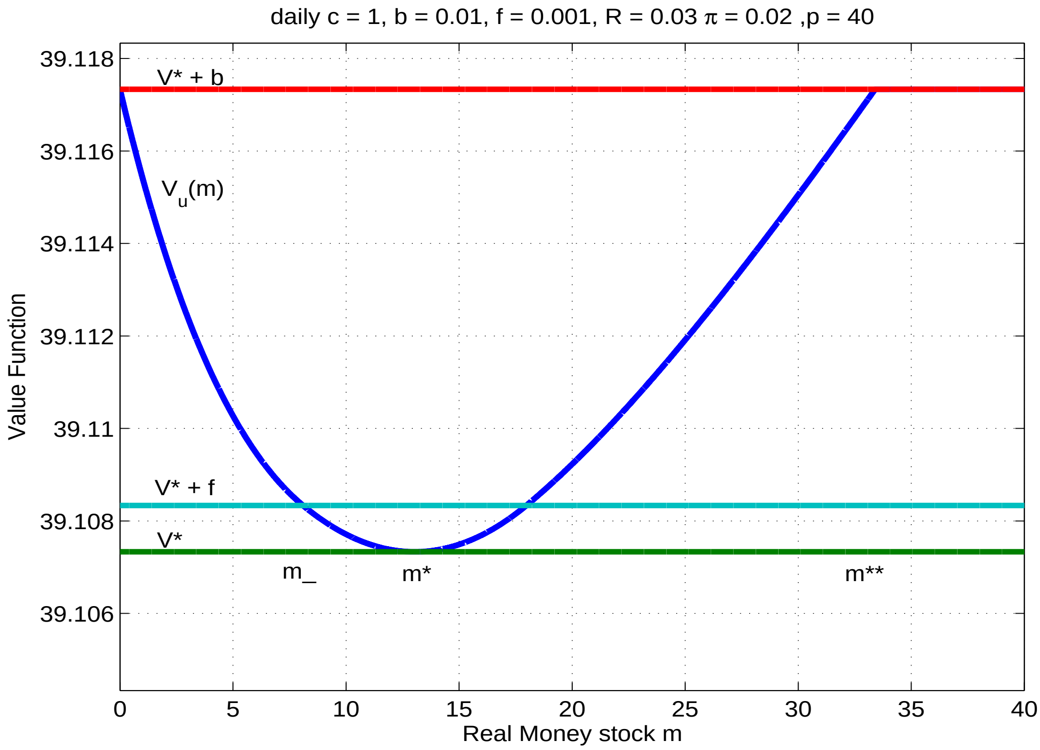
Proof. See appendix D.

Next we present a proposition about the determinants of the range of inaction $m^* - \underline{m}$, or equivalently the size of the minimum withdrawal.

Proposition 15. *The range of inaction $(m^* - \underline{m})$ relative to the drift of cash balances, $c + \pi m^*$, solves:*

$$\frac{f}{R(c + m^*\pi)} = \left(\frac{m^* - \underline{m}}{c + m^*\pi}\right)^2 \left[\frac{1}{2} + \sum_{k=1} \frac{1}{(k+2)!} \left(\frac{m^* - \underline{m}}{c + m^*\pi}\right)^k \Pi_{j=2}^{k+1}(r + j\pi)\right] \quad (48)$$

Figure 7: Value function for $f > 0$



Hence $(m^* - \underline{m}) / (c + m^*\pi)$ *is increasing in* f/R *(with elasticity smaller than* $1/2$*) and decreasing in* r*. Moreover it is decreasing (increasing) in* π *if* $\pi > 0$ *(*$\pi < 0$*).*

Finally, for small $f \;/\; [R\,(c + \pi m^*)]$ *we have*

$$\frac{m^* - \underline{m}}{c + m^*\pi} = \sqrt{\frac{2\ f}{R\ \ (c + \pi m^*)}} + o\left(\left(\frac{f}{R\ \ (c + \pi m^*)}\right)^2\right) \ . \tag{49}$$

Proof. See appendix D.

Importantly, this proposition says that the scaled range of inaction $(m^* - \underline{m}) / (c + m^*\pi)$ is *not* a function of p or b. Its approximation implies that

$$\frac{R}{(m^* - \underline{m})} \frac{\partial\,(m^* - \underline{m})}{\partial R}|_{\pi=0} = -\frac{1}{2} + \frac{1}{2}\left(R\frac{m^*}{c}\right)\frac{\partial \pi}{\partial R}.$$

The quantity $c + m^*\pi$ is a measure of the use of cash per period: the sum of cash consumption plus the opportunity cost πm^*. The quantity $m^* - \underline{m}$ also measures the size of the smallest withdrawal. Hence $(m^* - \underline{m}) / (c + m^*\pi)$ is a measure of the minimum withdrawal in terms of the cash used per period. We stress that the minimum withdrawal does not depend on p and b, and that, as the approximation above makes clear, it is analogous to the withdrawal of the BT model facing a fixed cost f and an interest rate R.

The next proposition examines the expected number of withdrawals n.

Proposition 16. *The expected number of withdrawals per unit of time, n is given by*

$$n = \frac{p}{(p/\pi)\log\left(1 + (m^* - \underline{m})\,\pi/c\right) + 1 - (1 + \underline{m}\pi/c)^{-\frac{p}{\pi}}} \tag{50}$$

and the fraction of agents with cash balances below $\underline{m}$ is given by

$$H(\underline{m}) = \frac{1 - (1 + \underline{m}\pi/c)^{-\frac{p}{\pi}}}{(p/\pi)\log\left(1 + (m^* - \underline{m})\,\pi/c\right) + 1 - (1 + \underline{m}\pi/c)^{-\frac{p}{\pi}}} \tag{51}$$

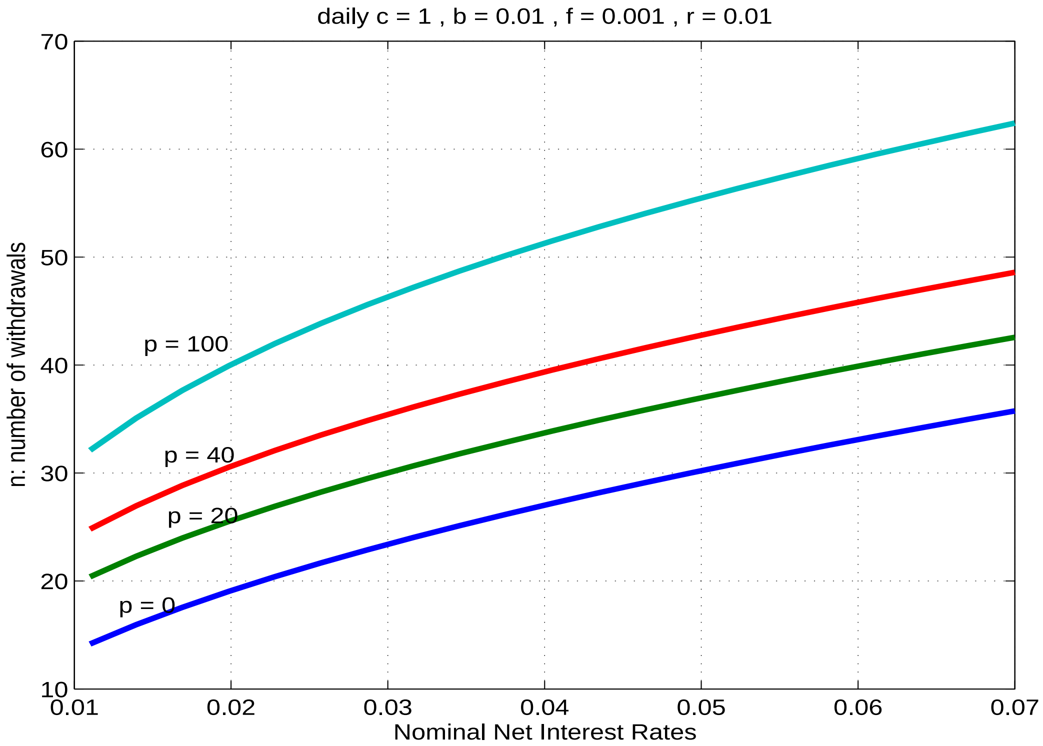
Proof. *See appendix D.*

Inspection of equation (50) confirms that when $m^* > \underline{m}$ the expected number of withdrawals (n) is no longer bounded below by p. Indeed, as $p \to \infty$ then $n \to \left[(1/\pi)\log\left(1 + (m^* - \underline{m})\,\pi/c\right)\right]^{-1}$, which is the reciprocal of the time that it takes for an agent that starts with money holding m^* (and consuming at rate c when the inflation rate is π) to reach real money holdings $\underline{m}$.

The next figure plots n against the nominal interest rate for several values of p. To highlight the role of $f > 0$ all the subsequent figures have the same parameter values for b, c and r that were used above for the case where $f = 0$.

Compare this figure with the one obtained for $f = 0$. Notice that the number

Figure 8: Number of withdrawals for $f > 0$



of trips associated to the same values for p and R are much smaller; in particular notice than in this case $n < p$. Given the parameters in this figure, a value of p of about 200 is required for the number of withdrawals to be similar to the ones in the case of $f = 0$ and $p = 40$, which are similar to the ones of households with an ATM card in our data set.

As in the case of $f = 0$, for any $m \in [0, \underline{m}]$ the density $h(m)$ solves the following ODE:

$$\frac{\partial h(m)}{\partial m} = \frac{(p - \pi)}{(\pi m + c)} h(m)$$

The reason for this is that in this interval the behavior of the system is the same as the one for $f = 0$. On the interval $m \in [\underline{m}, m^*]$ the density $h(m)$ solves the following ODE:

$$\frac{\partial h(m)}{\partial m} = \frac{-\pi}{(\pi m + c)} h(m)$$

The reason for this is that locally in this interval the chance meetings with the intermediary do not trigger a withdrawal, and hence it is as if $p = 0$.

Proposition 17. *The density $h(m)$ and CDF $H(m)$ for $m \in [0, \underline{m}]$ are given by:*

$$h(m) = A_0 \left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi} - 1} \tag{52}$$

$$H(m) = \frac{c}{p} A_0 \left[\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} - 1\right] \tag{53}$$

where

$$A_0 = \frac{p}{c} \frac{H(\underline{m})}{\left(1 + \frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}} - 1} \tag{54}$$

.

The density $h(m)$ and CDF $H(m)$ for $m \in [\underline{m}, m^*]$ are given by:

$$h(m) = A_1 \left(1 + \frac{\pi}{c} m\right)^{-1} \tag{55}$$

$$H(m) = \frac{c}{\pi} A_1 \log\left(\frac{1 + \frac{\pi}{c} m}{1 + \frac{\pi}{c} m^*}\right) + 1 \tag{56}$$

where

$$A_1 = \frac{\pi}{c} \frac{1 - H(\underline{m})}{\log\left(1 + \frac{\pi}{c} m^*\right) - \log\left(1 + \frac{\pi}{c}\underline{m}\right)} \tag{57}$$

Proof. *See appendix D.*

Using the previous density we compute average money holdings.

Proposition 18. *The average (expected) real money holdings are:*

$$M = \int_0^{\underline{m}} m h(m)\, dm + \int_{\underline{m}}^{m^*} m h(m)\, dm$$

or

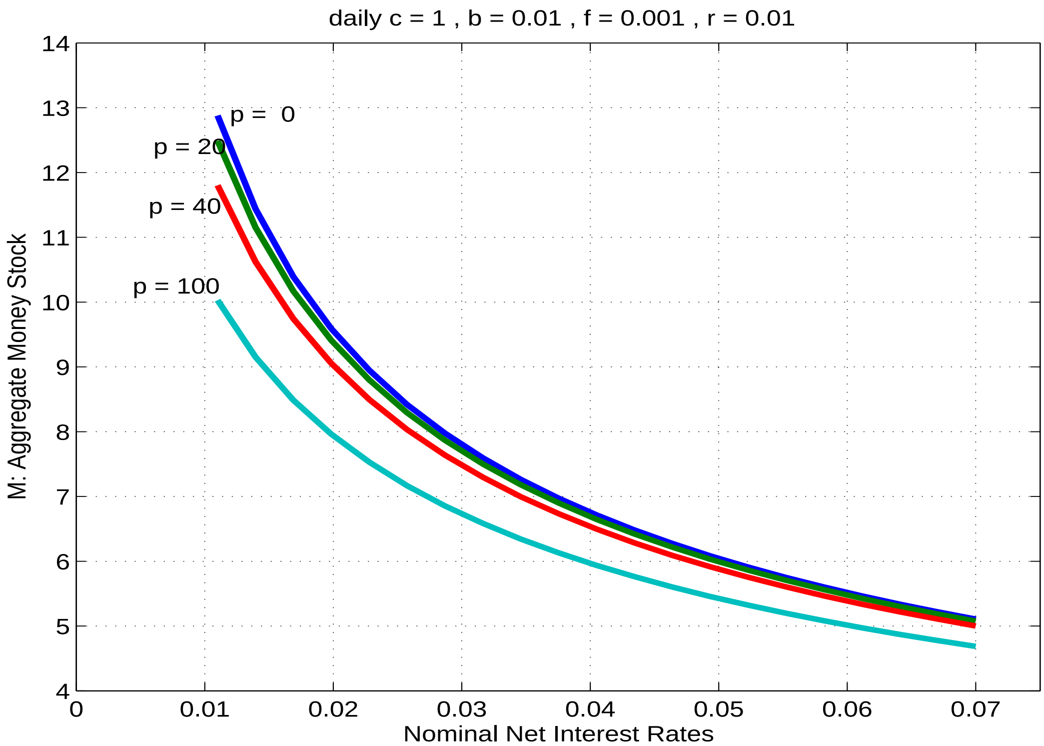
$$M = m^* - \frac{c}{p} A_0 \left[\frac{\left(1 + \frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1} - 1}{(p+\pi)/c} - \underline{m} \right] \tag{58}$$
$$-A_1 \left(\frac{c}{\pi}\right)^2 \left\{ \left(1 + \frac{\pi}{c} m^*\right) \left[\log\left(1 + \frac{\pi}{c} m^*\right) - 1\right] - \left(1 + \frac{\pi}{c}\underline{m}\right) \left[\log\left(1 + \frac{\pi}{c}\underline{m}\right) - 1\right] \right\}$$
$$+ \left(m^* - \underline{m}\right) \left(\frac{c}{\pi} A_1 \log\left(1 + \frac{\pi}{c} m^*\right) - 1 \right)$$

where A_0 and A_1 are given in (54) and (57).

Proof. *See appendix D.*

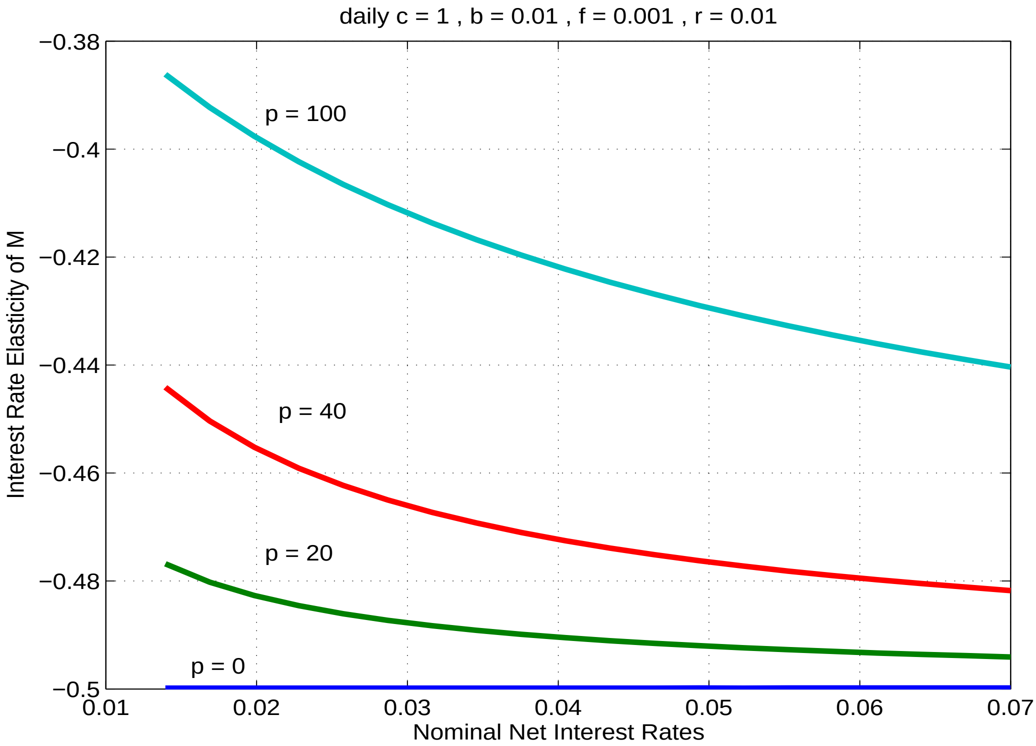
The next figures plot the level and elasticity of money demand for the same parameter values.

Figure 9: Money demand for $f > 0$



While, as shown in the previous figures, the introduction of $f > 0$ has a large effect on the average number of withdrawals, it has a much smaller effect on the level

Figure 10: Interest elasticity of money demand for $f > 0$



and on the interest rate elasticity of money demand. This is quite natural, since the effect of the fixed cost f on the number of withdrawals comes from eliminating the ones that are small in size.

As done in Section 5, we use the density to compute the average withdrawal:

Proposition 19. *The average withdrawal W is given by:*

$$W = m^* \left[1 - \frac{p}{n} H\left(\underline{m}\right)\right] + \left[\frac{p}{n} H\left(\underline{m}\right)\right] \frac{\int_0^{\underline{m}} \left(m^* - m\right) h\left(m\right) dm}{H\left(\underline{m}\right)} \tag{59}$$

where

$$\frac{\int_0^{\underline{m}} \left(m^* - m\right) h\left(m\right) dm}{H\left(\underline{m}\right)} = m^* - \underline{m} + \frac{\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1} - 1}{(p+\pi)/c} - \underline{m}}{\left(1 + \frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}} - 1}$$

To understand this expressions notice that $n - pH\left(\underline{m}\right)$ is the number of withdrawals in a unit of time that occur because agents reach zero balances, so if we divided it by the total number of withdrawals per unit of time, n, we obtain

$$\left[\frac{n - pH(\underline{m})}{n}\right] = 1 - \frac{p}{n} H(\underline{m})$$

i.e. the fraction of withdrawals that occur when agent reach zero balances. Each of these withdrawals is of size m^*. The complementary fraction gives the withdrawals that occur due to a chance meeting with the intermediary. Conditional on having money balances in $(0, \underline{m})$ then a withdrawal of size $(m^* - m)$ happens with frequency $h(m) / H(\underline{m})$.

By the same reasoning than in the $f = 0$ case, the average amount of money that an agent has at the time of withdrawal, $\underline{M}$, satisfies

$$\underline{M} = 0 \left[1 - \frac{p}{n} H(\underline{m})\right] + \left[\frac{p}{n} H(\underline{m})\right] \frac{\int_0^{\underline{m}} m\ h(m)\, dm}{H(\underline{m})} .$$

Simple algebra shows that:

$$\underline{M} = m^* - W. \tag{60}$$

Alternatively, inserting the definition of $\underline{M}$ into the expression for M we obtain

$$M = \frac{n}{p}\ \underline{M} + \int_{\underline{m}}^{m^*} mh(m)\, dm$$

or

$$p = n \quad \frac{\underline{M}/M}{1 - \left[\int_{\underline{m}}^{m^*} mh(m)\, dm\right] / M}.$$

7 Estimation of the model

This section estimates the structural parameters of the theoretical model presented above using the household data set described in Section 2. Our estimation procedure selects parameters values of (b, f, p) to produce values for $(M/c,\ W/c,\ n,\ \underline{M}/M)$

that are closest to the analogous quantities in the data, for each year, province and type of households, i.e. those with and without ATM cards. We investigate how our estimates of (b, f, p) relate to the empirical measures of financial innovations presented in Table 2 , and use them to asses if the pattern of financial development makes the cash-management more or less similar to the one of the Baumol-Tobin type . Finally, we use our model and the estimated parameters to compute the implied benefits of using an ATM card.

In the following discussion we fix a particular combination of year-province-type of household, where type is defined by the ownership of an ATM card. We let i index the household in that year-province-type combination. For a given year-province-type we assume that households can differ in their cash consumption c_i and their cost parameter b_i and f_i. We constraint these parameters so that $b_i = c_i\ b$ and $f_i = c_i\ f$. The rationale behind this assumption is that the cost b_i and f_i are mostly opportunity cost of the time, so that they are likely to be high for those household wit high cash-consumption. We assume that p is common for all households. Given the homogeneity of the optimal decision rules, these assumption allows us to aggregate the decisions of different households of a given year-province-type.

We assume that the variables M/c, W/M, n and $\underline{M}/M$, which we index as $j = 1, 2, 3$ and 4, are measured with a multiplicative error (additive in logs). Let x_i^j be the (log of the) $i - th$ observation on variable j, and $f^j(\theta)$ the (log of the) model prediction of the j variable for the parameter vector θ. The number N_j is the size of the sample of the variable j.[7]The idea behind this formulation is that the variable x_i^j is observed with a measurement error ε_i^j which has zero expected value and variance σ_j^2 so that

$$x_i^j = f^j(\theta) + \varepsilon_i^j$$

[7]Our data set have different number of observations for different variables. For instance, in year 2004 the question about the cash at withdrawal M was dropped from the survey. Additionally only a fraction of the households are asked about withdrawal amounts W.

where the errors ε_i^j are assumed to be independent across observations (i.e. households) and for now, across variables j. We estimate a the vector of parameters $\theta = (b, f, p)$ for each province-year-type by minimizing the following objective function:

$$F(\theta; x) \equiv \sum_{j=1}^{4} \left(\frac{N_j}{\sigma_j^2}\right) \left(\frac{1}{N_j} \sum_{i=1}^{N_j} x_i^j - f^j(\theta)\right)^2$$

where σ_j is an estimate of the variance of the measurement error for the variable j. We estimate σ_j^2 as the variance of the residual in a regression where we pull all the households of a given type, and include dummies for each province-year combination. Table 4 displays statistics for N_j across provinces and variables, as well as the values used for σ_j.

Table 4: Weights used in the estimation

	$log(M/c)$	$log(W/M)$	$log(n)$	$log(\underline{M}/M)$
	Households with ATM			
Standard dev. (σ_j)	0.78	0.72	0.76	0.97
Mean number of obs. per province-year(N_j)	32.2	18.0	28.9	27.1
	Households without ATM			
Standard dev. (σ_j)	0.73	0.80	0.87	1.01
Mean number of obs. per province-year(N_j)	26.3	20.3	18.6	20.9

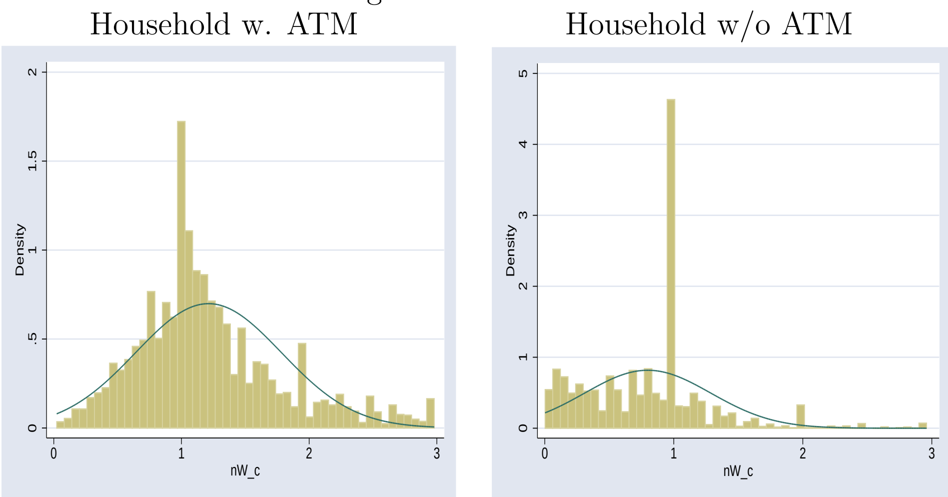
Notes: There is a total of 103 provinces and 6 years.

The estimation criterion assumes that the variables are affected by measurement error, a recurrent feature of survey data. For instance, measurement error in consumption has been documented in this data set by Battistin, Miniaci and Weber (2003). An illustration of the extent of the measurement error can be derived by assuming that the data should satisfy the identity for the cash flows:

$$c = n\, W - \pi M$$

which holds in a large class of models, as derived in Appendix E. Figure 11 reports a histogram of $n\ (W/c) - \pi\,(M/c)$ for each type of households. In the absence of measurement error, all the mass should be located at 1. It is clear that the data deviate from this value for many households. [8] At least for households with an ATM card, we view the histogram as well approximated by a normal distribution (in log scale).

Figure 11: Measurement error



While not strictly necessary, we add the assumption that the errors ε_j^j are normally distributed. Under the assumption that the measurement error is normally distributed the estimator coincide with the maximum likelihood estimator.

We also present estimates using an objective where we allow the variance of the $x^{j\prime}s$ to be non-diagonal. In this case the objective function is:

$$F(\theta, x) = [\ \bar{x} - \ f(\theta)]\ '\ \Omega^{-1}\ [\ \bar{x} - \ f(\theta)]$$

[8] Besides classical measurement error, which is probably important in this type of survey, there is also the issue of whether households have an alternatively source of cash. An example of such as source occurs if households are paid in cash. This will imply that they do require fewer withdrawals to finance the same flow of consumption, or alternatively, that they effectively have more trips per periods.

where $f(\theta) = (f^1(\theta), ..., f^4(\theta))$, $\bar{x} = (\bar{x}^1, ..., \bar{x}^4)$ with $\bar{x}^j = \left(\frac{1}{N_j}\sum_{i=1}^{N_j} x_i^j\right)$ and $\Omega = T\ \Sigma\ T'$ where Σ is the var-cov matrix of the x^j and $T = diag\left(\sqrt{N^j}\right)$.

7.1 Identification

In this section we discuss the features of the data that identifies our parameters. We argue with our date set we can identify (p, b) and test the model, but that the type of information that we have does not allow to identify f.

Let us consider first the version of the model with $f = 0$. As a first step we study how to select the parameters to match M/c and n only, as opposed to $(M/c, n, W/M, \underline{M}/M)$. To simplify the exposition here, assume that inflation is zero, so that $\pi = 0$. For the BT model, i.e. for $p = 0$, we have

$$W = m^*,\ c\ =\ m^*\ n,\ \text{and}\ M = m^*\ (1/2)$$

which implies

$$M/c = (1/2)\ /\ n\ .$$

Hence, if the data were generated by the BT model, M/c and n would have to satisfy this equation. Now consider the average cash balances generated by a policy like the one of the model of Section 4 with zero inflation, i.e. with $f = \pi = 0$, for an given value of p. We have:

$$M/c = \frac{1}{p}\left[n\ m^*/c - 1\right]\ \text{and}\ n = \frac{p}{1 - \exp\left(-pm^*/c\right)} \tag{61}$$

or, solving for M/c as a function of n :

$$M/c = \xi\left(n, p\right) \equiv \frac{1}{p}\left[-\frac{n}{p}\log\left(1 - \frac{p}{n}\right) - 1\right]\ .$$

For a given p,the $M/c = \xi\left(n,p\right)$, and n are the pairs that are consistent with a cash management policy of replenishing cash to some value m^{*} either when the balances reach zero, or when there is chance meeting with an intermediary and suffices to finance a consumption flow c. Notice first that setting $p = 0$ in this equation we obtain BT, i.e.

$$\xi\left(n,0\right) = \left(1/2\right) \ / \ n$$

Second, notice that this function is defined only for $n \geq p$. Furthermore, note that for $p > 0$ (see Appendix G for details):

$$\begin{aligned}
\frac{\partial \xi}{\partial n} &= \left(\frac{1}{p}\right)^{2}\left[\log\left(\frac{n}{n-p}\right) - \frac{p}{n-p}\right] \leq 0 \\
\frac{\partial^{2} \xi}{\partial n^{2}} &= \left(\frac{1}{p}\right)^{2}\frac{p}{\left(n-p\right)^{2}}\frac{p}{n} > 0 \\
\frac{\partial \xi}{\partial p} &= \frac{1}{p^{2}}\frac{n}{p}\left[2\log\left(1-\frac{p}{n}\right) + 1 + \frac{p/n}{1-p/n}\right] > 0
\end{aligned}$$

Consider plotting the target value of the data on the $(n,\ M/c)$ plane. For a given M/c, there is a minimum n that the model can generate, namely the value $\left(1/2\right)/\left(M/c\right)$. Given that $\partial\xi/\partial p > 0$, any value of n smaller than the one implied by the BT model cannot be made consistent with our model, regardless of the values for the rest of the parameters. By the same reason, any value of n higher than $\left(1/2\right)/\left(M/c\right)$ can be accommodated by an appropriate choice of p. This is quite intuitive: relative to the BT model, our model can generate a larger number of withdrawals for the same M/c if the agent meets an intermediary often enough, i.e. if p is large enough. On the other hand there is a minimum number of expected chance meetings, namely $p = 0$.

The previous discussion, show that p is identified. Specifically, fix a province-year-type of household combination, with its corresponding values for the averages of M/c and n, then solving $M/c = \xi\left(n, p\ \right)$ for p gives an estimate of p. Then, taking this value of p, M/c and n for this province-year-type of household combination, we use (20) to find the corresponding m^*/c as follows:

$$m^*/c = \frac{p\ M/c + 1}{n}.$$

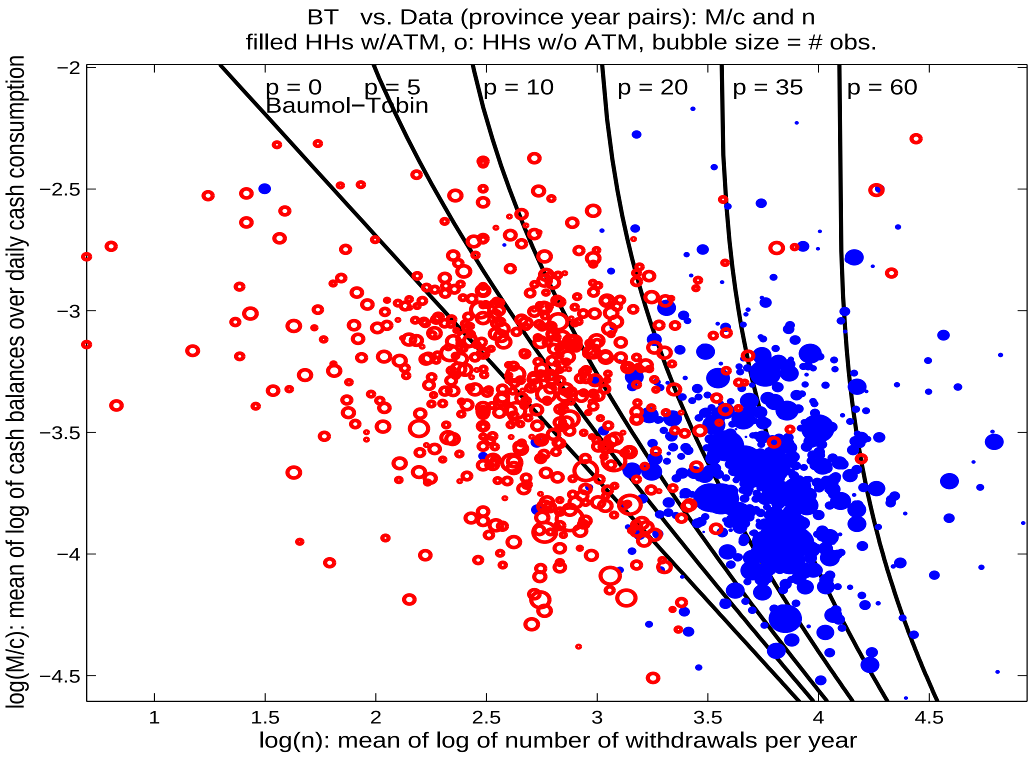
Finally, we use (92) to find the value of $\beta \equiv b/(cR)$ that rationalizes this choice. In particular, we specialize the expression in Appendix H to the case of $\pi = f = 0$ to obtain:

$$\beta \equiv \ \frac{b}{cR} = \left(\frac{1}{(r+p)^2}\right)(\exp\left((r+p)\ \ m^*/c\right) - \left[1 + (r+p)\left(m^*/c\right)\right]\ ) \tag{62}$$

(see the appendix G for details). To understand this expression, consider two pairs $(M/c, n)$, both pairs in the locus defined by $\xi(\cdot, p)$ for a given value of p. The pair with higher M/c and lower n corresponds to a higher value of β. This is quite intuitive: agents will economize on trips to the financial intermediary if β is high, i.e. if these trips are expensive relative to the opportunity cost of cash. Hence, data on M/c and n identify p and β. Using data on R for this province-year, we can estimate b/c.

Figure 12 plots the function $\xi(\cdot, p)$ for several values of p, as well as the average value of M/c and n for all households of a given type (i.e. with and without ATM cards) for each province-year in our data. Notice that 46 percent of province-year pairs for households without an ATM card are below the $\xi(\cdot, 0)$ line, so no parameters in our model can rationalize those choices. The corresponding value for those with an ATM card is only 3.5 percent of the pairs. The values of p required to rationalize the average choice for most province-year pairs for those households without ATM cards are in the range $p = 0$ to $p = 20$. The corresponding range for those with ATM cards is between $p = 5$ and $p = 60$. Inspecting this figure we can also see that the observations for households with ATM cards are to the south-east of those for households without ATM cards. Equivalently, we can see that for the same value of p, the observations that correspond to households with ATM tend to have lower values of β.

Figure 12: Theory vs. data



Now we turn to the analysis of the ratio of the average withdrawal to the average cash balances, W/M. As in the previous case, consider an agent that follows an arbitrary policy of replenishing her cash to a return level m^*, either as her cash balances gets to zero, or at the time of chance meeting with the intermediary. Again, to simplify consider the case of $\pi = 0$. Using the expression for for W/M (22), and replacing m^* from (61) we can define the function ζ as follows

$$\frac{W}{M} = \zeta\left(n, p\right) \equiv \left[\frac{1}{p/n} + \frac{1}{\log\left(1 - p/n\right)}\right]^{-1} - \frac{p}{n} \tag{63}$$

for $n \geq p$, and $p \geq 0$ (see appendix G for details). After some algebra one can show that

$$\begin{aligned}\zeta(n,0) &= 2,\ \zeta(n,n)=0, \\ \frac{\partial\zeta(n;p)}{\partial p} &< 0 \text{ and } \frac{\partial\zeta(n;p)}{\partial n}>0\end{aligned}$$

Notice that the ratio W/M is a function only of the ratio p/n.The interpretation of this is clear: for $p=0$ we have $W/M=2$, as in the BT model. This is the highest value that can can be achieved of the ratio W/M. As p increases for a fixed n, the replenishing level of cash m^*/c must be smaller, and hence the average withdrawal becomes smaller relative the average cash holdings M/c. Indeed, as n converges to p – a case where almost all the withdrawals are due to chance meetings with the intermediary–, then W/M goes to zero.

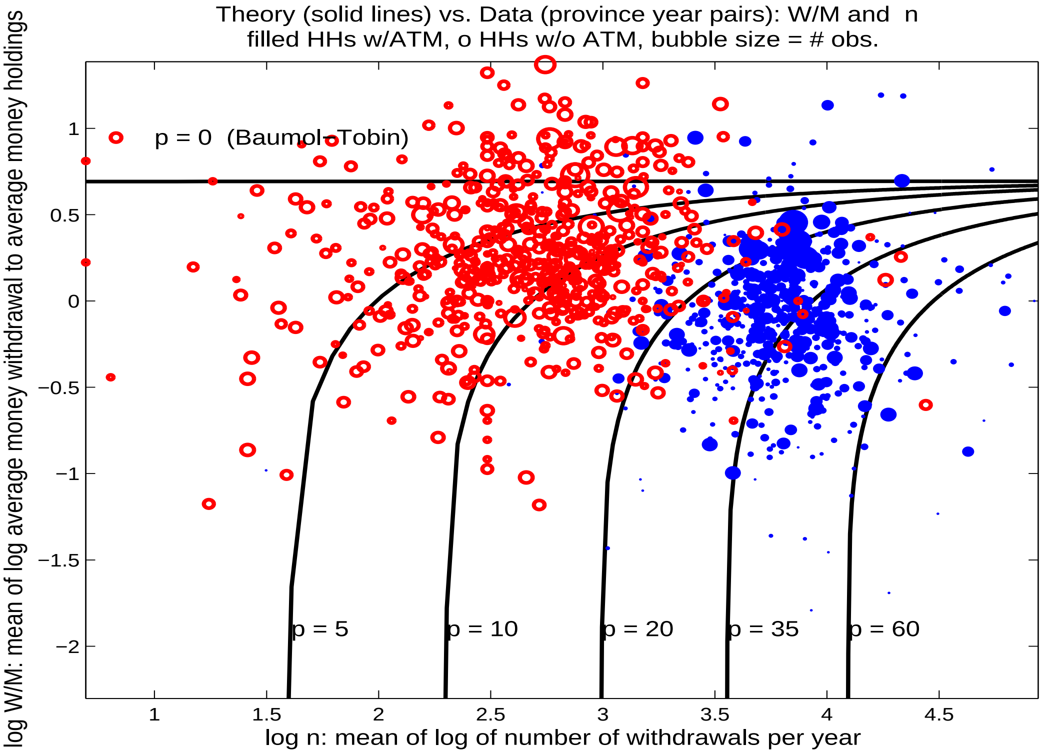
As in the previous case, given target values of W/M and n in the data we can use ζ to solve for the corresponding p. Then, using the values of $(W/M\ p,n)$ we can find a value of $(b/c)/R$ to rationalize the choice of W/M. To see how, notice that given $W/M,\ M/c$, and p/n, we can find the value of m^*/c using

$$\frac{W}{M}=\frac{m^*/c}{M/c}-\frac{p}{n}$$

With the values of $(m^*/c,\ p)$ we can find the unique value of $\beta=(b/c)/R$ that rationalize this choice, using (62). Thus, data on W/M and n identifies p.

Figure 13 plots the function $\zeta(n,p)$ for several values of p, as well as the average values of n and W/M for the different province-year-household type type combinations for our data set. We note that about 3 percent of the year province pairs of households with an ATM cards have W/M above 2, while for those without ATM card the corresponding value is 15 percent. In this this case, as opposed to the experiment displayed in Figure 11, no data on the average consumption flow c is used, thus it may be that these smallest percentages are due to larger measurement error on c. The implied values of p needed to rationalize these data are similar to the ones found using the information of M/c and n displayed in Figure 12. Also the implied values of β that corresponds to the same p tend to be smaller for households with an ATM card since the observations are to the south-east.

Figure 13: Theory vs. data



Finally we discuss the ratio between the average cash at withdrawals and the unconditional average cash: $\underline{M}/M$. In (24) we have derived that:

$$p = n \quad (\underline{M}/M) \, .$$

We use this equation as a way to estimate p. If $\underline{M}$ is zero, then p must be zero, as it is in the model with no randomness, such as the BT model –even if there are some "free" withdrawals. Hence, the fact that, as Table 1 indicates $\underline{M}/M > 0$ is an indication that our model requires $p > 0$. We can readily use this equation to estimate p since we have data on both n and $(\underline{M}/M)$. According to this formula a large value of p is consistent with either a large value of cash at withdrawals, $\underline{M}/M$, or a large number of total withdrawals, n. Also, for a fixed p, different combination of n and $\underline{M}/M$ that give the same produce are due to differences in $\beta = (b/c)/R$. If β is high, then agents economize in the number of withdrawals n and keep larger cash balances.

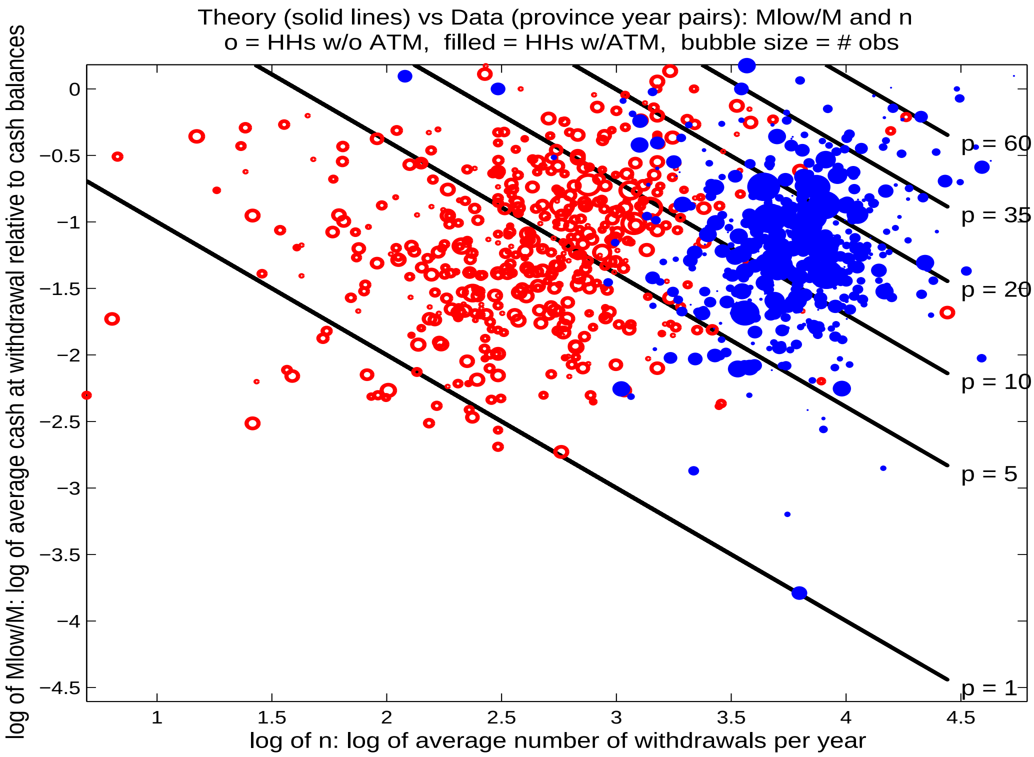
Figure 14 plots for each province/year type the average logarithm of $\underline{M}/M$ and n, as well as lines corresponding different hypothetical values of p. The fraction of province-years where $\underline{M}/M > 1$, is less than 3 percent for both households with and without ATMs. The ranges of values of p needed to rationalize the choices of households with and without ATM across the province-years is similar than the ones in the previous two figures. Also, as in the previous two figures, for the same p the observations corresponding to households with ATM correspond to lower values of β –they are to the south-east of those without ATM cards.

We have discussed how data on either of the pairs $(M/c, n)$, $(W/M, n)$ or $(\underline{M}/M,\ n)$ identify p and β. Of course, if the model has generated the data, the three ways of estimating $(p,\ \beta)$ should yield the same values. In other words, the model is overidentified. We will use this idea to report how well the model fits the data, or more formally, to test for the overidentifying restrictions. For instance, the model with $\pi = 0$ implies the following relationship between $\underline{M}/M$ and W/M:

$$\frac{W}{M} = \left[\frac{1}{(\underline{M}/M)} + \frac{1}{\log\left(1 - (\underline{M}/M)\right)}\right]^{-1} - (\underline{M}/M) \tag{64}$$

which is obtained by replacing $p/n = \underline{M}/M$ into $W/M = \zeta\,(1, p/n)$. This function is decreasing, reflecting that W/M is decreasing in p and $\underline{M}/M$ is increasing in p.

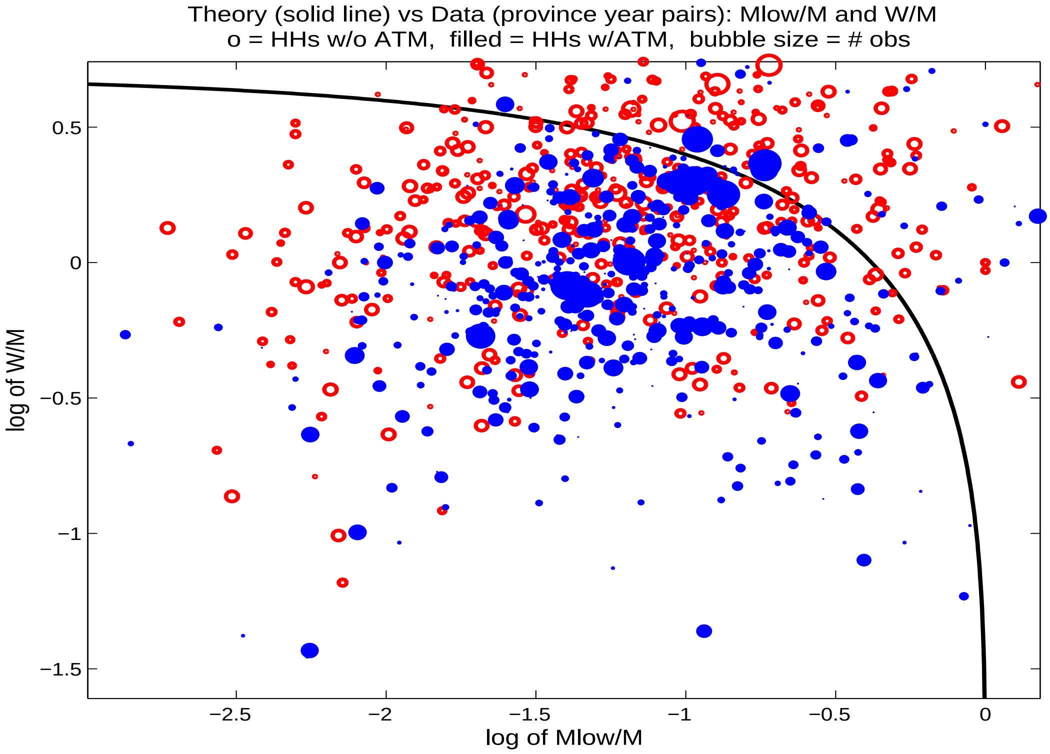
Figure 14: Theory vs. data



Indeed for p close to zero, the model tends to BT so $\underline{M}/M$ has to be close to zero and W/M has to be close to 2, and for p very large, W/M is close to zero and $\underline{M}/M$ is close to one. Figure 15 plots this theoretical relationship as well as the pair of W/M and $\underline{M}/M$ for different province year, all in logs. Different points on the theoretical curve defined by the function (64) correspond to different values of p.

It is apparent that lots of the province-year pairs are not close to the theoretical relationship, especially for the case of households with no ATM cards. Indeed the pairs of W/M and $\underline{M}/M$ for different province-years are positively correlated in the data, with correlations of 0.18 and 0.33 for those with ATM and without ATM cards, respectively. One potential reason for this counterfactual positive correlation is measurement error in M. Clearly, if there is substantial measurement error in M, the error ridden quantities W/M and $\underline{M}/M$ can be positively correlated even if they

Figure 15: Theory vs. data



were generated by a model where the relationship (64) holds.

Considering the case of $\pi > 0$ makes the expressions more complex, but, at least qualitatively, does not change any of the properties discussed above. Moreover, quantitatively, since the inflation rate in our data set is quite low, the expressions of the model for $\pi = 0$, approximates the relevant range for $\pi > 0$ very well. The estimates obtained below use the inflation rate π that corresponds to each year for Italy.

7.2 Estimation results

We estimate the $f = 0$ model for each province-year-type of household and report statistics of the estimates in Table 5. For each year we use the inflation rate corresponding to the Italian CPI for all provinces and fix the real return r to be 2% per year. The first two panels report the mean, median, 95th and 5th percentile of the estimated values for p and b/c across all province-years. The parameter p is measure in probability per unit of time, so that $1/p$ gives the average number of contacts per year. The parameter b is measure in units of daily cash-consumption. [9] We also report the mean and median values of the t statistics for these parameters. The standard errors are computed by solving for the information matrix. The results in this table confirm the graphical analysis of figures 12-14 discussed in the previous section: the median estimates of p are just where one will locate them in these figures. The t-statistics indicates that in average the values are precisely estimated. The difference between the mean and the median, as well as the 95/05 percentiles show that, as the figures show, there is a tremendous amount of heterogeneity across province-years. As expected, this table shows households with ATM cards have a higher mean and median value of p and correspondingly lower values of b. It is also the case that 95 percent of the province-year pairs the estimated value of p is higher for those with ATM, and for 93 percent of the pairs the estimated value of b is lower. Also the correlation between the estimated values of b for households with and without ATM across province-year pairs is 0.79. The same statistic for p is 0.37. These patterns are consistent with the hypothesis that households with ATM cards have access to a more efficient transactions system, and that the efficiency of the transaction technology in a given province-year is correlated for both ATM and non-ATM adopters. We find these reassuring since we have estimated the model for ATM holders and non-holders and for each province-year separately.

The third panel presents statistics that measure the goodness of fit of the model. We report the mean and median values of the minimized objective function F. Under the assumption of normal distributed errors, or as an asymptotic result, the minimized objective function is distributed as a $\chi^2_{(2)}$. According to the statistic reported

[9] See Table 1 for estimates of average annual cash-consumption, and ratios of cash-consumption to consumptio of non-durables and services.

Table 5: Estimation results for the $f = 0$ model

	Household w/o ATM	Household w. ATM
Parameter p		
Mean	7.2	23.2
Median	5.7	18.9
95^{th}percentile	18.3	50.1
5^{th} percentile	1.4	4.9
Mean t-stat	3.3	4.5
Median t-stat	3.1	3.9
Parameter b		
Mean	0.04	0.02
Median	0.02	0.005
95^{th} percentile	0.15	0.05
5^{th} percentile	0.002	0.001
Mean t-stat	3.8	4.0
Median t-stat	3.5	3.5
Goodness of fit		
Objective function $F(\theta, x) \sim \chi^2(2)$		
Mean	9.3	14.3
Median	4.1	6.0
Percentage of province-years where:		
- $F(\theta, x) < 4.6^*$	52.9%	44.9%
- Hp. $f = 0$ is rejected**	7.1%	35.4%
N. prov-year estimates	563	593

Notes: The table reports summary statistics for the estimates of (p, b, f) obtained from each of the 618 province-year where estimation was possible. All the lines except one (see note **) report statistics obtained from a model where the parameter f is equal to zero for all province years.
- * The statistics on this line report the percentage of province-year estimates where the overidentifying restriction test is not rejected at the 10 per cent confidence level.
- ** The statistics on this line are based on a comparison between the likelihood for the restricted model ($f = 0$) with the likelihood for a model where f is allowed to vary across province-years. The number reported is the percentage of province-year estimates where the null hypothesis of $f = 0$ is rejected by a likelihood ratio test at the 5% confidence level.

in the third line of this panel, in roughly half of the province-years the minimized objective function is smaller than the critical value corresponding to a 10% probability confidence level, i.e. the model is not rejected in about half of the cases, which is consistent with the information displayed in Figure 15 at the end of last section. We consider that the fit of the model is reasonable, given how simple it is.

As explained at the end of the previous section, the rejection of the model happens for two reasons: either there are no parameters for which the model can fit some of the observations (say $W/M > 2$) or the parameters needed to match one variable differ from the ones needed to match another variable (say, for instance, $\underline{M} = 0$, which implies $p = 0$ and $W/M < 1$,which requires $p > 0$). The rejections are due to each of these two reasons about half of the time.

Finally we examine the extent to which imposing the constraint that $f = 0$ diminishes the ability of the model to fit the data. To do so we reestimate the model letting f vary across province years, and compare the fit of the restricted ($f = 0$) with the unrestricted model using a likelihood ratio test. The last line of the panel reports the percentage of province-years pairs where the null hypothesis of $f = 0$ is rejected at a 5% confidence level. From this we conclude that while there is some improvement in the fit of the model by letting $f > 0$, this improvement is not that large.

In Appendix I we report the estimate of two variations of the model with $f > 0$. In one case, we fixed f at a positive value equal across all province years. In the other case, we let f vary across province-years. We argue that while there an improvement in the fit for a relatively small fraction of province-years of letting $f > 0$, as documented in the last line of Table 5, the variables in our data set does not contains the type of information that will allow us to identify the parameter f. Indeed, as documented in Tables 9 and 10 in Appendix I, when we let $f > 0$ and estimate the model for each province-year, the average as well as median t-statistic of the parameters (p, b, f) are very low, and the average correlation between the estimates is extremely high. Additionally, there is a extremely high variability in the estimated parameters across province-years. In the case where f is fixed at the same value for all province-years, the average t-statistic are higher, but the estimated parameters still vary considerably across province-years. We conclude that the information in our data set does not allows to estimate p, b and f with a reasonable degree of precision. As we explained when we introduced the model with $f > 0$, the reason to consider that model is to eliminate the extremely small withdrawals that the model with $f = 0$ implies. Hence, what will be helpful to estimate f is information on the minimum size of withdrawals, or some other feature of the withdrawal distribution.

In Table 6 we compute correlations of the estimates of p and b with indicators that measure the density of financial intermediaries. We have the following indicators that vary across province and years: number of Bank Branches per resident, as well as per square mile, and number of ATM per resident. We interpret these indicators as measures of the ease to which a household can withdraw money from her banking account, and hence we expect the estimated probability p to be positively correlated with these indicators, and the cost b to be negatively correlated with them. Strictly speaking the measure of ATM should not be relevant for household without an ATM card, but we include all correlations because these indicators may be crude proxies for the general degree of development of the financial system. We find that the estimated costs b are negatively correlated with these measures, and that the estimated p are positively correlated, but the latter correlation is small and very noisy.

Recall our discussion in Section 7.1 were we explained that our model identifies the cost of a withdrawal b/c relative to the interest rate R, i.e. it identifies $\beta \equiv (b/c/)\, R$. Of course, given data on R one can use the estimated β to back up the corresponding b/c, as we have done. In Table 7 we regress the estimated β against the log of the average level of cash consumption for that province year, the level of interest rate, and a measure of density of financial intermediaries. We want to answer two questions with this regression. The first question is to want extent the relative cost varies because of variations of interest rates. In one extreme, if the variation of

Table 6: Correlations between the estimates and financial development indices

	Household with ATM			
	p	b	$b \cdot p^2$	$V_u(0)$
Bank-branch per 1,000 head	0.07	-0.35**	-0.17**	-0.26**
Bank-branch per sq. mile	0.06	-0.14**	-0.04	-0.17**
ATM per 1,000 head	0.09*	-0.53**	-0.26**	-0.52**
	Household with No ATM			
	p	b	$b \cdot p^2$	$V_u(0)$
Bank-branch per 1,000 head	0.10	-0.37**	-0.23**	-0.28**
Bank-branch per sq. mile	0.11**	-0.21**	-0.08	-0.19**
ATM per 1,000 head	0.15**	-0.55**	-0.30**	-0.52**

Notes: Correlation coefficient between the estimated values of $(p, b, b{\cdot}p^2, V_u(0))$ and empirical diffusion measures of bank branches or ATM terminals. All variables are measured in logs. The sample size is 593 for HH w. ATM and 563 for HH without ATM. One or two asterisks indicate that the correlation coefficient p-value, computed assuming that the estimates are independent, is smaller than 5 or 1 per cent, respectively.

b will be independent of interest rates, we will obtain an elasticity of β with respect to R equal to -1 in this regression. Table shows a significant, negative, but small interest rate elasticity of β for households with and ATM card, and a very small, positive, but not significant elasticity for those without an ATM card. Hence, we find that b/c is systematically related to interest rates, a fact for which we have no good explanation for. The second question is to evaluate our assumption that the cost b is proportional to cash consumption c, which was used to be able to aggregate data across households up to the province level. To better understand this regression, assume that for each province year all the households have the same level of cash consumption. In this case, this regression can be used as a way to estimate the form of the cost b/c. A coefficient on log of cash consumption of -1 will indicate that the cost is fixed, independent of the level of cash consumption. The estimated value that we obtain is very close to -1, which calls into question our assumption that the cost is proportional to c. We leave further investigation of this issue for future work.

Table 7: Estimated $(b/c)/R$ vs. consumption, interest rates and technology

	Dependent variable: $\beta \equiv log((b/c)/R)$	
	Household w. ATM	Household wo. ATM
Log(interest rate)	-0.11	0.04
(t-stat)	(-3.4)	(1.6)
Log(cash consumption)	-0.94	-0.90
(t-stat)	(-6.5)	(-7.2)
Density of fin. intermediary*	-22.2	-23.7
(t-stat)	(-3.6)	(-4.1)
R^2	0.09	0.12
Number of observations	593	587

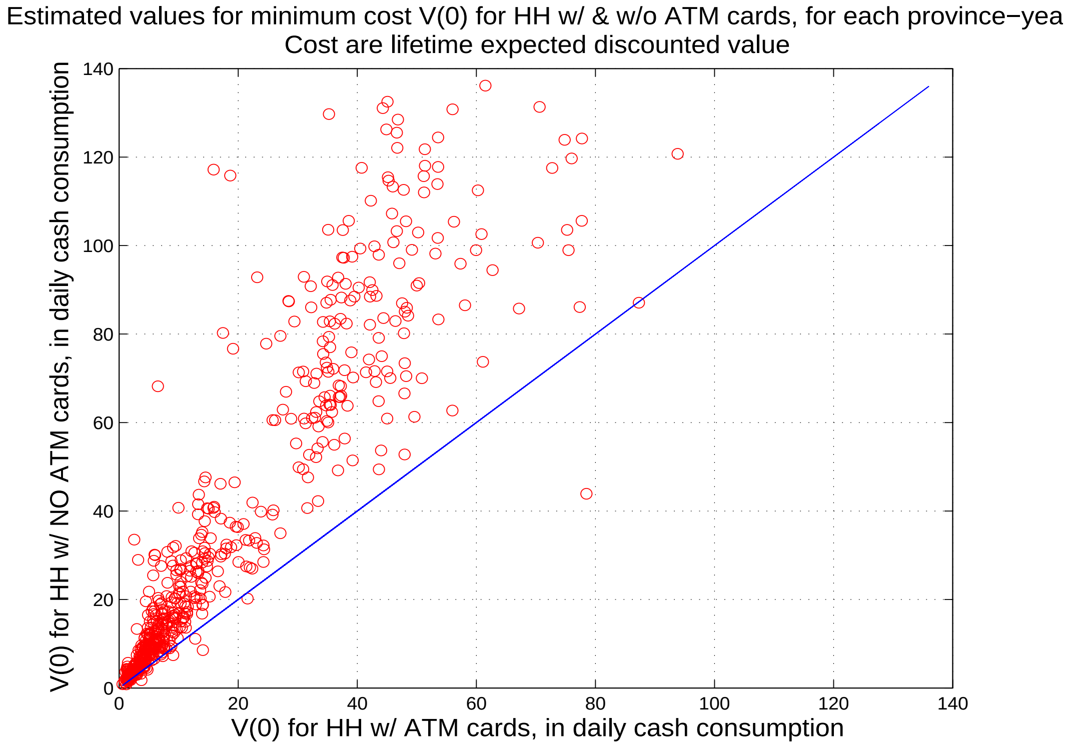
Notes: All regressions include a constant.
-* Number of bank branches (ATM terminals) per square mile for households without (with) ATM.

We use our estimates of p, b for households with and without ATM cards to estimate the implied benefits of owing an ATM card. To estimate this benefit in a given province year we subtract from the cost of financing the consumption stream c implied by the estimated parameters for households without an ATM card the cost implied by the estimated parameters of those that own ATM cards. Recall that these cost are given by the value function V_u and that we have shown in Proposition 3 that $V_{ut}(0) = m_t^* \; R/r + b_t$, where the subindex t denote the household type, i.e. $t = 1$ with and $t = 0$ without an ATM card. As explained in Proposition 2, m^* is a known function of the estimated parameter values. Hence, we define the benefit of owning an ATM card as:

$$V_{u0}(0) - V_{u1}(0) = [m_0^* \; R/r + b_0] - [m_1^* \; R/r + b_1]$$

Figure 16 plots V_{u1} vs. V_{u0}, where it can be seen that for 97.5 percent of the province-years this benefit is estimated to be positive. We remark that our estimates of the parameters for households with and without ATM are done independently, and hence we think that the finding that the estimated benefit is positive for most province years provides additional support for the model.

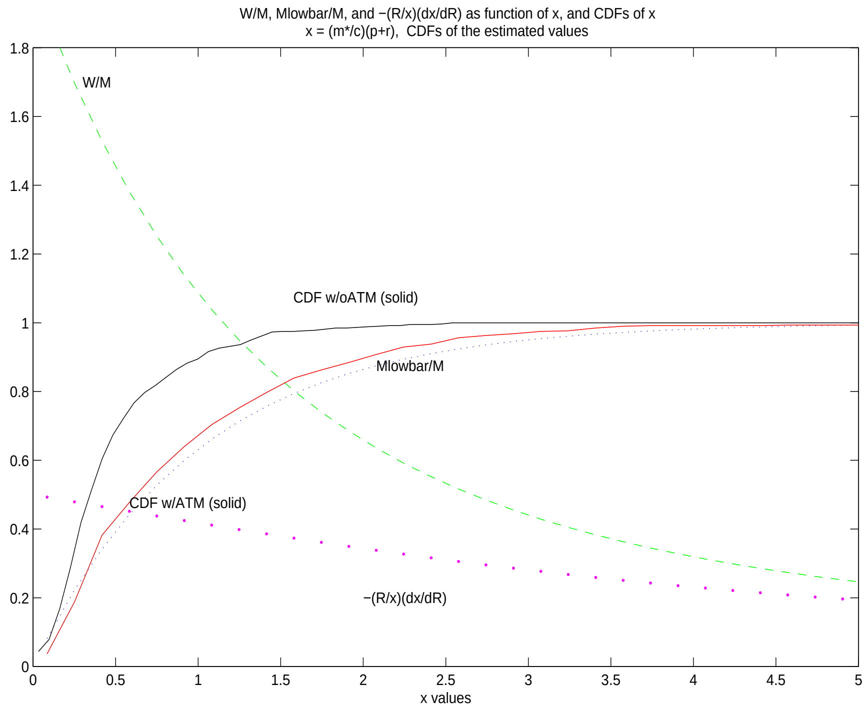
Figure 16: The benefit of ATM cards



The mean benefit across all province years equals 18 days of cash consumption, and its median is 9 days of cash consumption. Recall that these are stock measures of the gains, i.e. these are the gains of having an ATM card forever.

Finally we use our estimates to characterize how close is the cash-management behavior at the estimated parameter values to the BT benchmark. For this we use the results of Proposition 11. In Figure 17 we plot in the horizontal axis a range of values of the normalized cash consumption $x = (m^*/c)(r+p)$ and in vertical axis we plot the theoretical values that correspond to that x for W/M, $\underline{M}/M$ and the interest elasticity of $x\left(\hat{b}/r\right)$, which is essentially equal to the elasticity of the aggregate money demand (M/c). These functions are described in Proposition 11. We also plot the CDF for the estimated values of x across province-years, one for

Figure 17: A summary of estimation findings



those households with ATM cards and one for those without ATM cards. In this Figure it can be seen that for the median province-year for households without ATM x is about 0.35 and for those with ATM cards x is about 0.6. Recall that the BT model corresponds to $x = 0$, and that for higher x the cash-management behavior looks more and more different than the one in the BT model. For the median province-year for households with an ATM card, $W/M \cong 1.3$, $\underline{M}/M \cong 0.5$ and the interest rate elasticity of $M/c \cong 0.42$. While these values are different from the ones implied by the BT model, the difference is not that large, especially for the interest rate elasticity. Only for values of x that correspond to the tail of the estimated x, we find elasticities of M/c as low as 0.2.

Table 8 presents statistics on the average value of p and b across provinces for each year. Our goal with this table is to document the trends in the withdrawal

Table 8: Time series pattern of estimated model parameters

	1993	1995	1998	2000	2002	2004
	Households with ATM					
p	17	17	20	26	22	34
b	0.05	0.04	0.01	0.005	0.005	0.004
R	6.1	5.2	2.2	1.1	0.8	0.3
b/R	0.77	0.85	0.46	0.47	0.62	1.3
$(b/c)p^2$	0.04	0.04	0.01	0.01	0.01	0.01
$(b/c)p^2/R$	0.62	0.71	0.52	0.88	0.85	4.1
	Households without ATM					
p	4.8	4.7	7.4	8.3	7.5	8.6
b	0.10	0.09	0.03	0.02	0.01	0.005
R	6.1	5.2	2.2	1.1	0.8	0.3
b/R	1.6	1.7	1.5	1.3	1.4	1.5
$(b/c)p^2$	0.006	0.005	0.005	0.003	0.002	0.001
$(b/c)p^2/R$	0.1	0.1	0.23	0.25	0.22	0.3

Notes: Entries in table are the averages across provinces of the estimated parameter for each year. For the interest rate R they are the average of the observed values.

technology and the cash-management behavior of households. This table shows that p has approximately doubled, and that (b/c) has been approximately decreased 10 times. Both type of changes leads to lower money balances. To understand whether they lead to behavior that is closer or farther away from the one in the BT model, we use the results of Proposition 11 and compute the time trend of the index $\hat{b} = (b/c)\,p^2$. Since this index has declined through time, at the same level of interest rates, the cash-management behavior would have become closer to the one of the BT model. But since interest rates has declined, then $\hat{b}/R$,which determines the actual behavior, has increased. Indeed Table 8 shows that $\hat{b}/R$ has at least tripled its original value.

Finally, we comment on the absolute value of the parameter b. As Table 5 shows the average value of b across all province years is 2 per cent and 4 per cent of daily cash consumption, respectively for households with and without ATM cards. Using information from Table 1 these values correspond to approximately about 1 and 0.65 euros (in year 2000 prices). While these numbers are very small they are still within a reasonable range. As argued above, their absolute value crucially hinges on a precise measure of the level of the interest rate, especially at the very low interest rate levels reported at the end of the sample.

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