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Liquidity and Asset Prices

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I. Introduction

People in financial markets pay considerable attention to the behavior of the central bank. It is often said that there is a shortage of liquidity and that the central bank supplies liquidity to the market. Yet standard asset pricing models seem to have little scope for systematic analysis of the effect of monetary policy on liquidity and asset prices. Some well-known puzzles in the asset pricing literature, such as the low risk-free rate puzzle and the equity premium puzzle, seem to presume that the underlying economy is non-monetary. But such puzzles can be related to a traditional question in monetary economics: why do people hold monetary assets, even though the rate of return is low and often dominated by the return of some other assets?¹ This paper develops a canonical model of a monetary economy, in order to examine the interaction between circulation of monetary assets, resource allocation, asset prices, and monetary policy.

We broadly define monetary assets, or liquid assets, as the assets that can be readily sold in the market and can be held by a number of people in succession before maturity. When some asset circulates among many people as a means of short-term saving (liquidity), it also serves as a medium of exchange (money), because people hold it not for their own consumption at its maturity date but for future exchange. Thus we will use liquid asset and monetary asset interchangeably. We first ask in what environment is the circulation of monetary assets essential for the smooth running of the economy. Here, we contrast broad monetary assets and the other assets only in terms of liquidity; how quickly these assets can be sold in the market. That is, in this paper, all the assets are real; a broad liquid asset is not denominated by cash, and we ignore the issue of fiat money and inflation.²

¹ We explore the connection between these puzzles by developing a formal model. Although the robustness of the equity premium puzzle is debatable empirically, the puzzle of monetary assets appears to be robust. See [Mehra and Prescott(1985)], [Weil(1989)] and [Campbell, Lo and Mackinlay(1997)].

² In a companion paper [Kiyotaki-Moore(2001)], we consider an economy in which the only liquid asset is cash, considering issues of the nominal price level. In some popular monetary frameworks, such as the cash-in-advance model or the dynamic sticky-price model, the circulation of monetary assets is not indispensable for efficient resource allocation. Some other monetary frameworks, the over-lapping generations models and the random matching

In order to analyze the role of monetary assets for resource allocation, we consider an economy in which output is produced from two types of asset, capital and land. Capital stock can be accumulated through productive investment, while the supply of land is fixed. We depart from a standard model of a stochastic production economy with a representative agent (a real business cycle model) in two aspects. First, we assume that only a fraction of agents have a productive investment opportunity to accumulate capital stock at each point in time, even though agents are equally likely to find investment opportunities in the future. We also assume that there is no insurance contingent on the arrival of investment opportunity. Thus, the economy must transfer purchasing power through financial markets from those who do not have a productive investment opportunity to those who have.³ The second departure is that, at the time of productive investment, people can sell only a fraction of their capital stock (or equivalently, a claim to the future returns from capital stock). Thus capital stock is an asset with limited liquidity. Investing people may therefore face binding liquidity constraints. One interpretation of our model is that the productive investment opportunities disappear so quickly that investing agents do not have enough time to raise funds against their entire capital holding, nor to process an insurance claim, in order to finance new investment. In contrast, land is a liquid asset, and people can raise fund against the entire land

models, do explain why the circulation of money improves efficiency. These models, however, are not very easy to apply to an economy with well-developed financial markets. Perhaps the closest ancestors of this paper are [Townsend(1987)], and [Townsend and Wallace(1987)]. Although our model does not start from as fundamental assumptions as theirs, our framework is closer to standard business cycle models.

³ A large part of the asset pricing literature with credit constraints uses an endowment economy, in which the focus is on risk-sharing for households who face idiosyncratic utility shocks or income shocks. (See [Cochrane(2001)].) Here, we consider a production economy in which the role of financial markets is to transfer resources to those agents who have productive investment opportunity. [Holmström and Tirole(2000)] develops a liquidity-based asset pricing model of a three-period production economy with financial intermediaries in which the arrival of an investment opportunity is contractible. Our analysis largely abstracts from financial intermediation and contingent contracting of this kind in order to concentrate on the dynamic general equilibrium effects. Our framework is perhaps more comparable to a standard asset pricing model, given that in our economy, agents are identical *ex ante*, risk-averse and infinitely lived.

holdings at the time of investment.⁴

We show that the circulation of the liquid asset is essential for resource allocation, -- i.e. the economy is 'monetary', -- if, each agent rarely has a productive investment opportunity; investing agents can sell only a small fraction of capital; and the income share of land is small relative to capital. In the monetary economy, people with investment opportunities are liquidity constrained in the equilibrium. Also, there is a liquidity premium -- a gap in the expected rates of returns between the illiquid asset, capital, and the liquid asset, land. The expected rates of return on the liquid asset is lower than time preference rate. These phenomena are closely related. If people anticipate a binding liquidity constraint at the time of investment, they will hold the liquid asset in their portfolios even if its expected rate of return is dominated by that of the illiquid asset, and even if it is lower than their time preference rate, because the liquid asset is more valuable for financing the downpayment for investment than the illiquid asset. That is, the liquidity premium, the low liquid asset return, and the liquidity constraint for investing agents are all equilibrium features of the monetary economy.

In the later part of the paper, we extend the basic model in two directions. First, we introduce labor as a factor of production and workers as a new group of agents. Workers do not have productive investment opportunities, nor can they borrow against their future wage income. We show that, when the economy is monetary, the workers do not save, while entrepreneurs, who anticipate the arrival of an investment opportunity in the future, save with both the liquid asset and the illiquid asset. The saving behavior of workers and entrepreneurs are different because they have different expectations of productive investment opportunities. It is not because they have different preferences.

⁴ In reality, land (or a claim to the future returns on land) is often less liquid than capital. The term 'land' in this paper may be taken to represent the productive assets of old and well-established sectors of the economy. Such sectors consist of publicly-traded firms; their stock market is well-organized and their productive assets are relatively constant. In contrast, the term 'capital' might represent the productive assets of new and dynamic sectors of the economy, comprising less-established businesses.

The second extension is to introduce government. We consider the government as a large agent who does not have any productive investment opportunities. The government owns a fraction of capital stock and land, taxes workers, and purchases goods. The government behavior is taken to be exogenous. We examine how the portfolio policy and the fiscal policy of the government affect aggregate resource allocation and asset prices. We show that an open market operation to buy the illiquid asset and to sell the liquid asset, has expansionary effects on aggregate output even in the long run, because this operation increases the liquidity of the private sector, which facilitates the transfer of more resources from saving agents to investing agents. Again, the difference in saving behavior between workers and entrepreneurs, and the expansionary effect of the open market operation are both closely related to the other features of the monetary economy, such as the binding liquidity constraint faced by the investing entrepreneurs, the liquidity premium, and the low rate of return on the liquid asset. Because of the low rates of return on assets relative to the time preference rate, workers do not save. Entrepreneurs, anticipating the binding liquidity constraints, save in assets, despite the low rates of return on assets, in order to finance the downpayment for future investment. The government's open market operation increases the ratio of the liquid asset to the illiquid asset of the private sector. Such a policy stimulates aggregate output, because it is the private holding of the liquid asset in particular that lubricates the transfer of resources in the monetary economy.

II. Basic Model

Consider a discrete time economy with one homogeneous output. There is a continuum of infinitely-lived agents with population size of unity. The utility of an agent at date 0 is described as:

$$(1) \quad v_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t \ln c_t \right],$$

where c_t is consumption at date t , $\beta \in (0,1)$ is the constant discount factor, $\ln x$ is natural log of x , and $E_0[x]$ is expected value of x conditional on information at date 0.

The individual agent produces output from two types of homogeneous assets, capital stock and land, according to a constant returns to scale production function:

$$(2) \quad y_t = a_t k_t^\alpha \ell_t^{1-\alpha}, \quad 0 < \alpha < 1.$$

where y_t is output, k_t is capital stock and ℓ_t is land used for production. The variable $a_t > 0$ is aggregate productivity, which is common to all individuals. We assume that the aggregate productivity follows a stationary Markov process that fluctuates exogenously in the neighborhood of some constant level. The individual agents own capital and land, but can freely rent land for production at perfectly competitive rental market. The capital stock depreciates at a constant rate $1-\lambda$. The total supply of land is fixed at L .

Each agent meets an opportunity to invest in capital with probability Π at each period. The arrival of the investment opportunity is independent across periods and across agents. When an agent has the productive investment opportunity, the agent can convert homogeneous goods into capital one-for-one instantaneously. In order to finance this investment, the agent can sell land and capital. But the ownership of capital is not transferred until overnight.

Beforehand, the agent can abscond with a fraction $1-\theta$ of his capital holding, and can start a new life with a clear record. The transaction of capital must satisfy the incentive constraint for the seller to transfer capital properly. Thus, the investing agent can sell only a fraction θ of his capital held before the investment, and also can sell a fraction θ of the new capital just invested. We consider θ as an exogenous parameter which represents the limited liquidity of capital. An alternative interpretation of θ is a particular shape of transaction cost of capital, in which the cost of selling the first θ fraction is zero and the cost for the rest is infinite at the time of investment. On the other hand, the agent can sell all of his land, even if the agent cannot short sale land (because the agent cannot abscond with land, even if he can abscond without land.) Thus land (or a claim to the future rental income on land) is a liquid asset, which serves as money in our model. The value of θ also reflects how quickly the agent must exploit the investment opportunity once he finds such opportunity. Here, we consider that the opportunity disappears the day after the agent finds it, and that the agent does not have much time to sell the entire capital, even if the agent can sell all the capital eventually. We also assume that the arrival of the investment opportunity is not contractible so that the agent cannot arrange an insurance contract contingent on the arrival of the investment opportunity. (Again, we consider the environment in which the investment opportunity is gone by the time the agent may receive payment from the insurance company with verification of the opportunity.)⁵

Let p_t be the price of land, let q_t be price of capital installed, and let r_t^L be the rental price of land in terms of goods. The flow-of-fund constraint of the agent is:

$$(3) \quad c_t + i_t + q_t(k_{t+1} - i_t - \lambda k_t) + p_t(m_{t+1} - m_t) = y_t - r_t^L \ell_t + r_t^L m_t,$$

⁵ [Diamond-Dybvig(1983)] and [Hölmstrom-Tirole(1999)] style insurance arrangements are not incentive compatible here, because the individual agent can transact in financial markets without the insurance company detecting the transaction. See [Jacklin(1987)]. We will not consider the possibility of insurance payment after the investment opportunity is gone, because such insurance is not very useful for improving the resource allocation and we expect our main results unchanged with such insurance.

where i_t is investment, m_t is land and k_t is capital owned before the investment. The left hand side is the expenditure on consumption and investment as well as the net purchase of the ownership of capital and land. The right hand side is profit from production plus the rental income of land ownership. Because the agent can rent land for production, the ownership of land is not necessarily equal to land used for production, l_t . The incentive constraints for the transaction of the ownership of capital and land are:

$$(4) \quad k_{t+1} \geq (1-\theta)(\lambda k_t + i_t),$$

$$(5) \quad m_{t+1} \geq 0.$$

Inequality (4) implies that the agent cannot sell more than θ fraction of capital so that he has to retain at least $1-\theta$ fraction of capital. (5) means he cannot short sale land. Taken together, the agent faces liquidity constraints.

The aggregate state of the economy is going to be summarized by the aggregate capital stock and the aggregate productivity; $s_t = (K_t, a_t)$. The competitive equilibrium is described as price functions $p_t = p(s_t)$, $q_t = q(s_t)$, and $r_t^L = r^L(s_t)$ and quantities of consumption, investment, output, and the usage and ownership of land and capital $(c_t, i_t, y_t, l_t, m_t, k_t)$, such that (i) the individual agent chooses rules of consumption, investment, asset portfolio, and production to maximize the utility subject to the flow-of-funds constraints (3) and the liquidity constraints (4,5), taking the price functions as given; (ii) sum of the individual capital holding is equal to the aggregate capital stock; (iii) sum of the individual land holding is equal to total land supply; (iv) aggregate consumption and investment is equal to aggregate output, and (v) the rental market for land clears.

The production decision of the individual agent is straight-forward. The agent chooses land and output to maximize the profit, $y_t - r_t^L l_t$, subject to the production function. The profit is maximized when the marginal product of land is equal to the rental price of land, and the maximized profit is

proportional to the capital stock as:⁶

$$(6) \quad y_t - r_t^L l_t = r_t^K k_t, \text{ where } r_t^K = \alpha (a_t)^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{r_t^L} \right)^{\frac{1-\alpha}{\alpha}}.$$

When the agent does not have an opportunity to invest in capital stock, the flow of funds constraint (3) can be written as:

$$(7) \quad c_t^n + q_t k_{t+1}^n + p_t m_{t+1}^n = (r_t^K + \lambda q_t) k_t + (r_t^L + p_t) m_t = b_t^n,$$

where superfix 'n' represents the agent without the investment opportunity, and b_t is the total wealth.

When the agent has an opportunity to invest at date t , the agent has two ways of acquiring capital in equation (3); one through investment at the cost of one-for-one, and the other through market at price q_t . If the market price of capital q_t is smaller than 1 (the investment cost of capital), the agent will not invest. The agent is indifferent if $q_t = 1$. Thus, the flow-of-funds constraint is the same as (7), if the agent has an investment opportunity but q_t is smaller than or equal to 1. On the other hand, if q_t exceeds unity, then the agent will invest by selling capital as much as possible with the binding liquidity constraint, (4). Such an agent is liquidity constrained, because his investment is constrained by the available funds. The investment choice is similar to Tobin's q theory of investment in [Tobin(1969)]. Tobin's q may exceed unity here, because only a small fraction of agents have the opportunity to invest in capital stock at each point in time and they are liquidity constrained. When q_t exceeds unity, the flow-of-funds constraint of the investing agent is written from (3) and (4) with equality as:

⁶ If there were a rental market for capital in addition to the rental market for land, the rental price of capital would be equal to the gross profit rate of capital due to the constant returns to scale technology. The result of this paper will not change with the introduction of the rental market for capital.

$$(8) \quad c_t^i + \frac{1-\theta q_t}{1-\theta} k_{t+1}^i + p_t m_{t+1}^i = (r_t^K + \lambda)k_t + (r_t^L + p_t)m_t \equiv b_t^i,$$

where superfix 'i' represents the agent with a profitable investment opportunity. We can think of $[(1-\theta q_t)/(1-\theta)]$ as the cost of investment with maximum leverage, because when the agent sells the maximum fraction θ of the capital invested, he only needs to finance the required downpayment $(1-\theta q_t)$ from own funds for unit investment in order to retain $1-\theta$ units of capital. The capital in total wealth in the right-hand side is valued by the replacement cost λk_t rather than the market value $q_t \lambda k_t$, because the agent faces the liquidity constraint of capital in the market. Thus, the rate of return on capital from the last period becomes $[(r_t^K + \lambda)/q_{t-1}]$ instead of $[(r_t^K + \lambda q_t)/q_{t-1}]$, when the agent has a profitable investment opportunity at date t .

In the following, we concentrate on the case in which Tobin's q exceeds unity so that the liquidity constraint is binding for the investing agents, and we will later derive the condition that guarantees $q_t > 1$. When the agent who does not have an investment opportunity at date t chooses between consumption and saving in order to maximize the expected utility, the marginal utility of consumption must be equal to the expected marginal benefit of acquiring capital and land as:

$$(9a) \quad \frac{1}{c_t^n} = \frac{1}{q_t} \beta E_t \left(\Pi \frac{r_{t+1}^K + \lambda}{c_{t+1}^{ni}} + (1-\Pi) \frac{r_{t+1}^K + \lambda q_{t+1}}{c_{t+1}^{nn}} \right),$$

$$(9b) \quad \frac{1}{c_t^n} = \frac{1}{p_t} \beta E_t \left(\Pi \frac{r_{t+1}^L + p_{t+1}}{c_{t+1}^{ni}} + (1-\Pi) \frac{r_{t+1}^L + p_{t+1}}{c_{t+1}^{nn}} \right),$$

where c_{t+1}^{ni} is consumption of the agent who does not invest at date t and will invest at date $t+1$, and c_{t+1}^{nn} is consumption of the agent who does not invest at both dates. Because utility is the logarithm of consumption, the marginal utility is equal to the inverse of consumption in the left hand side of the above equations. In the right-hand side of equation (9a), the agent can

acquire $(1/q_t)$ units of capital by giving up one unit of consumption. With probability Π , the agent has an investment opportunity, and the return on capital becomes $r_{t+1}^K + \lambda$ with marginal utility of $(1/c_{t+1}^{ni})$. With probability $1-\Pi$, the agent does not have an investment opportunity, and the return on capital is $r_{t+1}^K + \lambda q_{t+1}$ with marginal utility of $(1/c_{t+1}^{nn})$. In the right hand side of equation (9b), the return on land is always equal to $r_{t+1}^L + p_{t+1}$, because land is liquid.⁷

The agent who has an investment opportunity chooses between consumption and investment with maximum leverage as:

$$(10) \quad \frac{1}{c_t} = \frac{1-\theta}{1-\theta q_t} \beta E_t \left(\Pi \frac{r_{t+1}^K + \lambda}{c_{t+1}^{ni}} + (1-\Pi) \frac{r_{t+1}^K + \lambda q_{t+1}}{c_{t+1}^{in}} \right).$$

In equation (10), the agent can acquire $(1-\theta)/(1-\theta q_t)$ units of capital by giving up one unit of consumption with maximum leverage. The investing agent will not hold any land, if the marginal cost of acquiring land exceeds the marginal benefit:

$$(11) \quad \frac{1}{c_t} > \frac{1}{p_t} \beta E_t \left(\Pi \frac{r_{t+1}^L + p_{t+1}}{c_{t+1}^{ii}} + (1-\Pi) \frac{r_{t+1}^L + p_{t+1}}{c_{t+1}^{in}} \right).$$

We will verify inequality (11) later.

From equations (7-11), we can show that

⁷ Immediately after productive investment, the agent holds only capital stock and no land. Thus the constraint of (4) may be binding, even for those who do not have investment opportunity. Then equation (9a) becomes an inequality. We will ignore such a possibility in examining the aggregate equilibrium, because the proportion of such agents is small when Π is not large. For the most of our comparative statics and dynamic analysis, we use the continuous-time approximation as a limit economy as the length of period becomes infinitesimal. Then, we can formally show that the effects of such agents on aggregate equilibrium is infinitesimal.

$$(12a) \quad c_t^n = (1-\beta)b_t^n,$$

$$(12b) \quad c_t^i = (1-\beta)b_t^i,$$

$$(13a) \quad m_{t+1}^i = 0,$$

$$(13b) \quad (1-\theta q_t)i_t = (r_t^K + \theta \lambda q_t)k_t + (r_t^L + p_t)m_t - c_t^i,$$

$$(14) \quad (1-\Pi) E_t \left[\frac{\frac{r_{t+1}^K + \lambda q_{t+1}}{q_t} - \frac{r_{t+1}^L + p_{t+1}}{p_t}}{(r_{t+1}^K + \lambda q_{t+1})k_{t+1}^n + (r_{t+1}^L + p_{t+1})m_{t+1}^n} \right]$$

$$= \Pi E_t \left[\frac{\frac{r_{t+1}^L + p_{t+1}}{p_t} - \frac{r_{t+1}^K + \lambda}{q_t}}{(r_{t+1}^K + \lambda)k_{t+1}^n + (r_{t+1}^L + p_{t+1})m_{t+1}^n} \right].$$

In equations (12), consumption is proportional to the total wealth.⁸ In equations (13), the investing agent holds no land and uses all the funds after consumption to finance the downpayment for investment. Equation (14) describes the portfolio decision of the agent who does not have an investment opportunity at date t , which is derived from (9). The numerator in the bracket in the left hand side represents how much the rate of return on capital exceeds that of land if there is no investment opportunity at date $t+1$. Because the marginal utility is equal to the inverse of the consumption (which is proportional to the total wealth from (12a)), the left hand side is the expected advantage of capital over land in terms of utility, if there is no investment opportunity in the next period. The right hand side is the expected advantage of land over capital when the agent meets an investment opportunity in the next period with probability Π .

Since the arrival of the productive investment opportunity is independent

⁸ The logarithmic utility in (1) is isomorphic to Cobb-Douglas utility function, as the expenditure share of the present consumption out of the total wealth is constant and equal to $1/(1+\beta+\beta^2+\dots) = 1-\beta$.

across agents, we can aggregate the individual's investment function (13b) as:

$$(15) \quad (1-\theta q_t)I_t = \Pi \left[(r_t^K + \theta \lambda q_t)K_t + (r_t^L + p_t)L - (1-\beta)[(r_t^K + \lambda)K_t + (r_t^L + p_t)L] \right],$$

where I_t is the aggregate investment. When investing agents are liquidity constrained, the aggregate investment is an increasing function of capital price and land price. This two-ways interaction between the aggregate investment and the asset prices is an important channel for the effects of the shock on aggregate output to propagate. (See for example [Kiyotaki-Moore(1987)].) Aggregate capital stock evolves over time as:

$$(16) \quad K_{t+1} = I_t + \lambda K_t.$$

The market clearing condition for capital holdings implies:

$$(17) \quad K_{t+1} = (1-\theta)(I_t + \Pi \lambda K_t) + K_{t+1}^n,$$

where K_{t+1}^n is the aggregate capital holdings of the agents who do not have investment opportunity. The first term in the right hand side is the aggregate capital holdings of the investing agents who face the binding liquidity constraint (4), which is $1-\theta$ fraction of capital invested (I_t) and capital owned from the previous period ($\Pi \lambda K_t$).

Through the competitive rental market of land and constant returns to scale production technology, both the marginal products of land and capital are equalized across producers. Thus the aggregate output Y_t becomes a function of aggregate capital stock as:

$$(18) \quad Y_t = a_t K_t^\alpha L^{1-\alpha}.$$

In the competitive rental market for land, the rental price of land is equal to the marginal product of land:

$$(19a) \quad r_t^L = (1-\alpha) a_t K_t^\alpha L^{-\alpha}.$$

Because the production function is constant returns to scale, the profit per unit of capital in (7) is equal to the marginal product of capital:

$$(19b) \quad r_t^K = \alpha a_t K_t^{\alpha-1} L^{1-\alpha}.$$

Aggregating the individual's consumption and investment, we have the goods market clearing condition as:

$$(20) \quad Y_t = I_t + (1-\beta) \left[(r_t^K + \pi\lambda + (1-\pi)\lambda q_t) K_t + (r_t^L + p_t) L \right].$$

All the land is held by the agents who do not have investment opportunities. Their portfolio behavior (14) is now as:

$$(21) \quad (1-\pi) E_t \left[\frac{\frac{r_{t+1}^K + \lambda q_{t+1}}{q_t} - \frac{r_{t+1}^L + p_{t+1}}{p_t}}{(r_{t+1}^k + \lambda q_{t+1}) K_{t+1}^n + (r_{t+1}^L + p_{t+1}) L} \right] = \pi E_t \left[\frac{\frac{r_{t+1}^L + p_{t+1}}{p_t} - \frac{r_{t+1}^K + \lambda}{q_t}}{(r_{t+1}^k + \lambda) K_{t+1}^n + (r_{t+1}^L + p_{t+1}) L} \right].$$

The competitive equilibrium of our economy is described recursively as prices (p_t, q_t, r_t^L, r_t^K) and aggregate quantities $(I_t, K_{t+1}, K_{t+1}^n, Y_t)$ as functions of the aggregate state (K_t, a_t) , which satisfy (15-21), with an exogenous evolution of the aggregate productivity a_t . Our model is almost as simple as standard asset pricing models of production economy, real business cycle

models, and IS-LM models. The main difference from standard asset production pricing models (such as [Merton(1975)] and [Brock(1982)]) is the portfolio behavior takes into an account of the illiquidity of capital stock in financing investment in (21). Our model differs from standard real business cycle models (such as [Kydland and Prescott(1982)]), because we have two-way interaction between the asset prices and the aggregate quantities. A typical real business cycles model can determine the quantities first, before deriving the implied prices. In this aspect, our model is closer to traditional IS-LM models. Our framework, however, differs substantively beyond the modeling strategy, because, while IS-LM compares cash and interest-bearing assets in determining the nominal interest rate, we contrast between the broad liquid asset and the illiquid asset in determining the liquidity premium.⁹

Before analyzing the dynamics, let us examine the steady state equilibrium for a constant aggregate productivity, i.e., $a_t = a$. In the steady state, capital stock is constant so that $I = (1-\lambda)K$, and

$$(22a) \quad (1-\theta q)(1-\lambda)K = \Pi \left[(r^K + \theta \lambda q)K + (r^L + p)L - (1-\beta)[(r^K + \lambda)K + (r^L + p)L] \right],$$

$$(22b) \quad aK^\alpha L^{1-\alpha} = (1-\lambda)K + (1-\beta) \left[(r^K + \Pi \lambda + (1-\Pi)\lambda q)K + (r^L + p)L \right],$$

$$(22c) \quad \frac{r^K + \lambda q}{q} - \frac{r^L + p}{p} = \Pi \lambda \frac{q-1}{q} \frac{r^L + p}{p} \frac{qK^n + pL}{(r^K + \lambda)K^n + (r^L + p)L},$$

$$(22d) \quad r^L = (1-\alpha)aK^\alpha L^{-\alpha},$$

$$(22e) \quad r^K = \alpha aK^{\alpha-1} L^{1-\alpha},$$

$$(22f) \quad K^n = [1 - (1-\theta)(1-\lambda + \lambda \Pi)]K.$$

From these conditions of the steady state equilibrium, we can show the

⁹ In this respect, we are close to [Keynes(1936)], which defines 'money' broadly as liquid assets, including treasury bills.

following proposition:

Proposition 1: Suppose that the parameters of the economy satisfies

$$\text{(Assumption 1)} \quad \theta < \theta^* \equiv 1 - \frac{\Pi}{1-\lambda+\Pi\lambda} \left(1 + \frac{1-\alpha}{\alpha} \frac{1-\beta\lambda}{1-\beta} \right).$$

Then, in the steady state equilibrium:

(i) Tobin's q exceeds 1, so that investing agents are liquidity constrained;

$$\text{(ii)} \quad \frac{r^L}{p} < \frac{r^K}{q} - (1-\lambda) < \frac{1-\beta}{\beta};$$

(iii) $K < K^*$, where K^* is the capital stock at the first-best steady state, which solves: $\alpha a(K^*)^{\alpha-1} L^{1-\alpha} = (1/\beta) - \lambda$.

(iv) Investing agents do not hold land at the end of period, $m_{t+1}^i = 0$.

(Proof in Appendix 1)

Assumption 1 implies that the fraction of capital which the agents can sell in order to finance the investment is smaller than the critical level θ^* . The value of θ^* is a decreasing function of the arrival rate of investment opportunity Π , and the ratio of the land value to capital under the first best allocation. Thus, Proposition 1(i) means that investing agents are liquidity constrained, if and only if capital is sufficiently illiquid in relation to the arrival rate of investment opportunity and the ratio of the land value to capital. Proposition 1(ii) implies that the rate of return on the liquid asset, land, is dominated by the rate of return on the illiquid asset, capital, which in turn is smaller than the time preference rate. The gap between the rates of returns on illiquid and liquid assets may be called as 'liquidity premium,' and the liquidity premium arises if and only if the

investing agents are liquidity constrained. From (22c), the magnitude of the liquidity premium is roughly equal to $\Pi(q-1)$, which can be substantial.¹⁰ Proposition 1(iii) says that there is under-investment relative to the first best allocation, if the credit constraint is binding so that not enough resources are transferred from saving agents to investing agents. All of these features, liquidity constraint, liquidity premium, low liquid asset return and under-investment are unique features of a "monetary economy," in which the circulation of the liquid asset is essential for the smooth running of the economy.¹¹

The reverse of Proposition 1 is also true. Suppose that $\theta \geq \theta^*$. Then, the steady state equilibrium achieves the first best allocation; Tobin's q is equal to 1 and the investing agents are not liquidity constrained; and the rates of returns on capital and land are both equal to the time preference rate. Notice that, if $\theta > (1-\Pi)(1-\lambda)/(1-\lambda+\Pi\lambda)$, then the economy achieves the first best allocation, even if liquid land is unimportant for production, $\alpha \approx 1$. Under such an environment, the economy ceases to be monetary, in the sense that the circulation of liquid monetary assets is no longer necessary for efficient allocation.

We now examine the dynamics of the economy with stochastic fluctuations of the aggregate productivity. For this purpose, we consider a continuous time approximation. Let the length of one period be Δ instead of one, and define the continuous time parameter as:

$$(23) \quad \beta = e^{-\rho\Delta}, \quad \lambda = e^{-\delta\Delta}, \quad 1-\Pi = e^{-\pi\Delta}.$$

¹⁰ For example, if Tobin's q is 5% above 1 and each agent has investment opportunity once every two years on average, then the liquidity premium is 2.5% annual rate. If the agents were heterogeneous in technology, then the arbitrage equation (14) holds only for those who hold both liquid and illiquid assets.

¹¹ [Holmström-Tirole(2000)] and [Kiyotaki-Moore(2000)] show a similar proposition in somewhat different environment.

The parameter ρ is time preference rate, δ is depreciation rate, and π is Poisson arrival rate of the investment opportunity to each agent. Define I_t and Y_t as the investment and output rates. Let r_t^L be the rental price of land and let r_t^K be the profit rate per unit of time. Let us take the limit of the aggregate equilibrium as Δ goes to zero. Then we have:

$$(24) \quad (1 - \theta q_t) I_t = \pi (\theta q_t K_t + p_t L),$$

$$(25) \quad \dot{K}_t = I_t - \delta K_t,$$

$$(26) \quad K_t^n = K_t,$$

$$(27) \quad Y_t = I_t + \rho (q_t K_t + p_t L),$$

$$(28) \quad \frac{r_t^K}{q_t} - \delta - \frac{r_t^L}{p_t} + \lim_{dt \rightarrow 0} E_t \left(\frac{1}{dt} \left(\frac{q_{t+dt}}{q_t} - \frac{p_{t+dt}}{p_t} \right) \frac{q_t K_t + p_t L}{q_{t+dt} K_t + p_{t+dt} L} \right) = \pi \frac{q_t - 1}{q_t} \frac{q_t K_t + p_t L}{K_t + p_t L}.$$

Equation (18) and (19) are unchanged except that a_t is aggregate productivity per unit of time. In equation (24), the investing agents use maximum sales of capital and land to finance the downpayment of investment. Because the fraction of the investing agents in an infinitesimal period $[t, t+dt]$ is equal to πdt (which is infinitesimal), effectively all the capital stock is owned by the agents who do not have investment opportunity in equation (26). The left hand side of (28) is the liquidity premium; the difference between the expected rates of returns on capital and land, taking into an account the risk aversion. The right hand side is the expected advantage of land over capital in financing the downpayment of investment. This advantage is proportional to the gap between Tobin's q and unity, which represents the tightness of the liquidity constraint when the agent has an investment opportunity. In Appendix 2, we lay out a model of a continuous time economy in order to derive the above equations directly, instead of taking a limit of the discrete time economy.

Here, we assume that the aggregate productivity follows a two-point Markov process:

(29) $a_t = \text{either } a(1+\Delta), \text{ or } a(1-\Delta), \text{ where } \Delta \text{ is an infinitesimal positive,}$

and the arrival of the productivity changes follows a Poisson process with arrival rate η . We also assume that Assumption 1 holds with strict inequality in the continuous time limit so that the investing agents are always liquidity constrained in the neighborhood of the steady state equilibrium. In order to concentrate our analysis on a realistic situation, we also assume that the arrival rate of an investment opportunity to each agent is larger than the depreciation rate:

(Assumption 2) $\pi > \delta.$

Assumption 2 is a mild condition. For example, if the depreciation rate is 10% a year, the arrival rate of investment opportunity to each agent is more frequent than once every 10 years.

Because it is easier to analyze the model in intensive form, let us define $i_t = I_t/K_t$ (investment rate, which is different from the investment of the individual agent), $v_t = P_t/K_t$ (land value - capital ratio) and $x_t = Y_t/K_t$ (output-capital ratio). Equations (24), (25), (18), (27) and (28) are now:

$$(30) \quad (1 - \theta q_t) i_t = \pi(\theta q_t + v_t),$$

$$(31) \quad \dot{K}_t/K_t = i_t - \delta,$$

$$(32) \quad x_t = a_t K_t^{\alpha-1} L^{1-\alpha} = i_t + \rho(q_t + v_t).$$

$$(33) \quad \left[\frac{\alpha}{q_t} - \frac{1-\alpha}{v_t} \right] x_t - i_t + \lim_{t \rightarrow 0} E_t \left(\frac{1}{dt} \left(\frac{q_{t+dt}}{q_t} - \frac{v_{t+dt}}{v_t} \right) \frac{q_t + v_t}{q_{t+dt} + v_{t+dt}} \right) = \pi \frac{q_t^{-1} q_t + v_t}{q_t 1 + v_t}$$

One way to analyze the dynamics is to examine a linearized system in the neighborhood of the steady state. Using notation \hat{x}_t as a proportional deviation of variable x_t from the steady state value x , i.e., $x_t \equiv (x_t - x)/x$, we can postulate the endogenous variables as functions of the state variables as:

$$(34a) \quad \hat{i}_t = \lambda \hat{x}_t + \mu \hat{a}_t,$$

$$(34b) \quad \hat{v}_t = \lambda_v \hat{x}_t + \mu_v \hat{a}_t,$$

$$(34c) \quad \hat{q}_t = \lambda_q \hat{x}_t + \mu_q \hat{a}_t,$$

From (29,31,32), we know that:

$$(35) \quad d\hat{x}_t = d\hat{a}_t - (1-\alpha)\delta \hat{i}_t dt,$$

Equation (35) means that the proportional change of output-capital ratio in an infinitesimal period $[t, t+dt]$ is equal to the proportional change in productivity plus the effect of capital accumulation through investment.

In Appendix 3, we show that there is a unique positive λ which satisfies the saddle-point stability of the dynamical system. Also for fairly mild regularity restrictions on parameters, we have

$$(36) \quad 0 < \lambda, \quad 0 < \lambda_v, \quad \lambda_q < \lambda_v,$$

$$(37) \quad \mu < 0, \quad \mu_v < 0, \quad \mu_q > 0,$$

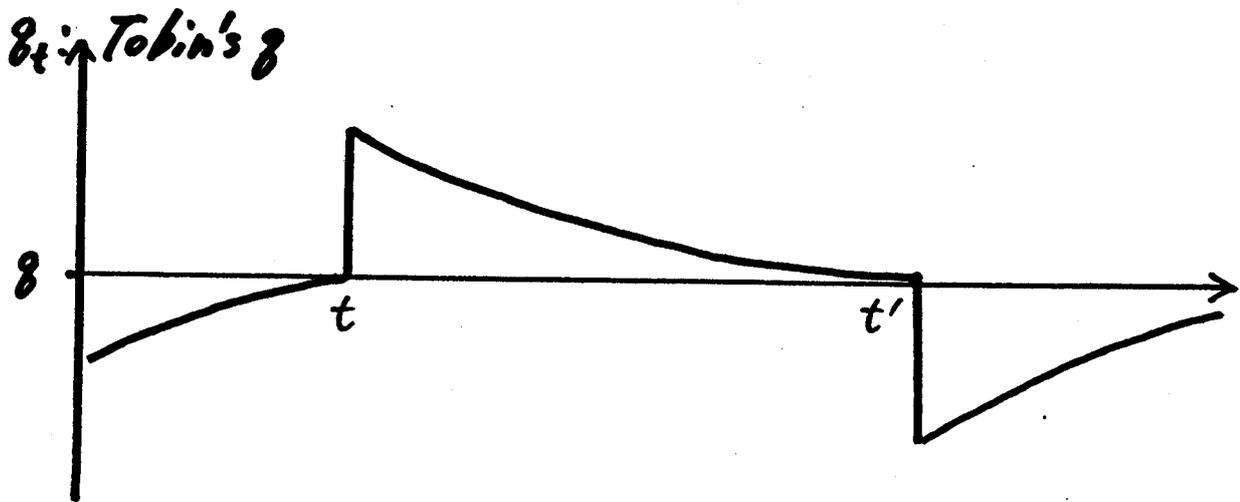
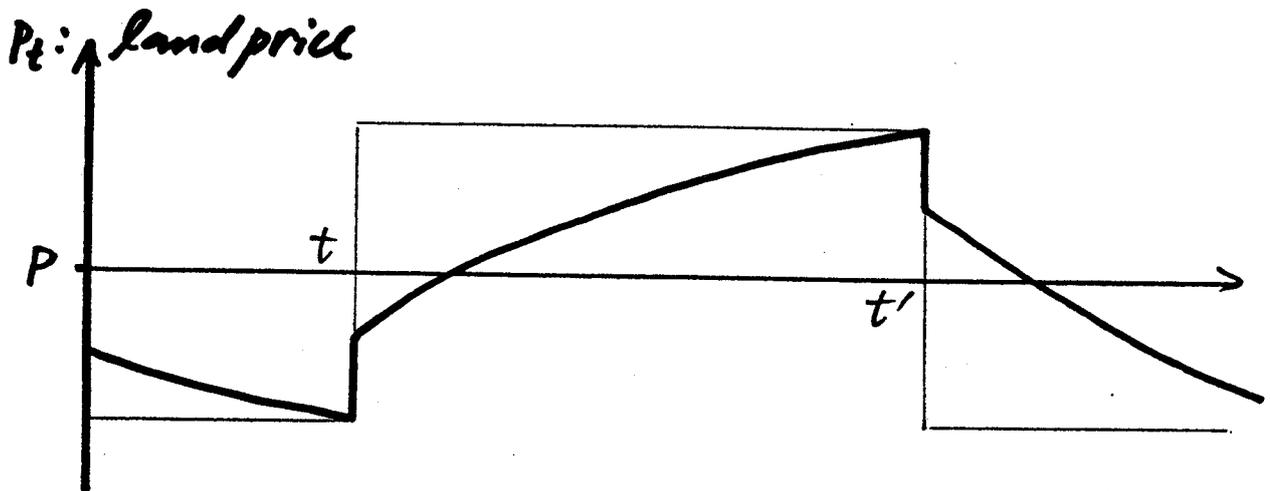
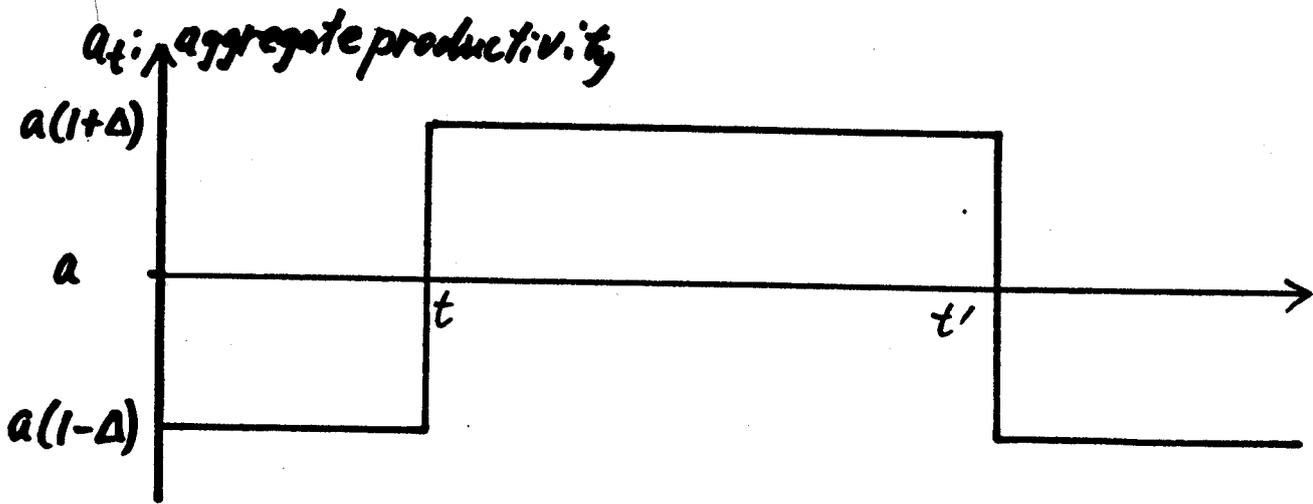
$$(38) \quad \text{all } \mu, \mu_v \text{ and } \mu_q \text{ are proportional to } \eta.$$

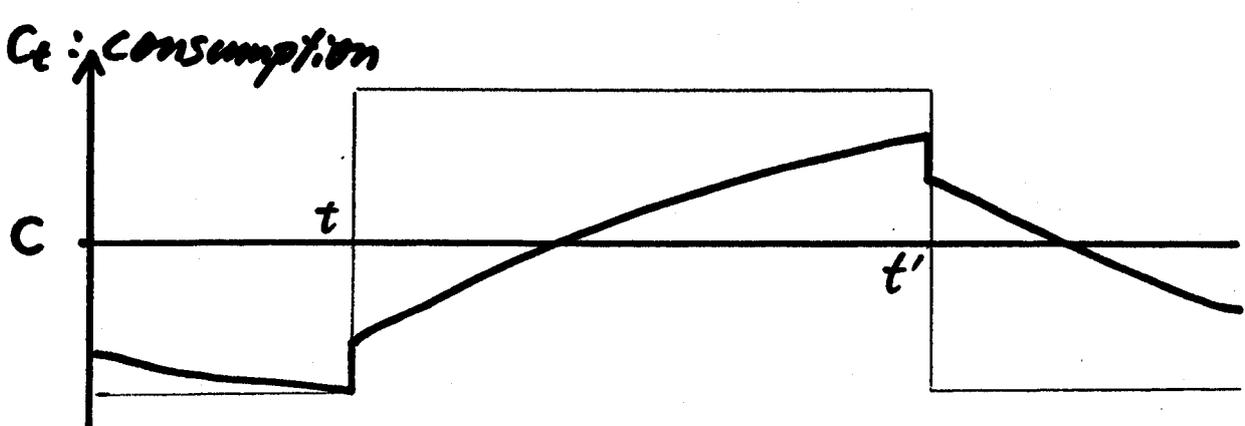
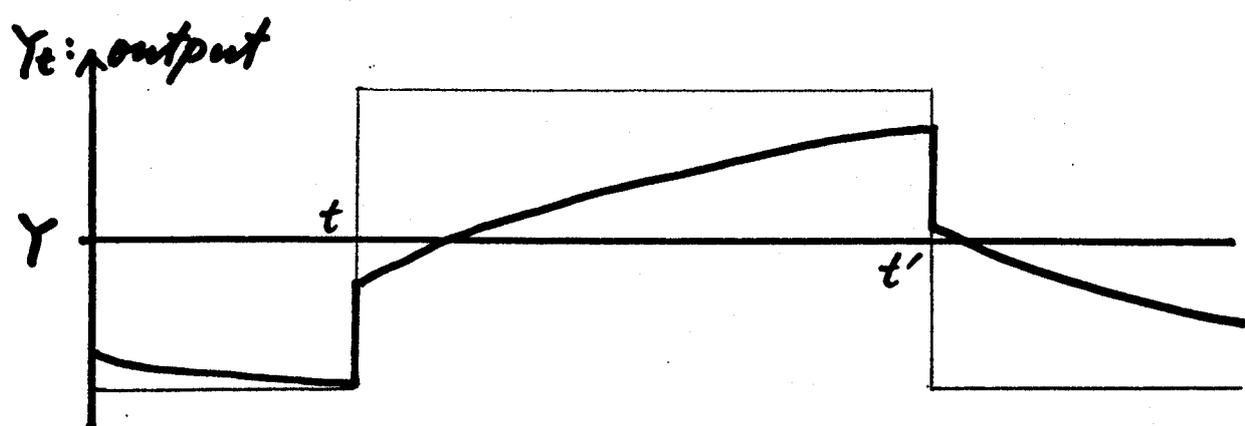
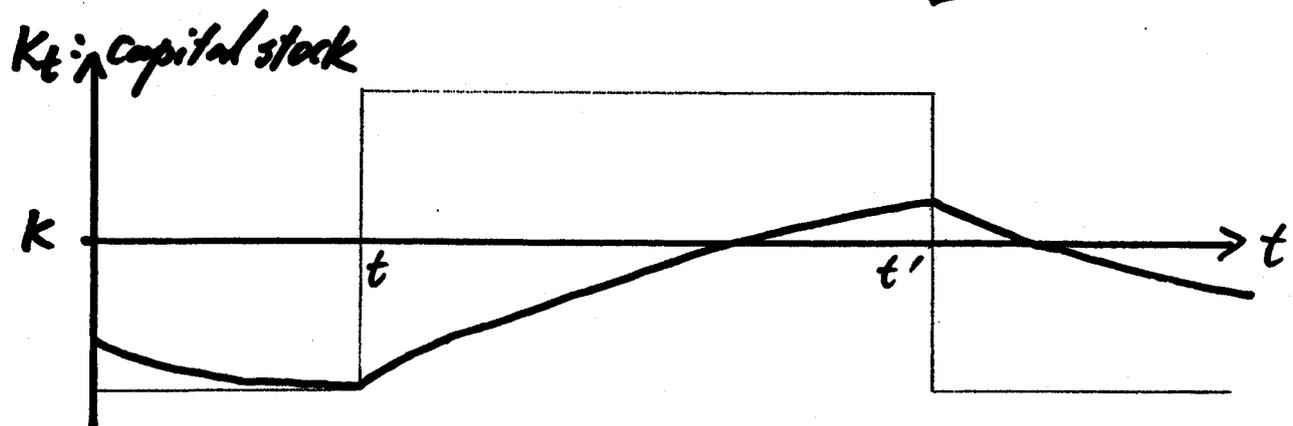
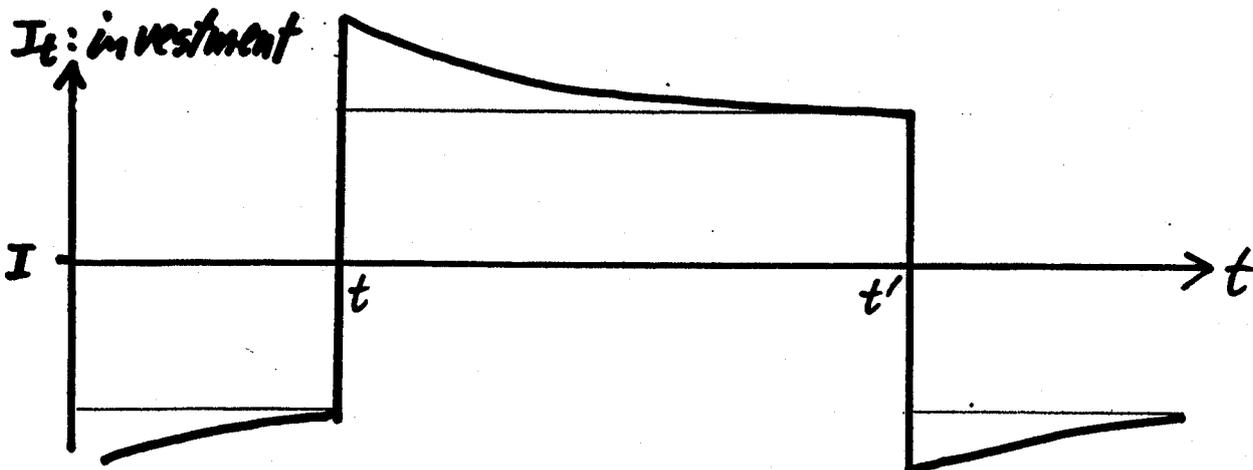
Notice that output-capital ratio x_t is a decreasing function of capital stock and an increasing function of productivity from (32). Inequalities (36) says that the investment rate (I_t/K_t) is an increasing function of the output-capital ratio, and thus a decreasing function of capital stock, which is the necessary condition for saddle-point stability. (36) also implies that the land value-capital ratio is an increasing function of output-capital ratio. Tobin's q also is most likely a decreasing function of capital stock. The effects of the aggregate productivity on endogenous variables are two fold: one through a direct change in output-capital ratio x_t , and the other through the change in expectations. The expectations effects of aggregate productivity on investment rate i_t , land value to capital ratio v_t and capital price are all negative in inequalities (37), because people expect aggregate productivity is mean-reverting. However, these expectation effects are in the order of the arrival rate of the productivity switch in (38), and we can show that the direct effect of productivity through x_t dominates the expectation effects.

The stochastic process of asset prices and aggregate quantities are described by the recursive rules (34) together with the evolution of aggregate productivity (29) and output-capital ratio (35). Figure 1 summarizes the typical fluctuations of aggregate quantities and prices. When the productivity rises to a higher level at date t , such a change is considered as good news, even if people anticipate occasional changes of the productivity. Both the prices of land and capital rise discontinuously. Investment increases vigorously, because the investing entrepreneurs can raise more funds against land and capital. Output and consumption also increase but not as much as investment in proportion. After date t , capital stock starts accumulating with the large investment. With capital accumulation, output, consumption and land price further increase. On the other hand, Tobin's q starts falling to a normal level as the liquidity constraint loosens with the capital accumulation, until another arrival of the productivity change.

In order to get some intuitions of dynamics, let us examine alternatively the local dynamics by phase diagram, assuming that the arrival of a productivity switch is rare. We can solve for (30) and (32) for i_t and v_t with respect to x_t and q_t as:

Figure 1 Responses to Productivity Shock





$$(39a) \quad i_t = \pi \frac{x_t - \rho(1-\theta)q_t}{\pi + \rho(1-\theta)q_t} \equiv i(x_t, q_t),$$

$$(39b) \quad v_t = \frac{(1-\theta q_t)(x_t - \rho q_t) - \pi \theta q_t}{\pi + \rho(1-\theta)q_t} \equiv v(x_t, q_t).$$

In (39a), investment rate is an increasing function of output-capital ratio x_t . It is also a decreasing function of capital price q_t for a given x_t , because the increase consumption with a higher q_t crowds out investment more than a higher q_t encourage investment through the flow-of-funds under Assumption 2. In (39b), land value-capital ratio is an increasing function of x_t and an decreasing function of q_t . If we substitute (39) into (31) and (33), we have a dynamical system with respect to output-capital ratio and Tobin's q as:

$$(40a) \quad \frac{dx_t}{x_t} = \frac{da_t}{a_t} - (1-\alpha) \left(i(x_t, q_t) - \delta \right) dt \equiv H(x_t, q_t),$$

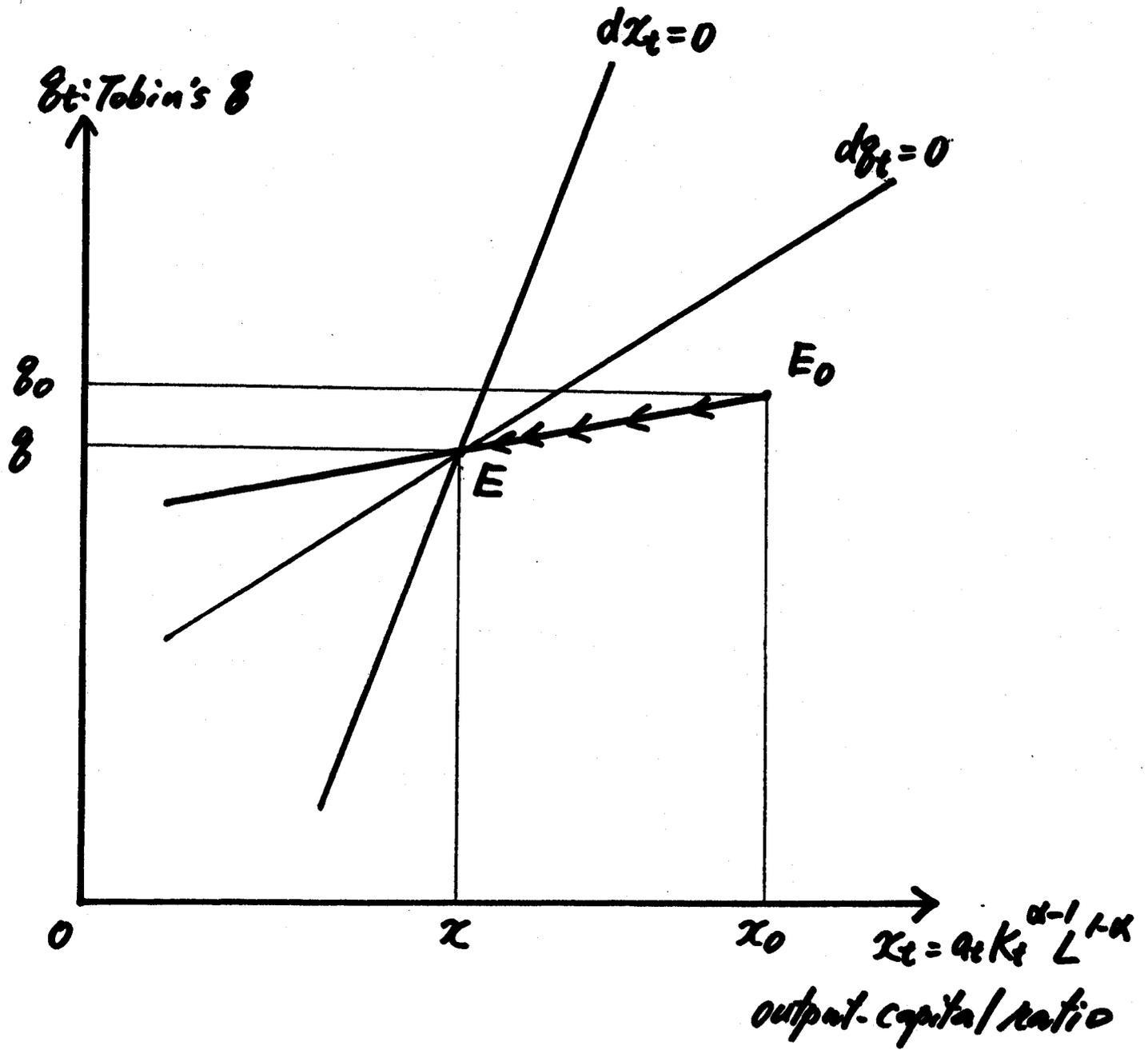
$$(40b) \quad (1-\varepsilon_{vq}) \left[\frac{q_t}{q_t} \right]_{da_t=0} = \left(\frac{1-\alpha}{v(x_t, q_t)} - \frac{\alpha}{q_t} \right) x_t + i(x_t, q_t) + \pi \frac{q_t^{-1} v(x_t, q_t) + q_t}{v(x_t, q_t) + 1}$$

$$- (1-\alpha)\varepsilon_{vx} [i(x_t, q_t) - \delta] + \eta [\varepsilon_{vx} + (\varepsilon_{vq} - 1)\varepsilon_{qa}] \left[\frac{da_t}{a_t} \right]_{da_t \neq 0} \equiv J(x_t, q_t),$$

where ε_{yz} is the elasticity of y with respect to z . (All the elasticities can be computed directly from (39), expect for $\varepsilon_{qa} \equiv \lambda_q + \mu_q$ in (36,38).) When the arrival rate of the productivity switch (η) is small, then we can analyze (40a) and (40b), using a phase diagram. We know $H_x < 0$ always, and $H_q > 0$. When we assume that the elasticities are relatively constant, we also see $J_q > 0$, and that J_x is negative if effect of x on i is not too large because $(1-\alpha)/v < (\alpha/q)$. Then the typical phase diagram looks like in Figure 2.

There is a unique saddle point path converging to the steady state for a

Figure 2



fixed aggregate productivity. Suppose that the economy is near the steady state for $a = a(1-\Delta)$, and that suddenly the aggregate productivity switches to high at $a = a(1+\Delta)$. Since the productivity switch is rare, this change is considered as largely unexpected. The output-capital ratio jumps from the steady state level x to a higher level x_0 . Then capital price q_0 jumps up from q to q_0 on the saddle point path. Both output and investment increase. After the initial date, the capital will gradually accumulate over time with further increase in aggregate output and declining Tobin's q along the saddle-point path from E_0 to E , until the economy will converge to the neighborhood of the steady state.

For applications of asset pricing, it is perhaps more natural to consider that the aggregate productivity follows a geometric Brownian motion, instead of two point Markov process:

$$(41) \quad da_t = \sigma a_t dz_t,$$

where z_t is a Wiener process and $\sigma > 0$ is standard deviation of innovation of the aggregate productivity. Such an economy can be considered as the economy of [Merton(1975)] with limited liquidity of capital stock. With this stochastic process of productivity, we no longer can presume that the investing agents are always liquidity constrained. Depending on the state of the economy, the investing agents may invest up to the maximum, may invest without liquidity constraint, or may not invest at all. In Appendix 4, we derive some preliminary results about this economy.

III. Workers and Government

In this section we introduce workers and government into the Basic Model. Suppose that output is now produced from homogeneous capital, land, and labor, according to the aggregate production function:

$$(42) \quad Y'_t = a'_t K_t^{\alpha'} L_t^{\gamma'} N_t^{1-\alpha'-\gamma'}, \quad 0 < \alpha', \gamma', 1-\alpha'-\gamma'$$

where Y'_t is aggregate output, K_t is capital, L_t is land and N_t is labor. Suppose also that there is a continuum of a new group of people, called workers, with population size of unity. The workers supply labor, but they do not have productive investment opportunities. The utility of workers at date 0 is described as:

$$(43) \quad E_0 \left(\sum_{t=0}^{\infty} \beta^t u \left(c_t - \frac{\omega}{1+\nu} N_t^{1+\nu} \right) \right), \quad 0 < \nu, \omega$$

where c_t is consumption, N_t is labor, and $u()$ is instantaneous utility function which satisfies $u' > 0$, $u'' < 0$, $u'(0) = \infty$ and $u'(\infty) = 0$. We also assume that the workers cannot borrow against their future wage income. We call the agents described in the Basic Model with occasional productive investment opportunities 'entrepreneurs.'

Beside workers and entrepreneurs, there is a large agent, called government. The government does not have productive investment opportunities, but can produce output according to the same production function as the private agent.¹² The government owns capital K_t^g and land L_t^g , taxes on workers T_t lump-sum, and purchases goods G_t . The budget constraint of government is:

¹² If there is a rental market for capital, the government does not need to produce by itself. See footnote 6.

$$(44) \quad G_t + q_t(K_{t+1}^g - \lambda K_t^g) + p_t(L_{t+1}^g - L_t^g) = T_t + r_t^K K_t^g + r_t^L L_t^g.$$

If the government is not subject to the short sales constraint because of its superior commitment technology, then the government can take a negative position on liquid asset, L_t^g , by issuing government security whose return is identical to land. Thus the government can act as a banker.¹³ In this paper, we take the behavior of the government as exogenous and only study their impact on the resource allocation. We assume that the government choice of holdings of capital and land (K_t^g, L_t^g) follows an exogenous stationary Markov process, and that the government adjust the fiscal policy $G_t - T_t$ to satisfy the budget constraint.

There is a competitive labor market in which the producers hire labor. Then, the real wage rate w_t will be equal to both marginal product of labor and marginal disutility of labor:

$$(45) \quad (1 - \alpha' - \gamma') a_t' K_t^{\alpha'} L_t^{\gamma'} N_t^{1 - \alpha' - \gamma'} = w_t = \omega N_t^\nu$$

Now we define the gross profit of producers Y_t as aggregate output minus aggregate wage for the workers. Then from (42), we now have:

$$(46) \quad Y_t \equiv Y_t' - w_t N_t = a_t K_t^\alpha L_t^\gamma, \quad \text{where}$$

$$\alpha \equiv \alpha' \frac{1 + \nu}{\alpha' + \gamma' + \nu}, \quad \gamma \equiv \gamma' \frac{1 + \nu}{\alpha' + \gamma' + \nu}, \quad \text{and } a_t \equiv (\alpha' + \gamma') \left(\frac{1 - \alpha' - \gamma'}{\omega} \right)^{\frac{1 - \alpha' - \gamma'}{\alpha' + \gamma' + \nu}} a_t' \frac{1 + \nu}{\alpha' + \gamma' + \nu}.$$

Because there is no income effect of labor supply of workers, we can write down the gross profit function as a function of capital and land. Because the marginal disutility of labor is an increasing function of labor ($\nu > 0$), the

¹³ In [Kiyotaki-Moore(2000)], we consider the role of bankers who have resource-consuming commitment technology.

gross profit function is a decreasing returns to scale in capital and land, i.e., $\alpha + \gamma < 1$.¹⁴

Concerning the entrepreneur's environment, the economy is exactly the same as Basic Model of the previous section. The only difference from the Basic Model is that, the rental price of land and the profit per unit of capital are equal to the marginal product of land and capital as:

$$(47a) \quad r_t^L = \gamma' \frac{Y_t'}{L} = \frac{\gamma}{\alpha + \gamma} a_t K_t^\alpha L^{\gamma-1},$$

$$(47b) \quad r_t^K = \alpha' \frac{Y_t'}{K_t} = \frac{\alpha}{\alpha + \gamma} a_t K_t^{\alpha-1} L^\gamma.$$

In the following, we restrict the analysis in the neighborhood of the steady state equilibrium, by assuming the aggregate productivity shock is small as in (29). We also conjecture that the steady state economy is monetary as in Proposition 1. We later derive the condition which guarantees the economy to be monetary, after describing the competitive monetary equilibrium. Then, the investing entrepreneurs face the liquidity constraint. Also, the equilibrium rates of returns on land and capital are both lower than the time preference rate. Then the workers will not save at all, and consume always the entire disposable wage income, $w_t N_t - T_t$, as long as the arrival of productivity switch is infrequent and the size of productivity shock is small so that the incentive for consumption smoothing is not too large. Here, the workers do not save, not because they are myopic or irrational, but because they do not expect future investment opportunities and because the rates of returns on land and capital are lower than the time preference rate in the monetary economy. In fact, the worker's time preference rate is the same as the entrepreneurs.

¹⁴ If tax on workers is proportional to wage income at rate τ_t instead of lump-sum, then ω should be replaced by $\omega/(1-\tau_t)$ in (46), which reduces the aggregate gross profit productivity a_t . In text, we avoid the distortionary tax in order to concentrate on the analysis of liquidity.

Now the aggregate state of the economy is summarized by aggregate capital stock, aggregate productivity, the government holdings of capital and land: $s_t = (K_t, a_t, K_t^g, L_t^g)$. The competitive equilibrium is described recursively by $\{p_t, q_t, r_t^L, r_t^K, I_t, Y_t, G_t - T_t\}$ as functions of s_t and the evolutions of s_t which satisfies the equilibrium conditions:

$$(48) \quad (1 - \theta q_t) I_t = \Pi \left([\beta r_t^K + \theta \lambda q_t - (1-\beta)\lambda] (K_t - K_t^g) + \beta (r_t^L + p_t) (L - L_t^g) \right),$$

$$(49) \quad Y_t = I_t + G_t - T_t + (1-\beta) \left([r_t^K + \pi \lambda + (1-\pi)\lambda q_t] (K_t - K_t^g) + (r_t^L + p_t) (L - L_t^g) \right),$$

$$(50) \quad (1-\Pi) E_t \left(\frac{\frac{r_{t+1}^K + \lambda q_{t+1}}{q_t} - \frac{r_{t+1}^L + p_{t+1}}{p_t}}{(r_{t+1}^K + \lambda q_{t+1}) K_{t+1}^n + (r_{t+1}^L + p_{t+1}) (L - L_t^g)} \right) = \Pi E_t \left(\frac{\frac{r_{t+1}^L + p_{t+1}}{p_t} - \frac{r_{t+1}^K + \lambda}{q_t}}{(r_{t+1}^K + \lambda) K_{t+1}^n + (r_{t+1}^L + p_{t+1}) (L - L_t^g)} \right).$$

$$(51) \quad K_{t+1} = K_{t+1}^g + K_{t+1}^n + (1-\theta) [I_t + \Pi \lambda (K_t - K_t^g)],$$

and (16), (44), (46), (47) and an exogenous law of motion of (a_t, K_t^g, L_t^g) . Equation (48) is the flow-of-funds constraint of the investing entrepreneurs. It is similar to the investment function before, except that only private holding of capital and land enter into the flow-of-funds of the investing entrepreneurs. Equation (49) describes the goods market clearing condition. Aggregate output is equal to the sum of investment, government purchase, and consumption of workers and entrepreneurs. Because the consumption of workers is equal to the disposable wage income, the goods market clearing is equivalent to the condition that the gross profit (aggregate output minus pre-tax wage income) is equal to investment, government primary deficit, and entrepreneur's consumption. Equation (50) is the portfolio choice of the entrepreneurs. Since the workers do not hold assets, the entrepreneurs' assets are equal to the entire private assets. Equation (51) is the market

clearing of the ownership of capital stock. It implies that the capital stock is owned by government, the entrepreneurs who are not investing, and the entrepreneurs who are investing.

Obviously, if the government reduces the lump-sum tax on workers by selling land now, and if it will increase the tax in future in order to buy back land, then the real allocation will change, because the workers' consumption is equal to their disposable income. Thus, Ricardian Equivalence Theorem between lump-sum tax finance and debt finance no longer holds here, because workers are credit constrained. A little less straightforward is the effect of government's portfolio policy, i.e., the monetary policy.

Suppose that the government does an open market operation to purchase capital by selling land (or the government liquid security whose return is identical to land.) Here, let us analyze the long-run effect in the steady state. It is more convenient to describe the steady state equilibrium as a continuous time approximation and in an intensive form as a ratio to the steady state capital stock. Define $\kappa = (K - K^g)/K$ (fraction of private capital holding), $l = (L - L^g)/L$ (fraction of private land holding), and $g = (G - T)/K$ (ratio of primary fiscal deficit to capital stock).¹⁵ The steady state equilibrium is described by the corresponding equations with (48, 49, 50, 44) with $i = \delta$ as:

$$(52) \quad (1 - \theta q)\delta = \pi(vl + \theta q\kappa),$$

$$(53) \quad x = \delta + g + \rho(vl + q\kappa),$$

$$(54) \quad \left(\frac{\alpha}{q} - \frac{\gamma}{v}\right) \frac{x}{\alpha + \gamma} - \delta = \pi \frac{q-1}{q} \frac{vl + q\kappa}{vl + \kappa},$$

$$(55) \quad g = \frac{\gamma}{\alpha + \gamma} x (1-l) + \left(\frac{\alpha}{\alpha + \gamma} x - \delta q\right) (1-\kappa)$$

¹⁵Notice that l is not labor of the individual entrepreneur in the production function (2). Even in the steady state, the capital and labor of the individual entrepreneur fluctuate due to the idiosyncratic arrival of the investment opportunity.

First, we can derive the condition under which the economy is monetary:

$$\text{(Assumption 3)} \quad \theta < \theta^{**} \equiv 1 - \frac{\pi}{\delta + \pi\kappa} \left(\frac{\rho - g}{\rho} + \frac{\gamma}{\alpha} \frac{\rho + \delta}{\rho} \right).$$

Assumption 3 is a generalization of Assumption 1 for the economy with workers and government. In particular, the threshold fraction of saleable capital at the time of investment is an increasing function of g and κ , taking an account of the implied value of ℓ from the budget constraint of the government. Thus, roughly speaking, the economy is more likely to be monetary, if the private agents hold a smaller fraction of land and capital instead of the government. In the following, we assume Assumption 3 so that the economy is monetary in the neighborhood of the steady state.

We now compare the two steady states for alternative government portfolio policies with the same primary fiscal deficit rate and the same aggregate productivity.¹⁶ Suppose that a 'new' steady state corresponds to a government portfolio of more capital holding and less land holding than 'old' steady state. In other words, in 'new' steady state, the supply of assets to the private agents is more liquid in the sense that the private agents holds a larger fraction (ℓ) of liquid land and a smaller fraction (κ) of illiquid capital. From equations (54-57), we can show that:

Proposition 2: Suppose that in the 'new' steady state, government holds more capital and less land than 'old' steady state for the same primary fiscal deficit ratio. Suppose also that both old and new steady state satisfy Assumption 3. Then, in the new steady state:

¹⁶ The government total asset is unstable in (44), because the real rates of returns on land and capital are positive in the neighborhood of steady state as in [Sargent and Wallace(1981)]. Thus, even if the productivity is constant, once-for-all an open market operation starting from a steady state economy will not converge to another steady state in the long run without adjustment of fiscal policy. The government needs to adjust g during the intermediate period in order to achieve the steady state with different portfolio with the same g in the long run.

(i) Tobin's q is smaller;

(ii) liquidity premium is lower;

than the old steady state, if the size of government is not very large. Moreover, if the economy is not too far from the first best allocation, then;

(iii) capital stock, employment and output are larger.

(Proof) in Appendix.

Intuitively, Proposition 2 implies that if the government holds more illiquid portfolio so that the private agent can hold more liquid portfolio, then the liquidity shortage of the economy is less severe in the new steady state. Tobin's q is smaller, liquidity premium is smaller, and aggregate capital, employment and output are most likely to be larger. An interesting aspect is the valuation of capital, q , which becomes lower when the private agents hold a smaller fraction of capital stock. This is because the gap between Tobin's q and unity arises because of the liquidity constraint, and a larger supply of liquidity mitigates the shortage of liquidity and reduces Tobin's q towards one. A Modigliani-Miller Theorem for open market operation in [Wallace(1983)] breaks down here, because the government portfolio choice directly affects the ratio of the liquid asset to the illiquid asset of the private agents, and it is the private liquid assets holding which particularly lubricates transferring resources from the savers to the investors. On the other hand, if Assumption 3 does not hold so that the economy achieves the first best allocation, then the rates of returns on liquid asset and illiquid capital would be equal, and the government open market operation would have no effect on the resource allocation.

Perhaps, one aspect of our model that is at odds with is the property that the workers do not save at all. If, instead of Poisson arrival of a productivity switch with infinitesimal size of shock, we have larger productivity shocks, or continual productivity shocks as in the case of

geometric Brownian motion, then the workers will save in order to smooth consumption, despite the fact that the expected rate of return on assets are smaller than the time preference rate as in buffer stock model. (See [Bewley(1977,1980,1983)], [Carroll(1992)] and [Deaton(1992)].) However, there is only one means of saving in the usual buffer stock model. Thus by itself, it does not explain why the workers do not save in illiquid capital. Alternatively, we can extend our model to allow workers to have investment opportunities of small size.

Suppose that each worker suffer a 'health' shock according to a Poisson process with arrival rate ϕ . With the shock, the worker has to spend ξ unit of goods instantaneously in order to maintain his human capital (capacity to supply labor). After arrival of one shock, there is a time interval T , during which the worker is immune against shock and $wT \gg \xi$ in the steady state. After this interval, the arrival rate of the shock goes back to ϕ . The shock is assumed to be not contractible. We restrict the attention to the case in which ϕ and ξ are sufficiently small so that the economy continues to be monetary by a slightly different condition than Assumption 3.

Then, we can show that, for a small enough θ , (i) in normal time, each worker holds exactly ξ units of liquid asset; (ii) with the arrival of a shock, the worker sells the entire liquid asset in order to meet the shock; (iii) the worker saves with the liquid asset at constant rate during the immune period in order to accumulate ξ units at the end of the immune period; and (iv) the worker does not hold the illiquid capital. Intuitively, the worker does not save more than ξ units of the liquid asset, because both the rate of returns on the illiquid and the liquid assets are lower than the time preference rate under the monetary economy. (See Proposition 1(ii).) The worker does not save in the illiquid capital even if the expected rate of return on capital dominates that of liquid land, because the worker needs more illiquid capital in order to meet the shock, which turns out to be 'more expensive', as:

$$(56) \quad \frac{1}{\theta} \left\{ \rho + \delta - E_t \left[\frac{r_t^K dt + dq_t}{q_t dt} \right] \right\} > \rho - E_t \left[\frac{r_t^L dt + dp_t}{p_t dt} \right].$$

Equation (56) implies that the opportunity cost of holding $1/\theta$ units of capital is larger than the opportunity cost of holding 1 unit of land, (remember that the agent can sell only θ fraction of capital in order to finance investment which includes overcoming the 'health' shock.) In the Appendix, we prove (56) holds in the neighborhood of steady state, if the economy is monetary and θ is small enough.

In the monetary economy, the expected returns on liquid assets and illiquid assets are both lower than the time preference rate. Despite these low returns on assets, people save because people expect to face liquidity constraint when they need to finance investment expenditures in the future. The entrepreneurs save substantial amounts of both liquid and illiquid assets, because their investment opportunity is large (or constant returns to scale here.) The workers save only a small amount of liquid assets, because their investment opportunity is small (or fixed size here.) If the workers were expecting large investment opportunity in future, say, buying a house or having their children educated in college, then these workers would save in substantial amounts like entrepreneurs.¹⁷ If people with large productive investment opportunities were not liquidity constrained, then their savings would not depend upon the expectation of future investment opportunities. Therefore, another unique feature of the monetary economy is that the agent's saving behavior depends upon his expectations of productive investment opportunities in future.

¹⁷In national income account, when households build and renovate houses, it counts as investment because such households engage in entrepreneurial activity. We can regard these households with large investment opportunities as entrepreneurs instead of workers.

References (Incomplete):

- Bewley, Truman, (1977): "The Permanent Income Hypothesis: A Theoretical Foundation," *Journal of Economic Theory*, 16(2), 252-292.
- _____, (1980): "The Optimum Quantity of Money," in *Models of Monetary Economies*, by J. Kareken and N. Wallace (eds.) Federal Reserve Bank of Minneapolis, Minneapolis, 169-210.
- _____, (1983): "A Difficulty with the Optimum Quantity of Money," *Econometrica*, 51, 1485-1504.
- Brock, William, (1982): "Asset Prices in a Production Economy," in J.J. McCall (ed.) *The Economics of Information and Uncertainty*, University of Chicago Press, Chicago IL, 1-43.
- Campbell, John, Andrew Lo, and Craig Mackinlay, (1997): *The Econometrics of Financial Markets*, Princeton University Press, Princeton, NJ.
- Carroll, Christopher, (1992): "The Buffer Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity*, 2, 61-135.
- Cochrane, John, (2001): *Asset Pricing*, Princeton University Press, Princeton, NJ.
- Deaton, Angus, (1992): *Understanding Consumption*, Oxford University Press, Oxford and New York.
- Diamond, Douglas, and Philip Dybvig (1983): "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy*, 91(3), 401-419.
- Holmström, Bengt, and Jean Tirole, (1998): "Private and Public Supply of Liquidity," *Journal of Political Economy*, 106, 1-40.
- _____, and _____, (2000): "LAPM: A Liquidity-based Asset Pricing Model," working paper, MIT and University of Toulouse.
- Jacklin, C.J., (1987): "Demand Deposits, Trading Restrictions and Risk Sharing," in *Contractual Arrangements for Intertemporal Trade*, by E. Prescott and N. Wallace (eds.), University of Minnesota Press, Minneapolis.
- Keynes, John Maynard, (1936): *The General Theory of Employment, Interest and Money*, Macmillan, London.
- Kiyotaki, Nobuhiro and John Moore, (1997): "Credit Cycles," *Journal of Political Economy*, 105, 211-248.
- _____, and _____, (2000): "Inside Money and Liquidity," working paper, London School of Economics.
- _____, and _____, (2001): "Liquidity, Business Cycles, and Monetary Policy," working paper, London School of Economics.
- Kydland, Finn, and Edward Prescott, (1982): "Time to Build and Aggregate Fluctuations," *Econometrica*, 50(6), 1345-1371.

Mehra, Rajnish, and Edward Prescott, (1985): "The Equity Premium Puzzle," *Journal of Monetary Economics*, 15, 145-161.

Merton, Robert, (1975): "An Asymptotic Theory of Growth Under Uncertainty," *Review of Economic Studies*, 42, 375-93.

Sargent, Thomas, and Neil Wallace, (1981): "Some Unpleasant Monetarist Arithmetic," *Quarterly Review*, Federal Reserve Bank of Minneapolis, 5(3), 1-17.

Tobin, James, (1969): "A General Equilibrium Approach to Monetary Theory," *Journal of Money, Credit and Banking*, 1(1), 15-29.

Townsend, Robert, (1987): "Asset Return Anomalies in a Monetary Economy," *Journal of Economic Theory*, 41(5), 219-47.

_____, and Neil Wallace, (1987): "Circulating Private Debt: an Example with Coordination Problem," in *Contractual Arrangements for Intertemporal Trade*, by E. Prescott and N. Wallace (eds.), University of Minnesota Press, Minneapolis.

Wallace, Neil, (1981): "A Modigliani-Miller Theorem for Open-Market Operation," *American Economic Review*, 71, 267-74.

Weil, Philippe, (1989): "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," *Journal of Monetary Economics* 24, 401-21.

Appendix 1: Proof of Propositions

Proof of Proposition 1: Define $x \equiv Y/K = aK^\alpha L^{1-\alpha}$ and $v \equiv pL/K$. From (22), we have:

$$(A1a) \quad (1-\theta q)(1-\lambda) = \Pi [\beta x + \theta \lambda q - (1-\beta)\lambda + \beta v],$$

$$(A1b) \quad \beta x = 1 - \lambda + \lambda \Pi (1-\beta) + (1-\beta)(1-\Pi)\lambda q + (1-\beta)v,$$

$$(A1c) \quad \left(\frac{\alpha}{q} - \frac{1-\alpha}{v} \right) x - (1-\lambda) = \Pi \lambda \frac{q-1}{q} \left(1 + \frac{1-\alpha}{v} x \right) \frac{\xi q + v}{\xi(\alpha x + \lambda) + (1-\alpha)x + v},$$

where $\xi \equiv K^\Pi/K = \lambda(1-\Pi) + \theta(1-\lambda + \lambda\Pi)$. Then we have,

$$(A2a) \quad x(q) = \frac{1}{\Pi\beta} \left((1-\beta + \Pi\beta)[1-\lambda + \lambda\Pi(1-\beta)] + (1-\beta)[\lambda\beta\Pi(1-\Pi) - (1-\lambda + \lambda\Pi)\theta]q \right),$$

$$(A2b) \quad v(q) = \frac{1}{\Pi} \left((1-\Pi)[1-\lambda + \lambda\Pi(1-\beta)] - [\lambda\Pi(1-\beta)(1-\Pi) + (1-\lambda + \lambda\Pi)\theta]q \right),$$

$$(A2c) \quad F(q; \theta) \equiv 1 - \lambda - x(q) \left(\frac{\alpha}{q} - \frac{1-\alpha}{v(q)} \right) + \Pi \lambda \frac{q-1}{q} \left(1 + \frac{1-\alpha}{v(q)} x(q) \right) \frac{\xi q + v(q)}{(\alpha \xi + 1 - \alpha)x(q) + \lambda \xi + v(q)}$$

$$= 0$$

We know $x'(q) > 0$ if $\Pi\beta\lambda(1-\Pi) > (1-\lambda + \lambda\Pi)\theta$, and $v'(q) < 0$.

Proof of Proposition 1(i): From (A2), we see F is an increasing function of q . Then there is a unique q which is larger than 1 if and only if $F(1; \theta) < 0$. Because F is an increasing function of θ , $F(1; \theta) < 0$, if and only if Assumption 1 holds. *Q.E.D.*

Proof of Proposition 1(ii): From the optimal portfolio condition (20c), the return on land is dominated by capital, $(r^L + p)/p < (r^K + \lambda q)/q$, if and only if q exceeds unity, or Assumption 1 holds. Also the left-hand side (LHS) of (A1c) is written as:

$$\text{LHS of (A1c)} = \left(\frac{\alpha}{q} x + \lambda - \beta \right) \left(1 + \frac{q}{v} \right) + \frac{1}{\beta} - 1 - \frac{1}{v} \left(x + q \left(\lambda - \frac{1}{\beta} \right) \right)$$

$$= \left(\frac{\alpha}{q}x + \lambda - \beta \right) \left(1 + \frac{q}{v} \right) + (q-1) \frac{1-\lambda+\lambda\Pi(1-\beta)}{\beta v}, \text{ (from (A1b)).}$$

Since we already proved $(r^K+\lambda q)/q > (r^L+p)/p$, we know $(r^K+\lambda)/q < (r^L+p)/p$ from (21) in the steady state. Also because $K^n < K$, we learn:

$$\text{the right-hand side (RHS) of (A1c)} < \Pi\lambda \frac{q-1}{q} \frac{q+v}{x+\lambda+v}.$$

Therefore, the sufficient condition for $(r^K+\lambda q)/q < 1/\beta$ is:

$$\Pi\lambda \frac{q+v}{x+\lambda+v} < q \frac{1-\lambda+\lambda\Pi(1-\beta)}{\beta v}.$$

Using (A1b), this is equivalent to:

$$\begin{aligned} \text{(A3)} \quad \frac{\lambda\beta\Pi[1+(v/q)]}{1-\lambda+\lambda\Pi(1-\beta)} &< 1 + \lambda/v + \frac{1}{v} \left(\frac{1-\lambda+\lambda\Pi(1-\beta)}{\beta} + \frac{(1-\beta)(1-\Pi)\lambda q}{\beta} \right) + \frac{1-\beta}{\beta}, \\ &= \frac{1}{\beta} + \frac{1}{\beta v} + (q-1) \frac{(1-\beta)(1-\Pi)\lambda}{\beta}. \end{aligned}$$

From (A2b), we know:

$$v(q) \leq v(1) = \frac{1}{\Pi} \left((1-\Pi)(1-\lambda) - (1-\lambda+\lambda\Pi)\theta \right) \leq \frac{(1-\Pi)(1-\lambda)}{\Pi}.$$

Then

$$\text{RHS of (A3)} \geq \frac{1}{\beta} \left(1 + \frac{\Pi}{(1-\Pi)(1-\lambda)} \right) = \frac{1-\lambda+\lambda\Pi}{\beta(1-\Pi)(1-\lambda)}.$$

$$\text{LHS of (A3)} \leq \frac{\beta\lambda(1-\lambda+\lambda\Pi)}{1-\lambda+\lambda\Pi(1-\beta)}.$$

But because $\beta^2\lambda(1-\Pi)(1-\lambda) < 1-\lambda+\lambda\Pi(1-\beta)$, we learn LHS of (A3) < RHS of (A3), which implies $(r^K+\lambda q)/q < 1/\beta$. Q.E.D.

Proof of Proposition 1(iii): From (A1c), we get

$$\begin{aligned} \text{LHS of (A1c)} &= \left(\frac{\alpha x}{q} + \lambda - \frac{1}{\beta} \right) \left(\frac{1}{q} + \frac{1}{v} \right) + \lambda - 1 - \frac{x}{v} + \left(\frac{1}{\beta} - \lambda \right) \left(\frac{1}{q} + \frac{1}{v} \right) \\ &= \left(r^K + \lambda - \frac{1}{\beta} \right) \left(\frac{1}{q} + \frac{1}{v} \right) - (q-1) \frac{(1-\beta)(1-\Pi)\lambda}{\beta v} - \left(\frac{1}{\beta} - \lambda \right) \frac{q-1}{q}, \text{ (from (A1b)).} \end{aligned}$$

But RHS of (A1c) > 0. Together with $q > 1$, we learn $r^{K+\lambda} > 1/\beta$. Q.E.D.

Proof of Proposition 1(iv): We know $m_{t+1}^i = 0$, iff (11) is true. From (8)(12) and (13b), we know that:

$$\begin{aligned} c_{t+1}^{ii} &= (1-\beta)(r_{t+1}^K + \lambda)k_{t+1}^i = (1-\beta)(r_{t+1}^K + \lambda) \frac{1-\theta}{1-\theta q_t} \beta b_t^i, \\ c_{t+1}^{in} &= (1-\beta)(r_{t+1}^K + \lambda q_{t+1})k_{t+1}^i = (1-\beta)(r_{t+1}^K + \lambda q_{t+1}) \frac{1-\theta}{1-\theta q_t} \beta b_t^i. \end{aligned}$$

Thus, together with (12b), the inequality (11) in the steady state is equivalent to:

$$1 > \left(1 + \frac{r^L}{p} \right) \frac{1-\theta q}{1-\theta} \left(\frac{\Pi}{r^{K+\lambda}} + \frac{1-\Pi}{r^{K+\lambda q}} \right),$$

or

$$\begin{aligned} \left(1 + \frac{r^L}{p} \right) \frac{1-\theta q}{(1-\theta)q} \left(1 + \frac{\Pi \lambda (q-1)}{r^{K+\lambda}} \right) &< \frac{r^{K+\lambda q}}{q} \\ &= \left(1 + \frac{r^L}{p} \right) \left(1 + \Pi \lambda \frac{q-1}{q} \frac{qK^n + pL}{(r^{K+\lambda})K^n + (r^L + p)L} \right), \text{ (from (22c)).} \end{aligned}$$

This is equivalent to:

$$1 > \Pi \lambda \frac{1-\theta q}{r^{K+\lambda}} - \Pi \lambda (1-\theta) \frac{qK^n + pL}{(r^{K+\lambda})K^n + (r^L + p)L}.$$

But, this is true because the first term in RHS is smaller than 1. Q.E.D.

Proof of Proposition 2: Combining equations (52-55) in the text, we can get the steady-state market clearing condition for goods market and asset market as:

$$(A4a) \quad \phi(q, x; \theta, \kappa) \equiv (\pi + \rho)\delta + \rho q[(1 - \theta)\pi\kappa - \theta\delta] + \pi g - \pi x = 0,$$

$$(A4b) \quad \psi(q, x; \theta, \kappa) \equiv \pi \frac{(1 - \frac{\alpha}{\alpha + \gamma}\kappa)x - g - (1 - \kappa)\delta q}{\delta - \theta(\delta + \pi\kappa)q} - \frac{\alpha}{\alpha + \gamma} \frac{x}{q} + \delta + \frac{\pi}{\delta + \pi\kappa} \frac{q - 1}{q} \frac{\delta + q[(1 - \theta)\pi\kappa - \theta\delta]}{1 - \theta q} = 0.$$

Combining these, we also get:

$$(A5) \quad \tilde{F}(q; \theta, \kappa) = \frac{\pi g + (\pi + \rho)\delta + \rho q[(1 - \theta)\pi\kappa - \theta\delta]}{\alpha + \gamma} \left(\frac{\alpha + \gamma - \alpha\kappa}{\delta - \theta q(\delta + \pi\kappa)} - \frac{\alpha}{\pi q} \right) + \delta - \pi \frac{g + (1 - \kappa)\delta q}{\delta - \theta q(\delta + \pi\kappa)} + \frac{\pi}{\delta + \pi\kappa} \frac{q - 1}{q} \frac{\delta + q[(1 - \theta)\pi\kappa - \theta\delta]}{1 - \theta q} = 0.$$

From (A5), we can see $\tilde{F}_q > 0$, $\tilde{F}_\theta > 0$, and $\tilde{F}_\kappa < 0$, if g is close to 0 (balanced fiscal budget) and κ is not too far from 1 (government does not own too much capital.)

Proof of Proposition 2(i): From (A5), we have:

$$(A6) \quad \frac{\partial q}{\partial \kappa} = - \frac{\tilde{F}_\kappa}{\tilde{F}_q} > 0,$$

Thus, a reduction of κ with the open market operation will reduce Tobin's q .

Proof of Proposition 2(ii): From the asset market equilibrium (A4b), the liquidity premium is equal to:

$$(A7) \quad \frac{r^K}{q} - \delta - \frac{r^L}{p} = \frac{\pi}{\delta + \pi\kappa} \frac{q - 1}{q} \frac{\delta + q[(1 - \theta)\pi\kappa - \theta\delta]}{1 - \theta q} \equiv \varphi(\kappa, q).$$

Then we have

$$(A8) \quad \frac{d\phi}{d\kappa} = \phi_{\kappa} + \phi_q \frac{dq}{d\kappa} > 0,$$

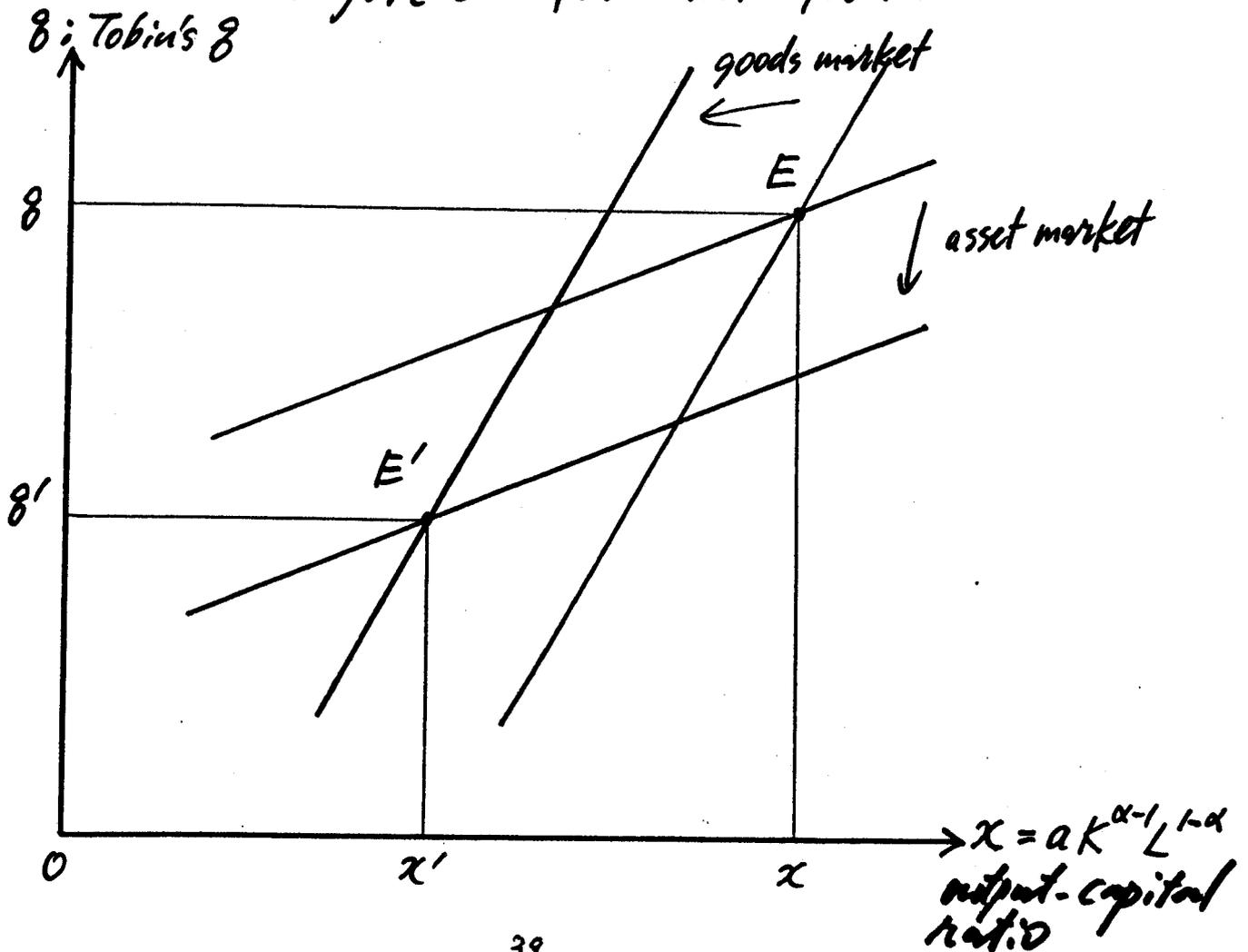
because $\phi_{\kappa}, \phi_q > 0$ and $\frac{dq}{d\kappa} > 0$ from (A6). Thus the open market operation to reduce κ will reduce the liquidity premium.

Proof of Proposition 2(iii): From (A4), market clearing conditions of goods market and asset market are described in Figure 3.

From (A4a), we have $\phi_{\kappa} > 0$. Also $\psi_{\kappa} < 0$, if the government is small (g is close to 0 and κ is close to 1) and θ is not too far from θ^{**} . Also from (A4a), we know $\phi_q > 0$ if and only if $\theta(\pi\kappa + \delta) < \pi\kappa$. From Assumptions 2 and 3 with $\kappa \cong 1$, we know $\theta(\pi\kappa + \delta) < \pi\kappa$. We also have $\phi_x < 0, \psi_q > 0$, and $\psi_x < 0$ if the government is small. Thus the open market operation shifts the equilibrium schedules as in Figure 3. Therefore, we learn x will decrease or capital stock will increase in the long run with the open market operation.

Q.E.D.

Figure 3 Open Market Operation



Proof of Inequality (56): The inequality (56) holds if and only if:

$$\rho - \frac{\gamma}{\alpha+\gamma} \frac{x}{v} < \frac{1}{\theta} \left(\rho + \delta - \frac{\alpha}{\alpha+\gamma} \frac{x}{q} \right),$$

from (47). When the government is negligible, $g=0$ and $\kappa=1$, then this is equivalent to

$$\left(\alpha\delta - \theta q [\alpha(\pi+\delta) + \gamma\pi] \right) \left((\pi+\rho)\delta + [\pi - \theta(\pi+\delta)]\rho q \right) < \pi(\alpha+\gamma)q[\delta + (1-\theta)\rho][\delta - \theta(\pi+\delta)q],$$

which we can verify from (A5) with a bit of algebra, if θ is small enough, $g \approx 0$, and $k \approx 1$. Q.E.D.

Appendix 2: Continuous-Time Formulation of the Agent's Behavior

Here, we describe the individual agent's behavior using a continuous-time formulation, assuming that Tobin's q exceeds unity. The agent's utility is given by:

$$(A9) \quad V_0 = E_0 \left[\int_0^{\infty} \ln c_t e^{-\rho t} dt \right],$$

where c_t is consumption rate at date t , ρ is a constant time preference rate. The individual agent can produce according to production function (2), where y_t is output per unit of time. The aggregate productivity a_t follows a continuous-time two-point Markov process (29). The depreciation rate of capital is δ .

Each agent meets an opportunity to invest in capital according to a Poisson process of arrival rate π . At the time of the investment, the agent can sell up to a fraction θ of the capital held before the investment and the newly invested capital. Also the agent cannot short sale on land. Thus, the flow-of funds constraint of the investing agent is:

$$(A10) \quad (1 - \theta q_t) i_t \leq p_t m_t + \theta q_t k_t,$$

where i_t is investment, m_t is land and k_t is capital owned before the investment. Let b_t be the total asset of the individual:

$$(A11) \quad b_t = p_t m_t + q_t k_t.$$

If the agent has a productive investment opportunity at date t , he can choose whether or not to invest the maximum amount subject to the flow-of-fund constraint. After the maximal investment, the agent has sold all the land, and holds $1-\theta$ fraction of capital previously held and capital newly invested. Thus, from (A10), the total asset after the investment is:

$$(A12) \quad b_{t+dt} = \text{Max} \left(q_t (1-\theta)(k_t + i_t), b_t \right)$$

$$= \text{Max} \left(\frac{(1-\theta)q_t}{1-\theta q_t} (p_t m_t + k_t), b_t \right).$$

Let r_t^L be the rental price of land and r_t^K be the profit per unit of capital per unit of time. Then the accumulation of the total asset of the agent between date t and $t+dt$ is:

$$(A13) \quad db_t = (r_t^L dt + dp_t) m_t + (r_t^K dt + dq_t) k_t - c_t dt \\ + \text{Max} \left(\frac{(1-\theta)q_t}{1-\theta q_t} (p_t m_t + k_t) - b_t, 0 \right) dM_t,$$

where $dM_t = 1$ if the agent has an investment opportunity at date t and $dM_t = 0$ otherwise.

In the following, we concentrate on the case in which Tobin's q exceeds unity so that the credit constraint is binding for the investing agents. Let $V(b,s)$ be the value function of individual with total asset b when the aggregate state is $s = (K,a)$. Then Bellman equation would be:

$$(A14) \quad \rho V(b,s) = \text{Max}_{c,m,k} \left\{ \ln c + V_b(b,s) \left[(r_t^L + \dot{p})m + (r_t^K + \dot{q})k - c \right]_{da=0} \right. \\ + \pi \left[V\left(\frac{(1-\theta)q}{1-\theta q}(pm+k), s\right) - V(b,s) \right] \\ \left. + \eta \left[V(p(s')m + q(s')k, s') - V(b,s) \right]_{da \neq 0} \right\},$$

subject to (A11) and (A12), where s is the aggregate state at date t and s' is the state at date $t+dt$. The first line in the right-hand side is the utility of consumption and the value of saving when there is no switch in aggregate productivity. With the same productivity, there is no discontinuous jump in the asset prices and \dot{x} denotes $\frac{dx}{dt}$. The second line is the expected gain in the value with the arrival of a productive investment opportunity, corresponding to equation (A12). The last term is the capital gains

associated with the switch of the aggregate productivity. The first order conditions for consumption and portfolio choice are:

$$(A15) \quad \frac{1}{c} = V_b(b, s),$$

$$(A16) \quad V_b(b, s) \left[\frac{r^L + \dot{p}}{p} - \frac{r^K + \dot{q}}{q} \right]_{da_t=0} + \eta V_b(p(s')l + q(s')k, s') \left[\frac{p(s')}{p} - \frac{q(s')}{q} \right]_{da_t \neq 0} \\ + \pi V_b \left(\frac{(1-\theta)q}{1-\theta q} (p\ell + k), s \right) \frac{(1-\theta)q}{1-\theta q} \left(1 - \frac{1}{q} \right) = 0.$$

Equation (A15) means the marginal utility of consumption must be equal to the marginal value of assets for the optimal consumption choice. Equation (A16) implies that the expected rate of returns on liquid land and illiquid capital should be equal in terms of utility for the optimal portfolio. In particular, the second line is the expected return for financing the downpayment of investment, in which the rate of return on illiquid capital is $(1/q_t)$ times as much as the rate of return on liquid land. Then, we have:

$$(A17) \quad V(b_t, s_t) = \tilde{V}(s_t) + \frac{1}{\rho} \ln b_t,$$

$$(A18) \quad c_t = \rho b_t,$$

$$(A19) \quad \left[\frac{r_t^L}{p_t} - \frac{r_t^K}{q_t} \right] + \pi \frac{q_t^{-1} p_t m_t + q_t k_t}{p_t m_t + k_t} \\ + \left[\frac{\dot{p}_t}{p_t} - \frac{\dot{q}_t}{q_t} \right]_{da_t=0} + \eta \left[\left(\frac{p_{t+dt}}{p_t} - \frac{q_{t+dt}}{q_t} \right) \frac{p_t m_t + q_t k_t}{p_{t+dt} m_t + q_{t+dt} k_t} \right]_{da_t \neq 0} = 0.$$

Because the instantaneous utility function is the log of consumption rate, the value function is a log linear function of total assets in equation (A10).

In the aggregate, between date t and $t+dt$, exactly a fraction πdt of agents invest with binding flow-of-funds constraint (A10), we get (24). From (A18), the goods market equilibrium is (27). From (A19) with (26), the asset market equilibrium is given by (28).

Appendix 3: Derivation of Local Dynamics:

From (34) and (35), we have,

$$(A20) \quad \lim_{t \rightarrow 0} E_t \left(\frac{1}{dt} \left(\frac{q_{t+dt}}{q_t} - \frac{v_{t+dt}}{v_t} \right) \frac{q_t^{1+\nu} v_t}{q_{t+dt}^{1+\nu} v_{t+dt}} \right) \\ = (\lambda_q - \lambda_v) [-(1-\alpha)\hat{\delta i}_t] + (\lambda_q - \lambda_v + \mu_q - \mu_v) \eta (-2\hat{a}_t).$$

The first term of RHS of (A20) is the difference in the rate of capital gains between capital and land due to capital accumulation. The second term is the difference in the expected rates of capital gains associated with the productivity change, noting that the arrival rate of the change is η and the proportional size of the change (\hat{da}_t) is equal to $-2\hat{a}_t$ by (29). Thus linearizing (30), (32) and (33) around the steady state, we have:

$$(A21) \quad (1-\theta q)\hat{\delta i}_t = \pi v \hat{v}_t + \theta(\pi+\delta)q \hat{q}_t,$$

$$(A22) \quad \hat{x}_t = \hat{\delta i}_t + \rho v \hat{v}_t + \rho q \hat{q}_t,$$

$$(A23) \quad \left[\frac{\alpha}{q} - \frac{1-\alpha}{v} \right] \hat{x}_t - \left[\frac{\alpha}{q} \hat{q}_t - \frac{1-\alpha}{v} \hat{v}_t \right] x - [1+(\lambda_q - \lambda_v)(1-\alpha)] \hat{\delta i}_t \\ - 2\eta(\lambda_q + \mu_q - \lambda_v - \mu_v) \hat{a}_t = \pi \left[\frac{1}{q} \frac{v+q^2}{v+1} \hat{q}_t - \frac{v}{q} \left(\frac{q-1}{v+1} \right)^2 \hat{v}_t \right]$$

Substituting (34) into (A21,A22,A23), these equations must hold for any \hat{x}_t and \hat{a}_t . Thus we have 6 unknown variables ($\lambda, \lambda_p, \lambda_q, \mu, \mu_p, \mu_q$) which satisfy 6 equations, two terms of \hat{x}_t and \hat{a}_t in each of (A21), (A22) and (A23).

Solving (A21,A22) for \hat{v}_t and \hat{q}_t with respect to \hat{i}_t and \hat{x}_t , we have:

$$(A24) \quad \begin{pmatrix} \hat{v}_t \\ \hat{q}_t \end{pmatrix} = \frac{1}{\rho[\pi - \theta(\pi + \delta)]} \begin{pmatrix} [\rho(1-\theta q) + \theta(\pi + \delta)]\delta q & -\theta(\pi + \delta)qx \\ -[\rho(1-\theta q) + \pi]\delta v & \pi vx \end{pmatrix} \begin{pmatrix} \hat{i}_t \\ \hat{x}_t \end{pmatrix}$$

$$= \begin{pmatrix} D_{vi} & D_{vx} \\ D_{qi} & D_{qx} \end{pmatrix} \begin{pmatrix} \hat{i}_t \\ \hat{x}_t \end{pmatrix}.$$

From (34) and (A24), we learn:

$$(A25) \quad \begin{pmatrix} \lambda_v(\lambda) & \mu_v(\mu) \\ \lambda_q(\lambda) & \mu_q(\mu) \end{pmatrix} = \begin{pmatrix} D_{vi} & D_{vx} \\ D_{qi} & D_{qx} \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ 1 & 0 \end{pmatrix}.$$

Substituting (A24,A25) into (A23), we have two equations for the terms of \hat{x}_t and \hat{a}_t as:

$$(A26) \quad 0 = \left(1 + [\lambda_q(\lambda) - \lambda_p(\lambda)](1-\alpha)\right) \delta\lambda - \left(\frac{\alpha}{q} - \frac{1-\alpha}{v}\right) x$$

$$+ \pi \frac{1}{q} \frac{v+q^2}{v+1} \lambda_q(\lambda) - \pi \frac{v}{q} \left(\frac{q-1}{v+1}\right)^2 \lambda_v(\lambda) \equiv \Phi(\lambda) \equiv \phi_2 \lambda^2 + \phi_1 \lambda + \phi_0,$$

$$(A27) \quad 0 = \left(1 + [\lambda_q(\lambda) - \lambda_p(\lambda)](1-\alpha)\right) \delta\mu + 2\eta[\lambda_q(\lambda) - \lambda_v(\lambda) + (D_{qi} - D_{vi})\mu]$$

$$+ \pi \frac{1}{q} \frac{v+q^2}{v+1} D_{qi} \mu - \pi \frac{v}{q} \left(\frac{q-1}{v+1}\right)^2 D_{vi} \mu \equiv \Psi(\mu, \lambda) \equiv \psi_1(\lambda) \mu + \psi_0(\lambda).$$

Under Assumption 1 and 2, we learn $\Phi(\lambda) = 0$ is quadratic equation of λ , with $\phi_2 < 0$ and $\phi_0 > 0$. Therefore, there is a unique λ which satisfies the the saddle-point stability condition $\lambda > 0$. From (A24,A25,A26), we can derive the properties of (36) in the text. Also from (A26), we see that $\Psi(\mu, \lambda) = 0$ is a linear equation of μ . We also learn μ is negative and proportional to η . Again from (A24,A25,A26,A27), we can learn the properties of (37,38) in the text.

Appendix 4: Basic Model with Brownian Productivity Shock

In this appendix, we layout the basic model with aggregate productivity shock which follows the geometric Brownian process as in (41). We ignore workers and government here. We first guess that the aggregate output-capital ratio $x_t = a_t K_t^{\alpha-1} L^{1-\alpha}$ and the aggregate productivity a_t summarizes the aggregate state of nature. We also guess that the land price - capital ratio, capital price, and investment rate are functions of x_t only:

$$(A28) \quad v_t = v(x_t), \quad q_t = q(x_t), \quad \text{and} \quad i_t = i(x_t).$$

Then we derive the equilibrium conditions which these functions must satisfy, and then verify that our guess is correct.

Using Ito's lemma with (31) and (32), we have:

$$(A29) \quad \frac{dx_t}{x_t} = (\alpha-1)[i(x_t) - \delta] dt + \sigma dz_t$$

Let us write down the rates of returns on land and capital as:

$$(A30a) \quad \frac{r^L dt + dp}{p} = \frac{(1-\alpha)x dt + dv(x)}{v(x)} + [i(x) - \delta] dt = r^P(x) dt + \omega^P(x) dz,$$

$$(A30b) \quad \frac{(r^K - \delta) dt + dq}{q} = \frac{\alpha x dt + dq(x)}{q(x)} - \delta dt = r^Q(x) dt + \omega^Q(x) dz.$$

Let b_t be the total asset as in (A3) and let h_t be the share of liquid asset in portfolio, $h_t = p_t m_t / b_t$. Then the budget constraint of the agent (A5) is:

$$(A31) \quad db_t = \left([h_t r_t^P + (1-h_t) r_t^Q] b_t - c_t \right) dt + [h_t \omega_t^P + (1-h_t) \omega_t^Q] b_t dz_t \\ + \text{Max} \left(\frac{1-\theta}{1-\theta q_t} (1-h_t + h_t q_t) b_t - b_t, 0 \right) dM_t,$$

Thus Bellman equation of the agent is:

$$(A32) \quad \rho V(b_t, x_t) = \text{Max}_{c_t, h_t} \left\{ \ln c_t + V_b(b_t, x_t) \left([h_t r_t^P + (1-h_t) r_t^Q] b_t - c_t \right) \right. \\ \left. + \frac{1}{2} V_{bb}(b_t, x_t) [h_t \omega_t^P + (1-h_t) \omega_t^Q]^2 b_t^2 \right. \\ \left. + \pi \max \left[V\left(\frac{1-\theta}{1-\theta q_t} (1-h_t + h_t q_t) b_t, x_t\right) - V(b_t, x_t), 0 \right] \right\}$$

The optimal investment rule is the same as Tobin's q theory of investment as in text. From the first order conditions, we get the optimal consumption rule and optimal portfolio rule as:

$$(A33) \quad c_t = \rho b_t,$$

$$(A34) \quad r_t^P - r_t^Q - (\omega_t^P - \omega_t^Q) [h_t \omega_t^P + (1-h_t) \omega_t^Q] + \pi \frac{\text{Max}(q_t - 1, 0)}{1-h_t + h_t q_t} = 0.$$

These are very similar to consumption rule (A10) and portfolio rule (A11), except for the effect of risk due to different stochastic process of the productivity shock. From (A34), we can see the optimal share of liquid asset is:

$$(A35) \quad h_t = h(r_t^P - r_t^Q, q_t, \pi, \omega_t^P, \omega_t^Q), \text{ where}$$

$$h_{(r^P - r^Q)} > 0, \quad h_{q_t} \geq 0, \quad h_{\pi} \geq 0, \quad h_{\omega^P} < 0 \quad \text{and} \quad h_{\omega^Q} > 0.$$

The difference from the standard portfolio theory is that the liquid asset ratio is a weakly increasing function of Tobin's q and the arrival rate of productive investment opportunity.

Now we can combine the individual's behavior with market clearing conditions and define the equilibrium as price functions $v(x)$, $q(x)$ and investment rate $i(x)$ which satisfy:

$$(A36) \quad [1 - q(x)] i(x) \leq \pi [v(x) + \theta q(x)], \text{ where}$$

$$\begin{cases} = \text{holds, if } q(x) > 1, \\ \leq \text{ holds, if } q(x) = 1, \\ i(y) = 0, \text{ if } q(x) < 1. \end{cases}$$

$$(A37) \quad x = i(x) + \rho [v(x) + q(x)],$$

$$(A38) \quad [r^p(x) - r^q(x)] - [\omega^p(x) - \omega^q(x)] \frac{v(x)\omega^p(x) + q(x)\omega^q(x)}{v(x) + q(x)} \\ + \pi \frac{\text{Max} [q(x) - 1, 0]}{q(x)} \frac{v(x) + q(x)}{v(x) + 1} = 0,$$

where the returns characteristics $r^p(x)$, $r^q(x)$, $\omega^p(x)$ and $\omega^q(x)$ are defined in (A30). The stochastic process of a_t and x_t follow (29) and (A29). Equation (A36) describes the behavior of the aggregate investment. Equation (A37) describes the goods market equilibrium, and (A38) describes the asset market equilibrium. In (A38), the first term is the difference of the expected rates of returns on the liquid asset and illiquid capital, the second is the effect of risk aversion, and the last is the expected advantage of the liquid asset over illiquid capital for financing the productive investment. This last term distinguishes our model from a standard capital asset pricing model. All the above equilibrium conditions are functions of the output-capital ratio x_t only, and the aggregate productivity and output-capital ratio are the sufficient statistics of the aggregate state of the economy. Thus our initial conjecture was correct.

This system is not much more complicated than the real business cycles model, or [Merton(1975)]. Thus, in principle, we can simulate the above system to examine the dynamics. Alternatively, if $v(x)$ and $q(x)$ were three-times differentiable, then we would know from Ito's Lemma that:

$$r^p(x) = \frac{(1-\alpha)x}{v(x)} + \left(1 + \frac{xv'(x)}{v(x)}(\alpha-1) \right) [i(x)-\delta] + \frac{1}{2} \sigma^2 \frac{x^2 v''(x)}{v(x)},$$

$$r^q(x) = \frac{\alpha x}{q(x)} - \delta + \frac{xq'(x)}{q(x)}(\alpha-1)[i(x)-\delta] + \frac{1}{2} \sigma^2 \frac{x^2 q''(x)}{q(x)},$$

$$\omega^p(x) = \sigma \frac{xv'(x)}{v(x)}, \quad \text{and} \quad \omega^q(x) = \sigma \frac{xq'(x)}{q(x)}.$$

However, such property may not hold here, because the pattern of investment differs qualitatively depending on the value of $q(x)$. Obviously, more study has to be done before we fully understand this economy.