# Heterogeneity, redistribution, and the Friedman rule*

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**Abstract**

We study several popular monetary models which generate a non-degenerate stationary distribution of money holdings. Across these environments, our principal finding is as follows: a monetary policy that sets long run nominal interest rates to zero (the Friedman rule) does not typically maximize ex-post social welfare if it can generate redistributive effects. An increase in the rate of growth of the money supply has the standard partial-equilibrium effect of making money a less desirable asset thereby decreasing the utility of all moneyholders. A second, general-equilibrium effect, is a transfer from one type of agent to the other. For each environment, when the rate of growth of

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the money supply is not too high, an increase in the latter away from the Friedman rule may produce a transfer effect that dominates the partial equilibrium effect thereby rendering the Friedman rule ex-post suboptimal.

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# 1 Introduction

Until 1951, the Federal Reserve System of the US explicitly pegged nominal interest rates on the Treasury's debt obligations. In March 1951, the Treasury-Fed Accord ended this explicit arrangement, ostensibly freeing the Fed to pursue an independent monetary policy. It was natural for economists to ask: how should an independent yet benevolent central bank conduct monetary policy? Milton Friedman (1969) offered a simple and yet deep answer (the Friedman rule): since money is an asset, the central bank ought to change the stock of outstanding money at a rate that causes the real rate of return on money to equal the real return rate on other physical assets. Over the next three decades, researchers have studied Friedman's dictum using the two main workhorse models for monetary theory, the infinitely-lived representative agent (ILRA) model and the overlapping generations (OG) model. In the context of infinitely-lived-*representative*-agent models, when lump-sum taxes and transfers are available, the Friedman rule is the optimal monetary policy.[1] The seminal reference on the Friedman rule in OG models is Wallace (1984). Wallace shows that once heterogeneity among agents is explicitly considered, it may be impossible for the central bank to settle on *one* monetary policy rule, including the Friedman rule, that benefits every agent.

By construction, monetary policy cannot have redistributive effects in representative-agent models. Yet these effects are known to be quantitatively

[1]See, for instance, Woodford (1990) and Ljunqvist and Sargent (2000). Note that Chari, Christiano and Kehoe (1996) and Correia and Teles (1996) extend this to the case in which other distortionary taxes are present.

significant and important (see, for example, Erosa and Ventura, 2002). For agents holding a disproportionate amount of money, their welfare is clearly negatively related to inflation. In contrast, agents holding relatively less money can realize welfare gains from higher inflation. This paper shows the following: If monetary policy can have redistributive effects then, a monetary policy that sets long-run nominal interest rates to zero – that is the Friedman rule – does not typically maximize ex-post social welfare, and in some cases, it does not maximize ex-ante welfare. Indeed, a necessary condition for the Friedman rule to be suboptimal ex-post is that changes in the rate of growth of the money supply have redistributive effects.

We study different monetary environments in which heterogeneity among agents produces a long-run non-degenerate distribution of money holdings. In particular, we study three different monetary economies: 1) the random-matching model of money due to Lagos and Wright (2002), 2) a turnpike model of the type introduced by Townsend (1980), and 3) an overlapping generations (OG) model with stochastic relocation as in Schreft and Smith (1997) and Smith (2002). In each model, agents have heterogeneous money holdings in equilibrium. In the LW frameworks the heterogeneity comes from differences in agents' preferences.[2] In the turnpike environment it arises from different endowment patterns. In the OG model agents alive in the same period may be from different generations. Our results are robust to these various ways of obtaining heterogenous money holdings in equilibrium.

In each of the models we study, an increase in the rate of growth of

[2]In the appendix we also study a textbook money-in-the-utility-function infinite horizon economy. In that model, as in the LW framework, heterogeneity of money holdings occurs because agents have different preferences.

the money supply away from the Friedman rule has two effects. First, the standard partial-equilibrium effect is to make money a less desirable asset thereby decreasing the utility of all agents. Second, the general-equilibrium effect is a redistributive transfer from one type of agent to the other. We show, for each environment, examples for which the transfer effect dominates the partial equilibrium effect when the rate of growth of the money supply is not too high.

Markets are assumed to be incomplete, implying it is impossible to undo transfers by means of non-distortionary fiscal policy. Deviating from the Friedman rule therefore produces multiple Pareto optimal yet non-comparable allocations. This assumption is crucial for our results. Indeed, if the central bank can levy type-specific lump-sum taxes, it is always best to implement the Friedman rule. This is because it is possible to offset any redistribution induced by monetary policy with an appropriate lump-sum tax or transfer. Type-specific lump-sum taxes and transfers are not the only way redistribution effects can be undone. Bhattacharya, Haslag, and Russell (2003), Haslag and Martin (2003), and Da Costa and Werning (2003) describe other mechanisms which produce the same result.

In the models we consider, the policy maker who chooses the rate of growth of the money supply is faced with different types of agents and can assign different weights to each type. Hence, we consider *ex-post* social welfare. Since it is possible to specify a social welfare function which puts enough weight on the type that benefits from such a deviation, it follows *ex-post* social welfare may not be maximized at the Friedman rule.[3]

[3]Under a different approach we could have appealed to a political economy criterion. For example, we could assume agents vote on their preferred policy. We can show, for

Our work is part of a growing literature studying environments (with heterogeneity) in which the Friedman rule is not optimal (see, for example, Levine 1991, Molico 1999, Deviatov and Wallace 2001, Smith (2002a,b), Edmonds 2002, Green and Zhou 2002). These papers consider an ex-ante welfare criterion and argue the Friedman rule might not be optimal from such a standpoint. In other words, their analysis assumes that agents "pick their preferred monetary policy under a "veil of ignorance", before knowing their true identity" [Ljungqvist and Sargent (2000)]. In contrast, we present results using a ex-post welfare criterion and can therefore better capture the "conflict of interest" between different types of agents that a benevolent policy maker has to consider. Our paper is also related to recent work that studies the impact of agent heterogeneity on monetary policy. Kocherlakota (2002) shows monetary policy should react to the degree of heterogeneity in the economy. Berentsen, Camera, and Waller show a one shot deviation from the Friedman rule might increase ex-ante welfare in a search economy of the type introduced by Lagos and Wright (2002).

The rest of the paper proceeds as follows. Section 2, 3, and 4 describe the search and matching economy, the turnpike economy, and the overlapping generations economy, respectively. Section 5 concludes. The money-in-the-utility function economy and proofs of certain results are relegated to the

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each of the economies we study, examples where at least 50 percent of the agents prefer a deviation from the Friedman rule. Alternatively, we could assume agents influence the choice of monetary policy by spending resources lobbying. For example, the "distance" between a group's preferred policy and the chosen policy could depend on the ratio of resources this group spends lobbying to the total resources spent lobbying. The Friedman rule will not be chosen for any such rule that implements a policy strictly between different group's preferred policies. In this paper, however, we limit ourselves to showing Pareto incomparable allocations can arise.

appendices.

## 2 A search economy

This section considers a search model of the type introduced by Lagos and Wright (2002, hereafter, LW).[4] Time is discrete and there is a continuum of mass 1 of infinitely-lived agents. Each period is divided in two sub-periods. It is assume that there are two types of goods: special goods which are traded in a decentralized market during the first sub-period and general goods which are traded in a centralized market during the second sub-period.

We consider each market in turn, starting with the centralized market. As in LW, preferences in the centralized market are assumed to be quasi-linear, so that the utility from consuming an amount $X$ and producing an amount $H$ is given by $U(X) - AH$, where $U$ is twice continuously differentiable, $U' > 0$, $U'' < 0$, and $A > 0$. As in LW, is also assumed $U$ is unbounded and $U'(X^*) = A$ for $X^* \in (0, \infty)$ with $U(X^*) > X^*$.

In the decentralized market, agents only consume and produce a subset of the goods. Agents do not produce the goods they like to consume. We assume there are no double coincidence of wants and denote by $\sigma$ the probability of a single coincidence of wants. Unlike the standard LW model, agents also differ in how much they value special goods relative to general goods. The utility derived by a type-$\alpha$ agent from consuming an amount $x$ and producing an amount $h$ is given by $\alpha u(x) - c(h)$, where $\alpha > 0$, $u$ and $c$ are at least thrice continuously differentiable, $u(0) = c(0) = 0$, $u' > 0$, $c' > 0$, $u'' < 0$, $c'' \geq 0$,

[4] We are indebted to Randy Wright for showing us how to work out the example in this section.

and $u(\bar{q}) = c(\bar{q})$ for some $\bar{q} > 0$.

A central bank can expands or contracts the money supply via lump-sum transfers or taxes, denoted by $\tau$, during the centralized market. The money supply evolves according to $M_t = (1+z)M_{t-1}$. Hence, $\tau = zM_{t-1}$.

Let $W_\alpha(m)$ be the value function for a type $\alpha$ agent entering the centralized market with money $m$ and $V_\alpha(m)$ be the value function for this agent entering the decentralized market with money $m$. The problem of an agent in the centralized market is

$$W_\alpha(m) = \max_{X,H,m^+} \{U(X) - AH + \beta V_\alpha(m^+)\}$$

subject to

$$X = \omega H + \phi(m + \tau - m^+), \tag{1}$$

where $\omega$ denotes the real wage, $\phi$ is the inverse of the price level, and $m^+$ is the money carried out of the market. We assume $\omega$ is fixed; for example, because of a linear technology.

Assuming an interior solution, we can substitute for $H$ to get

$$W_\alpha(m) = \max_{X,m^+} \left\{ U(X) - \frac{A}{\omega} \left[ X - \phi(m + \tau - m^+) \right] + \beta V_\alpha(m^+) \right\}.$$

The first order conditions for $X$ and $m^+$ are, respectively,

$$U'(X) = \frac{A}{\omega}, \tag{2}$$

$$\frac{A\phi}{\omega} = \beta V'_\alpha(m^+). \tag{3}$$

Also notice $W_\alpha(m)$ is linear since $W'_\alpha(m) = \frac{A\phi}{\omega}$ for all $m$. As in LW, the cost of producing $H$ is linear, $X$ and $m^+$ are independent of $m$. Hence, if there is only one type, all agents consume the same amount and leave the market with the same money holdings. With more than one type, however, $m^+$ may depend on $\alpha$ as can be seen from (3).

We now turn to the decentralized market. Let the joint distribution of money and types in this market be given by $F(\tilde{m}, \tilde{e})$. The above analysis shows $F$ is degenerate conditional on types. In other words, all type-$\alpha$ agents exit the centralized market with the same amount of money $m_\alpha^+$ and thus enter the decentralized market with the same amount $m_\alpha$. Hence, it is enough to know the distribution of types $G(\tilde{\alpha})$.

We can write the value function for an type-$\alpha$ agent entering the decentralized market with money $m$ as

$$V_\alpha(m) = \sigma \int \left\{-c[q_{\tilde{\alpha}}(m_{\tilde{\alpha}})] + W_\alpha[m + d_{\tilde{\alpha}}]\right\} dG(\tilde{\alpha})$$
$$+\sigma \left\{\alpha u[q_\alpha(m)] + W_\alpha[m - d_\alpha(m)]\right\} + (1 - 2\sigma)W_\alpha(m). \qquad (4)$$

This expression states that with probability $\sigma$, the agent is a seller who produces a quantity $q_{\tilde{\alpha}}(m_{\tilde{\alpha}})$ of special goods in exchange for $d_{\tilde{\alpha}}$ units of money. With probability $\sigma$ the agent is a buyer who consumes $q_\alpha(m)$ units of special goods acquired with $d_\alpha(m)$ units of money. In particular, we have assumed the terms of trade $q$ and $d$ depend on the buyer's but not the seller's money balances, as is standard in this kind of model. We have also assumed, as will be verified below, the terms of trade depend on the buyer's but not the seller's type. Now take the partial derivative of the above expression with respect to $m$:

$$\begin{aligned} V_\alpha'(m) = \quad & \sigma \textstyle\int +W_\alpha'[m + d_{\tilde{\alpha}}(m_{\tilde{\alpha}})]dG(\tilde{\alpha}) + \sigma\alpha u'[q_\alpha(m)]q_\alpha'(m) \\ & +\sigma[1 - d_\alpha'(m)]W_\alpha'[m - d_\alpha(m)] + (1 - 2\sigma)W_\alpha'(m). \end{aligned}$$

Recall $W'_\alpha(m) = \frac{A\phi}{\omega}$ for all $m$, so we can write

$$V'_\alpha(m) = \sigma\alpha u'[q_\alpha(m)]q'_\alpha(m) + [1 - \sigma d'_\alpha(m)]\frac{A\phi}{\omega}. \tag{5}$$

Hence, $V'_\alpha(m)$ depends on an agent's own type, $\alpha$, and money holding, $m$, but not on other agents type and money holdings.

As in LW, we assume the terms of trade are determined by the generalized Nash solution where the buyer has bargaining power $\theta$ and the threat points are given by the continuation values. First, note for any real $d$ and any type $\alpha$, $W_\alpha(m + d) - W_\alpha(m) = d\frac{A\phi}{\omega}$. It follows the terms of trade $(q, d)$ between a buyer of type $\alpha$ with money holding $m$ and a seller of any type is given by

$$\max_{q,d} \left[\alpha u(q) - d\frac{A\phi}{\omega}\right]^\theta \left[-c(q) + d\frac{A\phi}{\omega}\right]^{1-\theta}$$

subject to $d \leq m$. Thus, as claimed above, the terms of trade do not depend on the seller's type.

As in LW, it can be shown that in any equilibrium it must be the case $d = m$. In order to find $q$, we take the partial derivative of the above expression with respect to $q$ and set it equal to zero. This implies $q = q_\alpha(m)$ is the solution to

$$m\frac{A\phi}{\omega} = g_\alpha(q), \tag{6}$$

where $g_\alpha(q)$ is defined as

$$g_\alpha(q) \equiv \frac{\theta\alpha u'(q)c(q) + (1 - \theta)\alpha u(q)c'(q)}{\theta\alpha u'(q) + (1 - \theta)c'(q)}.$$

For example, if the buyer has all the bargaining power, so $\theta = 1$, this expression reduces to $g_\alpha(q) = c(q)$.

In the general case, implicit differentiation yields

$$q'_\alpha(m) = \frac{A\phi}{\omega g'_\alpha(q)}. \tag{7}$$

We can substitute this expression, as well as $d'_\alpha(m) = 1$, into $V'_\alpha(m)$, given by equation (5). Then, with equation (3), we get

$$\frac{A\phi}{\omega} = \beta \left\{ \sigma\alpha u'(q_\alpha^+) \frac{A\phi^+}{\omega g'_\alpha(q_\alpha^+)} + (1-\sigma)\frac{A\phi^+}{\omega} \right\},$$

where we use the superscript + to denote next period. Since we focus on steady states, we know $q$ is constant and $\phi = (1+z)\phi^+$. The above expression then reduces to

$$1 + z = \beta \left[ \sigma \frac{\alpha u'(q_\alpha)}{g'_\alpha(q_\alpha)} + (1-\sigma) \right]. \tag{8}$$

This expression determines the equilibrium value of $q_\alpha$ for an agent of type $\alpha$. From equation (6) we get $m_\alpha = \frac{\omega g_\alpha(q_\alpha)}{A\phi}$.

We can simplify expression (8) further. Define $\beta \equiv 1/(1+r)$ and the nominal interest rate $(1+i) = (1+z)(1+r)$. Then we can write

$$1 + \frac{i}{\sigma} = \frac{\alpha u'(q_\alpha)}{g'_\alpha(q_\alpha)}.$$

The price $\phi$ can be obtained through the money market clearing condition $\int m_\alpha dG(\alpha) = M$, since the $q_\alpha$'s are determined by equation (8).

Now let $\hat{m}_\alpha$ denote the money with which an agent of type $\alpha$ enters the centralized market. $\hat{m}_\alpha$ will depend on the type of meeting the agent was in during the previous decentralized market. In that market, the agent might have been either a seller, or a buyer, or no trade occurred. Hence,

$$\hat{m}_\alpha = \begin{cases} 0 & \text{with probability } \sigma, \\ m_\alpha & \text{with probability } 1-2\sigma, \\ m_\alpha + m_{\tilde{\alpha}} & \text{with probability } \sigma G(\tilde{\alpha}). \end{cases}$$

Since $\hat{m}_\alpha$ varies across agents according to their type, so must $H_\alpha$. Indeed, we can rewrite equation (1) as follows

$$\omega H_\alpha = X - \phi(\hat{m}_\alpha + \tau - m_\alpha^+).$$

A similar expression must hold for the average $\hat{m}_\alpha$ across type $\alpha$ agents, which is given by $\bar{m}_\alpha = (1-\sigma)m_\alpha + \sigma M$, where $M \equiv \int m_{\tilde{\alpha}} dG(\tilde{\alpha})$. Hence, we have

$$\begin{aligned} \omega \bar{H}_\alpha &= X - \phi(\bar{m}_\alpha + \tau - m_\alpha^+) \\ &= X - \phi(\sigma + z)(M - m_\alpha), \end{aligned} \tag{9}$$

where we have made use of the fact that $m_\alpha^+ = (1+z)m_\alpha$, in steady state, and $\tau = zM$.

Note $X$ is the same across type and is thus independent of the rate of growth of the money supply $z$. Assume $\phi$ is fixed for a moment, for any type $\alpha$ holding less than the average money balances $M$ an increase in $z$ will reduce expected hours $\bar{H}_\alpha$. Since $H$ enters the utility function linearly this increases expected utility. Clearly the opposite is true for any type holding more than the average money balances. An increase in $z$ might also change $V_\alpha(m)$. From inspection of equation (4), however, it is clear that this effect can be made as small as we want by reducing $\alpha$.

This can be illustrated by a simple example. Assume preferences are $\ln(X) - H$ in the centralized market and $\alpha_i \ln(x) - h$ in the decentralized market, with $\alpha_i \in \{\alpha_L, \alpha_H\}$, $\alpha_L < \alpha_H$. The two types $L$ and $H$ have equal mass. We also assume $\theta = 1$ so buyers make take-it-or-leave-it offers. Under this assumption, equation (6) implies

$$m\frac{\phi}{\omega} = q.$$

It is easy to verify that equation (8) becomes

$$1+z=\beta\left[\sigma\alpha\frac{1}{q_{\alpha}}+(1-\sigma)\right].$$

Thus the expressions for $q_{\alpha}$ and $m_{\alpha}$ are given by

$$q_{\alpha}=\frac{\alpha\beta\sigma}{[1+z-\beta(1-\sigma)]}$$

and

$$m_{\alpha}=\frac{\omega}{\phi}\frac{\alpha\beta\sigma}{[1+z-\beta(1-\sigma)]}.$$

The money market clearing condition is

$$M=\frac{1}{2}\frac{\beta\sigma}{[1+z-\beta(1-\sigma)]}\frac{\omega}{\phi}\left(\alpha_L+\alpha_H\right).$$

This determines the price $\phi$. With a little algebra we can write equation (9) for type $L$ as

$$\bar{H}_{\alpha_L}=1-\frac{1}{2}\frac{\beta\sigma(\sigma+z)}{[1+z-\beta(1-\sigma)]}\left(\alpha_H-\alpha_L\right).$$

Let $\Gamma\equiv(\sigma+z)/\left[1+z-\beta(1-\sigma)\right]$. Since $\alpha_H>\alpha_L$, average hours for type $L$ will decrease if $\partial\Gamma/\partial z>0$.

$$\frac{\partial\Gamma}{\partial z}=\frac{1+z-\beta(1-\sigma)-(\sigma+z)}{[1+z-\beta(1-\sigma)]^2},$$

which is positive for $\beta\in(0,1)$.

We can summarize these result in the following proposition.

**Proposition 1** *Increasing the rate of the money supply creates a transfer from types with high $\alpha$ to types with low $\alpha$ which can make types with low $\alpha$ better off.*

If $\phi$ is kept fixed, it is easy to see how the transfer operates; a higher $z$ allows types with low $\alpha$ to work less. Of course, in equilibrium, $\phi$ will decrease, which makes all types worse off. However, this partial equilibrium effect is more than compensated for by the transfer for $\alpha$ sufficiently low. Agents with such low values of $\alpha$ are made better off by a deviation from the Friedman rule. Hence, different values of $z$ give Pareto incomparable allocations. It is easy, by putting sufficient weight on the utility of types with a low $\alpha$ to write a social welfare function that is not maximized at $z = \beta - 1$.

This is true despite the fact that, as in LW, it can be shown from equation (8) that the Friedman rule generates the second best equilibrium in terms of efficiency. Further, if $\theta = 1$, so that there is no hold up problem associated with bargaining, then equation (8) yields $g_\alpha(q) = c(q)$ whenever $z = \beta$. This implies the first best, $\alpha u'(q_\alpha) = c'(q_\alpha)$.

Because increasing $z$ reduces the utility of both types of agents, the Friedman rule would be optimal if it were possible to achieve the transfer from types with high $\alpha$ to types with low $\alpha$ through other, less distorting, means. For example, assume type-specific lump-sum taxes and subsidies are available. It is possible to implement a transfer, by taxing high-$\alpha$ types and subsidizing low-$\alpha$ types, without increasing $z$. Hence a necessary condition for the Friedman rule to be suboptimal ex-post is that changes in the rate of growth of the money supply have redistributive effects.

# 3 A Turnpike Model

This section studies a version of the turnpike model developed in Townsend (1980) and described in Ljungqvist and Sargent (2000). Time is indexed by $t = 0, 1, 2, ...$, there is a single, perishable consumption good and a countably infinite number of infinitely-lived agents. There are two types of agents differing in their endowment patterns. Specifically, type-$E$ agents are endowed with 1 unit of the consumption good at even dates and nothing at odd dates. Type-$O$ agents are endowed with 1 unit of the consumption good at odd dates and nothing at even dates. Each type-$O$ agent is endowed with $M_0$ units of fiat money at date 0.

We restrict market participation in two ways. First, at each date $t$, there is a single pairing of one type-$E$ and one type-$O$ populating a market. Second, a type-$E$ will be paired with the specific type-$O$ agents only once.[5] These restrictions, combined with the absence of any common agent or intermediary, eliminates the possibility of debt issues. In what follows, the utility function $u$ is assumed to be CRRA and is described as

$$u(c) = \frac{c^{1-\rho}}{1-\rho} \tag{10}$$

where $c$ is consumption.

## 3.1 The type $E$ agent's problem

The problem of type $E$ agents can be written recursively as follows:

$$v(m) = \max u(c) + \beta u(c') + \beta^2 v(m''),$$

[5] We define $I_E = [1_E, 2_E, ...]$ and $J_O = [1_O, 2_O, ...]$. Let $i_E \in I_E$ and $j_O \in J_O$. Then $i_E$ is paired with $j_O$ only once.

subject to

$$c + (1+z)m' = 1 + \tau + m, \tag{11}$$

$$c' + (1+z)m'' = \tau + m', \tag{12}$$

where $c$ is consumption in even periods and $c'$ is consumption in odd periods, $m$ is the amount of real money balances the agent holds at the beginning of an even period, $\tau$ is a real lump-sum money transfer from the government which is positive if the money supply grows and negative if it shrinks, and $(1+z)$ is the rate of growth of the money supply which corresponds, in this static environment, to the inflation rate.

We know if $(1+z) > \beta$, then $m = m'' = 0$. The first order conditions imply

$$(1+z)u'(c) = \beta u'(c'), \tag{13}$$

which using (10) yields

$$\frac{c'}{c} = \left(\frac{\beta}{(1+z)}\right)^{\frac{1}{\rho}}, \tag{14}$$

while (12) yields

$$c + (1+z)c' = I \tag{15}$$

where $I \equiv 1 + \tau + (1+z)\tau$. It is easy to verify that

$$c = \frac{I}{1 + (1+z)\left(\frac{\beta}{(1+z)}\right)^{\frac{1}{\rho}}}, \tag{16}$$

$$c' = \left(\frac{\beta}{(1+z)}\right)^{\frac{1}{\rho}} \frac{I}{1 + (1+z)\left(\frac{\beta}{(1+z)}\right)^{\frac{1}{\rho}}}. \tag{17}$$

## 3.2 The type $O$ agent's problem

Similarly, the problem of type $O$ agents can be written as

$$v(\bar{m}) = \max u(\bar{c}) + \beta u(\bar{c}') + \beta^2 v(\bar{m}''),$$

subject to

$$\bar{c} + (1+z)\bar{m}' = \tau + \bar{m}, \tag{18}$$

$$\bar{c}' + (1+z)\bar{m}'' = 1 + \tau + \bar{m}', \tag{19}$$

where $\bar{c}$ is consumption in even periods and $\bar{c}'$ is consumption in odd periods, $\bar{m}$ is the real money balances the agent holds at the beginning of an even period, and $\tau$ and $(1+z)$ are as defined above.

We know if $(1+z) > \beta$, then $\bar{m}' = 0$. Standard aforedescribed arguments yield

$$\bar{c} = \frac{I}{(1+z) + \left(\frac{(1+z)}{\beta}\right)^{\frac{1}{\rho}}}, \tag{20}$$

$$\bar{c}' = \left(\frac{(1+z)}{\beta}\right)^{\frac{1}{\rho}} \frac{I}{(1+z) + \left(\frac{(1+z)}{\beta}\right)^{\frac{1}{\rho}}}. \tag{21}$$

We can combine equations (16) and (20) to get

$$\bar{c}\left[(1+z) + \left(\frac{(1+z)}{\beta}\right)^{\frac{1}{\rho}}\right] = c\left[1 + (1+z)\left(\frac{\beta}{(1+z)}\right)^{\frac{1}{\rho}}\right].$$

Feasibility requires that $c + \bar{c} = 1$. A little algebra yields

$$c = \frac{(1+z)^{\frac{1}{\rho}}}{\beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}}},$$

$$c' = \frac{\beta^{\frac{1}{\rho}}}{\beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}}}.$$

Clearly, $c \to c'$ as $(1+z) \to \beta$. Moreover, the gap between $c$ and $c'$ increases as $(1+z)$ increases. Why is there a gap between $c$ and $c'$? The reason has to do with the odd-even endowment pattern. If the return to money is less than unity, an agent is always better off receiving a unit of endowment earlier than later, since a unit invested in money earns less than the value of the investment itself. As $(1+z)$ increases, the return to money falls thereby increasing the gap between odd and even period consumption.

### 3.3 Evaluating type $E$ agents' welfare

Consider the start of an even date. At such a date, the type $E$ agent would prefer an increase in the money growth rate away from the Friedman rule, as long as this increase is not too large. A type $O$ agent prefers the Friedman rule. An increase in the rate of growth of the money supply means that money has less value. This allows type $E$ agent to give up less of their endowment and consume more. Neither agent likes positive inflation though.

Formally, the welfare of a type $E$ agent is given by

$$U^E = \sum_{t=0}^{\infty} \beta^{2t} \left[ u(c) + \beta u(c') \right] = \frac{1}{1-\beta^2} \left[ u(c) + \beta u(c') \right]. \tag{22}$$

For future use, note that

$$\frac{\partial u(c)}{\partial (1+z)} = \frac{\left[ \beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}} \right]^{-\rho}}{\left[ \beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}} \right]^{2(1-\rho)}} \frac{1}{\rho} \frac{1}{(1+z)^2} ((1+z)\beta)^{\frac{1}{\rho}}. \tag{23}$$

and

$$\frac{\partial u(c')}{\partial (1+z)} = -\frac{\left[ \beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}} \right]^{-\rho}}{\left[ \beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}} \right]^{2(1-\rho)}} \frac{1}{\rho} \frac{1}{(1+z)\beta} ((1+z)\beta)^{\frac{1}{\rho}}. \tag{24}$$

Since

$$\frac{\partial U^E}{\partial(1+z)} = \frac{1}{1-\beta^2}[\frac{\partial u(c)}{\partial(1+z)} + \beta\frac{\partial u(c')}{\partial(1+z)}], \tag{25}$$

we have

$$\frac{\partial U^E}{\partial(1+z)} = \frac{1}{1-\beta^2}\frac{\left[\beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}}\right]^{-\rho}}{\left[\beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}}\right]^{2(1-\rho)}}\frac{((1+z)\beta)^{\frac{1}{\rho}}}{\rho(1+z)}\left(\frac{1}{(1+z)} - 1\right). \tag{26}$$

**Proposition 2** *For all $\beta \in (0,1)$, $\rho > 0$, $\frac{\partial U^E}{\partial(1+z)} > 0$ if $(1+z) \in [\beta, 1)$.*

**Proof.** This is immediate from equation (26). ■

Proposition 2 shows type-$E$ agents benefit if the central bank chooses a money growth rate greater than that prescribed by the Friedman rule. Thus, if the central bank puts enough weight on the welfare of type $E$ agents, it will choose $1+z > \beta$.

As in the economy of the previous section an increase in the rate of growth of the money supply has two effects. First, it reduces the utility of all agents as it makes their consumption more volatile ($c$ deviates more from $c'$, which hurts any risk-averse agent). On the other hand, it creates a transfer from type $O$ to type $E$ agents. If the money stock does not grow too fast, the value of the transfer to type $E$ agents exceeds the cost in terms of volatility of consumption.

Also, as in the previous economy, the Friedman rule would be optimal if it were possible to make transfers that are less distorting. The type of transfer described in the previous section can be implemented if type-specific lump-sum transfers are feasible. Hence, again, a necessary condition for the Friedman rule to be suboptimal ex-post is that changes in the rate of growth of the money supply have redistributive effects.

# 4 A OG Model with random relocation

We consider a model economy in which money is valued because of limited communication across two spatially separated locations. Only a succinct description of the economic environment is provided; the interested reader is referred to Schreft and Smith (2002) and Bhattacharya, Haslag, and Russell (2004) for more details.

Time is discrete and denoted by $t = 1, 2...$ The world is divided into two spatially separated locations. Each location is populated by a continuum of agents of unit mass. Agents live for two periods, and receive an endowment of $\omega$ units of the single consumption good when young and nothing when old. There also is an initial old generation whose members are endowed with an amount of cash $M_0$. Only old-age consumption is valued. Let $c_t$ denote old-age consumption of the members of the generation born at date $t$; their lifetime utility is given by $u(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$, where $\rho \in (0, 1)$.

After receiving their endowment and placing it into a bank, agents learn whether they must move to the other location or not. Let $\alpha$ denote the probability that an individual will be relocated. We assume a law of large number holds so $\alpha$ is also the measure of agents that are relocated. $\alpha$ is the same on both islands so that moves across location are symmetric. Movers redeem their bank deposits in the form of money as this is the only way for them to acquire goods in the new location. In contrast, nonmovers redeem their deposits in the form of goods. Goods deposited in the bank can be used to acquire money from old agents belonging to the previous generation or put into storage. Each unit of the consumption good put into storage at date $t$ yields $x > 1$ units of the consumption good at date $t + 1$, where $x$ is a known constant.

The CB can levy lump-sum taxes $\tau$ on the endowment of agents by collecting the tax in the form of money balances removed from the economy. In contrast, a lump-sum subsidy is received in the form of a money injection. The money supply evolves according to $M_{t+1} = (1+z)\, M_t$ and $z$ is chosen by the CB in a manner that will be explained below. We assume $x > \frac{1}{1+z}$ implying that money is a bad asset. Let $p_t$ denote the time $t$ price level; in steady states, $p_{t+1} = (1+z)\, p_t$. Also, since we focus on steady-states, we drop the time subscript in what follows.

Agents deposit their entire after-tax/transfer endowments with a bank. The bank chooses the gross real return it pays to movers, $d^m$, and to non-movers, $d^n$. In addition, the bank chooses values $m$ (real value of money balances) and $s$ (storage investment) respectively. These choices must satisfy the bank's balance sheet constraint

$$m + s \leq \omega - \tau. \tag{27}$$

Banks behave competitively, so they take as given the return on their investments. In particular, the return on real money balances is $p_t/p_{t+1}$. If $x > p_t/p_{t+1}$ banks will want to hold as little liquidity as possible since money is dominated in rate of return. If $x = p_t/p_{t+1}$, banks are indifferent between money and storage. In this case, we consider the limiting economy as $p_t/p_{t+1} \to x$.

Banks must have sufficient liquidity to meet the needs of movers. This is captured by the following expression:

$$\alpha d^m (\omega - \tau) \leq \frac{m}{1+z}. \tag{28}$$

A similar condition for non-movers, who consume all the proceeds from the storage technology, is given by

$$(1-\alpha)d^n(\omega-\tau) \leq xs. \tag{29}$$

Banks maximize profits. Because of free entry, banks choose in equilibrium their portfolio in a way that maximizes the expected utility of a representative depositor. The bank's problem is written as

$$\max_{d^m, d^n} \quad \frac{(\omega-\tau)^{1-\rho}}{1-\rho}\left\{\alpha\left(d^m\right)^{1-\rho} + (1-\alpha)\left(d^n\right)^{1-\rho}\right\} \tag{30}$$

subject to equations (27), (28), and (29).

Let $\gamma \equiv \frac{m}{\omega-\tau}$ denote the bank's reserve-to-deposit ratio. Then, since equations (27), (28), and (29) hold with equality, the bank's objective function is to choose $\gamma$ to maximize

$$\frac{(\omega-\tau)^{1-\rho}}{1-\rho}\left\{\alpha^\rho\left[\frac{\gamma}{1+z}\right]^{1-\rho} + (1-\alpha)^\rho\left[(1-\gamma)x\right]^{1-\rho}\right\}. \tag{31}$$

Bhattacharya, Guzman, Huybens, and Smith (1997) show that the reserve to deposit ratio chosen by the bank is given by

$$\gamma = \frac{1}{\left[1+\frac{1-\alpha}{\alpha}\left\{(1+z)\,x\right\}^{\frac{1-\rho}{\rho}}\right]} \tag{32}$$

and that it increases as $1+z$ decreases. For the initial old, consumption is equal to the real value of money balances. Let $M_0$ denote the quantity of nominal money balances held by a member of the initial old generation. Then, $c_1^0 = \frac{M_0}{p_1}$, where $p_1 = \frac{(1+z)M_0}{\gamma(\omega-\tau)}$. Note, in equilibrium, the reserve-to-deposit ratio and the lump-sum tax are functions of the money growth rate. At a steady state, the central bank maximizes the following objective function

$$W(z) = (1-\beta)\frac{\left(\frac{M_0}{p_1}\right)^{1-\rho}}{1-\rho} + \beta\frac{\{\Omega(z)\}^{1-\rho}}{1-\rho}\Gamma(z)$$

where $\Omega(z) := \omega - \tau(z)$, and $\Gamma(z) := \alpha^{\rho}\left[\frac{\gamma(z)}{1+z}\right]^{1-\rho} + (1-\alpha)^{\rho}\left[(1-\gamma(z))x\right]^{1-\rho}$. This allows us to find the rate of growth of the money supply chosen by the central bank under different assumptions about the weight of the initial old generation and, in a steady state, all other generations. For example, if $\beta = 0$, then the central bank only considers the utility of the initial old. Conversely, as $\beta \to 1$, the weight of the initial old goes to zero and so the central bank maximizes the utility of a representative generation (in steady states) and completely ignores the initial old.

For future reference, note that, in steady states,

$$-\tau = \frac{M_t - M_{t-1}}{p_t} = -m\left(\frac{z}{1+z}\right)$$

$$m = \gamma(\omega - \tau) = \frac{\gamma\omega(1+z)}{(1+z)-\gamma z}$$

and hence,

$$\frac{M_0}{p_1} = \frac{M_1}{p_1}\frac{M_0}{M_1} = m\frac{1}{1+z} = \frac{\gamma\omega}{(1+z)-\gamma z}.$$

**Proposition 3** *The optimal rate of growth of the money supply is given by*

$$1+z = 1 + \frac{1-\beta}{\beta}\frac{\gamma}{\alpha^{\rho}},$$

*where $\gamma$ is computed from (32), along with the constraint that $1+z \geq \frac{1}{x}$.*

**Proof.** See Appendix. ■

It is straightforward to see that if $\beta \to 1$, then $1 + z \to 1$. As $\beta \to 0$, in the limit the weight is all on the initial old; the constraint $1 + z \geq \frac{1}{x}$ eventually binds and the central bank implements the Friedman rule.

Bhattacharya, Haslag, and Russell (2004) and Haslag and Martin (2003) study how an increase in the rate of growth of the money supply away from the Friedman rule creates a transfer from agents who hold money to those who do not. Indeed this effect may dominate the negative effects of a higher money growth rate and can render the Friedman rule suboptimal. As in the previous economies, here, deviations from the Friedman rule also come at a cost since the difference in consumption between movers and nonmovers increases as the rate of growth of the money supply increase. When the money stock does not grow too fast, the value of the transfer exceeds the cost created by the volatility in consumption. Edmond (2002) has a comparable result. Again, the sub-optimality of the Friedman rule hinges crucially on the assumption that it is not possible to undo the transfers created by changes in the rate of growth of the money supply. Bhattacharya, Haslag and Russell (2004) show that when these transfers can be undone, the Friedman rule is once again optimal. Hence, once again, a key component of the explanation for why the Friedman rule is suboptimal (ex-post) here is that changes in the rate of growth of the money supply have unremovable redistributive effects. Note also that unlike in the two economies studied above, here the Friedman rule is additionally sub-optimal ex-ante as shown in Smith (2002).

# 5 Summary and conclusion

In this paper, we consider steady state monetary policy in several alternative economic environments: two economies with infinitely-lived agents and an overlapping-generations economy.[6] We provide examples where the Friedman rule is not the ex-post welfare maximizing monetary policy in these economies. To varying degrees, these results are known in the literature. Certainly this is true in the overlapping generations economy. Our aim is to explain why the welfare maximizing policy is not the Friedman rule in each case. Indeed, our main contribution is to offer one common explanation to account for the shared monetary policy results.

An key characteristic of the models we considered is that agents have heterogenous money holding. This heterogeneity arises from differences in agents' preferences in the Lagos-Wright (2002) framework (as well as in the money-in-the-utility-function economy). It is generated by differences in endowment patterns in the turnpike economy. Finally, differences in the age of agents living during the same period are responsible for this heterogeneity in the OG framework (old agents may hold money while young agents do not). In each case, a change in the rate of growth of the money supply has a redistributive effect. As money growth rises, for example, agents who hold a disproportionately large amount of money see the real value of their money holdings diminish. At the same time agents who are holding comparatively less money benefit from the change in money growth because they are able to consume more. In each case, we provided an example where an increase in the rate of growth of the money supply creates Pareto incomparable allocations.

[6] In addition, we study a money-in-the-utility-function economy in the appendix.

Heterogeneity, therefore, plays an important role in explaining why the Friedman rule does not maximize ex-post steady-state welfare in a variety of economic settings. In an infinitely-lived-representative-agent economy, heterogeneity is suppressed; agents are identical and therefore hold the same levels of money balances in steady state. A second important assumption in each of the economies considered is that markets are incomplete so agents, or the monetary authority, are unable to undo the redistribution caused by an increase in the rate of growth of the money supply away from the Friedman rule. For example, in each of the environments we consider, the Friedman rule would be optimal if type-specific lump-sum taxes were available.

Appendix

# A A money-in-the-utility-function economy

This section considers a money-in-the-utility function economy with two types of agents who differ in the marginal utility they receive from real balances. Time is discrete and denoted by $t = 0, 1, 2, ...$ The economy is populated by a continuum, of unit mass, of infinitely-lived agents. As mentioned above, the population is divided into two groups. Let $\mu$ be the measure of agents that place a relatively high value on the services from real money holdings and $1 - \mu$ is the measure of agents that place a relatively low value on the services of real money holdings. This distinction will be made precise below.

There are two assets in the economy, money and capital. There is a single consumption good which is perishable. It can be transformed into capital at a one-for-one rate or used to acquire money balances. Agents can produce the consumption good with capital accumulated up to date-$t$, denoted $k_{t-1}$, using a technology $f(k_{t-1})$. The function $f$ has the following properties: $f'(k_{t-1}) > 0$, $f''(k_{t-1}) < 0$, $\lim_{k\to 0} f'(k_{t-1}) = \infty$, and $\lim_{k\to\infty} f'(k_{t-1}) = 0$. In addition, undepreciated capital can be converted into units of the consumption good at a one-for-one rate. We use $\delta$ to denote the capital depreciation rate.

The government has a balanced budget period by period. At each date $t \geq 0$ , the government finances a lump-sum tax or transfer, denoted $\tau$, by altering the money supply. Formally, the date-$t$ government budget constraint is: $\tau_t = \frac{M_t - M_{t-1}}{p_t}$, where $M_t$ denotes the per-capita quantity of nominal money at date $t$ and $p_t$ is the price level at that date. We assume the government follows a money supply rule such that $M_t = (1 + z_t) M_{t-1}$, where $z_t > -1$. The money supply expands if $z_t > 0$, so that $\tau_t > 0$. Conversely, the money supply contracts if $-1 < z_t < 0$, so that $\tau_t < 0$. In this stationary environment, the price level increases at the same rate as the money supply. Hence $p_t = (1 + z_t) p_{t-1}$.

All agents maximize the discounted sum of momentary utilities over an infinite horizon. Agents who place a relatively high value on the services of real money balances are referred to as type $H$, while those who place a relatively low value on the services are referred to as type $L$. The preferences of the type-$i$ where $i = H, L$ agents are represented by

$$U = \sum_{t=0}^{\infty} \beta^t u^i (c_t, m_t) \qquad i = H, L, \tag{33}$$

where $0 < \beta < 1$ is the agent's subjective time rate of preference, $c$ is the quantity of the consumption good and $m_t \equiv \frac{M_t}{p_t}$ is the quantity of real money balances. We assume $u^i_j > 0$ and $u^i_{jj} < 0$, $i = L, H$, $j = m, c$, where $u^i_j \equiv \frac{\partial u^i}{\partial j}$ and $u^i_{jj} \equiv \frac{\partial^2 u^i}{\partial j^2}$. There exists a satiation level of real money balances such that $u^H_m \left(c, m^{*H}\right) = u^L_m \left(c, m^{*L}\right) = 0$. Further, we assume $u^H_m (\hat{c}, \hat{m}) > u^L_m (\hat{c}, \hat{m})$, $\forall \hat{m} \leq m^{*H}$, $u^H_c (\hat{c}, \hat{m}) = u^L_c (\hat{c}, \hat{m})$ and $u^i_{cm} = 0$ for $i = L, H$. In words, for the same values of consumption and real balances, the type-$H$ derives greater marginal utility from the services associated with money than does a type-$L$ agent but there is no difference in the marginal utility of consumption. Lastly, we assume the momentary utility is separable in consumption and real money balances for both types of agents.

There are two problems, one for each type of agent. Formally, the maximization problem for the type-$i$ agents, $i = L, H$ is represented by

$$\sum_{t=0}^{\infty} \beta^t u^i \left(c_t^i, m_t^i\right)$$

$$\text{s.t. } f\left(k_{t-1}^i\right) + (1-\delta) k_{t-1}^i + \frac{m_{t-1}^i}{1+(1+z_t)} + \tau_t \geq c_t^i + k_t^i + m_t^i, \tag{34}$$

where $(1+z_t) \equiv \frac{p_t - p_{t-1}}{p_{t-1}}$ and stands for the (net) inflation rate. Let $a_t^i \equiv f\left(k_{t-1}^i\right) + (1-\delta) k_{t-1}^i + \frac{m_{t-1}^i}{1+(1+z_t)} + \tau_t, i = L, H$. The problem for an agent of type $i$ can be written in recursive form. Bellman's equation is:

$$V^i(a_t) = \max \left\{ u^i(c_t, m_t) + \beta V^i(a_{t+1}) \right\}, \quad i = H, L.$$

Since $k_t^i = a_t^i - c_t^i - m_t^i$, the unconstrained maximization problem can be written as

$$V^i(a_t) = \max \left\{ \begin{array}{c} u^i\left(c_t^i, m_t^i\right) \\ +\beta V^i \left[ f\left(a_t^i - c_t^i - m_t^i\right) + (1-\delta)\left(a_t^i - c_t^i - m_t^i\right) + \frac{m_t^i}{1+(1+z_{t+1})} + \tau_t \right] \end{array} \right\}$$

We also impose the following transversality conditions:

$$\lim_{T\to\infty} \beta^T u_c^i\left(c_T^i, m_T^i\right) k_T = \lim_{T\to\infty} \beta^T u_c^i\left(c_T^i, m_T^i\right) m_T = 0. \tag{35}$$

The first-order necessary conditions for agent $i's$ problem are:

$$u_c^i\left(c_t^i, m_t^i\right) - \beta\left[f'\left(k_t^i\right) + (1-\delta)\right] V_a^i\left(a_{t+1}^i\right) = 0, \tag{36}$$

$$u_m^i\left(c_t^i, m_t^i\right) - \beta\left[f'\left(k_t^i\right) + (1-\delta)\right] V_a^i\left(a_{t+1}^i\right) + \frac{\beta V_a^i(a_{t+1})}{1+(1+z_{t+1})} = 0, \tag{37}$$

and the envelope condition is

$$V_a^i(a_t) = u_c^i\left(c_t^i, m_t^i\right). \tag{38}$$

Update equation (38) by one period and substitute it into equations (36) and (37), to get

$$u_c^i\left(c_t^i, m_t^i\right) - \beta\left[f'\left(k_t^i\right) + (1-\delta)\right] u_c^i\left(c_{t+1}^i, m_{t+1}^i\right) = 0, \tag{39}$$

$$u_m^i\left(c_t^i, m_t^i\right) - \beta\left[f'\left(k_t^i\right) + (1-\delta)\right] u_c^i\left(c_{t+1}^i, m_{t+1}^i\right) + \frac{\beta u_c^i\left(c_{t+1}^i, m_{t+1}^i\right)}{1 + (1 + z_{t+1})} = 0. \tag{40}$$

We can solve the first-order necessary conditions specifically for a money demand function. Let $m^{dH}(\omega_t)$ and $m^{dL}(\omega_t)$ represent the money demand for type-$H$ and type-$L$ agents respectively. Here, $\omega$ is used to stand for the variables that the agents will take as given; namely, the state variables, the price level and the policy variables. Formally, $\omega_t = \left\{k_t^H, k_t^L, p_t, \tau_t, z_t\right\}$.

The money market clearing condition can be represented as follows:

$$M_t = p_t\left[\mu m^{dH}(\omega_t) + (1-\mu)\, m^{dL}(\omega_t)\right] \tag{41}$$

In steady state, consumption, capital, and real money balances are constant over time so that $c_t^i = \bar{c}^i$, $k_t^i = \bar{k}^i$, $m_t^i = \bar{m}^i$, for all $t$. In addition, the government fixes the money supply growth rate such that $z_t = z$ for all $t$. Hence, $\tau_t = \tau$. From equation (39), we have

$$u_c^i\left(\bar{c}^i, \bar{m}^i\right) = \beta\left[f'\left(\bar{k}^i\right) + (1-\delta)\right] u_c^i\left(\bar{c}^i, \bar{m}^i\right). \tag{42}$$

It follows $1/\beta = \left[f'\left(\bar{k}^i\right) + (1-\delta)\right]$. The implication is that in the steady state both type of agents hold the same amount of capital. To reduce notation, let $\bar{k} \equiv \bar{k}^H = \bar{k}^L$.

We now show the lump-sum transfers/taxes are given by $\tau = \frac{z}{1+z}\bar{m}$. We can express the government's lump-sum taxes as a function of the money growth rate: $\tau_t = \frac{(1+z_t)M_{t-1}-M_{t-1}}{p_t} = \frac{z_t M_{t-1}}{p_t}$. After multiplying the right-hand-side by $\frac{p_{t-1}}{p_{t-1}}$, we get $\frac{z_t M_{t-1}}{1+(1+z_t)}$. In steady-state, this expression becomes $\tau = \frac{z}{1+(1+z)}\bar{m}$.

We now derive the relationship between the money growth rate and the inflation rate. Update equation (41) by one period and rearrange to get,

$$\frac{p_{t+1}}{p_t} = \frac{\frac{M_{t+1}}{\mu m^{dH}(\bar{\omega})+(1-\mu)m^{dL}(\bar{\omega})}}{\frac{M_t}{\mu m^{dH}(\bar{\omega})+(1-\mu)m^{dL}(\bar{\omega})}} = 1 + z.$$

This follows because the quantity of real money balances demanded by both type-$H$ and type-$L$ agents are constant over time in steady-state. Hence, as noted above, the rate of increase of the price level is the same as the rate of increase in the money supply.

Next we combine equations (40), at the steady state, and (42), substituting for the inflation rate. This yields,

$$\frac{u_m^i\left(\bar{c}^i, \bar{m}^i\right)}{u_c^i\left(\bar{c}^i, \bar{m}^i\right)} = 1 - \frac{\beta}{1+z}. \tag{43}$$

Thus, agents allocate their resources between consumption, real money balances, and capital so the marginal return from each is equalized.

With the agent's budget constraint, it is possible to solve for steady-state real balances and consumption. More specifically, the steady-state budget constraint for a type-$L$ agent can be written as

$$f\left(\bar{k}\right) = \delta\bar{k} + \bar{c}^L + \frac{z}{1+z}\mu\left(\bar{m}^L - \bar{m}^H\right). \tag{44}$$

Solving for consumption, $\bar{c}^L$, yields

$$\bar{c}^L = f\left(\bar{k}\right) - \delta\bar{k} - \frac{z}{1+z}\mu\left(\bar{m}^L - \bar{m}^H\right). \tag{45}$$

It is straightforward to see that consumption by type-$L$ may be affected by a change in the money growth rate. Formally,

$$\frac{\partial \bar{c}^L}{\partial z} = -\frac{1}{(1+z)^2}\mu\left(\bar{m}^L - \bar{m}^H\right) - \frac{z}{1+z}\mu\left[\frac{\partial \bar{m}^L}{\partial z} - \frac{\partial \bar{m}^H}{\partial z}\right]. \tag{46}$$

Similar steps give the following expression for the consumption of type-$H$ agents, which is given by

$$\bar{c}^H = f\left(\bar{k}\right) - \delta\bar{k} - \frac{z}{1+z}(1-\mu)\left(\bar{m}^H - \bar{m}^L\right). \tag{47}$$

Note $\mu\bar{c}^H + (1-\mu)\bar{c}^L = f\left(\bar{k}\right) - \delta\bar{k}$. It follows that a change in $z$ will affect consumption through its effect on

$$\frac{z}{1+z}\left(\bar{m}^L - \bar{m}^H\right).$$

This effect is typically not zero, so a change in the rate of growth of the money supply creates a change in the amount of consumption enjoyed by each type.

We now show type-$L$ agents can benefit from an increase in the rate of growth of the money supply. First we note,

**Lemma 1** *In steady state, real money balances are decreasing in the money growth rate.*

**Proof.** Equations (43) and (45) simultaneously determine consumption and real money balances in steady state for type-$H$ agents. Totally differentiate both equations to obtain

$$\frac{u^H_{mm}}{u^H_c}dm^H - \frac{u^H_{mm}}{\left(u^H_c\right)^2}dc^H = \frac{\beta}{(1+z)^2}dz$$

and

$$dc^H = -\left(\frac{z}{1+z}\right)(1-\mu)\,dm^H - \frac{1}{(1+z)^2}(1-\mu)\left(m^H - m^L\right)dz$$

After rearranging and writing in $Ax = B$ form, it is straightforward to show that $\frac{dm^H}{dz} < 0$. Since equation (47) has the same structure as equation (45), it follows immediately that $\frac{dm^L}{dz} < 0$. ■

An increase in $z$ away from the Friedman rule has two effects on the welfare of agents in this economy. On the one hand it decreases real money balances, as indicated in lemma 1. This, in turn, decreases the utility of all agents. However, type-$L$ agents will enjoy more consumption because there is a transfer from type-$H$ to type-$L$ which is captured by the change in $\frac{z}{1+z}\left(\bar{m}^L - \bar{m}^H\right)$. If this second effect dominates the first, then an increase in $z$ produces Pareto incomparable allocations. In that case, it is easy to find a social welfare function that is not maximized at the Friedman rule.

To illustrate this can indeed happen we provide a simple example. Let $\mu = .5$ and assume both type have the same satiation level of money balances.[7] The effect of a change in $z$ on type $L$ 's utility is

$$u_c^L \frac{\partial c^L}{\partial z} + u_m^L \frac{\partial m^L}{\partial z} \frac{1}{2c^L}\left[\frac{\bar{m}^H - \bar{m}^L}{(1+z)^2} + \frac{z}{1+z}\left(\frac{\partial m^H}{\partial z} - \frac{\partial m^L}{\partial z}\right)\right]$$
$$+\lambda^L \frac{m^* - \bar{m}^L}{m^* \bar{m}^L} \frac{\partial m^H}{\partial z}. \tag{48}$$

**Lemma 2** *Given our assumptions on preferences, type-H agents will hold larger real money balances and consume less than type-L agents.*

[7]This is the case, for example, if preferences are given by $u^i\left(\bar{c}^i, \bar{m}^i\right) = \ln(\bar{c}^i) + \lambda^i\left(\ln(\bar{m}^i) - \frac{\bar{m}^i}{m^*}\right)$, where $m^*$ is the satiation level of real balances which is the same for both type. Thus, $u_c^i = 1/\bar{c}^i$ and $u_m^i = \left(\lambda^i/m^*\bar{m}^i\right)\left(m^* - \bar{m}^i\right)$.

**Proof.** The marginal rate of substitution on the left-hand side of equation (43) is equal to a constant on the right-hand side. Imagine that real balances and consumption were the same for both types. With $u_m^H(\bar{c}, \bar{m}) > u_m^L(\bar{c}, \bar{m})$, $u_c^H(\bar{c}, \bar{m}) = u_c^L(\bar{c}, \bar{m})$, the marginal rate of substitution is greater for the type-$H$ agents. Hence, equal values will not satisfy the agent's efficiency condition. Next, consider the case in which $\bar{m}^H = \bar{m}^L$, but $\bar{c}^H \neq \bar{c}^L$. Equations (45) and (47), imply that consumption by type-$H$ must equal consumption by type-$L$. Hence, we cannot have $\bar{m}^H = \bar{m}^L$with $\bar{c}^H \neq \bar{c}^L$.Next, consider $\bar{c}^H = \bar{c}^L$. For equation (43), the marginal utility of real balances must be equal across types. By concavity of the utility function, $\bar{m}^H > \bar{m}^L$ must hold. Thus, we can rule out the possibility that type-$H$ and type-$L$ will have equal real money balances or equal consumption levels. We also need to rule out the possibility that $\bar{m}^H < \bar{m}^L$. Note that with $\bar{m}^H < \bar{m}^L$, then $u_m^H\left(\bar{c}^H, \bar{m}^H\right) > u_m^L\left(\bar{c}^L, \bar{m}^L\right)$. Moreover, by equations (45) and (47), we know that $\bar{c}^H > \bar{c}^L$. By concavity, it follows that $u_c^H\left(\bar{c}^H, \bar{m}^H\right) < u_c^L\left(\bar{c}^L, \bar{m}^L\right)$. It follows immediately that $\frac{u_m^H\left(\bar{c}^H, \bar{m}^H\right)}{u_c^H\left(\bar{c}^H, \bar{m}^H\right)} > \frac{u_m^L\left(\bar{c}^L, \bar{m}^L\right)}{u_c^L\left(\bar{c}^L, \bar{m}^L\right)}$. So, equation (43) cannot be simultaneously satisfied for both type-$H$ and type-$L$ agents. ■

We can prove the following proposition.

**Proposition 4** *Welfare of the type-L agents can increase for values of $z > \beta$.*

**Proof.** It is enough to show $\frac{\partial m^H}{\partial z} - \frac{\partial m^L}{\partial z} > 0$, since in that case equation (48) is positive for small enough values of $\lambda^L$. We know both $\frac{\partial m^H}{\partial z}$ and $\frac{\partial m^L}{\partial z}$ are negative. Also, the satiation level is the same for each type and it is achieved whenever $z = \beta$. By lemma 2, $\bar{m}^H > \bar{m}^L$ if $z > \beta$. Thus, in a neighborhood of $z = \beta$, it must be the case that $\frac{\partial m^H}{\partial z} - \frac{\partial m^L}{\partial z} > 0$. ■

Below we provides another example for the utility functions $u^i\left(\bar{c}^i, \bar{m}^i\right) = \ln(\bar{c}^i) + \lambda^i \ln(\bar{m}^i)$. For that example, we explicitly derive the money demand functions so it is possible to obtain an expression for $\frac{\partial m^H}{\partial z}$ and $\frac{\partial m^L}{\partial z}$.
To summarize the argument, for each type the partial equilibrium effect of an increase in $z$ is to lower utility. However, there is also a general equilibrium effect; a transfer from type $H$ to type $L$ that can more than compensate type $L$ for the partial effect when $z$ is small.

The Friedman rule would be optimal if it were possible to achieve the transfer from type $H$ to type $L$ through other, less distorting, means. With type-specific lump-sum taxes and subsidies it is possible to implement a transfer from type-$H$ to type-$L$ agents, without increasing $z$. Hence a necessary condition for the Friedman rule to be suboptimal ex-post is that changes in the rate of growth of the money supply have redistributive effects.

A final example for the MIUF economy is presented below. Assume the utility functions are $u^i\left(\bar{c}^i, \bar{m}^i\right) = \ln(\bar{c}^i) + \lambda^i \ln(\bar{m}^i)$. In this case, $u_c^i = 1/\bar{c}^i$ and $u_m^i = \lambda^i/\bar{m}^i$. To find the expressions for $\bar{m}^H$ and $\bar{m}^L$, we solve the following set of equations:

$$\bar{c}^H \lambda^H = \left(1 - \frac{\beta}{1+z}\right) \bar{m}^H, \quad \bar{c}^L \lambda^L = \left(1 - \frac{\beta}{1+z}\right) \bar{m}^L,$$

$$\bar{c}^H = \Gamma(\bar{k}) - \frac{z}{1+z}\mu\left(\bar{m}^H - \bar{m}^L\right), \quad \bar{c}^L = \Gamma(\bar{k}) + \frac{z}{1+z}\mu\left(\bar{m}^H - \bar{m}^L\right),$$

where $\Gamma(\bar{k}) = f(\bar{k}) - \delta\bar{k}$. With a little algebra it is possible to get the following expression

$$\bar{m}^i = \Gamma(\bar{k})(1+z)\frac{2\mu z + \frac{1}{\lambda^i}(1+z-\beta)}{(1+z-\beta)\left[\mu z\left(\frac{1}{\lambda^H} + \frac{1}{\lambda^L}\right) + (1+z-\beta)\frac{1}{\lambda^H \lambda^L}\right]}.$$

From this it is easy to see $\bar{m}^H > \bar{m}^L \Leftrightarrow \lambda^H > \lambda^L$. Also,

$$\frac{\bar{m}^H}{\bar{m}^L} = \frac{2\mu z + \frac{1}{\lambda^L}(1+z-\beta)}{2\mu z + \frac{1}{\lambda^H}(1+z-\beta)}.$$

Since $\frac{\partial m^i}{\partial z} < 0$, if we show

$$\frac{\partial \frac{\bar{m}^H}{\bar{m}^L}}{\partial z} < 0,$$

it will imply $0 > \frac{\partial m^H}{\partial z} > \frac{\partial m^L}{\partial z}$ which in turn means $\frac{\partial m^H}{\partial z} - \frac{\partial m^L}{\partial z} > 0$. After rearranging, we obtain

$$\frac{\partial \frac{\bar{m}^H}{\bar{m}^L}}{\partial z} = -2\mu \frac{1-\beta}{\left[2\mu z + \frac{1}{\lambda^H}(1+z-\beta)\right]^2} \frac{\lambda^H - \lambda^L}{\lambda^H \lambda^L} < 0.$$

Thus in this case also, for sufficiently small values of $\lambda^L$ type $L$ agents are better off when $z > \beta$. Hence, a deviation from the Friedman rule will increase the utility of type $L$.

# B Proof of proposition 3

First, taking the derivative of equation (30) with respect to $\gamma$ and setting it equal to zero yields $\alpha^\rho \frac{1}{1+z} \left(\frac{\gamma}{1+z}\right)^{-\rho} - \left(\frac{1-\alpha}{1-\gamma}\right)^\rho x^{1-\rho} = 0$. We can rearrange this expression to get

$$\gamma = \left[1 + \frac{1-\alpha}{\alpha} \left((1+z)\, x\right)^{\frac{1-\rho}{\rho}}\right].$$

This is the choice of reserve-to-deposit ratio made by banks.

Note $\frac{M_0}{p_1} = \frac{\gamma\omega}{1+z-z\gamma}$. Substitute this into the central bank's objective function. Take the derivative with respect to $z$, obtaining

$$\begin{aligned}&\left(\frac{\gamma\omega}{1+z-z\gamma}\right)^{-\rho}\left[\frac{\gamma'\omega\left(1+z-z\gamma\right)-\gamma\omega\left(1-\gamma-z\gamma'\right)}{\left(1+z-z\gamma\right)^2}\right]\\&+\frac{\beta}{1-\beta}\frac{\Omega^{1-\rho}\Gamma}{1+z}\left[\frac{1+z}{\Omega}\frac{\partial\Omega}{\partial z}+\frac{1}{1-\rho}\frac{1+z}{\Gamma}\frac{\partial\Gamma}{\partial z}\right]=0\end{aligned}$$

It can be verified that

$$\frac{1}{1-\rho}\frac{1+z}{\Gamma}\frac{\partial\Gamma}{\partial z}=-\gamma$$

and

$$\frac{1+z}{\Omega}\frac{\partial\Omega}{\partial z}=\frac{\gamma+(1+z)z\gamma'}{1+z-\gamma z}.$$

After rearranging, we get

$$\begin{aligned}&\left(\frac{\gamma\omega}{1+z-z\gamma}\right)^{-\rho}\omega\left[\frac{\gamma'\left(1+z\right)-\gamma\left(1-\gamma\right)}{\left(1+z-z\gamma\right)^2}\right]\\=\;&-\frac{\beta}{1-\beta}\Gamma\frac{\left(\omega\left(1+z\right)\right)^{1-\rho}}{\left(1+z-z\gamma\right)^{1-\rho}}\frac{\left(1-\gamma\right)z}{\rho\left(1+z-z\gamma\right)}\end{aligned}$$

which simplifies to

$$\gamma^{1-\rho}=-\frac{\beta}{1-\beta}\Gamma\left(1+z\right)^{1-\rho}z$$

Note $\Gamma = \alpha^{\rho}\left(\frac{\gamma}{1+z}\right)^{1-\rho} + (1-\alpha)^{\rho}\left[(1-\gamma)\,x\right]^{1-\rho} = \left(\frac{\alpha}{\gamma}\right)^{\rho}\left[\frac{\gamma}{(1+z)^{1-\rho}}+\frac{1-\gamma}{x^{1-\rho}}\right]$. From the bank's maximization we have, $\alpha^{\rho}\frac{1}{1+z}\left(\frac{\gamma}{1+z}\right)^{-\rho} - \left(\frac{1-\alpha}{1-\gamma}\right)^{\rho}x^{1-\rho} = 0$, so that

$$\gamma=\frac{\beta}{1-\beta}z\alpha^{\rho}$$

which can be rewritten as

$$1+z=1+\frac{1-\beta}{\beta}\frac{\gamma}{\alpha^{\rho}}.$$

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