Questions

Q1. What does the pricing kernel look like?

- **Dispersion**: entropy
- **Dynamics**: $n$-period entropy and horizon dependence
- **Disasters**: entropy and high-order cumulants
  Illustration: the Vasicek model

Q2. How do these pricing kernels compare?

- Power utility
- Recursive preferences
- Habits
- Jumps and disasters
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- **Dynamics**: \(n\)-period entropy and horizon dependence “small”
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- Illustration: the Vasicek model

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### Facts about excess returns (% per month)

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Skewness</th>
<th>Excess Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.40</td>
<td>5.56</td>
<td>-0.40</td>
<td>7.90</td>
</tr>
<tr>
<td>Fama-French (small, low)</td>
<td>-0.30</td>
<td>11.40</td>
<td>0.28</td>
<td>9.40</td>
</tr>
<tr>
<td>Fama-French (small, high)</td>
<td>0.90</td>
<td>8.94</td>
<td>1.00</td>
<td>12.80</td>
</tr>
<tr>
<td>Pound Sterling</td>
<td>0.35</td>
<td>3.16</td>
<td>-0.50</td>
<td>1.50</td>
</tr>
<tr>
<td>5 year bond</td>
<td>0.15</td>
<td>1.90</td>
<td>0.10</td>
<td>4.87</td>
</tr>
</tbody>
</table>

- Also ... the nominal 60-month term spread is about 0.1%/month
Facts: summary

- Facts
  - “Big” excess returns, 1% $\gg$ equity premium
  - “Small” term spreads, ±0.1%
  - Skewness and kurtosis evident

- Each tells us something about the pricing kernel
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Entropy
Entropy

- Conditional entropy

\[ L_t(m_{t+1}) = \log E_t m_{t+1} - E_t \log m_{t+1} \]
Entropy

- Conditional entropy

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- Why “entropy”?

\[ L_t(m_{t+1}) = -E_t \log \left( \frac{q_{t+1}}{p_{t+1}} \right) \]
Entropy

- Conditional entropy

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- Why “entropy”?

\[ L_t(m_{t+1}) = -E_t \log (q_{t+1}/p_{t+1}) \]

- Applications

  - Entropy, \( EL_t(m_{t+1}) \)
  - Horizon dependence

\[ H(n) = \frac{n^{-1} EL_t(m_{t,t+n}) - EL_t(m_{t+1})}{\text{avg over } n \text{ periods}} \] one period
Properties of entropy

- Dispersion: entropy bound

\[ EL_t(m_{t+1}) \geq E(\log r_{t+1} - \log r_t^1) \]
Properties of entropy

- Dispersion: entropy bound

\[ EL_t(m_{t+1}) \geq E(\log r_{t+1} - \log r_t) \]

- Dynamics: horizon dependence

\[ H(n) = -E(y_t^n - y_t^1) \]
Properties of entropy

- Dispersion: entropy bound

\[ EL_t(m_{t+1}) \geq E(\log r_{t+1} - \log r_t^1) \]

- Dynamics: horizon dependence

\[ H(n) = -E(y^n_t - y^1_t) \]

- Disasters: high-order cumulants

\[ L_t(m_{t+1}) = \kappa_2(\log m_{t+1})/2! + \kappa_3(\log m_{t+1})/3! + \kappa_4(\log m_{t+1})/4! + \cdots \]

\[ \text{normal term} + \text{high-order cumulants} \]
What the pricing kernel looks like

- Dispersion
  - Entropy $\geq 0.01 = 1\%$ a month

- Dynamics
  - Horizon dependence $\leq 0.001 = 0.1\%$ a month

- Disasters
  - Something besides the normal distribution
Vasicek model: an example

- Pricing kernel

\[
\log m_{t+1} = \log m + a(B)w_{t+1} \\
= \log m + a_0 w_{t+1} + a_1 w_t + a_2 w_{t-1} + \cdots
\]

\(w \sim \text{NID}(0, 1)\)

- Interest rate

\[
y^1_t = -\log E_t(e^{\log m_{t+1}}) = -\log m - a_0^2/2 - a_1 w_t - a_2 w_{t-1} - \cdots
\]

- ARMA(1,1) for \(\log m_t\) is AR(1) [Vasicek] for the interest rate

\[
a_{j+1} = \phi a_j, \quad j \geq 1
\]
Vasicek model: properties

- Partial sums
  \[ A_n = a_0 + a_1 + a_2 + \cdots + a_n \]

- Entropy
  \[ EL_t(m_{t+1}) = a_0^2/2 = A_0^2/2 \Rightarrow a_0 \text{ “big”} \]

- Horizon dependence
  \[ H(n) = n^{-1} \sum_{j=1}^{n} (A_{j-1}^2 - A_0^2)/2 \Rightarrow a_j \text{ “small”} \]
Vasicek model: moving average coefficients
Vasicek model: horizon dependence

**Graph:**
- **Y-axis:** Entropy and Horizon Dependence
- **X-axis:** Time Horizon in Months

- **Lines:**
  - Entropy per period
  - Negative yield spread
  - Positive yield spread
  - Entropy lower bound
  - Horizon dependence upper bound
  - Horizon dependence lower bound

**Notes:**
- The graph illustrates the relationship between entropy and horizon dependence over time, with distinct periods and yield spreads highlighted.
Representative-agent models

- Additive power utility
- Recursive preferences
  - Bansal-Yaron with persistent consumption growth
  - ... and stochastic volatility
- Habits
  - Ratio habits
  - Difference habits
  - Campbell-Cochrane
- Jumps and disasters
Recursive preferences

Preferences

\[ U_t = \left[ (1 - \beta)c_t^\rho + \beta \mu_t(U_{t+1})^\rho \right]^{1/\rho} \]
\[ \mu_t(U_{t+1}) = \left( E_t U_{t+1}^\alpha \right)^{1/\alpha} \]
\[ \alpha, \rho \leq 1 \]

Interpretation

\[ EIS = 1/(1 - \rho) \]
\[ CRRA = 1 - \alpha \]
\[ \alpha = \rho \Rightarrow \text{additive power utility} \]
Consumption and pricing kernel

- Consumption growth

\[ \log g_t = g + \gamma(B) \nu^{1/2} w_t \]
\[ \{w_t\} \sim \text{NID}(0, 1) \]
Consumption and pricing kernel

- Consumption growth

\[ \log g_t = g + \gamma(B)v^{1/2}w_t \]
\[ \{w_t\} \sim \text{NID}(0, 1) \]

- Pricing kernel

\[ \log m_{t+1} = \text{constants} \]
\[ + \left[ (\rho - 1)\gamma_0 + (\alpha - \rho)\gamma(b_1) \right] v^{1/2}w_{t+1} \]
\[ + (\rho - 1)\gamma_1 v^{1/2}w_t + (\rho - 1)\gamma_2 v^{1/2}w_{t-1} + \cdots \]
Consumption and pricing kernel

Consumption growth

\[ \log g_t = g + \gamma(B)v^{1/2}w_t \]
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\[ + (\rho - 1)\gamma_1v^{1/2}w_t + (\rho - 1)\gamma_2v^{1/2}w_{t-1} + \cdots \]

Critical term: \( \gamma(b_1) = \gamma_0 + b_1\gamma_1 + b_1^2\gamma_2 + \cdots \)
Power and recursive preferences: moving average coefficients

![Graph showing moving average coefficients for different orders, with labels for Power Utility and Bansal–Yaron models.](image-url)
Recursive preferences: entropy and horizon dependence
Model summary

[Graph showing entropy and horizon dependence for different variables (V, PU, BY1, BY2, BY3, RH, DH, CC, PJR).]

- Entropy values range from approximately 0.00 to 0.05.
- Horizon dependence values range from approximately -6 to 6.

Legend:
- Entropy lower bound
- Horizon dependence upper bound
- Horizon dependence lower bound
Answers to questions

Q1. What does the pricing kernel look like?
   - Substantial dispersion: entropy $\geq 1\%$ monthly
   - Limited horizon dependence: $\leq 0.1\%$ monthly
   - Probably not normal
   - Useful diagnostics for any model

Q2. How do representative-agent models compare?
   - It is easy to get lots of entropy
   - But it often generates too much horizon dependence
   - All this conditional on parameters

Q3. What’s next?
   - Heterogeneous agents?
   - Business cycle models?
Related work (some of it)

- **Bounds**
  - Alvarez-Jermann, Bansal-Lehmann, Hansen-Jagannathan

- **Recursive preferences**
  - Preferences: Epstein-Zin, Kreps-Porteus, Weil
  - Asset pricing: Bansal-Yaron, Campbell, Hansen-Heaton-Li

- **Habits**
  - Abel, Campbell-Cochrane, Chan-Kogan, Constantinides, Heaton, Sundaresan

- **Jumps and disasters**
Derivation of the Entropy Bound

- **Fundamental Theorem of Asset Pricing**
  \[ E_t(m_{t+1} r_{t+1}) = 1, \]
  \[ E_t \log m_{t+1} + E_t \log r_{t+1} \leq \log(1) = 0, \text{ with equality iff } m_{t+1} r_{t+1} = 1 \]

- **Risk-free rate**
  \[ \log r^1_{t+1} = - \log E_t(m_{t+1}) = -L_t(m_{t+1}) - E_t \log m_{t+1} \]

- **Subtract from above:**
  \[ L_t(m_{t+1}) \geq E_t(\log r_{t+1} - \log r^1_{t+1}) \]

- **Unconditional entropy:**
  \[ L(m_{t+1}) = EL_t(m_{t+1}) + L(E_t(m_{t+1})) \]

- **Therefore,**
  \[ L(m_{t+1}) \geq E(\log r_{t+1} - \log r^1_{t+1}) + L(E_t(m_{t+1})) \geq E(\log r_{t+1} - \log r^1_{t+1}) \]

the bound is tighter
Entropy and HJ bounds (App A.2)

- **Entropy: High-return asset**
  \[ \log r_{t+1} = - \log m_{t+1} \]

- **Max excess return over time (iid)**
  \[ L(m_t, t+n) = n \left[ k^1(1) - \kappa_1 \right] \]

- **Excess log-return (normal)**
  \[ \log r_{t+1} \sim \mathcal{N} \left( \log r^1_{t+1} + \kappa_1 t, \kappa_2 t \right) \]
  \[ E_t(\log r_{t+1} - \log r^1_{t+1}) = \kappa_1 t \]

- **HJ: High-return asset**
  \[ r_{t+1} = \alpha_t - \frac{m_{t+1}}{\text{Var}_t(m_{t+1})^{1/2}} \]

- **Max SR over time (iid)**
  \[ \frac{\text{Var}(m_t, t+n)}{E(m_t, t+n)^2} = e^{n[k^1(2) - 2k^1(1)]} - 1 \]

- **SR (normal)**
  \[ SR_t = \frac{e^{\kappa_1 t + \kappa_2 t/2} - 1}{e^{\kappa_1 t + \kappa_2 t/2} \left( e^{\kappa_2 t} - 1 \right)^{1/2}} \]