Nominal Bonds and Interest Rates:
The Case of One-Period Bonds*

Shouyong Shi
University of Toronto

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Abstract

The primary question that I try to address here is: Why do government-issued, nominal bonds not circulate as a medium of exchange, while money does? A related question is: Why are such bonds sold for money at a discount, even if they bear no default risk? In the process of answering these questions, I integrate the microfoundation of monetary theory with the influential work of Lucas (1990). I examine two models of this integrated framework. In the first model, the government does not participate in the goods market. Then, there are a continuum of monetary equilibria where matured bonds circulate in the goods market as perfect substitutes for money. In the second model, the government participates in the goods market and accepts only money as payments for its goods, while private agents can trade among themselves using both money and bonds. This model has a unique equilibrium, and an arbitrarily small measure of government sellers is sufficient to drive matured bonds out of the circulation. In both models, newly issued bonds are sold at a discount for money and thus they bear positive interest. The effects of monetary policy differ in these two economies, some of which contrast sharply with the effects in traditional models.

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1. Introduction

The primary question that I try to address here is: Why do government-issued, nominal bonds not circulate as a medium of exchange, while money does? A related question is: Why do such bonds bear positive interest even if they are default-free? In the process of answering these questions, I integrate the microfoundation of monetary theory with the influential work of Lucas (1990), so that the theory can analyze standard monetary policy issues such as the nominal interest rate.

The questions raised here are long-standing challenges to monetary theory (see Hicks, 1939). The lack of a consistent model, rather than informal arguments, is responsible for the unresolved status of these questions. Many monetary models give money a unique role by putting money into the utility function, the production function, or the transaction function. Because these functions are specified exogenously, such models are incapable of answering the above questions. Recently, monetary theorists have started to construct the microfoundation of monetary theory. The most notable output along this line of research is monetary search theory, originated in Kiyotaki and Wright (1989, 1993), which uses decentralized exchanges to support a role for a medium of exchange. As Wallace (2001) forcefully argued, monetary search theory is suitable for monetary analysis. With recent developments in this theory, now it is now time to confront with the above questions regarding nominal bonds.

Another purpose of this paper is to develop an integrated framework for policy analysis. Monetary search theory has largely omitted nominal bonds. This omission is a major limitation of the theory for policy analysis. For example, one cannot use the theory to understand nominal interest rates or to examine open market operations. By eliminating this limitation, I integrate monetary search theory with Lucas's (1990) model of limited participation. This integrated framework will allow me to examine the effects of monetary policy, without imposing the cash-in-advance constraint in the goods market as Lucas did.

The framework has a centralized bonds market and a decentralized goods market, which are separated from each other during each period. In the bonds market, the government sells new nominal bonds and redeems matured bonds. In the goods market, exchanges are decentralized and modelled as bilateral matches. The lack of public record-keeping of agents' transaction histories and the lack of double coincidence of wants induce agents to use media of exchange to trade. Nominal bonds and money compete against each other to serve as such media. Both of them are fiat objects, in the sense that they do not yield direct utility or facilitate production. The only exogenous difference between them is that the government accepts money, but not bonds, as the means of payments. This legal restriction does not apply in a trade between two private agents, who can choose to use both money and bonds as payments.

I construct two models of this framework. In the first model, the government does not participate in the goods market, while in the second model the government does. Both models

generate positive discounting on newly issued bonds. This result holds regardless of whether such bonds, when matured, will circulate in the goods market. The positive nominal interest rate is an outcome of the temporary separation between the bonds market and the goods market, as in Lucas (1990). The temporary separation implies that newly issued bonds cannot be used in the goods market in the issuing period, and hence they are not perfect substitutes for money. To compensate for this one-period loss of liquidity, newly issued bonds must be discounted.

The two models have very different predictions on whether matured bonds circulate in the goods market as a medium of exchange. In the first model, where the government does not participate in the goods market, matured bonds circulate in the goods market as perfect substitutes for money. Agents are indifferent about how large a fraction of matured bonds to redeem. This indeterminacy makes the price level, the ratio of matured bonds to money, and the asset values all indeterminate. However, all these equilibria have the same real output/consumption and the same nominal interest rate.

Matured bonds do not circulate in the second model, where the government participates in the goods market and refuses to accept bonds as payments. The presence of an arbitrarily small measure of government sellers in the goods market is sufficient to drive matured bonds out of the circulation. This strong result arises from the decentralized nature of exchanges. With decentralized exchanges, a buyer in a match cannot exchange bonds for money instantaneously, nor switch costlessly from a match with a government seller to a match with a private seller. Since a buyer holding bonds will have a positive probability of meeting a government seller who refuses to accept bonds, such a buyer will have a smaller chance to trade than a buyer holding money. This wedge induces the households to redeem all matured bonds and use only money to buy goods. This argument is valid no matter how small the measure of government sellers in the goods market is, provided that it is positive. Therefore, an equilibrium with circulating bonds is not robust to the introduction of an arbitrarily small coverage of the legal restriction in the goods market. This seems a robust answer to the question why bonds do not circulate as money.

The key point here is not so much that there is a legal restriction, but rather that the coverage of the legal restriction can be arbitrarily small. If the coverage of the legal restriction is sufficiently large, then even a standard model with a centralized goods market will generate the result that bonds do not circulate as money. However, as the coverage becomes small, bonds will start to circulate as money in a standard model. In that case, the legal restriction merely shifts money from the market for private goods to the market for government goods.

I also analyze the effects of two types of monetary policy. One is an increase in the money growth rate and the other is an increase in the amount of bond sales in the open market, both of which are deterministic changes. I will summarize the policy effects in Section 6. Not surprisingly, the policy effects are very different depending on whether bonds circulate as money. Moreover, the policy effects in the second model contrast sharply with those in conventional monetary models,

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although the second model and conventional models both have no bonds circulating as money. Thus, it matters for policy analysis whether the outcome of no circulating bonds is endogenously generated or exogenously imposed.

The models constructed here share some important features of Lucas's (1990) model, such as a positive discount on newly issued bonds. The key distinctions from Lucas's analysis are that the value of money is supported by the description of the trading environment and that nominal bonds are not precluded from circulation as a medium of exchange.

The structure of the model builds on two previous papers (Shi, 1997, 1999), which in turned is rooted in the Kiyotaki-Wright model.[1] In this literature, Aiyagari et al. (1996) is the first attempt to analyze the coexistence of money and government bonds in a search model. They show that there exist two types of equilibria where money and interest-bearing bonds coexist. In one type, matured bonds and money are perfect substitutes in the goods market and in the other, matured bonds are discounted among private agents. These results have some resemblances to my results.[2] However, there are significant differences. First, I eliminate their assumption that money and bonds are indivisible, as indivisibility itself may generate spurious results on the coexistence of two nominal assets. Second, the models here are tractable for analyzing standard monetary policy, such as money growth and open market operations. Third, I assume that the bonds market is centralized. In contrast, Aiyagari et al. (1996) assume a decentralized bonds market, and so an agent who wants to redeem bonds may fail to do so with positive probability. By assuming centralized issuing and redemption of bonds, I allow for a greater degree of competition between bonds and money, and hence makes the results more robust.[3]

Before going into the details, let me clarify two issues about the analysis. First, the analysis in this paper is positive rather than normative. In particular, the legal restriction is exogenously imposed. Although it is interesting to investigate how the welfare of the society depends on the extent of the legal restriction, I do not take up this task here. Second, I will restrict attention to one-period nominal bonds. Both restrictions will be relaxed in a sequel (Shi, 2003a), where I will introduce bonds of longer maturity and examine the welfare implication of the legal restriction.

With the restriction to one-period bonds, the timing of events described in this paper implies that bonds can circulate only as matured bonds. Thus, the question is whether agents will choose

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[1]The original Kiyotaki-Wright search model assumes that goods and money are both indivisible. Shi (1995) and Trejos and Wright (1995) eliminate the assumption of indivisible money, Green and Zhou (1998) eliminate the assumption of indivisible money, and Molico (1997) and Shi (1997, 1999) eliminate both. Moreover, some search models have allowed for limited forms of competition between money and other means of exchange, such as bilateral credits (Shi, 1996) and middlemen (Li, 1998).

[2]In particular, for matured bonds to be perfect substitutes for money and yet newly issued bonds to be discounted in Aiyagari et al. (1996), the government must reject unmatured bonds with positive probability in trades.

[3]Another related model is by Kocherlakota (2003). He uses a model of spatial separation to examine the welfare-improving role of illiquid bonds. There, the illiquidity of bonds is a physical feature, rather than an equilibrium outcome as in my paper. He also assumes that matured bonds will perish if they are not redeemed immediately. This assumption precludes matured bonds from circulating as money.

not to redeem bonds at maturity and instead use them to buy goods in the future. One may wonder why any bond holder would choose to miss out on the redemption. This question is misguided by models where money is assumed to have an intrinsic value. In any model where money is intrinsically worthless, redemption of a nominal bond is a swap of one fiat object for another. Missing out on the redemption is not costly at all to a bond holder if bonds perform the role of a medium of exchange as well as money does.

In section 2 below, I will describe an economy in which the government participates in the bonds market but not in the goods market. In section 3, I will show that there are a continuum of equilibria in this economy and that bonds circulate at par with money. Then, in section 4, I will introduce government agents into the goods market and show that there is no equilibrium where bonds circulate in the goods market. The equilibrium in this economy will be characterized in section 5, where I will also analyze the effects of money growth and open market operations. I will conclude in section 6 and supply necessary proofs in the appendices.

2. A Search Economy with Nominal Bonds

2.1. Households, Matches, and Timing

Consider a discrete-time economy with many types of households. The number of households in each type is large and normalized to one. Households in each type are specialized in producing a specific good, which they do not consume, and exchange for consumption goods in the market. Goods are perishable between periods.[4] The utility of consumption is $u(.)$ from consumption goods and 0 otherwise. The cost (disutility) of production is $\psi(.)$. The utility function satisfies $u' > 0$, and $u'' \leq 0$. The cost function satisfies $\psi(0) = 0 = \psi'(0)$, $\psi'(q) > 0$ for all $q > 0$, and $\psi'' > 0$. Moreover, assume that $u'(0) > \psi'(0)$ and $u'(\infty) < \psi'(\infty)$. Thus, for any given $k \in (0, \infty)$, there is a unique $q \in (0, \infty)$ such that $u'(kq) = \psi'(q)$.

Agents meet their trading partners bilaterally and randomly in the market, as in Kiyotaki and Wright (1989, 1993). To emphasize the competition between money and nominal bonds, I assume that there is no chance for a double coincidence of wants in a meeting to support barter, or public record-keeping of transactions to support credit trades. As a result, every trade requires a medium of exchange, which can be money or nominal bonds. Money and bonds have no intrinsic value and can be stored without cost.

Nominal bonds are issued by the government and are default-free. Each unit of a bond can be redeemed for one unit of money at maturity and the maturity is restricted to one period (see the introduction for a discussion on the maturity). For convenience, I make two auxiliary assumptions. First, bonds can be redeemed only at the maturity. Second, an agent can bring money and bonds separately into matches but not together into a match. These assumptions are

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[4]See Shi (1999, 2001) for search models with capital accumulation and an endogenous fraction of sellers.

not critical, as I will show in section 4.3.

To specify the matching technology, let me call an agent in the goods market a *buyer* if he holds money or bonds, and a *seller* if he holds neither money nor bonds. Let σ be the (fixed) fraction of agents in the goods market who are sellers and $(1 - \sigma)$ the fraction of agents who are buyers. I call a match between two agents a *trade match* if there is a single coincidence of wants, i.e., if one (and only one) of the two agents can produce the partner's consumption goods. In this model, only a trade match can generate trade. A buyer encounters a trade match in a period with probability $\alpha\sigma$ and a seller with probability $\alpha(1 - \sigma)$, where $\alpha > 0$ is a constant.

Random matching can generate non-degenerate distributions of agents' money holdings and consumption. To maintain tractability, I assume that each household consists of a large number of members who share consumption each period and regard the household's utility function as the objective. This assumption makes the distribution of money holdings degenerate across households and hence allows me to focus on equilibria that are symmetric across households.[5]

A household consists of a measure σ of sellers and a measure $(1 - \sigma)$ of buyers. A seller produces and sells goods, while a buyer purchases consumption goods. The are two types of buyers, distinguished by what they hold. A buyer who carries money is called a *money holder* and a buyer who carries bonds a *bond holder*. Let n be the measure of money holders in the household and $(1 - \sigma - n)$ be the measure of bond holders. This division between the two types of buyers, captured by the choice n, is endogenous, although the division between buyers and sellers is fixed.

In contrast to the goods market, the bonds market is centralized and has a much lower transaction cost. To simplify, I abstract from such a transaction cost altogether by assuming that trades in the bonds market take zero measure of members. Also, following Lucas (1990), I assume that the government sells bonds only for money. Each newly issued bond is sold at a market price S and can be redeemed for one unit of money at (and only at) maturity. Thus, $1/S$ is the gross nominal interest rate.

The events in an arbitrary period t unfold as in Figure 1, where the subscript t is suppressed. At the beginning of the period the household has an amount m of money and an amount b of matured bonds. These are the bonds that were matured in previous periods but not redeemed. The household divides money into a fraction a to be taken into the goods market and a fraction $(1 - a)$ into the bonds market. The household also divides buyers into money holders and bond holders, by choosing $n \in [0, 1 - \sigma]$. Thus, each money holder holds an amount of money, am/n, and each bond holder holds an amount of matured bonds, $b/(1 - \sigma - n)$. The household does not carry any matured bonds into the bonds market because new bonds are sold only for money.

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[5]The assumption of large households, used by Shi (1997, 1999), is a modelling device extended from Lucas (1990). Lagos and Wright (2001) use a different set of assumptions to achieve essentially the same purpose of risk smoothing.

At the time of choosing the portfolio divisions (a, n), the household also chooses the quantities of trade (q, x), which I will describe later.

Next, the goods market and the bonds market open simultaneously. In the bonds market, the household purchases an amount d of new bonds, using the money allocated to the bonds market. In the goods market, the agents trade according to the quantities (q, x) prescribed by the household. Then, the goods market closes. The household pools the receipts from the trades and allocates the same amount of consumption to every member.

After consumption, the bonds purchased at the beginning of the current period mature and the household chooses the fraction of such bonds to be redeemed (in the centralized market).[6] Let $r \in [0, 1]$ denote this *redemption fraction*. By the earlier auxiliary assumption, bonds that are not redeemed at maturity cannot be redeemed in the future. After redeeming bonds, the household receives a lump-sum monetary transfer L and time proceeds to the next period.

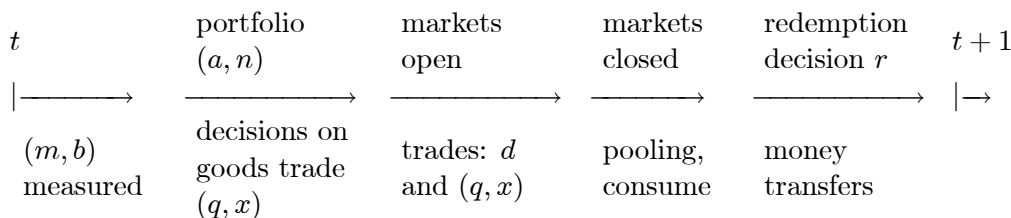| t	portfolio (a, n)	markets open	markets closed	redemption decision r	$t + 1$
$\vert\!\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\vert\!\rightarrow$
(m, b) measured	decisions on goods trade (q, x)	trades: d and (q, x)	pooling, consume	money transfers	

Figure 1. Timing of events in a period

This timing sequence highlights the temporary separation of the bonds market and the goods market, as in Lucas (1990). A household cannot shift resources between the two markets within a period, although it can do so over time. This is a critical feature, because it makes it costly to bring money into the bonds market.

There are two main differences between this model and Lucas's. First, the goods market is decentralized here, in contrast to the centralized goods market. With decentralized trades, fiat objects like money and nominal bonds can have positive values in equilibrium, even when the current model is extended to allow agents to barter. This endogenous role of a medium of exchange is absent when the goods market is Walrasian. Second, I allow households to use bonds, as well as money, to buy goods, while Lucas assumes that money is the only medium of exchange. Lucas's assumption amounts to imposing $r = 1$ a priori.

To focus on these new elements, I keep other aspects of the model as close as possible to Lucas's (1990). In particular, I adopt Lucas's assumption that monetary transfers maintain the money growth rate constant (In Appendix C I will relax this assumption). Let M be the aggregate

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[6] An alternative timing structure is that bonds issued in this period become mature at the beginning of the next period at which time they can be redeemed for money, before other activities take place. This alternative structure does not make any difference in the results from the timing structure assumed here. In both cases, the amount of money obtained from redeeming the bonds cannot be used until the next period.

money holdings per household at the beginning of a period and M_{+1} the holdings in the next period. The (gross) rate of money growth between the two periods is $\gamma_{+1} = M_{+1}/M$. Monetary transfer maintains $\gamma_{+1} = \gamma$ as a constant. This particular process of transfer helps isolating the effects of open market operations from the effects of money growth.

2.2. Quantities of Trade in the Goods Market

Pick an arbitrary household as the representative household. Normalize the measure of members in the household to one. Lower-case letters denote the decisions of this household, while capital-case letters denote other households' decisions or aggregate variables. Also, suppress the generic time subscript t, denoting the subscript $t \pm j$ as $\pm j$, for $j \geq 1$.

The household chooses the quantities of money and goods in each trade match. To simplify the analysis, I assume that the buyer in a trade match makes a take-it-or-leave-it offer.[7] Consider a buyer holding asset i, where $i = m, b$. The offer specifies the quantity of goods that the buyer wants the seller to supply, q^i, and the quantity of asset i that the buyer gives, x^i. If a match is not a trade match, the household instructs its members to not trade.[8]

To describe the decisions on (q^i, x^i), let $\beta \in (0,1)$ be the discount factor and $v(m,b)$ the household's value function, where I suppressed the dependence of v on aggregate variables. Let ω^i be the household's marginal value of asset i ($= m, b$) in the next period, discounted to the current period. That is,

$$\omega^m \equiv \beta v_1(m_{+1}, b_{+1}), \qquad \omega^b \equiv \beta v_2(m_{+1}, b_{+1}), \tag{2.1}$$

where the subscripts of v indicate partial derivatives and the subscript $+1$ indicates "the next period". Other households' values of the two assets are Ω^m and Ω^b, respectively.

When choosing the quantities (q^m, x^m), the household anticipates the following constraints:

$$x^m \leq am/n, \tag{2.2}$$

$$\psi(q^m) \leq \Omega^m x^m. \tag{2.3}$$

The first constraint says that the amount of money traded cannot exceed the amount the buyer carries into the trade. This is necessary because the matches are separated from each other. The second constraint says that the offer cannot make the seller worse off than not trading, where $\Omega^m x^m$ is the value of money that the seller's household gets by agreeing to the trade.

Similarly, the following constraints apply when the buyer holds bonds:

$$x^b \leq b/(1 - \sigma - n), \tag{2.4}$$

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[7]More general bargaining schemes are analyzed in a similar environment elsewhere (Shi, 2001).

[8]I omit the possible trades between a money holder and a bond holder in the goods market, because such trades are immaterial in the current model. As it will become clear later, whenever there are matured bonds in the goods market, they are traded at par with money.

$$\psi(q^b) \le \Omega^b x^b. \tag{2.5}$$

I will call (2.2) and (2.4) the *trading constraints* in the goods market. When (2.2) binds, I say that money yields *liquidity* or *service* in the goods market. Similarly, matured bonds yield liquidity or service in the goods market when (2.4) binds.

2.3. The Representative Household's Decision Problem

The household's choices in each period are the portfolio divisions (a, n), the quantities of trade (q^m, x^m, q^b, x^b), the amount of new bonds to purchase d, consumption c, the redemption fraction r, and future asset holdings (m_{+1}, b_{+1}). Taking other households' choices and aggregate variables (i.e., the capital-case letters) as given, the household solves the following problem:

$$(PH) \quad v(m, b) = \max \left\{ u(c) - \alpha\sigma \left[N\psi(Q^m) + (1 - \sigma - N)\psi(Q^b) \right] + \beta v(m_{+1}, b_{+1}) \right\}.$$

The constraints are as follows:

(i) the constraints in the goods market, (2.2)–(2.5), and

$$c = \alpha\sigma \left[nq^m + (1 - \sigma - n) q^b \right]; \tag{2.6}$$

(ii) the money constraint in the bonds market:

$$Sd \le (1 - a)m; \tag{2.7}$$

(iii) the laws of motion of asset holdings:

$$m_{+1} = m + (r - S)d + \alpha\sigma \left(NX^m - nx^m \right) + L, \tag{2.8}$$

$$b_{+1} = b + (1 - r) d + \alpha\sigma \left[(1 - \sigma - N) X^b - (1 - \sigma - n) x^b \right]; \tag{2.9}$$

(iv) and other constraints:

$$0 \le a \le 1, \ 0 \le n \le 1 - \sigma, \ 0 \le r \le 1. \tag{2.10}$$

The disutility of production in the objective function is calculated as follows. The total number of trades which involve the household's sellers is $\alpha\sigma$. In a fraction N of such trades the trading partners are money holders, who ask for Q^m units of goods; in the remaining fraction of trades the trading partners are bond holders, who ask for Q^b units of goods. The amount of consumption given by (2.6) can be explained similarly. The constraint (2.7) in the bonds market is self-explanatory, with S being the price of newly issued bonds and d the amount of such bonds purchased by the household.

To explain the law of motion of money, (2.8), note that the household's money holdings can change between the current period and next period for three reasons: purchasing and redeeming

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newly issued bonds, trading in the goods market, and receiving monetary transfers. In particular, the household gets a net amount $(r - S)d$ from purchasing d units of money and redeeming a fraction r of them. Adding these three changes to the household's current money holdings gives the household's money holdings at the beginning of the next period.

The law of motion of matured bonds, given by (2.9), can be explained similarly. Note that bonds acquired in the goods market are matured bonds that have already passed their maturity and hence, under an earlier auxiliary assumption, they cannot be redeemed for money in the current or any future period.

To characterize optimal decisions, let ρ be the Lagrangian multiplier of the money constraint in the bonds market, (2.7), λ^m of the money constraint in the goods trade, (2.2), and λ^b of the bond constraint in the goods trade, (2.4). To simplify the equations, multiply λ^m by the number of trades that involve the household's money holders, $\alpha\sigma n$. Similarly, multiply λ^b by $\alpha\sigma(1-\sigma-n)$. Incorporating these constraints, I can rewrite the objective function in (PH) as follows:

$$
\begin{aligned}
v(m, b) \quad &= \max \Big\{ \beta v(m_{+1}, b_{+1}) + u \left(\alpha\sigma \left[nq^m + (1 - \sigma - n) q^b \right] \right) \\
&\quad - \alpha\sigma \left[N\psi(Q^m) + (1 - \sigma - N)\psi(Q^b) \right] + \rho \left[(1 - a) m - Sd \right] \\
&\quad + \alpha\sigma n \lambda^m \left(\tfrac{am}{n} - x^m \right) + \alpha\sigma \left(1 - \sigma - n \right) \lambda^b \left(\tfrac{b}{1-\sigma-n} - x^b \right) \Big\},
\end{aligned}
$$

where x^m and x^b satisfy (2.3) and (2.5) with equality (provided $\omega^m, \omega^b > 0$); that is,

$$
x^i = \psi(q^i)/\Omega^i, \text{ for } i = m, b. \tag{2.11}
$$

The following conditions are necessary for the decisions to be optimal:

(i) For q^i:
$$
u'(c) = \left(\omega^i + \lambda^i \right) \psi'(q^i)/\Omega^i, \ i = m, b. \tag{2.12}
$$

(ii) For (r, a, d, n):
$$
\omega^m = \omega^b \quad \text{if } r \in (0, 1), \tag{2.13}
$$

$$
\alpha\sigma\lambda^m = \rho \quad \text{if } a \in (0, 1), \tag{2.14}
$$

$$
r\omega^m + (1 - r)\omega^b = (\omega^m + \rho) S \quad \text{if } d \in (0, \infty), \tag{2.15}
$$

$$
q^m = q^b \quad \text{if } n \in (0, 1 - \sigma). \tag{2.16}
$$

In each of these conditions, the variable attains the lowest value in the specified domain if the corresponding equality is replaced by "<", and the highest value if ">".

(iii) For (m, b) (envelope conditions):
$$
\omega_{-1}^m/\beta = \omega^m + (1 - a)\rho + a\alpha\sigma\lambda^m, \tag{2.17}
$$

$$
\omega_{-1}^b/\beta = \omega^b + \alpha\sigma\lambda^b. \tag{2.18}
$$

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The condition (2.12) requires that the net gain to the buyer's household from asking for an additional amount of goods be zero. To induce the seller to produce one additional unit of good, the buyer must pay an additional amount $\psi'(q^i)/\Omega^i$ of asset i (see (2.11)). By giving one additional unit of the asset, the buyer foregoes the discounted future value of the asset, ω^i, and causes the asset constraint in the trade to be more binding. Thus, $(\omega^i + \lambda^i)$ is the shadow cost of one unit of asset i to the buyer's household. The cost of getting an additional unit of good from the seller, given by the right-hand side of (2.12), must be equal to the marginal utility of consumption.

In (ii), (2.13) says that matured bonds must have the same value as money in the next period if the household chooses not to redeem all bonds at maturity. The condition (2.14) says that for the household to allocate money to both the goods market and the bonds market, money must generate the same marginal "service" or liquidity in the two markets by relaxing the trading constraints in these markets. The condition (2.15) says that the expected future value of newly issued bonds must be equal to the cost of money that is used to acquire such bonds. Because the household may not redeem all bonds at the end of the period, the expected future value of newly issued bonds is the weighted average of the future values of money and matured bonds, where the weights are r and $(1 - r)$.

The condition (2.16) says that for the household to send both money holders and bond holders to the goods market, the two types of buyers must obtain the same quantity of goods in a trade match. To explain this condition, note that a buyer of type i $(= m, b)$ obtains a surplus or net gain from a trade, $[u'(c)q^i - (\omega^i + \lambda^i)\, x^i]$, where $(\omega^i + \lambda^i)$ is the marginal cost of asset i to the buyer (as explained above). Since $(\omega^i + \lambda^i)x^i = u'\psi(q^i)/\psi'(q^i)$ by (2.11) and (2.12), a type-i buyer's surplus from trade is $u'(c)\left[q^i - \psi(q^i)/\psi'(q^i)\right]$. For the optimal choice for n to be interior, the two types of buyers must obtain the same surplus in a trade. This requires $q^m = q^b$, because $[q - \psi(q)/\psi'(q)]$ is a strictly increasing function.[9]

Finally, the envelope conditions require the current value of each asset to be equal to the sum of the future value of the asset and the expected service generated by the asset in the current markets. Take money for example. The current value of money is given by the left-hand side of (2.17), where ω_{-1}^m is divided by β because ω_{-1}^m is defined as the value of money discounted to one period earlier. The right-hand side of (2.17) consists of the future value of money, ω^m, the service generated by money in the current bond market, ρ, and the service generated by money

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[9]Throughout this paper, the term "surplus" or "net gain" is the marginal gain to the household, derived from a single trade. One should not confuse this surplus with the total surplus that the household obtains from all matches of a given type that involve the household's buyers. It is the marginal surplus that is relevant for the household's decisions, and hence for the assets' values in equilibrium, because the household has a large number of buyers each contributing only marginally to the household's utility. One way to explicitly distinguish the marginal surplus from the total surplus is to index each match by, say, j and express the choices in such a match as $(q^i(j), x^i(j))$ (see Shi, 1997). These choices across the index j interact with each other only through their effects on (c, ω^i), which are negligible when the household has a large number of trades in each period.

in the current goods market, $\alpha\sigma\lambda^m$. The services in the two markets are weighted by the division of money into the two markets.

3. Symmetric Equilibrium

3.1. Definition, Focus, and the Nominal Interest Rate

A (symmetric) monetary equilibrium consists of a sequence of a representative household's choices

$$(a_t, n_t, q_t^m, x_t^m, q_t^b, x_t^b, d_t, c_t, r_t, m_{t+1}, b_{t+1})_{t=0}^{\infty},$$

the implied shadow prices $\left(\omega_t^m, \omega_t^b, \rho_t, \lambda_t^m, \lambda_t^b\right)_{t=0}^{\infty}$, and other households' choices such that the following requirements are met:

(i) Optimality: given other households' choices, the household's choices solve (PH) with given initial holdings (m_0, b_0);

(ii) Symmetry: the choices (and shadow prices) are the same across households;

(iii) Clearing in the bonds market: $d_t = zM_t$, where M_t is money holdings per household at the beginning of period t and zM_t is the amount of new bonds issued in period t, with $z > 0$;

(iv) Positive and finite total value of each asset: $0 < \omega_t^m m_t < \infty$ and $0 < \omega_t^b b_t < \infty$ for all t if $m, b > 0$.

In this definition I have restricted bond issuing to be a constant fraction of the money stock. I have also restricted the total value of each asset to be positive and finite, in order to examine the coexistence of money and bonds.[10]

For the issues in this paper, I restrict attention further to equilibria that have the following features. First, money serves as a medium of exchange. This requires $a > 0$ and $n > 0$. (Note also that $a < 1$ as a result of the bonds market clearing condition, and so $\rho = \alpha\sigma\lambda^m$.) Second, the household redeems a positive fraction of matured bonds, i.e., $r > 0$. Third, the equilibrium is stationary and, in particular, the bond-money ratio (b/m) and the value of the total stock of money $(\omega_{-1}^m m/\beta)$ are constant. Since the money stock grows at a constant rate γ, then $\omega_{-1}^m/\omega^m = m_+/m = \gamma$, which is also the gross rate of inflation in the stationary equilibrium.

Under these restrictions, the equilibrium is one of the following two types:

(a) $0 < r < 1$ and $0 < n < 1 - \sigma$: In this case $\omega^m = \omega^b$, $q^m = q^b$, and bonds circulate in the goods market as perfect substitutes for money;

(b) $(1 - r)(1 - \sigma - n) = 0$: In this case $q^m > q^b$ and bonds do not circulate.

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[10] The bounded total value of each asset is necessary for the household's optimal decisions to be indeed characterized by the first-order conditions obtained in the previous section.

In both cases, the gross nominal interest rate, $1/S$, can be derived from (2.15) as

$$1/S = 1 + \alpha\sigma\lambda^m/\omega^m. \tag{3.1}$$

Note from (2.17) that $\alpha\sigma\lambda^m/\omega^m = \omega^m_{-1}/(\beta\omega^m) - 1$. Since $\omega^m_{-1}/\omega^m = \gamma$, the standard Fisherian relationship holds between the nominal interest rate $1/S$, inflation (γ) and the real interest rate $1/\beta$:

$$1/S = \gamma/\beta. \tag{3.2}$$

Newly issued bonds are sold at a discount and hence the net nominal interest rate is positive, provided $\gamma > \beta$. This result is reminiscent of Lucas's (1990), but it holds even when bonds circulate in the goods market as a medium of exchange. Although bonds can become perfect substitutes for money one period after they are issued, they are not perfect substitutes for money at the time of issuing. This is because the temporary separation between the bonds market and the goods market prevents newly issued bonds from being used to purchase goods in the *same* period as they are issued. Thus, there is a one-period loss of liquidity to the amount of money allocated to purchase newly issued bonds. To compensate for this loss of liquidity, a positive discount on newly issued bonds is necessary; otherwise the household will not allocate any money to the bonds market.

The liquidity value of money is the extent to which money relaxes the trading constraints in the goods market, as captured by the term $\alpha\sigma\lambda^m$ in (3.1). This liquidity value is positive only if $\gamma > \beta$. When $\gamma = \beta$, money generates a rate of return equal to the rate of time preference, in which case a household is indifferent between spending a marginal unit of money on goods and keeping it for its store of value. In the remainder of this paper, I will impose $\gamma > \beta$.

3.2. Equilibria with Bonds Circulating in the Goods Market

Matured bonds can circulate in the goods market as perfect substitutes for money. In fact, there are a continuum of such equilibria, as described below.

Proposition 3.1. *Assume $z \in (0, \gamma/\beta)$. If $\gamma > 1$, the redemption fraction r is indeterminate. For each $r \in (0,1)$, there exists a stationary equilibrium in which matured bonds circulate in the goods market as perfect substitutes for money. The price level and the bond-money ratio are lower in an equilibrium with a higher r than with a lower r. Also, the values of assets and the division n depend on r. However, the quantity of goods traded in a match, the level of consumption, the division of money a and the nominal interest rate are all independent of r. If $\gamma \le 1$, there is no equilibrium with a bounded and positive ratio b/m unless $r = 1$.*

It is useful to prove the proposition by constructing the continuum of equilibria in the proposition. Fix r at any arbitrary level in $(0,1)$. For this interior choice to be optimal, matured bonds

and money must have the same value, i.e., $\omega^b = \omega^m$ (see (2.13)). For matured bonds to circulate, the household must allocate a positive measure of members to trade with such bonds. However, for the choice $n \in (0, 1 - \sigma)$ to be optimal, matured bonds and money must exchange for the same quantity of goods in a trade, i.e., $q^m = q^b$ (see (2.16)). Since the two assets have the same value and trade for the same quantity of goods, they are perfect substitutes in the goods market.

The quantities q^m and q^b are independent of r. To show this, suppress the superscripts of q^m and q^b since there are equal to each other. Then, (2.12) yields $\lambda^m = \lambda^b$ and $\lambda^m/\omega^m = u'(c)/\psi'(q) - 1$, where $c = \alpha\sigma(1 - \sigma)q$. Substituting these results into (3.1) and using (3.2), I get

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma \left[\frac{u'\left(\alpha\sigma(1 - \sigma)q\right)}{\psi'(q)} - 1 \right]. \tag{3.3}$$

There is a unique solution for q for all $\gamma \geq \beta$ and the solution is independent of r. Thus, real consumption and real output are independent of r.

Also independent of r are the price of newly issued bonds, the nominal interest rate, and the division of money between the two markets, a. The price of newly issued bonds is $S = \beta/\gamma < 1$, as shown before, and the fraction a is given by

$$a = 1 - Sd/m = 1 - z\beta/\gamma. \tag{3.4}$$

Clearly, $a \in (0, 1)$ if and only if $0 < z < \gamma/\beta$, a condition imposed in Proposition 3.1.

The redemption fraction affects other variables. First, the bond-money ratio, obtained from the law of motion of bonds (2.9), is

$$\frac{b}{m} = \frac{(1 - r)z}{\gamma - 1}. \tag{3.5}$$

Given $r \in (0, 1)$, this ratio is positive and finite if and only if $\gamma > 1$. Also, it is a decreasing function of r when $\gamma > 1$. Second, to keep the money stock growing at the constant rate γ, (2.8) requires monetary transfers to satisfy:

$$\frac{L}{m} = \gamma - 1 + z\left(\frac{\beta}{\gamma} - r\right). \tag{3.6}$$

Third, because $\gamma > \beta$, (3.3) implies $u'(c) > \psi'(q)$, and so $\lambda^m = \lambda^b > 0$. That is, $x^m = am/n$ and $x^b = b/(1 - \sigma - n)$. Dividing these two equations and using $x^i = \psi(q)/\omega^i$, I get

$$n = (1 - \sigma) \bigg/ \left[1 + \left(\frac{1 - r}{\gamma - 1}\right)\left(\frac{z}{1 - z\beta/\gamma}\right)\right]. \tag{3.7}$$

This fraction indeed lies in $(0, 1 - \sigma)$ for $\gamma > 1$ and it is an increasing function of r. Fourth, the values of money and bonds are:

$$\omega^m = \psi(q)/x^m = n\psi(q)/(am); \\ \omega^b = \psi(q)/x^b = (1 - \sigma - n)\psi(q)/b. \tag{3.8}$$

Finally, nominal prices of goods are:

$$p^m \equiv x^m/q = x^b/q \equiv p^b. \tag{3.9}$$

Because $x^m = am/n$ and $dn/dr > 0$, the price level is a decreasing function of r. This completes the construction of the equilibrium for an arbitrary $r \in (0, 1)$.

The redemption fraction is indeterminate because matured bonds are perfect substitutes for money in the goods market. By choosing not to redeem a unit of matured bond, the household misses out on the payment of one unit of money. However, the retained bond can purchase exactly the same amount of goods as does a unit of money. Thus, it does not matter to the household how much of the matured bonds to be redeemed.

Different redemption fractions lead to different levels of total "moneyness" in the economy, $(m + b)$. A higher redemption fraction reduces the bond-money ratio and hence reduces the total moneyness in the goods market. Since real output is the same for all r, the higher redemption fraction reduces the price level and increases the values of the assets. Also, as the bond-money ratio falls, households shift some members from holding bonds to holding money; i.e., n increases. This shift is necessary for maintaining perfect substitutability between the two assets, by ensuring that money and bonds yield the same services in the goods market.

It is remarkable that (matured) nominal bonds circulate in the goods market in equilibrium, given that the model has imposed a few restrictions that seem to reduce the desirability of bonds relative to money. For example, the government does not accept matured bonds as payments for newly issued bonds and it refuses to redeem bonds that have passed their maturity. In the end, however, the fiat nature of bonds makes them equally acceptable in the goods market as the other fiat object – money.

The indeterminacy of equilibria in the current model differs from that in a conventional monetary model where the real money balance appears in the utility function. There, the price level is indeterminate when the utility function depends on the real money balance in particular ways. Here, the real money balance does not appear in the utility function. Instead, the substitution between money and other money-like assets in the goods market is the key to the indeterminacy.

Notice that the continuum of equilibria requires $\gamma > 1$. If $\gamma \leq 1$ and $r < 1$, then there is no equilibrium with a positive and bounded bond-money ratio (see (3.5)). However, this is an artifact of the assumption $b > 0$. If the government can lend as well as borrow, the continuum of equilibria can exist for $\gamma < 1$ as well. Alternatively, if the government is willing to redeem bonds that have passed the maturity, then one can show that a continuum of equilibria exist for $\gamma < 1$ (as well as for $\gamma > 1$), provided $1 > r > \max\{0, 1 - \gamma\}$.

3.3. Effects of Open Market Operations

The continuum of equilibria makes policy analysis difficult, because a change in a policy may induce the economy to switch from one equilibrium to another. To illustrate, let us examine a tightening open market operation, modelled as an increase in z. Let $r < 1$ and $\gamma > 1$.

Assume first that the redemption fraction does not respond to the increase in z. Then, the tightening operation increases the bond-money ratio and *increases* the price level. To explain this seemingly paradoxical result, note that new bonds are sold for money at a discount. Each unit of bond is sold for less than one unit of money but, when the bond matures, it will circulate in the goods market as a perfect substitute for one unit of money. Thus, an increase in the bond sale increases the total moneyness in the goods market in the future and pushes up the price level.

The households absorb the increased quantity of bonds by allocating more buyers to transact with matured bonds, thus maintaining the perfect substitutability between the two assets in the goods market. The bond price and the nominal interest rate do not change. This is because the households fully anticipate the increase in z, and so they increase the amount of money allocated to the bonds market to completely offset the increase in the new issues.[11] Real consumption and output do not respond to the increase in bond issuing, either.

The effect on the price level can be quite different if the redemption fraction responds to the tightening operation, a possibility that can occur given the continuum of equilibria. For example, if the redemption fraction increases sufficiently with the tightening operation, then the bond-money ratio and the price level can fall, rather than rise, with the increase in z. Thus, whether the tightening open market operation will increase or decrease the price level depends on the direction and magnitude in which the redemption fraction responds to the operation.[12] Real consumption and output, however, are still invariant with respect to the operation.

4. Government in the Goods Market

Now I introduce the government into the goods market. Assume that government goods are perfect substitutes for private goods. This assumption is not critical, as discussed later.

4.1. Government Traders and Private Households' Decisions

The government has a measure N_g of buyers and a measure σ_g of sellers. Each government buyer holds M_g/N_g units of money. The total measure of buyers in the economy is $(1 - \sigma + N_g)$ and the total measure of sellers is $(\sigma + \sigma_g)$. With random matching, a buyer (private or government)

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[11]If the households must allocate money between the two markets before knowing the change in z, then the nominal interest rate will change, as in Lucas (1990).

[12]One can push the above argument further by introducing sunspots into the economy that serve as a device to coordinate the households' decisions on the redemption ratio. Then, it is possible that the price level can remain invariant with respect to open market operations. Such invariant prices, of course, have nothing to do with any cost associated with adjusting nominal prices.

gets a trade match with a private seller with probability $\alpha\sigma$, and with a government seller with probability $\alpha\sigma_g$. Similarly, a seller gets a trade match in a period with a private money holder with probability αN, with a (private) bond holder with probability $\alpha(1 - \sigma - N)$, and with a government buyer with probability αN_g.[13]

An important assumption is that the government sells goods only for money and buys goods only with money. More specifically, I assume that government agents trade in the following reasonable but exogenous ways:[14]

(i) A government buyer. A government buyer carries only money. In a trade match with a private seller, the government buyer buys goods at the price which a private money holder pays to a private seller. This price, denoted P^m, is the price averaged over all trades between two private agents who use money as payments, not including the trades that use bonds as payments. In addition, the government buyer spends all his money in the trade, and so the quantity of good he purchases is $M_g/(N_g P^m)$.[15]

(ii) A government seller. A government seller accepts only money for trade. In a trade match with a private money holder, a government seller sells goods only at the average price P^m, but leaves the quantity of goods for the private money holder to decide. Denote this quantity of goods as q^g.

Notice that individual agents, private or government, take the price P^m as given. Also, I abstract from the trades between two government agents, since such trades are inconsequential to private households' behavior.

The government's refusal to accept money as payments is a legal restriction. Let me clarify two aspects of this restriction. First, the legal restriction is only partially enforced, namely, only in trades involving government agents. When two private agents trade with each other, they can choose whether or not to use matured bonds as the payments. Second, the legal restriction requires only that bonds should not be used *directly* to pay for government goods. Of course, agents can redeem the bonds for money first and then use the receipt to pay for government goods. However, this indirect payment takes one period of time in the model.

I now modify a household's consumption to incorporate the trades with government agents:

$$c = \alpha \left\{ n \left[\sigma q^m + \sigma_g q^g \right] + \sigma \left(1 - \sigma - n \right) q^b \right\}. \tag{4.1}$$

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[13]To generate these matching probabilities, the aggregate number of matches is $\hat{\alpha}(\sigma+\sigma_g)(1-\sigma+N_g)/(1+\sigma_g+N_g)$, where $(1+\sigma+N_g)$ is the total number of agents in the market and $\hat{\alpha} > 0$ is a constant. The matching probabilities described here are generated under random matching and the normalization $\alpha = \hat{\alpha}/(1 + \sigma_g + N_g)$.

[14]Another way to model government agents' trading behavior is to specify the government's objective function. I do not take this approach because it is not clear what the appropriate objective of the government is.

[15]In order for a private seller to be willing to sell a quantity as high as $M_g/(N_g P^m)$, his surplus must be non-negative at this quantity, i.e., $\omega^m M_g/N_g \geq \psi \left(M_g/ \left(N_g P^m \right) \right)$. In a later proposition (Proposition 5.1), I will provide a condition under which this requirement is satisfied for all policy experiments conducted in this paper.

The household also has a net receipt of money from trades with the government, $\alpha(\sigma M_g - \sigma_g n P^m q^g)$. Incorporating this additional term, the law of motion of the household's money holdings becomes:

$$m_{+1} = m + (r - S)d + \alpha\sigma\left(NX^m - nx^m\right) + \alpha\left(\sigma M_g - \sigma_g n P^m q^g\right) + L. \tag{4.2}$$

The law of motion of the household's bond holdings is still (2.9) and the money constraint in the bonds market is still (2.7).

The household's additional choice is q^g, the quantity of goods that a private buyer asks a government seller to supply. The corresponding money constraint in such a trade is:

$$P^m q^g \le am/n. \tag{4.3}$$

Let λ^g be the Lagrangian multiplier of this constraint and multiply λ^g by the number of trades between a household's buyers and government sellers, $\alpha\sigma_g n$. The optimal choice of q^g satisfies:

$$u'(c) = (\omega^m + \lambda^g) P^m. \tag{4.4}$$

The quantities of goods and assets traded in matches between two private agents still satisfy (2.11) and (2.12). The amount of purchases of new bonds satisfies (2.15) and the redemption fraction satisfies (2.13). Moreover, the optimal choice of n still obeys (2.16).[16]

There are two revisions to the household's optimal conditions. First, when choosing the division of money between the two markets, a household takes into account the service generated by money in trades with government sellers, $\alpha\sigma_g\lambda^g$. So, the condition for a changes from (2.14) to the following equation:

$$\alpha(\sigma\lambda^m + \sigma_g\lambda^g) = \rho \text{ if } a \in (0,1). \tag{4.5}$$

Second, for the same reason, I revise the envelope condition for money as follows:

$$\omega_{-1}^m/\beta = \omega^m + (1-a)\rho + a\alpha\left(\sigma\lambda^m + \sigma_g\lambda^g\right). \tag{4.6}$$

With these revisions, I can adapt the previous definition of a symmetric equilibrium to the current economy. In the current environment, symmetry between households requires an additional condition:

$$P^m = p^m \equiv x^m/q^m. \tag{4.7}$$

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[16]To see this, note that a private money holder's surplus from a trade with a government seller, $u'(c)q^g - (\omega^m + \lambda^g) P^m q^g$, is zero under (4.4). Thus, a private buyer gets a positive surplus from trade entirely from matches with private sellers, just as a bond holder does. So, (2.16) continues to hold. Clearly, this result depends on the assumption that a government seller sells goods at a fixed price P^m. In addition to simplifying the algebra, this assumption strengthens the later result that matured bonds do not circulate in the goods market. If a money holder obtained a positive surplus from a trade with a government seller, as well as from a private seller, there would be an additional advantage to holding money relative to bonds.

As before, let γ be the gross rate of growth of the private sector's money holdings. Again, I focus on equilibria with $0 < a < 1$, $n > 0$, $0 < r \leq 1$ and $\gamma > \beta$. Then, the nominal interest rate is:

$$\frac{1}{S} = \frac{\gamma}{\beta} = 1 + \alpha \left(\sigma \frac{\lambda^m}{\omega^m} + \sigma_g \frac{\lambda^g}{\omega^m} \right). \tag{4.8}$$

The net nominal interest rate is positive for all $\gamma > \beta$. Also, the constraint (2.7) binds and implies $a = 1 - z\beta/\gamma$.

Notice that

$$P^m = x^m/q^m = \psi(q^m)/(\omega^m q^m) < \psi'(q^m)/\omega^m, \tag{4.9}$$

where the inequality follows from the fact that $\psi(q)/q < \psi'(q)$ for all $q > 0$. Then, (2.12) and (4.4) imply $\lambda^g > \lambda^m \geq 0$. That is, the money constraint between a private buyer and a government seller always binds. This yields:

$$q^g = \frac{am}{nP^m} = q^m \frac{am}{nx^m}. \tag{4.10}$$

If the money constraint also binds in a trade between two private agents, then $q^g = q^m$.

As in previous sections, monetary transfer keeps the money holdings per household growing at a constant rate γ. I also assume that the government's money holdings grow at the constant rate, γ, so that the ratio between private and government money holdings remains constant. This is achieved by money creation. Let T be the amount (per household) of money creation at the end of a period. (If $T < 0$, then T is the amount of money destroyed.) The government's budget constraint is:

$$T = (M_{g+1} - M_g) + [(r - S)zM + \alpha (\sigma M_g - \sigma_g nP^m q^g) + L]. \tag{4.11}$$

The term in [.] is equal to the change in the representative household's money holdings between two adjacent periods (see (4.2)). This change is $(\gamma - 1)M$. Therefore, to keep M_g and M both growing at γ, money creation is $T = (\gamma - 1)(M + M_g)$.

4.2. Matured Bonds Do Not Circulate in the Goods Market

Matured bonds do not circulate in this economy, as stated below (see Appendix A for a proof):

Proposition 4.1. *For $\gamma > \beta$, there is no stationary equilibrium where $0 < r < 1$.*

This proposition does not depend on the size of government sellers (σ_g), provided $\sigma_g > 0$. That is, even when the measure of government sellers is arbitrarily small, their presence in the goods market is sufficient to drive matured bonds out of the circulation. I take this result as a robust answer to the question posed in the introduction on why matured bonds do not circulate in the goods market, although the legal restriction is exogenously imposed in this model. I will argue that, with the same legal restriction, a Walrasian model will not deliver the same result. Before doing so, however, let me explain why Proposition 4.1 holds.

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Bonds exit from the goods market in this model because of decentralized exchanges and the government's trading policy. To explain this, let us examine two equilibrium requirements that must be met in order for matured bonds to circulate.

First, for matured bonds to circulate in the goods market, a household must be indifferent between money and matured bonds. This requires that the two assets have the same value, i.e., $\omega^m = \omega^b$. Since each asset derives a value from the services that it generates in the markets by relaxing the trade constraints, these services must be the same for the two assets. That is, $\alpha\sigma\lambda^b/\omega = \alpha\sigma\lambda^m/\omega + \alpha\sigma_g\lambda^g/\omega$, where ω is the common value of money and matured bonds. However, money can relax the trading constraint in more trades than bonds do. Facing this inferior chance of trade, bonds can have the same value as money only if they can relax the asset constraint in each trade to a greater extent than money does, i.e., only if $\lambda^b > \lambda^m$. Notice that the extent to which the asset constraint binds in a match is reflected by the quantity of goods traded in that match – a lower quantity of goods traded reflects a more severely binding constraint. Therefore, $\lambda^b > \lambda^m$ if and only if $q^b < q^m$.

Second, for matured bonds to circulate in the goods market, each household must also allocate a positive measure of members (or time) to trade using matured bonds. For this decision to be optimal, a bond holder must obtain an expected surplus from trade that is equal to or greater than what a money holder obtains. However, a bond holder encounters fewer trades than a money holder does, because government sellers refuse to accept bonds as payments. To satisfy the requirement on the expected surplus, a bond holder's surplus in each trade must be at least as high as a money holder's in a similar trade. Since a buyer's surplus increases in the quantity of goods in the trade, the requirement becomes $q^b \geq q^m$.[17] This contradicts the previous requirement $q^b < q^m$. Therefore, matured bonds do not circulate in the goods market.

The government's policy of selling goods for only money is necessary, but *not* sufficient, for the exit of bonds from the goods market. Decentralized exchanges are also important for the result. By implying an implicit (time) cost of trading, decentralized exchanges force a household to consider the number of trades, as well as the quantity in each trade, when allocating assets to the goods market. The government's refusal to bond payments drives a wedge between the trading costs of using money and matured bonds, and hence makes the two assets imperfect substitutes in the goods market. Faced with such imperfect substitutability, households redeem all matured bonds for money.

To appreciate the importance of decentralized trades, let me argue that the same legal restriction does not prevent bonds from circulating in a Walrasian goods market. To make the argument more general, let a household's utility function be $u(c, g)$, rather than $u(c + g)$, where g is consumption of government goods. So, government goods may be indispensable to the house-

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[17]The inequality need not be strict because the surplus a money holder obtains in a trade with a government seller is zero under the particular description of government agents' trading strategies, as discussed earlier.

hold, rather than being perfect substitutes for private goods. Let the price level of private goods be P^m and of government goods P^g, measured in terms of money. Then, the household faces the following trading constraints in the goods market:

$$P^m c + P^g g \leq m + b,$$

$$P^g g \leq m. \tag{4.12}$$

The constraint (4.12) does not bind when the household's expenditure on government goods does not exceed its money holdings. In this case, money and matured bonds both circulate as perfect substitutes for each other, despite the legal restriction. The government's insistence on money as the only means of payment for its goods does not drive bonds out of the Walrasian goods market; rather, it merely shifts money from purchases on private goods to government goods. Private agents can still use bonds to purchase goods from other private agents.[18]

Of course, if the household's expenditure on government goods is so high that it exceeds the household's money holdings, then (4.12) binds and matured bonds are not perfect substitutes for money in the goods market. In this case, a household will redeem all its holdings of matured bonds. Therefore, in a Walrasian market with the legal restriction, matured bonds do not circulate only if the government is sufficiently large or, more accurately, only if the coverage of the legal restriction is sufficiently wide. In contrast, in the economy with decentralized exchange, matured bonds do not circulate even if the coverage is arbitrarily small.

Proposition 4.1 may seem striking, especially when compared with the result in previous monetary search models that different assets can co-exist as media of exchange (e.g., Shi, 1995). However, matured nominal bonds are not just another asset; rather, they are assets whose exchange rate with money is fixed at one by the redemption process. Thus, what Proposition 4.1 states in general is that, when two assets can be costlessly exchanged at a fixed exchange rate of unity, the one with a less favorable chance of trade in the decentralized goods market will not circulate as a medium of exchange. This result seems easy to understand.

4.3. Robustness

In this subsection I show that Proposition 4.1 is robust to two variations of the model. The first uses a different assumption on when bonds can be redeemed and the second on how assets can be used in the goods market.

First, let me relax the assumption that matured bonds can be redeemed only when they just become matured. Instead, assume that agents can redeem matured bonds in any period at or

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[18]The above argument is robust to allowing for arbitrage between money and bonds, either indirectly through goods or directly. Let b' be the amount of matured bonds that the household holds after the arbitrage, with the constraint $b' \geq 0$. Then, (4.12) becomes $P^g g \leq m + (b - b')$. This constraint reduces back to (4.12) in any symmetric equilibrium, because $b' = b$. Also, the argument is robust to the arbitrage between private and government goods, which merely determines the relative price P^g / P^m.

after maturity. This alternative assumption seems to reduce the opportunity cost of carrying bonds to the goods market. If such bonds fail to trade for goods, they can be redeemed for money. Despite this enlarged role of bonds, Proposition 4.1 continues to hold. For an interior choice of r to be optimal, it must be true that $\omega^m = \omega^b$. Then, the same argument following Proposition 4.1 leads to a contradiction to $r \in (0,1)$ and $n \in (0, 1-\sigma)$.

Now I retain the assumption that bonds can only be redeemed at maturity, but relax the assumption that money and bonds must be used in separate matches. Instead, assume that a buyer can carry money and bonds together into each match. In this alternative setup, a household does not need to divide buyers into money holders and bond holders. Rather, each buyer carries $(am + b)/(1 - \sigma)$ units of assets into the goods market.[19]

To show that Proposition 4.1 still holds, suppose to the contrary that there is a symmetric equilibrium with $r \in (0,1)$. Then, $\omega^m = \omega^b$. Denote this common value as ω. I show that this leads to a contradiction.

To proceed, I need to revise the asset constraints in trade. Use superscripts mb to indicate a trade match between two private agents (where the buyer can use both m and b), and a superscript g to indicate a trade match between a private buyer and a government seller (where the buyer can only use money). The asset constraint facing the buyer in these two matches is, respectively,

$$\frac{am + b}{1 - \sigma} \geq x^{mb} = \frac{\psi(q^{mb})}{\Omega}, \tag{4.13}$$

$$\frac{am}{1 - \sigma} \geq x^g = \frac{\psi(q^g)}{\Omega}. \tag{4.14}$$

Let λ^{mb} be the Lagrangian multiplier of (4.13) and λ^g of (4.14). Similar to (2.12), the optimal choice of q^{mb} satisfies:

$$u'(c) = (\omega + \lambda^{mb})\psi'(q^{mb})/\Omega.$$

The quantity q^g still satisfies (4.4), once the average price is redefined as $P^{mb} = X^{mb}/Q^{mb}$. As before, $\lambda^g > \lambda^{mb}$ and so $\lambda^g > 0$. The envelope conditions for money and matured bonds require that the two assets yield the same service in the market. When $\omega^m = \omega^b$, as supposed, this requirement becomes $\alpha\sigma\lambda^{mb} = \alpha(\sigma\lambda^{mb} + \sigma_g\lambda^g)$, which contradicts $\lambda^g > 0$. Therefore, there is no symmetric equilibrium where matured bonds circulate in the goods market.

5. Equilibrium and Policy Effects

Now that the only stationary equilibrium is one in which bonds do not circulate, it is useful to characterize the equilibrium and examine its properties. This equilibrium has $r = 1$, $n = 1 - \sigma$ and $b = 0$.

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[19]There is no need to distinguish money and bonds in a buyer's holdings because they will be identical to each other for any trading partner and because any matured bonds that are not traded away in this period can be redeemed at par for money.

5.1. Characterization

There are two cases of an equilibrium, depending on whether the money constraint in a trade between a private money holder and a private seller binds, i.e., whether (2.2) binds. If this constraint does not bind, then $\lambda^m = 0$; if this constraint binds, then $\lambda^m > 0$.

Case 1. $\lambda^m = 0$: In this case, $q^g \geq q^m$ (see (4.10)) and $\psi'(q^m) = u'(c)$ (see (2.12)). Denote q^m in this case as q_1, q^g as q_1^g, and c as c_1. Then,

$$c_1 = u'^{-1}\left(\psi'(q_1)\right), \quad q_1^g = \frac{1}{\sigma_g}\left[\frac{c_1}{\alpha(1-\sigma)} - \sigma q_1\right].$$

To solve for q_1, use (4.8), (4.4) and (4.9) to derive the following equation:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma_g\left(\frac{q_1\psi'(q_1)}{\psi(q_1)} - 1\right). \tag{5.1}$$

Case 2. $\lambda^m > 0$: In this case, $q^g = q^m$ (see (4.10)) and $\psi'(q^m) < u'(c)$ (see (2.12)). Denote q^m in this case as q_2, q^g as q_2^g, and c as c_2. Then,

$$q_2^g = q_2, \quad c_2 = \alpha(1-\sigma)(\sigma + \sigma_g)q_2.$$

To solve for q_2, use (4.8), (2.12), (4.4) and (4.9) to derive the following equation:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c_2)}{\psi'(q_2)} - 1\right) + \alpha\sigma_g\left(\frac{u'(c_2)q_2}{\psi(q_2)} - 1\right). \tag{5.2}$$

To describe the existence of the equilibrium, define q_0, γ_0 and $\underline{\gamma}$ as follows:

$$\psi'(q_0) = u'(\alpha(1-\sigma)(\sigma + \sigma_g)q_0), \tag{5.3}$$

$$\frac{\gamma_0}{\beta} - 1 = \alpha\sigma_g\left(\frac{q_0\psi'(q_0)}{\psi(q_0)} - 1\right), \tag{5.4}$$

$$\underline{\gamma} = \beta\left[1 + \alpha\sigma_g\left(\Psi - 1\right)\right] \text{ and } \Psi \equiv \lim_{q \to 0} q\psi'(q)/\psi(q). \tag{5.5}$$

In both cases of the equilibrium, $\lambda^m \geq 0$, which requires $\psi'(q) \leq u'(c)$. Thus, the equilibrium must satisfy $q \leq q_0$ in both cases. The following proposition establishes the existence of the equilibrium (see Appendix B for a proof).

Proposition 5.1. *Assume that $M_g/N_g < m/(1-\sigma)$ and that $q\psi'(q)/\psi(q)$ is a non-decreasing function for all $q > 0$. A stationary equilibrium with $r = 1$ exists iff $\gamma \geq \underline{\gamma}$ and $0 < z < \frac{\gamma}{\beta}\left(1 - \frac{M_g/N_g}{m/(1-\sigma)}\right)$. The equilibrium is unique if $\gamma > \underline{\gamma}$. Moreover, if $[q\psi'(q)/\psi(q)]$ is a strictly increasing for all $q > 0$, then $\underline{\gamma} < \gamma_0$ and both cases can occur: Case 1 occurs for $\underline{\gamma} < \gamma \leq \gamma_0$ and Case 2 for $\gamma > \gamma_0$. If $[q\psi'(q)/\psi(q)]$ is a constant, then $\gamma_0 = \underline{\gamma}$ and only Case 2 can occur (for $\gamma > \underline{\gamma}$).*

The condition $M_g/N_g < m/(1 - \sigma)$ and the upper bound on z are required to ensure that a private seller is willing to accept the entire holdings of money by a government buyer, an assumption that I have maintained implicitly. To understand the assumption that the function $[q\psi'(q)/\psi(q)]$ is non-decreasing in q, note that the real price of goods is $\omega^m P^m = \psi(q)/q$ (see (4.9)). Thus, the assumption requires that the marginal cost of production be at least as sensitive to changes in the quantity as the price function is. This assumption seems reasonable and it is satisfied by a wide class of functional forms including the common specification $\psi(q) = q^\Psi(\psi_0 + \psi_1 q)$, where $\Psi > 1$, $\psi_0 > 0$, and $\psi_1 > 0$.
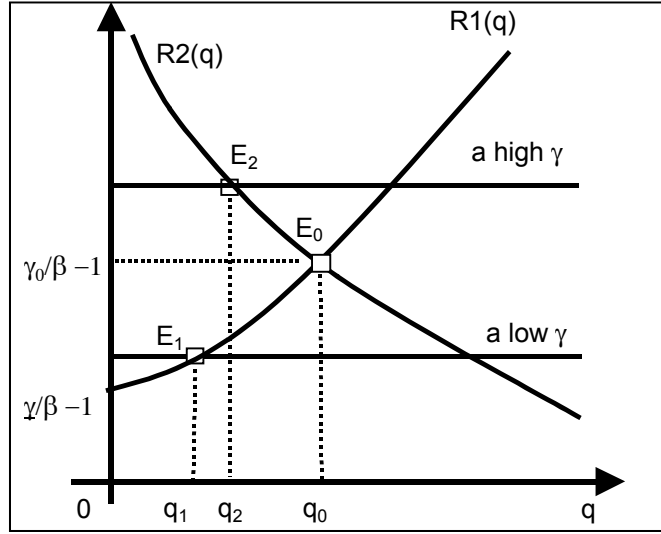

Figure 2.

Figure 2 depicts the two cases of the equilibrium for a strictly increasing function $[q\psi'(q)/\psi(q)]$. The two horizontal lines are $(\gamma/\beta - 1)$ at two levels of γ. The curve $R1(q)$ depicts the right-hand side of (5.1), which is increasing when $[q\psi'(q)/\psi(q)]$ is increasing. The curve $R2(q)$ depicts the right-hand side of (5.2), which is always decreasing. The two curves intersect at point E_0, where $\gamma = \gamma_0$ and $q = q_0$. As stated earlier, the equilibrium level of q must satisfy $0 < q \leq q_0$ in both cases. When $\gamma \leq \underline{\gamma}$, no such solution for q exists. When $\underline{\gamma} < \gamma \leq \gamma_0$, the solution is depicted by a point like E_1, which is Case 1. When $\gamma > \gamma_0$, the solution is depicted by a point like E_2, which is Case 2.[20]

An interesting feature of the equilibrium is that the money constraint does not always bind in a trade between two private agents, although the net nominal interest rate is positive. When trading with a government seller, a private buyer faces a binding constraint on money, provided $\gamma > \beta$. However, when trading with a private seller, a private buyer's money constraint binds only when $\gamma > \gamma_0$. This is a consequence of the assumption that a government seller always sells at the price P^m, while a private seller takes into account the increasing marginal cost of

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[20]If $[q\psi'(q)/\psi(q)]$ is constant for all $q > 0$, then the curve $R1(q)$ becomes a horizontal line going through E_0, and so an equilibrium exists only for $\gamma = \underline{\gamma}$.

production. This difference implies that a private seller sells a smaller quantity of goods than a government seller does, i.e., $q^m < q^g$. Since a private buyer has the same amount of money when trading with these two types of sellers, he is more likely to face a binding money constraint when trading with a government seller than with a private seller.

In both cases of the unique equilibrium, $b = 0$ (since $r = 1$). The fraction of money taken to the goods market is $a = 1 - z\beta/\gamma$. The price level, and the value of money are, respectively,

$$P^m = \frac{am}{(1-\sigma) q^g}, \quad \omega^m = \frac{(1-\sigma) \psi(q^m)}{am}. \tag{5.6}$$

5.2. Effects of Monetary Policy

I examine two types of monetary policy – an increase in the money growth rate and an increase in the amount of new bond sales. As stated before, the government's budget constraint is always satisfied and the ratio M_g/M is constant. To simplify the exposition here, let me assume that $[q\psi'(q)/\psi(q)]$ is a strictly increasing function, so that both Case 1 and Case 2 of the equilibrium can occur in different intervals of the money growth rate. Before examining the effects of the policy, it is useful to distinguish private and public consumption/output, as listed below:

	consumption	output
private	$c = \alpha (1 - \sigma) (\sigma q^m + \sigma_g q^g)$;	$y = \alpha \sigma (1 - \sigma) \left(q^m + q^g \frac{M_g}{am} \right)$
government	$c_g = \alpha \sigma (1 - \sigma) q^g \frac{M_g}{am}$;	$y_g = \alpha (1 - \sigma) \sigma_g q^g$
aggregate	$c + c_g = y + y_g = \alpha (1 - \sigma) \left(\sigma q^m + \sigma_g q^g + \sigma q^g \frac{M_g}{am} \right)$.	

Examine first the effects of an increase in the money growth rate γ. An increase in the money growth rate affects q^m in cases 1 and 2 in opposite directions. This can be seen from Figure 2, where the solution for q^m is q_1 for $\gamma < \gamma_0$ (Case 1) and q_2 for $\gamma > \gamma_0$ (Case 2). When $\gamma < \gamma_0$, the money constraint does not bind in a trade between two private agents, but binds in a trade between a private money holder and a government seller. An increase in γ shifts the purchasing power of money from the binding trades with government sellers to the non-binding trades between private agents, thus increasing q^m. When $\gamma > \gamma_0$, the money constraint binds in all trade matches. By reducing a buyer's real balance, an increase in γ reduces q^m. Therefore, the amount of goods traded between two private agents increases with γ for $\gamma < \gamma_0$ and decreases with γ for $\gamma > \gamma_0$. At $\gamma = \gamma_0$, the quantity of goods in such trades reaches the maximum.

Despite the above non-monotonicity, an increase in the money growth rate always reduces the quantity of goods traded between a private buyer and a government seller, q^g, and hence reduces government output. This is because money growth reduces the real money balance and tightens the binding money constraint in such a trade. Moreover, the reduction in q^g dominates the change in private consumption. That is, even if a higher money growth rate increases the quantity of goods traded between private agents, q^m, it always reduces private consumption.

Higher money growth also reduces government consumption, through a price effect. To understand this effect, notice that an increase in money growth increases a, the fraction of money that a household takes to the goods market. This re-allocation of money drives up the normalized price level, P^m/M; that is, the price level increases by more than the increase in the money stock. As a result, a government buyer can buy less with his money holdings, which reduces government consumption.

Because private consumption and government consumption both fall with a higher money growth rate, both consumption and output fall at the aggregate level.

In contrast, private output may either rise or fall. Private output consists of two parts. The first part is goods sold to private agents and the second part is goods sold to government agents. Because the first part responds to money growth non-monotonically and the second part always decreases with money growth, the overall response of private output to money growth depends on both the level of money growth and the ratio between the two parts of private output. If the ratio of government money holdings to private holdings is low, i.e., if M_g/m is significantly smaller than $a\sigma_g/\sigma$, then most of private output is sold to private agents. In this case, private output responds to money growth in a hump-shaped pattern, although the hump does not necessarily occur at γ_0. If $M_g/m \geq a\sigma_g/\sigma$, private output decreases with money growth for all $\gamma > \underline{\gamma}$.

I now examine a tightening open market operation, modelled as an increase in z with corresponding changes in transfers to maintain the money growth rate at γ. A tightening operation shifts money from the goods market to the bonds market; i.e., it reduces a. Given the money growth rate, however, this shift has no effect on (q^m, q^g). So, private consumption and government output remain the same as before. However, the price level falls because there is less money in the goods market. The lower price enables the government to purchase more goods from the private sector, leading to higher private output and aggregate output.

These real effects were absent in the previous model where the government does not participate in the goods market. They are also absent in models where the goods market is Walrasian, e.g., the certainty counterpart of Lucas (1990). There, a fully anticipated increase in open market operations changes only the price level and other nominal variables, provided that monetary transfers keep the money growth rate constant.

Noticing that government consumption increases to respond to the tightening operation in this model, one may argue that a standard cash-in-advance model can also generate real effects of the operation if government spending is assumed to increase with the tightening open market operation. The question, of course, is why government spending necessarily increases to accompany a tightening operation. The current model provides a mechanism that is consistent with the model's assumption that the government uses only money to purchase goods. Moreover, government spending in a standard model does not generate the same effects as in the current model. For example, government spending is likely to crowd out private consumption in a standard model,

which does not occur in the current model.

6. Conclusion

In this paper I construct two search models to analyze the competition between fiat money and default-free nominal bonds. In both models, newly issued bonds are sold at a discount for money and thus they bear positive interest. However, the two models have different predictions on whether nominal bonds will circulate as a medium of exchange. In the first model, the government does not participate in the goods market. In this case, there are a continuum of equilibria in which matured bonds circulate in the goods market as perfect substitutes for money. In the second model, the government participates in the goods market and refuses to accept money as payments, although private agents can choose to trade among themselves using both money and bonds. In this case, there is a unique monetary equilibrium. In this equilibrium, matured bonds do not circulate, even when the measure of government sellers in the goods market is arbitrarily small. This result provides an answer to the question why matured bonds do not act as a medium of exchange. In contrast, the same legal restriction in the conventional (Walrasian) goods market can drive bonds out of the circulation as money only if the restriction is sufficiently large.

The two models yield different policy effects. Take, for example, a permanent increase in the issuing of bonds in the open market, accompanied by a change in the lump-sum monetary transfer to maintain the growth rate of the money stock constant. In both models, the open market operation shifts money from the goods market to the bonds market. In the first model, this shift of money between the two markets has no effect on real output/consumption or the nominal interest rate. If households do not change the redemption fraction of matured bonds, the tightening operation increases the bond-money ratio and *increases* the total amount of media of exchange in the goods market, thus increasing the price level. If households change the redemption fraction to respond to the open market operation, the response of the price level is difficult to predict. In the second model, however, the shift of money between the two markets reduces the price level. Also in contrast to the first model, real output and government consumption increase with the tightening operation, through a price effect, while private consumption and government output remain unchanged.

Money growth also has different effects in the two models. When bonds circulate as a medium of exchange (i.e., in the first model), higher money growth reduces real output and consumption. When bonds do not circulate (i.e., in the second model), real output rises with money growth if the money growth rate is initially low, and falls when the money growth rate is high. However, higher money growth increases the nominal interest rate in both models.

The results here cast doubts on the policy effects obtained in traditional models that assume money as the unique medium of exchange. When there is no restriction on bonds as payments

for goods, as in the first model here, bonds will circulate as money and a continuum of equilibria will arise. The continuum presents a clear difficulty for predicting the effects of monetary policy. To avoid the continuum of equilibria, one can introduce the legal restriction in the goods market, as is done in the second model in this paper. Although the legal restriction does induce the outcome that bonds do not circulate as money, an outcome that is exogenously imposed in conventional monetary models, the explicit model reveals some policy effects that differ from those in conventional monetary model. For example, an increase in bond issuing leads to higher private output through a price effect.[21]

As stated in the introduction, one purpose of this paper is to integrate the microfoundation of monetary theory with the fruitful analysis of Lucas (1990). The resulting framework combines a decentralized goods market with a centralized bonds market. It is easy to see that this framework can be used to examine any centrally traded asset, not just nominal bonds. However, there is still quite a distance to go toward a full integration of the two classes of models. In two other papers, I have extended the framework in two directions. The first is to allow bonds to have longer maturities (see Shi, 2003a). This extension does not alter the results on matured bonds, because the same analysis shows that a small legal restriction can succeed in driving matured bonds out of the circulation. However, the legal restriction does not necessarily drive unmatured bonds out of the circulation. If unmatured bonds circulate in the goods market, they are imperfect substitutes for money. This introduces deeper discounting on newly issued, long-term bonds, and hence generates a non-trivial term structure of interest rates. The second extension is to make the model stochastic (see Shi, 2003b). There, I examine the liquidity effect of (stochastic) open market operations on nominal interest rates, as Lucas (1990) does. Since the operations also have non-trivial real effects like those in the second model here, there is an interesting interaction between these real effects and the liquidity effect.

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[21]Using a model of spatial separation, Williamson (2002) illustrates the sensitivity of the results in limited participation models to the introduction of private money.

A. Proof of Proposition 4.1

For $0 < r < 1$ to be consistent with an equilibrium, matured bonds must be perfect substitutes for money in the private goods market, i.e., $\omega^b = \omega^m$ (see (2.13)). The constraint (2.4) must bind, i.e., $\lambda^b > 0$; otherwise (2.18) would require $\omega_{-1}^m/(\beta\omega^m) = 1$, which would in turn require $\gamma = \beta$. For $\lambda^b > 0$, it must be true that $n < 1 - \sigma$, because $b/(1 - \sigma - n) = \infty$ otherwise. The choice $n \in (0, 1 - \sigma)$ is optimal iff $q^m = q^b$ (see (2.16)). Denote this common value of q^m and q^b as q. Substituting λ^b from (2.12) into (2.18) and using $\omega_{-1}^m/\omega^m = \gamma$, I have:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c)}{\psi'(q)} - 1\right). \tag{A.1}$$

Clearly, $\psi'(q) < u'(c)$ for all $\gamma > \beta$. So, (2.12) implies $\lambda^m > 0$, which in turn implies $q^g = q$ by (4.10). Substituting λ^m from (2.12) and λ^g from (4.4) into (4.8), I get:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c)}{\psi'(q)} - 1\right) + \alpha\sigma_g\left(\frac{u'(c)q}{\psi(q)} - 1\right). \tag{A.2}$$

Because $\psi(q)/q < \psi'(q) < u'(c)$, (A.1) and (A.2) cannot both hold. Therefore, there does not exist a stationary equilibrium with $0 < r < 1$. **QED**

B. Proof of Proposition 5.1

Let me first validate the maintained assumption that a private seller in a match with a government buyer is always willing to produce enough goods to exchange for all the money the government buyer holds. This behavior is indeed optimal for the private seller iff $\omega^m M_g/N_g \geq \psi(M_g/(N_g P^m))$. Substituting $P^m = \psi(q^m)/(\omega^m q^m)$, I write the required condition as $k \geq \psi(kq^m/\psi(q^m))$, where $k = \omega^m M_g/N_g$. For any given $A > 0$, let $k^*(A)$ be the positive solution to the equation $k = \psi(kA)$. It is easy to verify that this solution exists and is unique. Moreover, $k \geq \psi(kA)$ iff $k \leq k^*(A)$. Notice that, when $A = q^m/\psi(q^m)$, $k = \psi(q^m)$ satisfies the equation $k = \psi(kq^m/\psi(q^m))$. Since the solution $k^*(A)$ is unique for given A, then $k^*(q^m/\psi(q^m)) = \psi(q^m)$. Therefore, it is optimal for a private seller to produce enough goods for all the money that a government buyer has if and only if $\omega^m M_g/N_g \leq \psi(q^m)$. Substituting the equilibrium results $\omega^m = (1-\sigma)\psi(q^m)/(am)$ and $a = 1 - z\beta/\gamma$, I can rewrite this condition as

$$z \leq \frac{\gamma}{\beta}\left(1 - \frac{M_g/N_g}{m/(1-\sigma)}\right).$$

This upper bound on z is imposed in Proposition 5.1, together with the condition that ensures its positivity, $M_g/N_g < m/(1-\sigma)$. Note that $a = 1 - z\beta/\gamma > 0$.

Next, I show that the solutions to (5.1) and (5.2) exist under the conditions described in the proposition. Notice that a unique solution to (5.2), denoted q_2, exists for all $\gamma > \beta$. In contrast, a unique (positive) solution to (5.1), denoted q_1, exists only when $\gamma \geq \underline{\gamma}$ and when $q\psi'(q)/\psi(q)$

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is strictly increasing for all $q > 0$. If $q\psi'(q)/\psi(q)$ is constant over q, then (5.1) is satisfied only at $\gamma = \underline{\gamma}$, in which case $\gamma_0 = \underline{\gamma}$ and $q = q_0$. To determine which of the two cases characterizes the equilibrium, recall that the equilibrium requires $0 < q \leq q_0$. Suppose first that $[q\psi'(q)/\psi(q)]$ is strictly increasing for all $q > 0$. For $\gamma \leq \gamma_0$, the solutions to (5.1) and (5.2) satisfy $q_1 \leq q_0 < q_2$, and so Case 1 characterizes the equilibrium. For $\gamma > \gamma_0$, the solutions to (5.1) and (5.2) satisfy $q_2 < q_0 < q_1$, and so Case 2 characterizes the equilibrium. Suppose now that $[q\psi'(q)/\psi(q)]$ is constant over $q > 0$. Then, at $\gamma = \underline{\gamma}$, the equilibrium is Case 1 with $q = q_0$. For $\gamma > \underline{\gamma}$, the equilibrium is given by Case 2.

To show that these solutions constitute a unique equilibrium, I need to verify that an individual household does not have incentive to deviate to $r < 1$ when all other households choose $r = 1$, $n = 1 - \sigma$, and trade according to the described quantities. Consider the following deviation by an individual household. The household keeps an amount, ε, of bonds that matured in the *previous* period and uses them to trade for goods in the current period. All other households continue to use the equilibrium strategies and so, if they receive any bond in the trade, they will value it at $\Omega^b < \Omega^m$.

In order to trade bonds for goods, the deviating household must allocate some buyers to carry the bonds to the goods market. Let Δ be the measure of the household's buyers who carry the bonds into the goods market, each carrying an amount ε/Δ. Each of these bond holders has a trade match with a private seller with probability $\alpha\sigma$.

The deviation yields a benefit from potential trades with private sellers. When meeting a private seller in a trade match, a bond holder can offer the amount of bonds for q_ε units of goods. Because the seller will accept the trade as long as $\Omega^b \varepsilon/\Delta \geq \psi(q_\varepsilon)$, the bond holder will offer q_ε such that $\Omega^b \varepsilon/\Delta = \psi(q_\varepsilon)$, i.e., $q_\varepsilon = \psi^{-1}\left(\Omega^b \varepsilon/\Delta\right)$. Because each bond holder gets such a trade with probability $\alpha\sigma$, the total number of such trades is $\alpha\sigma\Delta$ and the total utility generated from such trades is $\alpha\sigma\Delta u'(c)q_\varepsilon$. The amount of bonds that the household fails to trade away is $(1 - \alpha\sigma)\Delta(\varepsilon/\Delta) = (1 - \alpha\sigma)\varepsilon$. Thus, the total benefit of the deviation is

$$\alpha\sigma\Delta u'(c)q_\varepsilon + (1 - \alpha\sigma)\varepsilon\omega^b.$$

The deviation has two costs. The first cost is the value of money that the household would have if it did not deviate in the previous period. In that case, the amount ε of bonds would be redeemed for ε units of money, which would have a value $\varepsilon\omega^m_{-1}/\beta$. The second cost is that allocating Δ buyers to hold bonds takes them away from trading with money. Because each trade by a money holder generates a surplus $u'(c)[q^m - \psi(q^m)/\psi'(q^m)]$ (recall that the surplus from trading with a government seller is zero), the second cost of the deviation is $\alpha\sigma\Delta$ times this amount. Therefore, the total cost of the deviation is

$$\varepsilon\frac{\omega^m_{-1}}{\beta} + \alpha\sigma\Delta u'(c)\left(q^m - \frac{\psi(q^m)}{\psi'(q^m)}\right).$$

The net gain from the deviation, divided by Δ, is less than the following amount:

$$\alpha\sigma u'(c)q_\varepsilon - \left(\frac{\gamma}{\beta} - 1 + \alpha\sigma\right)\psi(q_\varepsilon) - \alpha\sigma u'(c)\left(q^m - \frac{\psi(q^m)}{\psi'(q^m)}\right),$$

where I have substituted $\omega^b\varepsilon/\Delta = \psi(q_\varepsilon)$, $\omega^b < \omega^m$, and $\omega_{-1}^m/\omega^m = \gamma$. Denote the above expression temporarily as $f(q_\varepsilon)$. For the deviation to be not profitable, it is sufficient that $f(q_\varepsilon) \le 0$ for all $q_\varepsilon \ge 0$. Since f is concave, it is maximized at q_ε^* which solves:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c)}{\psi'(q_\varepsilon^*)} - 1\right).$$

Using this definition to substitute $(\gamma/\beta - 1)$ in $f(q_\varepsilon^*)$, I can show that $f(q_\varepsilon^*) \le 0$ iff $q_\varepsilon^* - \psi(q_\varepsilon^*)/\psi'(q_\varepsilon^*) \le q^m - \psi(q^m)/\psi'(q^m)$. Because $[q - \psi(q)/\psi'(q)]$ is an increasing function of q, $f(q_\varepsilon^*) \le 0$ iff $q_\varepsilon^* \le q^m$. In turn, $q_\varepsilon^* \le q^m$ iff

$$\frac{\gamma}{\beta} - 1 \ge \alpha\sigma\left(\frac{u'(c)}{\psi'(q^m)} - 1\right).$$

This is satisfied in both cases 1 and 2 (see (5.1) and (5.2)). Therefore, the deviation is not profitable. **QED**

C. A Different Assumption on Monetary Transfer

In the text I have assumed that, when there is an open market operation, the government adjusts monetary transfer to keep the money growth rate unchanged. This helped me isolating the effects of the open market operations from the effects of a change in money growth. Still, one may want to know the features of equilibria when the transfer does not neutralize the effect of the operation on money growth. I examine this alternative transfer process in this appendix. To economize on space, I analyze only the economy where the government does not participate in the goods market (i.e., the one examined in section 3).

Let monetary transfer be a constant fraction, l, of the money stock. The price of newly issued bonds is still $S = \beta/\gamma$. The equations (3.3), (3.4) and (3.5) continue to hold. The gross rate of money growth, still defined as $\gamma = m_{+1}/m$, is now endogenous. With $L/m = l$, (3.6) implies:

$$\gamma - 1 - l = z\left(r - \frac{\beta}{\gamma}\right). \tag{C.1}$$

This equation solves for γ for given (l, z, r). In the following proposition I detail the existence of the solution and the implied properties of the equilibria. A discussion on the proposition will follow the proof.

Proposition C.1. *Assume $z > 0$ and $l > -1$. The condition $l > (\beta - r)z$ is necessary and sufficient for there to be a solution for γ that satisfies $\gamma > 1$ and induces $a > 0$. Under this*

condition, the solution for γ is unique for given (l, z, r). There are a continuum of equilibria that differ from each other in the redemption fraction r. Between equilibria, the following properties hold:

$$\frac{d\gamma}{dr} > 0, \ \frac{dS}{dr} < 0, \ \frac{da}{dr} > 0, \ \frac{dn}{dr} > 0, \ \frac{dq}{dr} < 0, \ \frac{d(b/m)}{dr} < 0.$$

In each equilibrium, i.e., for each given r, an increase in z reduces the money growth rate γ if r is not too large. In this case,

$$\frac{dS}{dz} > 0, \ \frac{da}{dz} < 0, \ \frac{dn}{dz} < 0, \ \frac{dq}{dz} > 0, \ \frac{d(b/m)}{dz} > 0.$$

Also, if either u is sufficiently concave or ψ sufficiently convex, then p/m is lower in an equilibrium with a higher r than with a lower r, and an increase in z increases p/m.

Proof. Rewrite (C.1) as follows:

$$\gamma^2 - (1 + l + rz)\gamma + \beta z = 0. \tag{C.2}$$

Temporarily denote the left-hand side as $f(\gamma)$. Recall that $a > 0$ iff $\gamma > \beta z$. Also, the bond-money ratio is interior iff $\gamma > 1$. So, I look for the solution that satisfies $\gamma > \gamma_A \equiv max\{1, \beta z\}$.

Notice that $f(1) = (\beta - r)z - l$ and $f(\beta z) = \beta z[(\beta - r)z - l]$. So, $f(\gamma_A) < 0$ iff $l > (\beta - r)z$. Also, by computing $f'(1)$ and $f'(\beta z)$, I have $f'(\gamma_A) = 2\gamma_A - 1 - l - rz$. Consider the case $l \leq (\beta - r)z$ first. In this case, $f(\gamma_A) \geq 0$. Also,

$$f'(\gamma_A) = 2\gamma_A - 1 - l - rz \geq 2\gamma_A - 1 - \beta z \geq 0.$$

This implies that, for all $\gamma > \gamma_A$, $f(\gamma) > f(\gamma_A) \geq 0$. There is no solution to $f(\gamma) = 0$ that satisfies $\gamma > \gamma_A$. Thus, a necessary condition for the desired solution to exist is $l > (\beta - r)z$.

The condition $l > (\beta - r)z$ is also sufficient for a unique solution $\gamma > \gamma_A$ to exist. To see this, note that $f(\gamma_A) < 0$ when $l > (\beta - r)z$. Since $f(\gamma)$ is a convex, quadratic expression, its two roots lie on the opposite of γ_A. The unique solution that satisfies $\gamma > \gamma_A$ is

$$\gamma = \frac{1}{2}\left\{1 + l + rz + \left[(1 + l + rz)^2 - 4\beta z\right]^{1/2}\right\}.$$

It is easy to verify that this solution satisfies $d\gamma/dl > 0$ and $d\gamma/dr > 0$. Under this solution for γ, there are a continuum of values of r each leading to an equilibrium.

Because $d\gamma/dr > 0$, (3.2) implies $dS/dr < 0$; (3.5) implies $d(b/m)/dr < 0$; (3.7) implies $dn/dr > 0$; (3.4) implies $da/dr > 0$; and (3.3) implies $dq/dr < 0$. After substituting a from (3.4), n from (3.7) and $x^b = am/n$, the price level of goods normalized by the money stock is

$$\frac{p}{m} = \frac{1}{(1 - \sigma)q(\gamma - 1)}\left[\gamma - 1 + (1 - r - \beta)z + \frac{\beta z}{\gamma}\right].$$

From (C.2), I get $\beta z / \gamma = 1 + l + rz - \gamma$. Thus,

$$\gamma - 1 + (1 - r - \beta)z + \frac{\beta z}{\gamma} = l + (1 - \beta)z.$$

Therefore,

$$\frac{p}{m} = \frac{l + (1 - \beta)z}{(1 - \sigma)(\gamma - 1)q}.$$

The normalized price level falls with r iff $(\gamma - 1)q$ increases with r. This requires that either the cost function ψ be sufficiently convex or the utility function u be sufficiently concave. More precisely, computing $dq/d\gamma$ from (3.3), I have:

$$\frac{d(p/m)}{dr} < 0 \iff \frac{q\psi''}{\psi'} + \frac{(-u'')c}{u'} > \frac{\gamma - 1}{\gamma - \beta + \alpha\sigma\beta}.$$

The effects of z can be analyzed similarly. From the above solution for γ, I have $d\gamma/dz < 0$ iff $\gamma < \beta/r$. Because γ is an increasing function of r, then $\gamma < \beta/r$ is satisfied for at least small r (and possibly for all $r \leq 1$.) Thus, for small r, $d\gamma/dz < 0$. In this case, (3.2) implies $dS/dz > 0$; (3.5) implies $d(b/m)/dz > 0$; (3.7) implies $dn/dz < 0$; (3.4) implies $da/dz < 0$; and (3.3) implies $dq/dz > 0$. The normalized price level increases with z if either the cost function ψ or the utility function u has sufficient curvature. **QED**

As in section 3, there are still a continuum of equilibria distinguished from each other in the redemption fraction. Also similar to section 3, different equilibria have the following differences. First, a higher redemption fraction reduces the ratio of matured bonds to money in the economy. That is, the total moneyness in the goods market, normalized by the money stock, is lower when households redeem a larger fraction of matured bonds. Thus, a higher redemption fraction exerts a downward pressure on the price level normalized by the money stock. Second, a higher redemption fraction induces each household to allocate more buyers to carry money, in order to absorb the increased money stock relative to bonds.

In contrast to section 3, a higher redemption fraction increases the money growth rate here, and hence generates the following additional effects. First, the bond price is lower and the nominal interest rate higher. Second, the fraction of money taken to the goods market is higher. Third, the quantity of goods exchanged in a trade is lower, because higher money growth reduces the value of the real money balance. The last effect provides an upward pressure on the price level, but it is weak when the utility function is sufficiently concave or the cost function sufficiently convex. In this case, the dominating effect on the price level is the reduction in the total moneyness. That is, a higher redemption fraction reduces the price level normalized by the money stock, as in section 3.

A tightening open market operation has effects similar to those in section 3. In particular, the bond-money ratio rises to raise the total moneyness in the goods market and the fraction

of buyers (money holders or money holders) falls. However, because the operation reduces the money growth rate, the nominal interest rate falls and the quantity of goods in a trade increases. Again, the change in the quantity of goods affects the price level only slightly, and so the price level normalized by the money stock increases with the operation.

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