Money as a Mechanism in a Bewley Economy*†

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Abstract

We study what features an economic environment might possess, such that it would be Pareto efficient for the exchange of goods in that environment to be conducted on spot markets where those goods trade for money. We prove a conjecture that is essentially due to Bewley [1980, 1983]. Monetary spot trading is nearly efficient when there is only a single perishable good (or a composite commodity) at each date and state of the world; random shocks are idiosyncratic, privately observed, and temporary; markets are competitive; and the agents are very patient. This result is a fairly close analogue, for trade using outside, fiat money, of a recent characterization by Levine and Zame [2002] of environments in which spot trade using inside money, in the form of one-period debt payable in a commodity, is nearly Pareto efficient. We also study examples where nonmonetary and expansionary monetary mechanism Pareto dominates laissez-faire or contractionary monetary mechanism in an environment with impatient agents.

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1. Introduction

The research reported here is prompted by the debate in monetary economics regarding Friedman's [1969] provocative suggestion that an optimal monetary policy should generate negative seignorage. While Friedman does not necessarily equate optimality of a policy with ex ante Pareto efficiency of equilibrium under the policy, subsequent research has made this identification.

This debate has focused on whether a rate of negative seignorage as extreme as recommended by Friedman would be compatible with existence of an equilibrium (cf. Hellwig [1982] and Bewley [1983]), and on whether the opportunity for self-insurance that is Friedman's grounds for his recommendation is overshadowed by the loss of some insurance that inflation implicitly provides (cf. Levine [1991])[1] or the suboptimal incentives for agents on both sides of a market to expend effort in a search for trading partners (cf. Shi [1995]). However, the various critics of Friedman's proposal seem to share an implicit assumption that one should look for an efficient *monetary* mechanism, rather than looking for an efficient mechanism within the potentially broader class of mechanisms that fit the environmental constraints that the use of money suggests must exist.

Our work stands in contrast to this tradition of ignoring nonmonetary mechanisms. We admit such mechanisms, and study what features an economic environment might possess, such that it would be Pareto efficient for the exchange of goods in that environment to be conducted on spot markets where those goods trade for money. We prove a conjecture that is essentially due to Bewley [1980, 1983].[2] Spot trading using money that pays zero interest is nearly efficient when there is only a single good (or a composite commodity) at each date and state of the world; random shocks are idiosyncratic, privately observed, and temporary; markets are competitive; and the agents are very patient. (When agents

[1]There is a small body of literature on the potential beneficial effect of social insurance that expansionary monetary policy can provide. Levine [1991] and Kehoe, Levine and Woodford [1992] study two-state Markov equilibrium in an environment where two types of agents switch their preferences stochastically and equilibrium distribution of money balances is degenerate. Deviatov and Wallace [2001] study the issue in a search theoretic model of money where money is indivisible and agents can hold at most two units of it. Molico [1997] and Edmond [2002] provide numerical examples of expansionary monetary policy dominating other policies, the former in a random-matching model of money, and the latter in an overlapping generation setting with money-in-utility-function.

[2]In [1980], Bewley conjectures that full risk sharing is achieved in the limit if a gross interest rate virtually as high as the inverse of agents' discount factor is paid, regardless of what value the discount factor has. In [1983], he shows that setting the interest rate virtually at that level generally precludes equilibrium from existing, but that the interest rate on money can be set arbitrarily close to the inverse of the discount factor as the discount factor approaches unity.

are patient, zero interest on money is close to the Friedman rule.) This result is a close analogue, for trade using outside, fiat money, of a recent characterization by Levine and Zame [2002] of environments in which spot trade using inside money, in the form of one-period debt payable in a commodity, is nearly Pareto efficient.

Bewley's results, and his discussion of them, also make it clear that monetary spot trading does not achieve full insurance if traders are impatient and risk averse. We sharpen Bewley's negative observation here.

Bewley's negative observation is put in perspective by the research of Atkeson and Lucas [1992], who characterize the symmetric, Pareto efficient long-term contract in an environment closely similar to Bewley's. That contract, which achieves an upper bound of what any economic institution in Bewley's environment could achieve, falls short of full insurance. If the question to be resolved is whether or not (or in what circumstances) monetary spot trading is an efficient economic institution, then the relevant comparison would seem to be between the equilibrium allocations of Bewley's sequence of spot markets, on the one hand, and Atkeson and Lucas' contract on the other.[3] Nevertheless this comparison, like Bewley's, indicates that monetary spot trading is inefficient. That is, Atkeson and Lucas' contractual allocation cannot be implemented by monetary spot trading.

Kocherlakota [2002] suggests that even Atkeson and Lucas' allocation may not be the most appropriate candidate for comparison with Bewley's equilibrium allocation. Kocherlakota argues that monetary spot trade has two features that would make it obviously inefficient in many economic environments, and he concludes that the equilibrium allocation of monetary spot trading should be compared with the equilibrium allocation of an efficient mechanism *in an environment where constraints impose the two limitations on all mechanisms.* One of the two features is almost complete anonymity, in the sense that each agent must be treated on the basis of his current characteristics and behavior and a one-dimensional summary statistic of his past characteristics and behavior. The other feature is the ability of an agent at any time to consume his own endowment without interference or nonpecuniary punishment. This second feature entails that a mechanism cannot induce agents to behave efficiently by threatening them with lower-than-autarky levels of expected discounted utility otherwise. Both Levine and Zame's spot-trading allocation with inside money and also Atkeson and Lucas' contractual allocation can be implemented

[3] Mas Colell and Vives [1993] show that Atkeson and Lucas' contractual allocation can be implemented by a mechanism, that is, by a game form that has feasible outcomes at all out-of-equilibrium message profiles, as well as in equilibrium.

subject to the constraint on dimensionality of information or memory. However, neither of those allocations can be implemented without using nonpecuniary punishments for enforcement, since both allocations involve some agents being in date-event situations where their expected discounted utility falls below the autarkic level. Kocherlakota shows that a random-matching environment resembling those of Shi [1995] and Trejos and Wright [1995] constrains a feasible mechanism to possess both features of monetary spot trading, but that nevertheless there is a mechanism with an equilibrium allocation that Pareto dominates monetary spot trading ex ante.

We study monetary spot trading in an environment that combines the constraints represented in the two bodies of research that we have just discussed. First, each agent's preferences among net trades in the current spot market and his current endowment are private information (as in Bewley, Levine and Zame, and Atkeson and Lucas). Second, agents can consume their own endowments without restriction or nonpecuniary punishment (as in Kocherlakota). Because an agent's characteristics are assumed to be private, a feasible mechanism cannot condition the agent's treatment directly on those characteristics as the mechanism formulated by Kocherlakota does. Despite this constraint, we construct an example of a Bewley environment in which a nonmonetary mechanism provides a strictly higher level of ex ante expected discounted utility than is provided by a monetary mechanism that does not involve continuous policy intervention by the monetary authority or planner. We also construct an example of an environment in which an expansionary monetary mechanism is Pareto superior to a laissez-faire or contractionary mechanism.

2. The environment

The economy is an infinite horizon exchange economy. Time is discrete and denoted by $t = 0, 1, 2, \ldots$. There is a continuum $(I, \mathcal{I}, \mu)$ with measure 1 of infinite-lived agents. At each date, there is a single perishable good with which agents are endowed, and that they trade and consume.

Agents' endowments and preferences fluctuate. For a generic agent i, his date-t state θ_{it} is a sequence of independent, identically distributed random variables taking values in a finite state space Θ.[4] Each θ_{it} has distribution π on Θ. Each agent's state follows his

[4] Bewley [1980, 1983] and Levine and Zame [2002] model each agent's shocks as Markovian, and study price-taking equilibrium in an economy with finitely many traders. Bewley assumes time is infinite in the past as well as the future, which avoids there being an initial condition and ensures existence of a stationary equilibrium. Levine and Zame study an equilibrium that is not stationary in general. In the markovian

own independent process. We assume that the realization of the sequence of profiles of individual agents' states $\langle\{\theta_{it}\}_{i\in I}\rangle_{t=0}^{\infty}$ is an i.i.d. process of random variables with distribution π defined on $(I, \mathcal{I}, \mu)$, almost surely with respect to the probability space on which the random states of all agents are defined.[5] To make explicit the mathematical structure just described, we denote this probability space by $(\Omega, \mathcal{B}, P)$. That is, for every i and t, $\theta_{it}\colon \Omega \to \Theta$; and for every $n \in \mathbb{N}$ and every 1–1 mapping $f\colon \{0,\ldots,n\} \to I \times \mathbb{N}$ and every mapping $g\colon \{0,\ldots,n\} \to \Theta$, $P(\bigcap_{m\leq n}\{\omega|\theta_{f(m)}(\omega) = g(m)\}) = \Pi_{m\leq n}\pi(\{g(m)\})$. The formal statement of our assumption is that, for every ω in an event $B \in \mathcal{B}$ with $P(B) = 1$, for every $n \in \mathbb{N}$ and every 1–1 mapping $f\colon \{0,\ldots,n\} \to \mathbb{N}$ and every mapping $g\colon \{0,\ldots,n\} \to \Theta$, $\mu(\bigcap_{m\leq n}(\{i|\theta_{if(m)}(\omega) = g(m)\}) = \Pi_{m\leq n}\pi(\{g(m)\})$.

At each date, an agent with state $\theta \in \Theta$ receives endowment $e(\theta)$ and enjoys period utility $u(c,\theta)$ if he consumes c units of good. The endowment good is perishable. $\mathsf{E}[e(\theta)] > 0$. The consumption set at each date, and on each sample path, is the set $[0,\infty)$ of nonnegative real numbers. The bounded function $u\colon \mathbb{R}_+ \times \Theta \to [0,b]$ is weakly increasing, continuous, and concave in c. It is assumed that, when $u(c,\theta)$ is regarded as a function of c, it has a positive, finite supergradient at $e(\theta)$.[6] Agents maximize the discounted expected utility of their future consumption streams, with common discount factor β.

Agents exchange endowments according to a trading mechanism that must be feasible with respect to some informational constraints in the environment. Competitive trading using money can be implemented by a mechanism that meets these constraints. First we discuss the constraints and define a trading mechanism in general terms, and then we will specify the mechanism that implements competitive monetary trade.

Each agent i privately learns his own realization of θ_{it} at date t. Each agent i delivers a quantity $z_{it} \in \mathbb{R}_+$ of the endowment good to a resource pool at the planner's disposition and also sends a message $m_{it} \in \mathbb{R}$ to the planner. The planner maintains a one-dimensional summary statistic (that is, a real number) w_{it} regarding i's history, as will be described fully below. The planner uses the summary statistics and messages of all agents and the amounts contributed by all agents to update the summary statistic of each agent i and to reallocate a quantity y_{it} of the endowment good from the resource pool to i. Agent i consumes $c_{it} = e(\theta_{it}) - z_{it} + y_{it}$. The quantities w_{it}, m_{it}, z_{it}, y_{it}, c_{it}, and $w_{i(t+1)}$ are observed

case, the stationary joint distribution of money balances and individual shocks is statistically dependent. Since we will treat the initial distribution of money balances as part of the mechanism, and since we confine attention to stationary equilibrium, we must restrict attention to i.i.d. shock processes.

[5]This assumption is not a theorem of probability, but it is a logically consistent extension of probability theory. Cf. Green, [1994].

[6]That is, for some $g > 0$, $\forall c \in \mathbb{R}_+\ \ u(c,\theta) \leq u(e(\theta),\theta) + g(c - e(\theta))$.

by agent i and the planner, but not by the other agents.

The planner's limited memory and the agents' inability to observe or communicate with one another are important features of the environment. The planner is not able to recall the entire history of his dealings with agent i prior to date t, but only the one-dimensional statistic w_{it}. Because the agents are ignorant of other's histories, states and reports, which are reflected in the planner's decisions, in principle an agent might draw inferences about other agents from observing the planner's decisions. Although the stationarity and "law-of-large-numbers" assumptions regarding the particular environment studied here make such inference uninformative, for logical clarity we will not suppress past decisions of the planner as arguments of an agent's decision rule.

Another feature that we emphasize heavily (following Kocherlakota [2002]) is the planner's limited enforcement power. The planner cannot impose any nonpecuniary penalty on an agent for sending or failing to send a particular message, or for not following an instruction given in the planner's message. The worst that the planner can do is to give the agent nothing in the current period when the endowment pool is reallocated, and then update the agent's summary statistic to a value that encodes the fact that the prohibited message has been sent or that the instruction has been flouted, and then to treat the agent ungenerously in the future as a result of the summary statistic having that unfavorable value. In particular, the worst outcome that the planner can impose on an agent is autarky. (The planner would impose autarky on agent i by setting $y_{it} = 0$ for the current and all future t. Faced with this planner's policy, i would optimally set $z_{it} = 0$ for all future t.)

We will denote the set of profiles of summary statistics of all agents by F, the set of profiles of agents' contributions to the resource pool by P, and the set of profiles of agents' messages to the planner by G. Formally, let F be the set of measurable functions from I to $\mathbb{R}$, let P be the set of nonnegative-valued functions in F, and let $G = F$.[7] If $f \in F$, then we use f_i to denote $f(i)$, and so forth with elements of other spaces of functions on I. A trading mechanism consists of an initial $w_0 \in F$ and time-indexed sequences of updating rules $W = \langle W_t \colon F \times G \times P \to F \rangle_{t \in \mathbb{N}}$ and reallocation rules $Y = \langle Y_t \colon F \times G \times P \to P \rangle_{t \in \mathbb{N}}$. We assume that the planner is able to assign w_0 according to any distribution in a way that is independent of all θ_{it} considered as a random variables defined on $(I, \mathcal{I}, \mu)$, almost surely with respect to $(\Omega, \mathcal{B}, P)$.[8] If $w_t \in F$, $m_t \in G$, and $z_t \in P$, are the profiles of agents' summary statistics,

[7] An agent can report a real number to the planner. Alternatively, if $\mathbb{R}$ is mapped onto Θ, then the mapping provides a semantics by which an agent can report his current state.

[8] Formally, we require that the planner observes a uniformly distributed r.v. $U \colon I \to [0, 1]$ such that, for every ω in an event $B \in \mathcal{B}$ with $P(B) = 1$, the following condition holds. For every probability

messages, and endowment contributions at date t; and if $w_{t+1} \in F$ and $y_t \in P$ are the profiles of the planner's updated summary statistics for the agents and reallocations of endowment to them; then $w_{t+1} = W_t(w_t, m_t, z_t)$, and $y_t = Y_t(w_t, m_t, z_t)$. The reallocation rule Y_t must satisfy the materials-balance condition that $\int_I Y_{it}(w_t, m_t, z_t)\, d\mu \leq \int_I z_{it}\, d\mu$.

Agent i's strategy consists of time-indexed sequences of functions $M = \langle M_{it} \rangle_{t \in \mathbb{N}}$ and $Z = \langle Z_{it} \rangle_{t \in \mathbb{N}}$ that specify i's message and the quantity of the endowment good that he delivers, respectively, at date t. Agent i has full recall of his own history, including the histories of his states, the values of the summary statistic that the planner has assigned him, and his endowment-good deliveries and messages to the planner. Because i can recursively reconstruct his past deliveries and messages from the other data, those past actions do not have to be explicit arguments of his current decision functions. We can thus represent $M_{it}\colon (\mathbb{R} \times \Theta)^{t+1} \to \mathbb{R}$ and $Z_{it}\colon (\mathbb{R} \times \Theta)^{t+1} \to \mathbb{R}_+$. That is, $m_{it} = M_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it})$ and $z_{it} = Z_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it})$. There is a feasibility constraint that i cannot deliver more than his endowment, that is, $Z_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it}) \leq e(\theta_{it})$.

Now we represent a competitive trading arrangement using a constant nominal stock of fiat money as such a mechanism. We suppose that agents hold money as account balances rather than as physical inventories of a fiat object. Indeed, an agent's money wealth (that is, the amount of money in his account) is the summary statistic that the planner will initially assign and subsequently update. We require that $\int_I |w_{i0}| d\mu < \infty$. At every date t, the planner essentially operates a spot market according to the rules of a Shapley-Shubik [1977] trading game. The planner interprets each agent's message as a bid to spend money to acquire other traders' endowment, disregarding messages that are negative or that exceed the sender's balance. That is, the planner considers $\tilde{m}_{it} = \max(0, \min(m_{it}, w_{it}))$ to be the money bid of agent i. These money bids and the agents' contributions z_{it} determine the spot price $p_t = \int_I \tilde{m}_{it}\, d\mu / \int_I z_{it}\, d\mu$. The planner redistributes $\tilde{m}_{it}/p_t$ quantity of the endowment pool to each trader i and adds $p_t z_{it} - \tilde{m}_{it}$ to the wealth w_{it} of agent i. That is, if we represent the profile of $\tilde{m}_{it}$ by defining $B\colon F \times G \to P$ according to $\forall i \;\; B_i(w, m) = \max(0, \min(m_i, w_i))$, then

$$Y_{it}(w_t, m_t, z_t) \;\; = \;\; B_i(w_t, m_t) \frac{\int_I z_{it}\, d\mu}{\int_I B_i(w_t, m_t)\, d\mu} \tag{1}$$

measure ψ on $\mathbb{R}$, interval $[a, b] \subseteq [0, 1]$, $n \in \mathbb{N}$, and every mapping $f\colon \{0, \ldots, n\} \to \mathbb{N}$ and every mapping $g\colon \{0, \ldots, n\} \to \Theta$, $\mu(\{i | a \leq U(i) \leq b\} \cap \bigcap_{m \leq n}(\{i | \theta_{if(m)}(\omega) = g(m)\}) = (b - a)\Pi_{m \leq n}\pi(\{g(m)\})$. Then, given an arbitrary measure ψ on $\mathbb{R}$ that the planner wants to make the distribution of w_0 and letting f be the c.d.f. of ψ, he can define $w_{i0} = \min\{x | U(i) \leq f(x)\}$.

$$W_{it}(w_t, m_t, z_t) \;\; = \;\; w_{it} + z_{it}\frac{\int_I B_i(w_t, m_t)\, d\mu}{\int_I z_{it}\, d\mu} - B_i(w_t, m_t). \tag{2}$$

We call a mechanism of this form a *laissez-faire monetary mechanism*, since the planner does not pay interest on money nor tax money nor adjust the nominal money stock after date 0, but merely operates a market on which the agents trade competitively. Note that the specifications of Y and W just given are part of the definition of the class of laissez-faire monetary mechanisms. That is, laissez-faire monetary mechanisms differ from one another only in how the initial summary statistics (that is, agents' initial money balances) w_{i0} are assigned.

We call a monetary mechanism *stationary expansionary* (resp. *stationary contractionary*) if there is a $\tau > 0$ (resp. $\tau < 0$),

$$W_{it}(w_t, m_t, z_t) = \tau Q + (1-\tau)\Big(w_{it} + z_{it}\frac{\int_I B_i(w_t, m_t)\, d\mu}{\int_I z_{it}\, d\mu} - B_i(w_t, m_t)\Big) \tag{3}$$

where $Q = \int_I w_{it}\, d\mu$ is the aggregate money balance in the economy. That is, with a stationary expansionary mechanism, an agent's summary statistics is updated as if his after-trade money holdings is inflated at a constant rate τ, and the seignorage is distributed as a lump-sum transfer. In contrast, with a contractionary monetary mechanism, an agent's summary statistics is updated as if he receives interest payment on his money holdings at a rate τ which is financed by a lump-sum tax on the population.

3. Definition of equilibrium

We focus on symmetric equilibrium, in which all agents use the same strategy (M, Z). (That is, M and Z are infinite sequences of functions with the domains and ranges specified above. Agents may take different actions from one another because their individual states are distinct points of the domains of these decision functions.) A competitive equilibrium is represented by a strategy that each trader is assumed to follow. A strategy is an equilibrium strategy if each agent acts optimally by following it, when he takes it as parametric that the other traders will follow the strategy.

It is well known that such an equilibrium can be characterized by dynamic programming. Consider a mechanism (w_0, W, Y), where each of W and Y is a time-indexed sequence of functions. Consider a strategy (M, Z), where each of M and Z is a time-indexed sequence of functions, and consider the value function of a trader i participating in the mechanism, who takes it as parametric that the other traders will all follow (M, Z). Let w_t be the profile of all agents' summary statistics at the beginning of date t. For $j \neq i$, define $m_{jt} = M_{jt}(w_{j0}, \theta_{j0}, \ldots, w_{jt}, \theta_{jt})$ and $z_{jt} = Z_{jt}(w_{j0}, \theta_{j0}, \ldots, w_{jt}, \theta_{jt})$. Then define $m^*(m)$ to be the message profile that results from i sending message m while every other agent j sends the message m_{jt} specified by strategy M. Formally, define $m^*\colon \mathbb{R} \to G$ by $[m^*(m)](i) = m$ and $\forall j \neq i \quad [m^*(m)](j) = m_{jt}$ and define $z^*\colon \mathbb{R} \to P$ by $[z^*(z)](i) = z$ and $\forall j \neq i \quad [z^*(z)](j) = z_{jt}$. Now the value function $V_t^*\colon \mathbb{R} \times F \to \mathbb{R}$ of i at t can be defined as

$$\begin{aligned} V_t^*(w_{it}, w_t) \quad = \quad & \mathsf{E}\Big[\max_{z,m}\Big\{u(e(\theta) - z + [Y_{it}(w_t, m^*(m), z^*(z))], \theta) \\ & + \beta V_{t+1}^*(W_{it}(w_t, m^*(m), z^*(z)), W_t(w_t, m^*(m), z^*(z)))\Big\}\Big]. \end{aligned} \tag{4}$$

The expectation on the right side is taken with respect to the measure π on Θ. Standard reasoning about the fixed point of a contraction mapping establishes that the sequence $V_0^*, V_1^*, \ldots$ is uniquely defined. The initial profile of summary statistics w_0, the statistic-updating rules W_t, and a strategy (M, Z) determine a sequence of summary statistics w_t. The strategy (M, Z) is an equilibrium strategy if, for all t and for all w in the range of w_t, Z and M specify the optimizing values of z and m in the expression on the right side of the value function.

For an equilibrium strategy (M, Z), define the value function sequence of the equilibrium by $V_t(w_{it}) = V_t^*(w_{it}, w_t)$. In particular, in the case of a monetary mechanism with stationary policy τ, Y_t and W_t are defined in terms of the price

$$p_t = \frac{\int_I M_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it})\, d\mu}{\int_I Z_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it})\, d\mu}.$$

Utilizing these observations, the value to an agent of having the summary statistic w at date t is a function $V_t\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$V_t(w) = \mathsf{E}\Big[\max_{z \in [0, e(\theta)],\, m \in [0, w]} \Big\{u(e(\theta) - z + \frac{m}{p_t}, \theta) + \beta\, V_{t+1}\Big(\tau Q + (1 - \tau)(w + p_t z - m)\Big)\Big\}\Big] \tag{5}$$

Restricting m to the interval $[0, w_t]$ is justified by the fact that $\tilde{m} = w_t$ if $m > w_t$, and $\tilde{m} = 0$ if $m < 0$.

We conclude this section by defining stationary Markov competitive equilibrium of a monetary mechanism, the existence of which will be investigated in Section 4. Define the current-date projection mapping $\gamma\colon \bigcup_{t \in \mathbb{N}} (\mathbb{R} \times \Theta)^{t+1} \to \mathbb{R} \times \Theta$ by $\gamma(w_0, \theta_0, \ldots, w_t, \theta_t) = (w_t, \theta_t)$. A sequence $\langle H_t\colon (\mathbb{R} \times \Theta)^{t+1} \to \mathbb{R}\rangle_{t \in \mathbb{N}}$ is stationary Markov if for each t, $H_t = H_0 \circ \gamma$. An equilibrium (M, Z) is a stationary Markov competitive equilibrium if the sequences

M and Z are stationary Markov and almost surely with respect to $(\Omega, \mathcal{B}, P)$, w_0 and w_1 are identically distributed random variables on I. These are sufficient conditions for $\langle w_{it}, \theta_{it}, c_{it} \rangle_{t \in \mathbb{N}}$ to be almost surely a stationary Markov process on I and for the spot price $\int_I \tilde{m}_{it}\, d\mu / \int_I z_{it}\, d\mu$ to be constant over time.[9]

Given any such equilibrium, clearly there is another monetary mechanism for which the time-invariant price is 1 and the equilibrium allocation is identical to that of the original mechanism. The new mechanism is obtained simply by dividing w_{i0} by the equilibrium price p_0, for each trader i. The equilibrium strategy in the mechanism is obtained from that of the old one by the same normalization. In a stationary Markov competitive equilibrium with price 1, the definition of equilibrium can be simplified by defining the net trade $x_t = z_t - m_t$. Then the Bellman equation can be rewritten as

$$V(w) = \mathsf{E}\Big[\max_{x \in [-w, e(\theta)]} \Big\{ u(e(\theta) - x, \theta) + \beta V\Big(\tau Q + (1-\tau)(w + x)\Big)\Big\}\Big]. \tag{6}$$

4. Existence of a laissez-faire monetary mechanism having a stationary Markov competitive equilibrium

In this section we prove that, for any environment satisfying the assumptions in Section 2, there is a laissez-faire monetary mechanism that has a stationary Markov competitive equilibrium.[10] This is done by studying an auxiliary optimization problem of an autarkic agent who can store the endowment good without depreciation, and by applying information about the solution of this problem to construct the equilibrium.

Consider an environment identical to that of Section 2 except in three respects: there is only one agent rather than a continuum, he receives an endowment of size $w_0 + e(\theta_0)$ at date 0, and he can store without depreciation the endowment that he has received. Other aspects of the model are the same. That is, the agent's endowment and utility are functions of an i.i.d. process $\langle \theta_t \rangle_{t \in \mathbb{N}}$ taking values in a finite set Θ and having distribution π. He receives endowment $w_0 + e(\theta_0)$ at date 0 and $e(\theta_t)$ at each date $t > 0$. The agent chooses

[9] Note that the function sequences W and Y of a laissez-faire monetary mechanism are stationary Markov. The definition of stationary equilibrium given here is the appropriate definition, in view of this fact. An example of a monetary mechanism that is not itself stationary Markov is one in which each agent receive a so-called "helicopter drop," that is, a fixed amount of newly created fiat money, proportional to the current aggregate nominal money stock, in each period. The mechanism is not stationary Markov because the amount received, which grows geometrically, is a time-dependent, additively separable term of W. The appropriate definition of stationary Markov equilibrium for this mechanism would focus on time invariance of the distribution of agents' real balances, rather than of their nominal balances.

[10] The proof can be easily extended to the case of stationary expansionary monetary mechanism.

date-0 consumption c_0 from $[0, w_0 + e(\theta_0)]$ and, for $t > 0$, chooses date-t consumption c_t from $[0, w_t + e(\theta_t)]$ (where $w_t = w_{t-1} + e(\theta_{t-1}) - c_{t-1}$) as a function of previous history. He maximizes expected discounted utility $\mathsf{E}[\sum_{t \in \mathbb{N}} \beta^t u(c_t, \theta_t)]$, and his utility function $u(c, \theta)$ is bounded, and strictly increasing and concave in c.

Standard dynamic programming results (cf. Lucas and Stokey [1989]) provide the following information.

Lemma 1. *For the auxiliary problem, there is a decision function* $C : \mathbb{R}_+ \times \Theta \to \mathbb{R}_+$ *such that the agent's optimal choice at every date* t *is that* $c_t = C(w_t, \theta_t)$. *There is a strictly concave, increasing value function* $V : \mathbb{R}_+ \to [0, b/(1-\beta)]$ *such that, for all* w *and* θ,

$$C(w, \theta) = \arg \max_{c \in [0, w+e(\theta)]} [u(c, \theta) + \beta V(w + e(\theta) - c)] \tag{7}$$

and $V(w) = \mathsf{E}[u(C(w, \theta), \theta) + \beta V(w + e(\theta) - C(w, \theta))]$. *There is a probability measure* ψ *on* $\mathbb{R}_+$ *such that* $\langle (w_t, \theta_t) \rangle_{\theta \in \mathbb{N}}$ *is a Markov process that has stationary transition probabilities and that converges weakly to a stationary asymptotic distribution such that the marginal distribution of* w *is* ψ.

For this specific optimization problem, Lemma 1 can be sharpened by showing that ψ has bounded support.

Lemma 2. *For the stationary asymptotic marginal distribution* ψ *of Lemma 1, there exists* $\bar{w} \in \mathbb{R}_+$ *such that* $\psi([0, \bar{w}]) = 1$.

Proof. Since V is concave, for every $w \in \mathbb{R}_+$, there is a supergradient $g_w \in \mathbb{R}_+$ satisfying, for all $x \in \mathbb{R}_+$, $V(x) \leq V(w) + (x - w)g_w$. Setting $x = 0$ and noting that $0 \leq V(0) \leq V(w) \leq b/(1-\beta)$, the supergradient inequality yields $g_w \leq b/(w(1-\beta))$. For each $\theta \in \Theta$, consider $u(c, \theta)$ as a function of c and let $h_\theta \in \mathbb{R}_+$ be a supergradient of the function at $e(\theta)$. If $\bar{w} > \beta b/((1-\beta) \min_{\theta \in \Theta} h_\theta)$, then equation (7) implies that $C(w, \theta) > e(\theta)$ for all $w \geq \bar{w}$ and for all θ. Thus $w_t > \bar{w}$ implies that $w_{t+1} < w_t$. Equation (7) also shows, in conjunction with the fact (established in Rockafellar [1970], Theorem 24.3) that every selection from the superdifferential of a continuous concave function is nonincreasing, that $w_t \leq \bar{w}$ implies $w_{t+1} \leq \bar{w}$. That is, w_t first decreases monotonically to a level not exceeding $\bar{w}$ if $w_0 > \bar{w}$, and then does not escape from the interval $[0, \bar{w}]$. Therefore, since ψ is the marginal of a stationary distribution, $\psi([0, \bar{w}]) = 1$. ∎

Now we apply this information regarding solution of the auxiliary problem to specifying a laissez-faire monetary mechanism that has a stationary Markov competitive equilibrium.

Proposition 1. *In an environment such as has been described in Section 2, and where the utility function u is strictly concave in c for each θ, there is a laissez-faire monetary mechanism that has a stationary Markov competitive equilibrium.*

Proof. This mechanism is specified by distributing w_0 according to the stationary marginal distribution ψ in the solution of the auxiliary problem. Clearly ψ has finite mean, since μ is a finite measure and ψ has bounded support by Lemma 2. The agents' stationary strategy is defined in terms of the decision function C of Lemma 1. Specifically for every agent i, $M_{it}(w_{i0}, \theta_{i0}, \dots w_{it}, \theta_{it}) = \max(0, C(w_{it}, \theta_{it}) - e(\theta_{it})$ and $Z_{it}(w_{i0}, \theta_{i0}, \dots w_{it}, \theta_{it}) = \max(0, e(\theta_{it}) - C(w_{it}, \theta_{it}))$. By induction on t, the joint distribution of w_t and θ_t (as random variables on $(I, \mathcal{I}, \mu)$) is the same as the stationary distribution of w and θ in the auxiliary problem.[11] Thus, by stationarity of that distribution, the equilibrium price p_t is 1 and the distribution of w_{t+1} is also ψ. Since $p_t = 1$ for all t almost surely, the decision problem of an agent in this equilibrium is isomorphic to the agents' decision problem in the auxiliary problem. Thus M and Z are an equilibrium strategy because C is the optimal strategy in the auxiliary problem. ∎

Two points are worth mentioning. First, we impose strict concavity of u in Lemma 1 and Proposition 1 so that the optimal strategy C given in equation (7) is continuous and the asymptotic distribution ψ is stationary (cf. Lucas and Stokey [1989]). Second, autarky is obviously also an equilibrium of this mechanism. We do not know whether or not there are multiple non-autarkic equilibrium. But given the way that the equilibrium is constructed, it Pareto dominates all other equilibrium ex ante.

5. Equilibrium of a laissez-faire monetary mechanism is nearly efficient if agents are sufficiently patient

In this section we show that stationary Markov competitive equilibrium of a laissez-faire monetary mechanism is nearly ex ante Pareto efficient in an environment of sufficiently patient traders. To do so, consider a family of environments that are identical in all respects except for the value β of the agents' discount factor. We will show that, as β approaches 1, the equilibria constructed in the proof of Proposition 1—in which each trader's optimization problem is isomorphic to that of an autarkic agent whose endowment is perfectly storable—are nearly efficient.

[11] This assertion holds almost surely with respect to $(\Omega, \mathcal{B}, P)$.

The concept of near efficiency that we study is a variant of Debreu's [1951] coefficient of resource utilization. A mechanism in an environment is *δ-efficient*, for $\delta \in (0,1]$, if it has an equilibrium allocation that all agents would weakly prefer ex ante to the full-risk-sharing allocation of the environment in which the endowment of the actual environment is shrunken to any scalar replica of proportion smaller than δ.

Formally, fix a stochastic process θ, endowment function e, and utility function u satisfying the requirements of Proposition 1, so that stationary Markov competitive equilibrium is assured to exist. For $\beta \in (0,1)$ and $\delta \in (0,1]$, define $\mathcal{E}_{\beta\delta}$ to be the environment with stochastic process θ in which all agents' preferences are characterized by utility function u and discount factor β, and in which each trader i receives endowment $\delta e(\theta_{it})$ at date t. Let $r_\delta \colon \Theta \to \mathbb{R}_+$ be a mapping such that $\mathsf{E}[r_\delta(\theta) - \delta e(\theta)] = 0$ and also such that there is a common supergradient of $\{u(r_\delta(\theta), \theta)\}_{\theta\in\Theta}$.[12] The allocation implied by r_δ is the complete risk sharing allocation in economy $\mathcal{E}_{\beta\delta}$, for every β. For every β and δ, define $U_\delta = \sum_{\theta\in\Theta} \pi(\theta)u(r_\delta(\theta), \theta)$. $U_\delta/(1-\beta)$ is the ex ante expected discounted utility of consumption in a full-risk-sharing allocation of environment $\mathcal{E}_{\beta\delta}$. Note that the consumption levels $r_\delta(\theta)$ and the expected utility U_δ per period do not depend on β. By the assumption of Proposition 1 that each $u(c,\theta)$ is strictly concave in c, $\delta < \varepsilon$ implies that $\forall\theta\ \ r_\delta(\theta) < r_\varepsilon(\theta)$. Thus, because a strictly concave, increasing function on $\mathbb{R}_+$ is strictly increasing, $\delta < \varepsilon$ implies that $U_\delta < U_\varepsilon$. Define V_β to be the ex ante expected value of consumption in the stationary Markov competitive equilibrium of the laissez-faire monetary mechanism constructed in the proof of Proposition 1. (That is, $V_\beta = \mathsf{E}_\psi V(w_0)$, where V is the value function for the auxiliary problem of Lemma 1 with discount factor β.) Then the laissez-faire monetary mechanism in environment $\mathcal{E}_{\beta 1}$ is δ-efficient if $\delta = \sup\{\varepsilon | V_\beta \geq U_\varepsilon/(1-\beta)\}$.

Proposition 2. *For any $\delta < 1$, there is a $\beta < 1$ such that the laissez-faire monetary mechanism is an δ-efficient mechanism of the environments with discount factors in $[\beta, 1)$.*

Proof. We set $\varepsilon = (1+\delta)/2$, and we construct a strategy that asymptotically provides the full-risk-sharing allocation in $\mathcal{E}_{\beta\varepsilon}$. The expected discounted utility that this strategy yields is a lower bound for V_β, which is the expected discounted utility that an agent's optimal strategy yields. We prove the proposition by using the strategy to show that, for sufficiently large β, the lower bound is sufficiently close to $U_\varepsilon/(1-\beta)$ that $V_\beta \geq U_\delta/(1-\beta)$.

As in the proof of Lemma 2, we define the strategy in terms of the consumption func-

[12] If $u'(r_\delta(\theta), \theta)$ exists for each t, then the condition that this derivative has the same value for all θ is equivalent.

tion that it implies. Define $\Gamma\colon \mathbb{R}_+ \times \Theta \to \mathbb{R}_+$ by $\Gamma(w, \theta) = \min(w + e(\theta), r_\varepsilon(\theta))$. That is, the agent attempts to replicate the consumption that he would enjoy in the full-risk-sharing allocation in $\mathcal{E}_{\beta\varepsilon}$, subject to the constraint that the laissez-faire monetary mechanism in the actual economy $\mathcal{E}_{\beta 1}$ places on his choice. The strategy for agent i implied by this consumption function is that $M_t^*(w_{i0}, \theta_{i0}, \dots w_{it}, \theta_{it}) = \max(0, \Gamma(w_{it}, \theta_{it}) - e(\theta_{it}))$ and $Z_t^*(w_{i0}, \theta_{i0}, \dots w_{it}, \theta_{it}) = \max(0, e(\theta_{it}) - \Gamma(w_{it}, \theta_{it}))$. The wealth-updating rule of the laissez-faire monetary mechanism entails that $w_{i(t+1)} - w_{it} = (1-\varepsilon)e(\theta_{it}) + \varepsilon e(\theta_{it}) - \Gamma(w_{it}, \theta_{it})$. Define $v_{it} = (1-\varepsilon)(e(\theta_{it}) - \mathsf{E}[e(\theta_{it})]) + \varepsilon e(\theta_{it}) - r_\varepsilon(\theta_{it})$. Note that $\langle v_{it} \rangle_{t \in \mathbb{N}}$ is i.i.d., $\mathsf{E}[v_{it}] = 0$, and $w_{i(t+1)} - w_{it} \geq v_{it} + (1-\varepsilon)\mathsf{E}[e(\theta_{it})]$. Applying a law of the iterated logarithm (Breiman [1968],Theorem 13.25) to the sums $\sum_{\tau < t}(-v_{i\tau})$ establishes that $\lim_{t \to \infty} w_{it} = \infty$ almost surely. Therefore, almost surely $\exists \tau \ \forall t \geq \tau \ \Gamma(w_{it}, \theta_{it}) = r_\varepsilon(\theta_{it})$.

There is a number $\varphi > 0$ such that $U_\varepsilon - \varphi b > U_\delta$. By the preceding argument, there is date τ such that $P(\{\omega | \forall t \geq \tau \ \Gamma(w_{it}, \theta_{it}) = r_\varepsilon(\theta_{it})\}) > 1 - \varphi/2$.

Let $D = \{\omega | \forall t \geq \tau \ \Gamma(w_{it}, \theta_{it}) = r_\varepsilon(\theta_{it})\}$. Then, for $t \geq \tau$,

$$\begin{aligned}
\mathsf{E}[u(\Gamma(w_{it}, \theta_{it}))] &= \int_D u(r_\varepsilon(\theta_{it}))\, dP + \int_{\Omega \setminus D} u(\Gamma(w_{it}, \theta_{it}))\, dP \\
&\geq \int_D u(r_\varepsilon(\theta_{it}))\, dP + \int_{\Omega \setminus D} [u(r_\varepsilon(\theta_{it})) - b]\, dP \\
&= \mathsf{E}[u(r_\varepsilon(\theta_{it}))] - bP(\Omega \setminus D) \\
&> U_\varepsilon - (\varphi/2)b.
\end{aligned}$$

Therefore $V_\beta \geq \mathsf{E}[\sum_{t \geq \tau} \beta^t u(\Gamma(w_{it}, \theta_{it}))] > \beta^\tau (U_\varepsilon - (\varphi/2)b)/(1-\beta)$, so $V_\beta > U_\delta/(1-\beta)$ if $\beta \geq [(U_\varepsilon - \varphi b)/(U_\varepsilon - (\varphi/2)b)]^{1/\tau}$. ∎

6. When agents are impatient

The *approximate* efficiency of laissez-faire policy with very patient agents does not preclude other policy or mechanism from being even better. The potential efficiency loss might be large when agents are impatient. In this section, we study two examples of a specialization of the environment discussed above. The first one shows that the equilibrium of the laissez-faire monetary mechanism is not an efficient allocation subject to the material balance and incentive constraints. Specifically, the mechanism can be modified by adding an insurance arrangement that intuitively is nonmonetary, and that provides agents with a higher level of expected discounted utility ex ante. The second example shows that the equilibrium of a expansionary monetary mechanism is efficient while a laissez-faire or contractionary monetary mechanism is not. It remains as a question whether in some environment a nonmonetary mechanism dominates any monetary mechanism.

6.1. *An example where a nonmonetary mechanism Pareto dominates a laissez-faire monetary mechanism*

Consider an environment where agents' marginal utility fluctuates between high (state h) and low (state l) over time, $\Theta = \{h, l\} \subseteq \mathbb{R}_+$ and $0 < l < h$, but they all receive a constant endowment $e(\theta) \equiv e$ for all $\theta \in \Theta$ every period. For agent i, θ_{it} is i.i.d. with a Bernoulli(1/2), that is, the probability of $\theta_{it} = h$ is 1/2 for all $t \geq 0$. Agents have a satiation level of consumption ζ each period, $e < \zeta \leq 2e$. An agent with an individual state θ derives period utility

$$u(c, \theta) = \theta \min\{c, \zeta\} \tag{8}$$

from consuming c units of endowment.[13]

Given that preference shocks are independent across agents, and each agent's preference shock follows a Bernoulli process, at each period, half of the population have high marginal utility and the other half have low marginal utility. The first-best outcome (efficient allocation subject only to material balance constraint) in this environment is to have agents with high marginal utility consume up to satiation level ζ, and agents with low marginal utility consume the rest $e - (\zeta - e)$ units of endowment per person. Hence, the first-best welfare level is given by

$$\mathcal{W}_{\text{first-best}} = \frac{1}{1-\beta}\Big[\frac{h}{2}\,\zeta + \frac{l}{2}(e - (\zeta - e))\Big]. \tag{9}$$

At autarky, an agent consumes his own endowment every day. The corresponding welfare is

$$\mathcal{W}_{\text{autarky}} = \frac{1}{1-\beta}\frac{h+l}{2}\,e. \tag{10}$$

In this environment, consider a laissez-faire monetary mechanism where the trading price is normalized to 1. That is, for any $t \geq 0$, any profiles of agents' summary statistics

[13] In this specification, the utility function is not strictly concave and the agent is satiated at consumption level ζ. These simplifying assumptions are not crucial to the results derived here. We could define $u(c, \theta) = \theta \min\{c, \zeta\} + f(c)$, where $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly concave, increasing function having very small right derivative at 0, and our arguments would remain sound. The utility function so defined would be strictly concave and increasing in consumption in every state.

$w_t \in F$, messages $m_t \in G$, and endowment contributions $z_t \in P$, for any agent i,

$$Y_{it}(w_t, z_t, m_t) = \max(0, \min(m_{it}, w_{it})) \tag{11}$$

$$W_{it}(w_t, z_t, m_t) = w_{it} + z_{it} - m_{it} \tag{12}$$

We are going to show that for some parameter values, there is a stationary equilibrium of the mechanism at which agents accumulate only up to $S = \zeta - e$ units of money. The equilibrium strategy is as follows. When an agent's marginal utility is high ($\theta_{it} = h$), he spends up to S units of money if he can afford to get as close to satiation level ζ as possible. When his marginal utility is low ($\theta_{it} = l$), he spends S units of money if he has at least $2S$ units; keeps S units and spends the rest if his money balance is between S and $2S$; and sells $S - w_{it}$ units of endowment to get his end-of-period money balance to S if he has less than S units of money. That is,

$$Z_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it}) = \begin{cases} S - w_{it} & \text{if } \theta_{it} = l, \quad w_{it} < S \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

$$M_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it}) = \begin{cases} S & \text{if } \theta_{it} = h, \quad w_{it} \geq S \\ w_{it} & \qquad\qquad\quad w_{it} < S \\ S & \text{if } \theta_{it} = l, \quad w_{it} \geq 2S \\ w_{it} - S & \qquad\qquad\quad S \leq w_{it} < 2S \\ 0 & \qquad\qquad\quad w_{it} < S \end{cases} \tag{14}$$

At this equilibrium, agents' money balances are shuffling back and forth between zero and S units. No one holds any other quantity of money. (If they do, they will get back to zero or S in finite time.) That is, the support of the money balance distribution for this potential equilibrium is $\{0, S\}$. Given that θ_{it} follows a Bernoulli process for each agent,

$$\mu(h, 0) = \mu(h, S) = \mu(l, 0) = \mu(l, S) = \frac{1}{4}. \tag{15}$$

The corresponding value function is then

$$V(0) = \frac{1}{2}(h\, e + \beta V(0)) + \frac{1}{2}(l(2e - \zeta) + \beta V(S)) \tag{16}$$

$$V(S) = \frac{1}{2}(h\, \zeta + \beta V(0)) + \frac{1}{2}(l\, e + \beta V(S)) \tag{17}$$

for $w \in (0, S)$,

$$V(w) = \frac{1}{2}(h(e + w) + \beta V(0)) + \frac{1}{2}(l(2e - \zeta + w) + \beta V(S)) \tag{18}$$

for $w \in (S, 2S]$,

$$V(w) = \frac{1}{2}(h\,\zeta + \beta V(w + e - \zeta)) + \frac{1}{2}(l(2e - \zeta + w) + \beta V(S)) \tag{19}$$

and for $w > 2S$,

$$V(w) = \frac{1}{2}(h\,\zeta + \beta V(w + e - \zeta)) + \frac{1}{2}(l\,\zeta + \beta V(w + e - \zeta)). \tag{20}$$

The value function on the support $V(0)$ and $V(S)$ can be solved from (16) and (17),

$$V(0) = \frac{1}{4(1-\beta)}\Big[2l(2e - \zeta) + 2h\,e + \beta(h + l)(\zeta - e)\Big] \tag{21}$$

$$V(S) = \frac{1}{4(1-\beta)}\Big[2h\,\zeta + 2l\,e - \beta(h + l)(\zeta - e)\Big]. \tag{22}$$

Given (21) and (22), the value function off the equilibrium support can be computed using equations (18), (19), and (20). It is easy to show that the value function is piecewise linear, continuous, and strictly increasing, and that its restriction to the set of integer multiples of $S = \zeta - e$, $\{0, S, 2S, \ldots\}$ is strictly concave on that lattice.

Given the value function, we can now verify that, under certain condition, the conjectured equilibrium strategy (M, Z) given in equation (13) and (14) is indeed optimal.

PROPOSITION 3. *The strategy* (M, Z) *is optimal if*

$$\frac{\beta^2}{4 - 2\beta - \beta^2} \leq \frac{l}{h} \leq \frac{\beta}{2 - \beta}. \tag{23}$$

The proof is in the appendix. Condition (23) is derived from an agent's incentive regarding trade and consumption when current marginal utility is low. The first half of the condition guarantees that an agent does not sell more endowment when his money balance is S since the marginal value of holding more money above S is below that of consuming today l. The second half of the condition ensures that an agent sells his endowment when his money balance is below S because the marginal value of postponing consumption to some future date is higher than that of consuming today l.

The welfare level of this monetary equilibrium is exactly halfway between the welfare levels of autarky and that of the first-best.

$$\begin{aligned} \mathcal{W}_{\text{money}} = \frac{1}{2}V(0) + \frac{1}{2}V(S) &= \frac{1}{1-\beta}\Big[\frac{h+l}{2}\,e + \frac{h-l}{4}(\zeta - e)\Big] \\ &= \frac{1}{2}(\mathcal{W}_{\text{autarky}} + \mathcal{W}_{\text{first-best}}). \end{aligned} \tag{24}$$

This equilibrium is not efficient in the sense that a planner can modify the mechanism slightly in a way that has a Pareto preferred equilibrium allocation to the monetary mechanism. One modification is to have agents with low marginal utility and zero money balance to give a small $\varepsilon > 0$ amount of their endowment to agents with high marginal utility and zero money balance, and the rest of the strategy remains the same.

The initial profile of agents' summary statistics w_0 in the modified mechanism is identical to the initial profile in the monetary mechanism;

$$\hat{Y}_{it}(w_t, z_t, m_t) = \max(0, \min(m_{it}, w_{it})) \tag{25}$$

$$\hat{W}_{it}(w_t, z_t, m_t) = \begin{cases} S & \text{if } z_{it} = S + \varepsilon \text{ or } \{w_{it} = S \text{ and } m_{it} = 0\} \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

We show that the following strategy is an equilibrium strategy of the mechanism.

$$\hat{Z}_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it}) = \begin{cases} S + \varepsilon & \text{if } w_{it} = 0 \text{ and } \theta_{it} = l; \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

$$\hat{M}_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it}) = \begin{cases} S & \text{if } w_{it} = S \text{ and } \theta_{it} = h \\ \varepsilon & \text{if } w_{it} = 0 \text{ and } \theta_{it} = h; \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

Proposition 4. *Suppose that condition (23) is satisfied. Then, there exists $\hat{\varepsilon}$ such that*

$$\frac{l}{h} = \frac{\beta(\zeta - e - \hat{\varepsilon})}{(2 - \beta)(\zeta - e) + (4 - \beta)\hat{\varepsilon}} \tag{29}$$

for any $\varepsilon \in (0, \hat{\varepsilon}]$, (i) *the welfare level achieved by adopting strategy $(\hat{M}, \hat{Z})$ for mechanism $(\hat{Y}, \hat{W})$, $\mathcal{W}^{\varepsilon}_{\text{money}}$, is higher than that of the monetary equilibrium, $\mathcal{W}_{\text{money}}$; and* (ii) $(\hat{M}, \hat{Z})$ *is incentive compatible.*

The proof of Proposition 4 is in the appendix. The reason that the monetary allocation can be improved upon is simple. With a monetary mechanism, the only way an agent can smooth consumption over time is by accumulating money. When money is run out, he losses the ability to shift future consumption to today, at which his current marginal utility is high. If a social planner can reallocate a small amount of endowment from the low marginal utility agents to those with high marginal utility but no money, welfare should be improved. Indeed, this can be accomplished by taking some endowment away from the low-marginal-utility zero-money-balance agents and give it to high-marginal-utility zero-money-balance agents. By doing so, the value of having both no money and S unit of money are increased, but the difference between them is decreased, making it easier for the incentive constraint to satisfy.

To summarize, this example shows that, in general, a laissez-faire monetary mechanism fails to be ex ante Pareto efficient within the class of all allocation mechanisms.

6.2. *An example where expansionary policy Pareto dominates laissez-faire*

Now consider the same environment as in the last example except that the satiation level $\zeta > 2e$. The first-best outcome in this environment is to have agents with low marginal utility transfer all endowment to agents with high marginal utility. Moreover, because utility is linear on $[0, \zeta]$, any such transfer that does not exceed state-h traders' satiation levels is efficient. We show that under some parameter restriction, such an outcome can be achieved as an equilibrium of a stationary expansionary monetary mechanism. The efficiency of expansionary policy in this example is fragile. It depends crucially on the local risk-neutrality just mentioned. Nevertheless, it is a robust feature (cf. footnote 13) that this policy is superior to laissez-faire..

Consider a stationary monetary mechanism specified by policy τ and trading price normalized to 1. That is, for any $t \geq 0$, any profiles of agents' summary statistics $w_t \in F$, messages $m_t \in G$, and endowment contributions $z_t \in P$, for any agent i,

$$Y_{it}(w_t, z_t, m_t) = \max(0, \min(m_{it}, w_{it})) \tag{30}$$

$$W_{it}(w_t, z_t, m_t) = \tau Q + (1-\tau)(w_{it} + z_{it} - m_{it}) \tag{31}$$

where $Q = \int_I w_{it} d\mu$. We are going to show that the following strategy is an equilibrium strategy of the mechanism,

$$Z_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it}) = \begin{cases} e & \text{if } \theta_{it} = l; \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

$$M_{it}(w_{i0}, \theta_{i0}, \ldots, w_{it}, \theta_{it}) = \begin{cases} 0 & \text{if } \theta_{it} = l \\ w_{it} & \text{otherwise} \end{cases} \tag{33}$$

That is, an agent spends all his money on consumption when marginal utility is high ($\theta_{it} = h$), and sells all his endowment e when his marginal utility is low ($\theta_{it} = l$). Such an outcome is efficient.

Following this strategy, agents' money balances (summary statistics) are concentrated on a set $\{\alpha_n\}_{n=0}^{\infty}$, where α_n is an agent's money balance after n consecutive sales since his last purchase,

$$\alpha_0 = \tau Q \tag{34}$$

$$\forall n \geq 1 \qquad \alpha_n = \tau Q + (1-\tau)(\alpha_{n-1} + e). \tag{35}$$

Recursively applying (35), for $n \geq 1$,

$$\alpha_n = \frac{1}{\tau}\Big[\tau Q + e(1-\tau) - (\tau Q + e)(1-\tau)^{n+1}\Big]. \tag{36}$$

Given that the environment is stationary, and that agents' taste shock follows a Bernoulli process, for all $n \geq 0$, the measure of agents whose money balances are α_n is

$$\mu\{w_{it} = \alpha_n\} = \frac{1}{2^{n+1}}. \tag{37}$$

Then

$$Q = \sum_{n=0}^{\infty} \alpha_n \mu\{w_{it} = \alpha_n\} = \frac{1}{\tau}\Big[\tau Q + e(1-\tau) - (\tau Q + e)\frac{1-\tau}{1+\tau}\Big]. \tag{38}$$

Solving Q from (38), we have

$$Q = e. \tag{39}$$

That is, at this equilibrium, aggregate real money balance at any date (which is also per capita real money balance given that the measure of agent is 1) equals to an agent's endowment. By (36) and (38), for $\tau \in (0,1)$,

$$\lim_{n\to\infty} \alpha_n = \frac{e}{\tau}. \tag{40}$$

Given the satiation level ζ, the optimality of strategy for $\theta_{it} = h$ (spending all money on consumption) requires that $e + \alpha_n \leq \zeta$ for all $n \geq 0$. Therefore, a necessary condition for the optimality of strategy $(M,\, Z)$ is

$$e + \frac{e}{\tau} \leq \zeta. \tag{41}$$

The value function on $\{\alpha_n\}_{n=0}^{\infty}$ is defined as follows. For all $n \geq 0$,

$$V(\alpha_n) = \frac{1}{2}\Big(h(e+\alpha_n) + \beta V(\alpha_0)\Big) + \frac{1}{2}V(\alpha_{n+1}). \tag{42}$$

The solution to this system of equations can be expressed recursively as follows.

$$V(\alpha_0) = \frac{1}{1-\beta}\frac{he(1+\tau)}{2-\beta(1-\tau)} \tag{43}$$

$$\forall n \geq 1 \qquad V(\alpha_n) = V(\alpha_{n-1}) + \frac{he(1+\tau)(1-\tau)^n}{2-\beta(1-\tau)}. \tag{44}$$

By (36), (39) and (44),

$$\frac{V(\alpha_n)-V(\alpha_{n-1})}{\alpha_n-\alpha_{n-1}}=\frac{he(1+\tau)(1-\tau)^n}{2-\beta(1-\tau)}\Big/\Big(e(1+\tau)(1-\tau)^n\Big)=\frac{h}{2-\beta(1-\tau)} \tag{45}$$

which is a constant. Hence, the value function V is affine on $[e\tau,\, e/\tau)$ with slope given by (45).

Given the value function, we can verify that the conjectured strategy $(M,\, Z)$ given in (32) and (33) as equilibrium strategy.

Proposition 5. *Strategy $(M,\, Z)$ given in (32) and (33) is optimal if the parameters of the model β, l, h and policy variable τ satisfy the following condition,*

$$\frac{e}{\zeta-e}\le\tau\le\frac{\beta(h+l)-2l}{\beta(h+l)}. \tag{46}$$

Proof. The first inequality of condition (46) is a restatement of condition (41). We need only to show the second half of the condition.

When $\theta_{it}=h$, strategy $(M,\, Z)$ specifies the optimal net trade to be $x^*=-w_{it}$. This is optimal if for any $\varepsilon>0$, and any $x\in[-w_{it},\, e]$ such that $x-\varepsilon\in[-w_{it},\, e]$, the expected value of net trade x is lower than that of $x-\varepsilon$, that is,

$$h(e-x)+\beta V(\tau e+(1-\tau)(w_{it}+x))\le h(e-x+\varepsilon)+\beta V(\tau e+(1-\tau)(w_{it}+x-\varepsilon)). \tag{47}$$

Given that the value function V is affine with slope $h/(2-\beta(1-\tau))$, this inequality is equivalent to

$$\frac{\beta h}{2-\beta(1-\tau)}\le\frac{h\varepsilon}{(1-\tau)\varepsilon} \tag{48}$$

which always holds. That is, given the first half of condition (46), $x^*=-w_{it}$ when $\theta_{it}=h$ is optimal.

When $\theta_{it}=l$, strategy $(M,\, Z)$ specifies the optimal net trade to be $x^*=e$. This is optimal if for any $\varepsilon>0$, any $x\in[-w_{it},\, e]$ such that $x+\varepsilon\in[-w_{it},\, e]$, the expected value of net trade x is lower than that of $x+\varepsilon$, that is,

$$l(e-x)+\beta V(\tau e+(1-\tau)(w_{it}+x))\le l(e-x-\varepsilon)+\beta V(\tau e+(1-\tau)(w_{it}+x+\varepsilon)). \tag{49}$$

This inequality is equivalent to

$$\frac{l\varepsilon}{(1-\tau)\varepsilon}\le\frac{\beta h}{2-\beta(1-\tau)}$$

or the second half of condition (46). That is, $x^* = e$ when $\theta_{it} = l$ is optimal if the second half of condition (46) is satisfied. ∎

By Proposition 5, the efficient allocation in this environment is achieved by the equilibrium of a stationary expansionary monetary mechanism since policy $\tau > e/(\zeta - e) > 0$. Any policy with $\tau \leq 0$, i.e., laissez-faire or contractionary monetary mechanism, would not accomplish the task. With an expansionary policy, all agents' money balances are bounded by e/τ given that they are constantly inflated away at a rate τ. So "rich" people can never get too rich to not sell. If $\tau \leq 0$, however, selling whenever an agent's marginal utility is low is no longer an optimal strategy. Let $\hat{t}$ be the smallest t such that $\beta^t h < l$, so for all $t \geq \hat{t}$, the discounted marginal utility of consumption in state h after t periods is lower than the marginal utility of consuming in today's l state. Then when an agent in state l today has money balances $t(\zeta - e)$, $t \geq \hat{t}$, he will consume rather than selling his endowment, contrary to strategy (M, Z) given in (32) and (33), as well as the prescription to achieve the efficient allocation.

7. Conclusion

We consider a class of environments where there is a stringent restriction on the amount of information that can be kept regarding the history of each agent, where an agent's endowment cannot be taken from him forcibly or by threat of nonpecuniary punishment, and where an agent's current characteristics are his private information. We suggest that this class of environments formalizes the assumptions under which, according to previous conjectures, spot trade using fiat money can be an exactly or approximately efficient allocation mechanism if monetary policy is set appropriately. Within this class of environments, we provide an explicit definition of a monetary mechanism and particularly of a monetary mechanism governed by laissez-faire policy. We show that a laissez-faire monetary mechanism is nearly efficient, in terms of a criterion in the spirit of Debreu's coefficient of resource utilization for ex ante Pareto efficiency, in an environment within our class where agents are sufficiently patient. We also provide an example that shows that, in an environment within our class where agents are impatient, an expansionary monetary mechanism can Pareto dominate any laissez-faire or contractionary monetary mechanism.

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Appendix

The proof of Proposition 3.

Given that the time-invariant price is 1 at an stationary equilibrium, agents choose optimal strategy (M, Z) is equivalent to choose net trade $X = Z - M$. For convenience, we work with net trade here. At each period t, an agent chooses his net trade x after observing his preference θ, $x \in [-w,\, e]$. Let $U(w, \theta; x)$ be the expected discounted utility of choosing net trade x, when money balance is w and preference shock is θ,

$$U(w, \theta; x) = \theta \min\{e - x,\, \zeta\} + \beta V(w + x). \tag{50}$$

Depending on the value of w and θ, there are four cases to consider.[14]

The first case is when $\theta = h$ and $w \leq S$. Since $x \geq -w \geq -S = e - \zeta$, $e - x \leq \zeta$. By definition (50), $U(w, h; x) = h(e - x) + \beta V(w + x)$.

$$\frac{dU(w, h; x)}{dx} = -h + \beta V'(w + x) \leq -h + \frac{1}{2}\beta(h + l) < 0.$$

So $x^* = -w$.

[14]When a derivative is taken at a kink in the analysis below, the relevant left or right derivative applies.

When $\theta = h$ and $w > S$,

$$\frac{dU(w,h;x)}{dx} = \begin{cases} \beta V'(w+x) > 0 & \text{if } x < -S \\ -h + \beta V'(w+x) \le -h + \beta(h+l)/2 < 0 & \text{otherwise} \end{cases}$$

due to V being strictly increasing. Hence, $x^* = -S = e - \zeta$.

When $\theta = l$ and $w \le 2S$,

$$\frac{dU(w,h;x)}{dx} = \begin{cases} \beta V'(w+x) = \beta(h+l)/2 > 0 & \text{if } x < -S \\ -l + \beta V'(w+x) & \text{otherwise.} \end{cases}$$

When $x \ge -S$, there are two more separate cases depending on whether $w + x < S$.

$$\frac{dU(w,h;x)}{dx} = -l + \beta V'(w+x) \begin{cases} = -l + \beta(h+l)/2 & \text{if } -S \le x < S - w \\ \le -l + \beta[l + \beta(h+l)/2]/2 & \text{if } x \ge S - w. \end{cases}$$

The upper expression $-l + \beta(h+l)/2 \ge 0$ if the second inequality of (23) holds. The lower expression $-l + \beta[l + \beta(h+l)/2]/2 \le 0$ if the first inequality of (23) holds. That is, if condition (23) holds, the objective function $U(w,h;x)$ reaches its peak at $S - w$, hence $x^* = S - w$.

When $\theta = l$ and $w > 2S$,

$$\frac{dU(w,h;x)}{dx} = \begin{cases} \beta V'(w+x) > 0 & \text{if } x < -S \\ -l + \beta V'(w+x) \le -l + \beta[l + \beta(h+l)/2]/2 & \text{otherwise} \end{cases}$$

Again, the lower expression $-l + \beta[l + \beta(h+l)/2]/2 \le 0$ if the first inequality of (23) holds. So, if condition (23) holds, the objective function $U(w,h;x)$ reaches its peak at $-S$, hence $x^* = -S = e - \zeta$.

The solution x^* in the above four cases corresponds exactly to $x^* = Z_{it} - M_{it}$ where (M, Z) is given in (13) and (14). ∎

The proof of Proposition 4.

If agents adopt strategy $(\hat{M}, \hat{Z})$, the corresponding value function $\hat{V}$ is then

$$\hat{V}(0) = \frac{1}{2}(h\,e + h\varepsilon + \beta\hat{V}(0)) + \frac{1}{2}(l(2e - \zeta) - l\varepsilon + \beta\hat{V}(S)) \tag{51}$$

$$\hat{V}(S) = \frac{1}{2}(h\,\zeta + \beta\hat{V}(0)) + \frac{1}{2}(l\,e + \beta\hat{V}(S)). \tag{52}$$

From (51) and (52), $\hat{V}(0)$ and $\hat{V}(S)$ satisfies

$$\hat{V}(S) - \hat{V}(0) = \frac{1}{2}\Big[(h+l)(\zeta - e) - (h-l)\varepsilon\Big] \tag{53}$$

$$\hat{V}(S) + \hat{V}(0) = \frac{1}{1-\beta}\Big[(h+l)\,e + \frac{h-l}{2}(\zeta - e + \varepsilon)\Big]. \tag{54}$$

The welfare level of this modified allocation $\mathcal{W}^{\varepsilon}_{\text{money}}$ is then

$$\mathcal{W}^{\varepsilon}_{\text{money}} = \frac{1}{2}\hat{V}(0) + \frac{1}{2}\hat{V}(S) = \frac{1}{1-\beta}\Big[\frac{h+l}{2}\,e + \frac{h-l}{4}(\zeta - e + \varepsilon)\Big] \tag{55}$$

which is higher than that of the monetary equilibrium $\mathcal{W}_{\text{money}}$ given in (24) if $\varepsilon > 0$.

Next we want to show that if $\varepsilon < \hat{\varepsilon}$, $(\hat{M}, \hat{Z})$ is incentive compatible, that is, no agent wants to misreport his current marginal utility.[15] At steady state, there are four types of agent each date depending on one's money balance (0 or S) and preference shock θ (h or l). So there are four truth-telling constraints,

$$\begin{aligned}
IC_{h0}: &\qquad h(e+\varepsilon) + \beta\hat{V}(0) \geq h(2e - \zeta - \varepsilon) + \beta\hat{V}(S)\\
IC_{hS}: &\qquad h\,\zeta + \beta\hat{V}(0) \geq h\,e + \beta\hat{V}(S)\\
IC_{l0}: &\qquad l(2e - \zeta - \varepsilon) + \beta\hat{V}(S) \geq l(e+\varepsilon) + \beta\hat{V}(0)\\
IC_{lS}: &\qquad l\,e + \beta\hat{V}(S) \geq l\,\zeta + \beta\hat{V}(0).
\end{aligned}$$

Condition IC_{hS} implies IC_{h0}, and condition IC_{l0} implies IC_{lS}. Conditions IC_{hS} and IC_{l0} can be rewritten as

$$\begin{aligned}
IC_{hS}: &\qquad h(\zeta - e) \geq \beta(\hat{V}(S) - \hat{V}(0))\\
IC_{l0}: &\qquad \beta(\hat{V}(S) - \hat{V}(0)) \geq l(\zeta - e + 2\varepsilon).
\end{aligned}$$

Substituting (53) into IC_{hS} and IC_{l0}, condition IC_{hS} holds automatically, and condition IC_{l0} can be rewritten as

$$\frac{l}{h} \leq \frac{\beta(\zeta - e - \varepsilon)}{(2-\beta)(\zeta - e) + (4-\beta)\varepsilon}. \tag{56}$$

Condition (56) holds with equality when $\varepsilon = \hat{\varepsilon}$. Given that the right hand side of (56) is a decreasing function of ε, for all $\varepsilon \in (0, \hat{\varepsilon}]$, all four incentive constraints are satisfied.

That is, for all $\varepsilon \in (0, \hat{\varepsilon}]$, the equilibrium allocation for mechanism $(\hat{Y}, \hat{W})$ Pareto dominates the equilibrium allocation of the monetary mechanism (Y, W) and is incentive compatible. ∎

[15] According to the equilibrium strategy, $m_{it} = 0$ constitutes a report that $\theta_{it} = l$ and any other message constitutes a report that $\theta_{it} = h$.