The Distribution of Money Balances and the Non-Neutrality of Money*

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Abstract

We construct a model where agents hold money for transactions purposes, and trade on a sequence of spot markets during a period. Agents choose spending strategies and cash holdings taking into account expected cash flows. Because buying and selling opportunities are idiosyncratic, intra-period heterogeneity in money holdings emerges. Due to this distribution, unexpected money transfers may increase output and welfare, in the short-run. We also show that deterministic but asymmetric injections of money may increase aggregate output and welfare permanently.

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1 Introduction

In this paper we construct a monetary model with a non-degenerate distribution of real balances to study the issue of monetary neutrality. We show that increases in the money growth rate are detrimental to economic activity. Yet because of the non-degenerate distribution of real balances, an unexpected monetary injection can have beneficial effects in the short-run on aggregate output and welfare even though it is neutral in the long-run. We also show that deterministic but asymmetric injections of money may alter the distribution of real balances in such a way that aggregate output and welfare are permanently increased.

The notion of long-run money neutrality but short-run non-neutrality dates back to Hume and has given rise to a large body of theoretical and empirical research (see Lucas, 1996). It is clear from this research that the answer one obtains regarding the neutrality of money hinges on the trading environment and the manner in which the money supply changes. Monetary injections can be non-neutral if prices are rigid, there are informational frictions, transfers are asymmetric, or there is a non-degenerate distribution of money.[1] The problem with many of these models is that money is either assumed to be needed for trading or, if it is essential for trade, the models cannot be studied analytically or only examine special transaction patterns.

In addition to the issue of non-neutrality, there is also the question of whether monetary injections improve efficiency and welfare. For example, if some agents are cash constrained, monetary injections would appear to provide much needed liquidity. Unfortunately, the Friedman rule requires withdrawals of liquidity from agents. This tension between relaxing liquidity constraints and the Friedman rule was summarized by Lucas (1980) in his seminal model on cash-in-advance at the first Models of Money conference:

[1] Informational frictions generate non-neutralities for example in Lucas (1972), Katzman, Kennan, and Wallace (2004), and Wallace (1997). Examples where non-neutralities occur because of asymmetric injections are Kiyotaki-Wright (1991, 1993) type models, limited participation in financial markets (Lucas (1990), Fuerst (1992), Williamson (2004)), segmented markets (Alvarez, Atkinson and Kehoe (2002)), or overlapping generation models (Samuelson, 1958). Heterogeneity in money holdings play a role in for example in Scheinkman and Weiss (1986), Levine (1991), İmrohoroğlu (1992), Camera and Corbae (1999), Molico (1999), Zhu (2003), Berentsen, Camera, Waller (2004) and Bhattacharya, Haslag and Martin (2004).

> In any period, there will be some households ... with large real balances accumulated but no particular urgency to spend them. There will be others... with (low) balances and a high marginal utility of current consumption. ... Can this gap been filled by a government-engineered deflation, in which currency is withdrawn from the system via lump-sum taxes and a positive real yield thereby created? Clearly not, though by some efficiency criteria this policy may be utility increasing. (Lucas 1980, p.144)

Developing monetary models that allow us analytically to study the non-neutrality and possible effiency gains from monetary injections is fruitful. However, doing so requires constructing a model where money is essential. Wallace (1998) forcefully argued that we should construct models that explain *why* money is necessary and then proceed to study *how* monetary changes affect the economy. There is now a growing literature in monetary economics that follows this strategy. We contribute to this literature by building a model emphasizing the medium of exchange role of money, in the tradition of Kiyotaki and Wright (1989, 1993), to address the issue of monetary non-neutrality and efficiency.

Our model combines aspects of the monetary economies in Lagos and Wright (2002) and Aruoba, Waller and Wright (2003). The key difference is that we allow a non-degenerate distribution of money balances to exist in steady state within each period. We also have a stochastic money supply process that allows us to study how random shocks to the money growth rate affect consumption and production.

In our model, infinitely lived agents trade on a sequence of three competitive spot markets within a period and face idiosyncratic production and consumption shocks in some of these markets. There is anonymous trade, imperfect record-keeping and no contract enforcement. These frictions, combined with the absence of double coincidence of wants, are sufficient to make money necessary for trade.

The randomness of preference shocks gives rise to intra-period heterogeneity in money holdings; buyers deplete their money balances while sellers acquire them. This intra-period distribution of money holdings is the key innovation of the model and it drives the results we obtain regarding short-run non-neutrality and welfare. To make the money distribution analytically tractable, we assume that everyone can produce and consume in the last market.

This allows agents to adjust their money holdings in such a way that the distribution of money holdings becomes degenerate in the last market under appropriate conditions.

With regard to money neutrality, we show that, in accordance with many monetary models, there is long-run neutrality from changes in the money stock and the optimal behavior of the money supply is to make the expected return on money equal to the real interest rate. Under the Friedman rule, we achieve the first-best allocation and monetary injections are neutral. In short, under the Friedman rule, holding money is costless so agents are never cash constrained no matter what money shock prevails.[2]

However, away from the Friedman rule, random monetary injections are non-neutral even though all prices change proportionately. An unexpectedly high money growth rate causes aggregate output to increase if the average inflation rate is sufficiently low. For sufficiently high inflation rates, aggregate output is unaffected but individual consumption patterns are affected. These results occur even though injections are symmetric across agents. More importantly, these non-neutralities only exist if the distribution of real balances is not degenerate.

Essentially what high money injections do is provide consumption insurance against states in which agents hold little liquidity. This raises the question of whether monetary policy can be used to provide consumption insurance in a deterministic fashion. We therefore consider a deterministic version of the model where agents receive asymmetric transfers. We show that changes in the asymmetry of transfers have no real effects in the low inflation economy. In contrast, in the high inflation economy giving the poor a larger share of the monetary injection raises aggregate output permanently and is welfare improving from the perspective of the representative agent under appropriate conditions.

[2] Alternatively, agents hold the optimum quantity of money which is finite. This differs from Bewley's (1980) model because agents do not face an infinite sequence of random consumption opportunities. The Lagos-Wright framework gives agents a chance to readjust their money balances after a finite number of trades. Thus, any finite stock of money is optimal under the Friedman rule.

2 The Environment

The basic environment is that of Lagos and Wright (2002) modified as in Aruoba, Waller and Wright (2003). Time is discrete and in each period a $[0,1]$ continuum of infinitely-lived agents trade on three Walrasian markets, that open and close sequentially. Only one market, denoted by $j = 1, 2, 3$, is open at any one time.

One perishable good is produced and consumed by all agents. Before entering the first two markets an agent receives one of two equally probable consumption/production shocks. He may want to consume or produce but not both. As a result, there is an equal number of consumers and producers in each market. Agents get utility $u(q)$ from consuming $q > 0$ in the first two markets, where $u'(q) > 0$, $u'(0) = \infty$, $u'(+\infty) = 0$, $u''(q) < 0$ and $u'''(q) \geq 0$. In the last market all agents consume and produce, getting utility $U(q)$ from q consumption, with $U'(q) > 0$, $U'(0) = \infty$, $U'(+\infty) = 0$ and $U''(q) < 0$.[3] Production of q output generates disutility q. The discount factor across dates is $\beta \in (0, 1)$.

To rule out credit and motivate fiat monetary exchange, we assume anonymous trading, no record-keeping and no enforcement of contracts. This is sufficient to make fiat currency essential for trade. In order to study the non-neutralities of money we want to see how unexpected changes of the money supply affect consumption and production. To this end, we assume that the law of motion of the money stock is $M_t = z_t M_{t-1}$, where z_t is a random variable such that

$$z_t = \begin{cases} z^H = \mu\left(1 + \varepsilon^H\right) & \text{with probability} \quad \pi \\ z^L = \mu\left(1 - \varepsilon^L\right) & \text{with probability} \quad 1 - \pi. \end{cases}$$

We assume $\mu, \varepsilon^L, \varepsilon^H > 0$ and $\pi = \frac{\varepsilon^L}{\varepsilon^H + \varepsilon^L}$ so that $E(z_t) = \mu$.

We also assume money is injected via lump-sum transfers $\tau_t = (z_t - 1)M_{t-1}$, after the closing of market 1 in period t, but prior to the realization of individual trading shocks. In short, at the beginning of the second market one of two states, denoted $i = H, L$, can be realized. In one state money growth is large, z^H, in the other it is small, z^L.

We refer to M_{t-1} as the beginning-of-period money supply for date t. This is the money supply existing in the economy before the shock takes place. We refer to M_t as the money

[3] The difference in preferences over the good sold in the last market is a technical device we use to obtain a degenerate distribution of money holdings, at the beginning of a period.

supply present in the market at the end of period t, after the shock is realized. This is the money stock that will be available for trade at the beginning of in period $t+1$.

2.1 Sequential Market Trades in a Stationary Equilibrium

In period t, let $p_{j,t}$ be the nominal price in market j, and $\phi_t = 1/p_{3,t}$ be the real price in the last market. We study equilibria where end-of-period real money balances are time-invariant

$$\phi_t M_t = \phi_{t+1} M_{t+1}. \tag{1}$$

We refer to it as a stationary equilibrium. For this reason we omit the time subscript when understood, and study a representative period working backwards from last to first market, within the period. In the steady state it then follows that prices change instantly and proportionately since

$$\frac{\phi^H}{\phi_{-1}} = \frac{1}{z^H} \text{ and } \frac{\phi^L}{\phi_{-1}} = \frac{1}{z^L} \tag{2}$$

We also have that average inflation is $E\left(\frac{\phi_t}{\phi_{t+1}}\right) = E\left(z_t\right) = \mu$, while the average gross real return on money is

$$R = E_t\left[\frac{\phi_{t+1}}{\phi_t}\right] = E_t\left[\frac{M_t}{M_{t+1}}\right] = \frac{1}{\mu}\frac{1+\varepsilon_H-\varepsilon_L}{\left(1+\varepsilon_H\right)\left(1-\varepsilon_L\right)} \equiv \frac{1}{\gamma},$$

which is negatively associated with expected inflation μ. In our analysis we will focus on γ rather than μ since γ is proportional to μ.

Let $V_j(m_j)$ denote the expected value from trading in market j with m_j money. Let q_{jb} and q_{js} respectively denote the quantities bought or sold by an agent trading in market j. We let q_3^* be the solution to $U'(q_3) = 1$ and q^* the solution to $u'(q_j) = 1$ for $j = 1, 2$.

2.1.1 The last market

In the last market agents can produce and consume. They choose how much to buy, q_{3b}, how much to sell, q_{3s}, and how much money to take into the next period, $m_{1,+1}$. As a result, the representative agent's program is

$$\begin{aligned} V_3\left(m_3\right) \quad &= \max_{q_{3b},q_{3s},m_{1,+1}} \left[U\left(q_{3b}\right) - q_{3s} + \beta V_1\left(m_{1,+1}\right)\right] \\ &\text{s.t.} \quad q_{3b} + \phi m_{1,+1} = q_{3s} + \phi m_3 \end{aligned}$$

where $m_{1,+1}$ is the money taken into period $t+1$. Substituting for q_{3s} yields

$$V_3(m_3) = \phi m_3 + \max\left[U(q_{3b}) - q_{3b} - \phi m_{1,+1} + \beta V_1(m_{1,+1})\right] \tag{3}$$

where $(q_{3b}, m_{1,+1})$ are choice variables, hence the conditions for maximization are

$$\begin{aligned} &U'(q_{3b}) = 1 \\ &-\phi + \beta V_1'(m_{1,+1}) \leq 0 \qquad (= 0 \text{ if } m_{1,+1} > 0) \end{aligned} \tag{4}$$

so that $\beta V_1'(m_{1,+1}) = \phi$ in a stationary monetary equilibrium, since $m_{1,+1} > 0$.

There are two key results. First, trades are always efficient in the last market, since $q_{3b} = q_3^*$ always and for every agent. Second, and most importantly, the distribution of beginning-of-period money holdings is degenerate. This is because $m_{1,+1}$ is chosen independently of m_3. The reason is $V_3(m_3)$ is linear, so the equilibrium marginal value of money in the last market is independent of the agent's holdings, i.e.

$$V_3'(m_3) = \phi. \tag{5}$$

It follows that in equilibrium everyone exits the last market with identical money holdings, regardless of how much money they brought into the last market. Those who bring excessive money into the last market, spend some on goods, while those with too little money sell output.[4] This feature of the Lagos and Wright model makes the distribution of money degenerate at the beginning of market one.[5]

2.1.2 The second market

Conditional on the realization of the shock z_t, an agent who has m_2 money balances at the opening of the second market, at any date t, has expected lifetime utility

$$V_2(m_2) = \frac{1}{2}\left[u(q_{2b}) + V_3(m_2 - p_2 q_{2b})\right] + \frac{1}{2}\left[-q_{2s} + V_3(m_2 + p_2 q_{2s})\right]. \tag{6}$$

Here $p_2 q_{2b}$ is the amount of money spent when buying q_{2b} goods, and $p_2 q_{2s}$ is the money received when selling q_{2s} goods.

[4] Conditions need to be imposed to ensure $q_{s3} \geq 0$. See later.

[5] Thus, we avoid analytical intractabilities and the need to solve numerically for the distribution of money as done, for example, in Molico (1999) or İmrohorğlu (1992). See Shi (1997) for an alternative way to obtain a degenerate distribution.

The agent chooses quantities to buy and sell, taking the price p_2 as given. Specifically, as a seller, the agent chooses q_{2s} to maximize $-q_{2s}+V_3\left(m_2+p_2q_{2s}\right)$. This yields the first-order condition

$$p_2V_3'\left(m_2+p_2q_{2s}\right)=1 \quad \Rightarrow \quad p_2=p_3=\frac{1}{\phi} \tag{7}$$

where we have used (5). That is prices in the last two markets must be equal and are pinned down by the agent's value of money in the last market.

The intuition is that the seller can acquire a unit of money in the second or the third market and will do so at the lowest cost. Since sellers have linear production costs, if $p_2>p_3$ it is cheaper to acquire money in the second market and vice versa if $p_2<p_3$. At price $p_2=p_3$ sellers are indifferent as to which market they sell in to acquire money. This also implies that they are willing to supply all that is demanded, so the supply curve in the second market is flat.

As a buyer, the agent chooses q_{2b} to maximize his expected utility $u\left(q_{2b}\right)+V_3\left(m_2-p_2q_{2b}\right)$, given his cash constraint $p_2q_{2b}\leq m_2$. Letting $\lambda_2\geq 0$ be the multiplier on the cash constraint, the conditions for maximization are

$$\begin{aligned} u'\left(q_{2b}\right)&=p_2V_3'\left(m_2-p_2q_{2b}\right)+p_2\lambda_2 \\ \lambda_2(m_2-p_2q_{2b})&=0 \end{aligned} \tag{8}$$

We can state the following

Lemma 1. *Let $m^*=q^*/\phi$. In equilibrium, if*

(i) $m_2<m^*$ then $\lambda_2>0$, $q_{2b}=\phi m_2<q^*$ and $V_2\left(m_2\right)$ is strictly increasing and concave;

(ii) $m_2\geq m^*$ then $\lambda_2=0$, $q_{2b}=q^*\leq\phi m_2$ and $V_2\left(m_2\right)$ is strictly increasing and linear.

The key implication is that trades in the second market are inefficient, $q_{2b}<q^*$, if the buyer is cash constrained, $m_2<m^*$. Otherwise, they are efficient. To see why, in the appendix we show that if the constraint is binding,

$$V_2'\left(m_2\right)=\frac{1}{2p_2}\left[u'\left(q_{2b}\right)+1\right]>\phi \tag{9}$$

whereas if it is not binding

$$V_2'(m_2) = \phi. \tag{10}$$

Intuitively, if a buyer is not cash constrained, $m_2 \geq m^*$, then he spends only part of his money and carries the rest into the last market. Thus, the marginal value of money for an agent entering market 2 with $m_2 \geq m^*$ is simply ϕ, or the value of money in the last market. The reason is, whether he ends up buying or selling, the agent will not spend all his balances.

If the agent enters market 2 with less than m^*, however, he is cash constrained. Therefore, the marginal value of money is greater than ϕ. Here, the marginal value of money has two components. With probability one half, the agent sells so he does not spend any money and values an extra dollar at the prevailing market price $\frac{1}{p_2}$. With probability one half the agent buys, in which case an extra dollar buys $\frac{1}{p_2}$ goods whose marginal utility is $u'(q_{2b})$. Since the buyer optimally spends all his money, the marginal value of money must be greater than the value of simply holding onto it, $\frac{1}{p_2}$. Thus, $V_2'(m_2) > \frac{1}{p_2} = \phi$.

2.1.3 The first market

An agent starting a period with m_1 money has expected lifetime utility

$$V_1(m_1) = \frac{1}{2}\left[u(q_{1b}) + EV_2(m_1 - p_1 q_{1b} + \tau)\right] + \frac{1}{2}\left[-q_{1s} + EV_2(m_1 + p_1 q_{1s} + \tau)\right] \tag{11}$$

where $p_1 q_{1s}$ and $p_1 q_{1b}$ are, respectively, the amounts of money received as a seller and spent as a buyer. Notice that agents take into account that they will receive a random transfer τ at the beginning of market 2.

As a seller, the agent chooses q_{1s} to maximize his expected profit $-q_{1s}+ EV_2(m_1 + p_1 q_{1s})$, taking the price p_1 as given. This yields the first-order condition

$$p_1 EV_2'(m_1 + p_1 q_{1s} + \tau) = 1. \tag{12}$$

Production takes place until the marginal value of money, $EV_2'(m_1 + p_1 q_{1s} + \tau)$, equals its real price $1/p_1$. This money can be used to buy consumption in markets that open later.

As a buyer the agents chooses q_{1b} to maximize $u(q_{1b}) + EV_2(m_1 - p_1 q_{1b} + \tau)$ subject to the cash constraint $p_1 q_{1b} \leq m_1$. The conditions for maximization are

$$\begin{aligned} u'(q_{1b}) &= p_1 E V_2'(m_1 - p_1 q_{1b} + \tau) + p_1 \lambda_1 \\ \lambda_1 (m_1 - p_1 q_{1b}) &= 0 \end{aligned} \tag{13}$$

where $\lambda_1 \geq 0$ is the multiplier on the cash constraint. It follows

Lemma 2. *In equilibrium $\lambda_1 = 0$, $q_{1b} = q_{1s} = q_1 \leq q^*$, $q_1 < m_1/p_1$, and $V_1(m_1)$ is strictly increasing and concave.*

The main implication of Lemma 2 is that agents never spend all of their money in the first market, $q_1 < m_1/p_1$. The marginal value of consuming even a little bit in the second market is very high for every agent, should a consumption opportunity arise. Consequently, for precautionary reasons, agents always want to carry some cash into the second market. In short, agents self-insure against consumption risk.

We also have

$$V_1'(m_1) = \frac{1}{2p_1}\left[u'(q_{1b}) + 1\right] \tag{14}$$

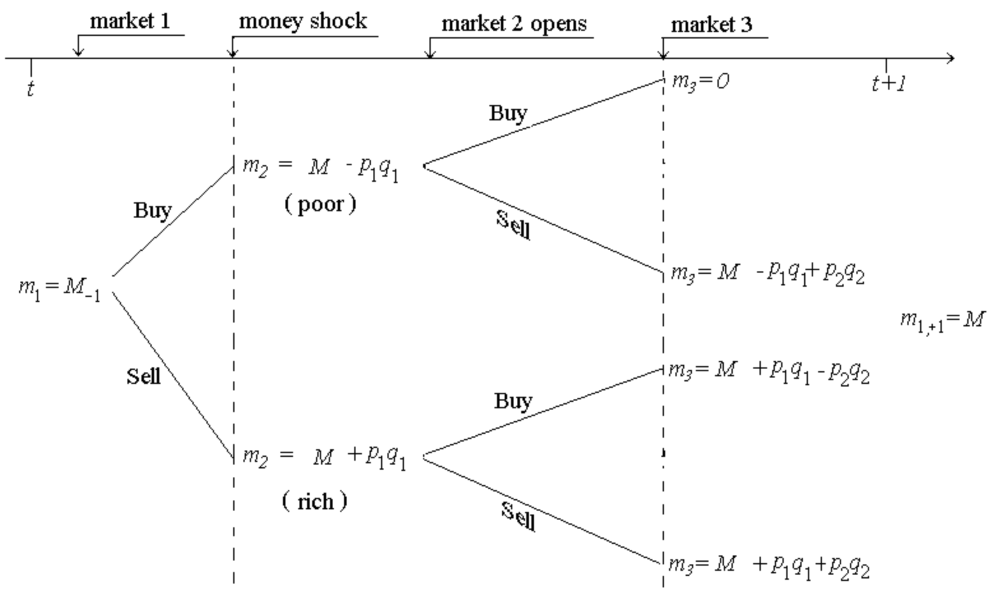
that is, the marginal value of money at the opening of the first market is given by an expression similar to (9). The difference is that $1/p_2$ is equal to ϕ whereas $1/p_1$ may not. This possible price dispersion across markets play a key role in some of our later results.

3 Equilibria

A key feature of our model is that the idiosyncratic consumption and production shocks generate intra-period heterogeneity in money balances. As we demonstrate later, the existence of a non-degenerate distribution of money holdings is what opens the door to possible beneficial effects of money creation in the short-run.

More precisely, every agent enters a period with $m_1 = M_{-1}$ money, i.e. the money stock from the prior period (see Figure 1). Then, agents are randomly divided into buyers and sellers in market 1. Buyers reduce their money holdings by $p_1 q_1$ and sellers acquire $p_1 q_1$. Then the injection occurs. Consequently, when the second market opens half of the agents will be 'poor', holding $M - p_1 q_1$ units of money, and half will be 'rich', holding $M + p_1 q_1$.

Then, agents will once more be divided into sellers and buyers. Since marginal cost is constant in equilibrium sellers are indifferent of how much to produce. For simplicity, we assume that all sellers produce the same amount.[6] Therefore, when market 3 opens the support of the distribution of money will have four mass points. However, all agents leave market 3 with the same money holdings $m_{1,+1} = M$.



From what we have learned so far, consumption may differ across buyers only in the second market, due to heterogeneity in money holdings. Thus, let (q_2^{pi}, q_2^{ri}) and $(\lambda_2^{pi}, \lambda_2^{ri})$ denote the values of consumption and cash constraint multipliers of, respectively, poor and rich buyers in market 2, contingent on the realization of state $i = H, L$. If, by a small abuse in notation, we let $q_2 = (q_2^{pH}, q_2^{pL}, q_2^{rH}, q_2^{rL})$, $\lambda_2 = (\lambda_2^{pH}, \lambda_2^{pL}, \lambda_2^{rH}, \lambda_2^{rL})$, and let $\bar{m}_j$ denote the vector of possible money holdings at the opening of market j we can state the following

Definition. *A stationary monetary equilibrium is a time-invariant list* $\{p_j, q_j, \bar{m}_j\}_{j=1}^{3}$ *and* $\{\lambda_1, \lambda_2\}$ *that satisfy (1)-(4), (6)-(8), and (11)-(13).*

In the proof of Proposition 1 we show that there exist critical values $\gamma_1 \leq \gamma_2 \leq \gamma_3$ such that the following is true.

[6] This indifference would vanish if we had increasing marginal cost. However, it would greatly complicate the analysis without changing the basic results.

Proposition 1 *A stationary monetary equilibrium exists only if $\gamma \geq \beta$. An equilibrium exists and is unique for $\gamma \in (\beta, \gamma_2]$ where*

i) for $\gamma \in (\beta, \gamma_1]$, $q_2^{pL} < q_2^{pH} = q^*$ and $q_2^{rL} = q_2^{rH} = q^*$;

ii) for $\gamma \in (\gamma_1, \gamma_2]$, $q_2^{pL} < q_2^{pH} < q^*$ and $q_2^{rL} = q_2^{rH} = q^*$.

For $\gamma > \gamma_2$ if an equilibrium exists then

iii) for $\gamma \in (\gamma_2, \gamma_3]$, $q_2^{pL} < q_2^{pH} < q^*$ and $q_2^{rH} < q_2^{rL} = q^*$;

iv) for $\gamma > \gamma_3$, $q_2^{pL} < q_2^{pH} < q^*$ and $q_2^{rH} < q_2^{rL} < q^*$.

Since γ is monotonically increasing in the expected gross inflation rate μ, Proposition 1 also characterizes the monetary equilibrium as a function of μ. When $\gamma < \gamma_2$ we refer to this economy as the *low* inflation economy, and when $\gamma \geq \gamma_2$ we call it the *high* inflation economy. In the low inflation economy rich buyers are never constrained. In the high inflation economy they can be constrained. Although we cannot prove existence of equilibrium when inflation is high, for particular utility functions we can show that it exists.

From Proposition 1 the following is true:

Corollary 2 *All quantities less than q^* are strictly decreasing in γ and approach q^* as $\gamma \to \beta$. Consequently, the Friedman rule attains the first-best allocation.*

Corollary 3 *Shocks to the money supply are non-neutral in the short run.*

Corollary 2 is a standard result in these types of models that money is not superneutral. An increase of the money growth rate decreases the value of money which reduces consumption (e.g. see Lagos and Wright (2002) or Shi (1997)). In contrast to Lagos and Wright (2002) we obtain the first-best under the Friedman rule because we have competitive markets and not bilateral bargaining.[7] Under the Friedman rule, the expected opportunity cost of holding money is zero so agents take enough money to buy the efficient quantity in all markets for all states. Thus, they are never cash constrained.

[7] Note that the Friedman rule here requires less deflation than in a deterministic model, i.e. to set the average gross inflation rate μ above β. In fact, the optimal policy may require a positive average rate of inflation. For example, if $\varepsilon = \varepsilon^H = \varepsilon^L$, then one can show that the optimal average rate of inflation is strictly positive if $\varepsilon > \sqrt{1-\beta}$.

Corollary 3 summarizes the main result of our paper: monetary shocks have real effects for individual agents in all equilibria. The non-neutrality is a direct consequence of the distribution of money holdings. However, there is no persistence on quantities from these shocks. So all real effects are temporary. It can be shown that random injections in Lagos and Wright (2002) are neutral regardless of when they occur. The same is true in our model if they occur when the distribution is degenerate, i.e. in markets 1 or 3.

The intuition for this result is straightforward. After any shock to the money supply prices change proportionately. When money is higher than expected, the price increase reduces the real balances of every agent, acting as a *proportional* tax on their money holdings. Since in market 2 money holdings are heterogeneous, those who hold less cash are taxed less than those who hold more. This allows poor buyers—who are cash strapped—to increase their consumption in market 2 because the lump-sum transfer more than offsets the inflation tax. In contrast, rich buyers lose real wealth even after accounting for the lump-sum transfer. This does not affect their consumption when inflation is low because in this case rich buyers are not constrained by their cash holdings. However, if inflation is high the inflation tax created by the surprise injection reduces the consumption of rich buyers because they are also cash constrained in market 2. Effectively, an unanticipated increase in the money stock redistributes real wealth from those with more to those with less through the price increase.

Proposition 4 *Consider an unexpected increase in the money supply. For $\gamma < \gamma_3$ aggregate output is higher than average. For $\gamma \geq \gamma_3$ aggregate output is unaffected by the money supply shock.*

It is clear that aggregate output is increasing in the low inflation economy since rich buyers do not change their consumption while poor buyers consume more. In contrast, in the high inflation economy when all agents are constrained, i.e. case (iv) in Proposition 1, aggregate output is unaffected by the monetary shock since rich buyers reduce their consumption by the same amount as the poor buyers increase theirs.

In summary, the short-run non-neutralities of our model hinge on three key elements. First, monetary injections must be unanticipated. Second, such injections must take place when agents hold different amounts of money. Third, inflation cannot be out of hand, otherwise every buyer would be cash constrained and so aggregate output is unaffected. What our results do *not* hinge on are any price rigidities, information frictions, or asymmetric injections across agents.

4 Consumption insurance

Clearly, the high money shock is welfare improving. It raises consumption of poor buyers without affecting the consumption of the rich buyers in the low inflation equilibrium. Although it lowers their consumption in the high inflation economy, it also increases the poor buyers' consumption by the same amount. Since the rich have a lower marginal utility of consumption than the poor there is still a potential for welfare gains from this redistribution. Unfortunately, the low money supply shock does just the opposite. So it is hard to imagine that these random injections improve welfare on average.

From the perspective of the representative agent at the start of market 1, a high money shock acts like consumption insurance. This suggests that a scheme that transfers real balances from agents when they are rich to when they are poor in all periods would be welfare improving. In the following we explore this issue, i.e. we ask whether a monetary scheme exists that provides consumption insurance in all periods.

To explore this question, let us assume that $z^H = z^L = \gamma$ so that the money supply is deterministic. With this process the only possible equilibrium allocations are the ones described in (ii) and (iv) of Proposition 1. From (ii) in Proposition 1 rich buyers are unconstrained and from (iv) they are constrained.

Assume further that the perfectly anticipated lump-sum transfer received by agents depends on their trading state in the first market as follows. Each agent who drew a consumption opportunity in the first market gets the transfer $\tau_t^p = (\gamma - 1)xM_t$ and each agent who drew a production opportunity gets the transfer $\tau_t^r = (\gamma - 1)(2 - x) M_t$ where $x \in [0, 2]$. This allows us to consider, for example, symmetric transfers ($x = 1$) as in the previous section, transfers only to the poor ($x = 2$) and transfers only to the rich ($x = 0$).

Our new assumptions do not affect the equations in the second and last markets. In the first market the changes are as follows. An agent starting a period with m_1 money has expected lifetime utility

$$V_1(m_1) = \frac{1}{2}\left[u(q_{1b}) + V_2(m_1 - p_1 q_{1b} + \tau^p)\right] + \frac{1}{2}\left[-q_{1s} + V_2(m_1 + p_1 q_{1s} + \tau^r)\right] \qquad (15)$$

Here, the perfectly anticipated transfers τ^p and τ^r are new in equation (15). As a seller, the agent chooses q_{1s} to maximize his expected profit $-q_{1s} + V_2(m_1 + p_1 q_{1s} + \tau^r)$, taking the price p_1 as given. This yields the first-order condition

$$p_1 V_2'(m_1 + p_1 q_{1s} + \tau^r) = 1.$$

As a buyer the agents chooses q_{1b} to maximize $u(q_{1b}) + V_2(m_1 - p_1 q_{1b} + \tau^p)$ subject to the cash constraint $p_1 q_{1b} \leq m_1$. The conditions for maximization are

$$u'(q_{1b}) = p_1 V_2'(m_1 - p_1 q_{1b} + \tau^p) + p_1 \lambda_1$$
$$\lambda_1(m_1 - p_1 q_{1b}) = 0$$

where $\lambda_1 \geq 0$ is the multiplier on the cash constraint.

In the following we analyze how a change in x affects steady state production and consumption in markets 1 and 2.

Proposition 5 *In the low inflation economy, changes in x have no effect on individual or aggregate consumption in markets 1 and 2. In the high inflation economy if $\gamma \neq 1$, changes in x are non-neutral.*

A change in x does not have any real effects when inflation is low. The only effect is that the steady state value of real money balances decreases. In contrast, if inflation is high changes in x are non-neutral.

What is the intuition for this result? Changes in x have different effects on the relative prices across markets. In the low inflation economy all prices are the same, i.e. $p_1 = p_2 = p_3 = 1/\phi$. Thus, any changes in x causes all prices to change proportionately. Since relative prices between markets 1 and 2 are unaffected, market 1 consumption does not change. But then, by the inter-market Euler equation, consumption of the poor buyers in market 2 cannot change. Finally, since the rich buyers continue to consume q^* in market 2, their consumption is not affected.

In the high inflation economy there is price dispersion across markets since $p_1 < p_2 = p_3$. The price in market 1, p_1, depends on the marginal value of money in market 2, which in contrast to the low inflation economy is non-linear in x. Then, changes in x change the relative price between markets 1 and 2. As a consequence, agents change their consumption patterns. More intuitively, an increase in x reduces the money holdings of rich agents in market 2 and, because they are cash constrained in this equilibrium, this increases their marginal value of money. Since they are the sellers in market 1, choose to sell more in market 1 to acquire additional cash. By selling more, they lower p_1 relative to p_2 and so buyers find it optimal to consume more in market 1.

The following Proposition gives us insight how the consumption pattern in the high inflation economy changes.

Proposition 6 *Consider a small change of x from $x = 1$ in the high inflation economy. Then, if $\frac{1+u'(q)}{u'(q)} > -\frac{qu''(q)}{u'(q)}$ for all feasible q, for $\gamma > 1$, q_1 and q_{2b}^p increase while q_{2b}^r decreases. If $\gamma < 1$, the opposite happens.*

Effectively, an increase in x lowers p_1 relative to p_2 if $\gamma > 1$. The reverse is true for $\gamma < 1$ because in this case lump-sum taxes are being imposed. This causes q_1 to change inversely. Then, the inter-market Euler condition for buyers implies that q_{2b}^p changes in the same direction as q_1 and the sellers first-order condition in market 1 causes q_{2b}^r to move in the opposite direction.

We define welfare as the life-time expected utility of the representative agent at the beginning of the period.

Corollary 7 *For $\gamma > 1$ increasing x increases welfare. For $\gamma < 1$, welfare increases by decreasing x if the equilibrium exists.*

The reason welfare increases is that an increase in x provides consumption insurance in market 2 - agents give up consumption when they are rich but increase it when they are poor. Given our assumptions on preferences, this insurance lowers the expected marginal utility of consumption in market 2 which induces agents to increase consumption in market 1.

Finally, note that in our model the beneficial effects of asymmetric transfers are only due to consumption insurance unlike other Kiyotaki-Wright models where asymmetric transfers are welfare improving because they change the composition of buyers and sellers.

5 Conclusion

We have presented a framework in which a monetary expansion, while neutral in the long-run, can have beneficial effects in the short-run. The objective is to take a first step in complementing the large literature of the effects of money creation, by building on a recent research program that emphasizes the medium of exchange role of money.

The key feature of our model is that agents trade on a sequence of competitive markets while being subject to idiosyncratic shocks. For this reason, there is equilibrium heterogeneity in money balances, so that one-time monetary transfers can be used to redistribute purchasing power from rich to poor. Since an unexpectedly high money growth rate redirects consumption to those who most value it, welfare is positively affected.

The short-run non-neutralities of our model hinge on three key elements. First, monetary injections must be unanticipated. Second, such injections must take place when agents hold different amounts of money. Third, inflation cannot be out of hand, otherwise aggregate output is unaffected. What our results do *not* hinge on are any price rigidities, information frictions, or asymmetric injections across agents.

Finally, we show that by providing consumption insurance fully anticipated asymmetric lump-sum transfers increase aggregate output and welfare in the high inflation economy permanently. Surprisingly, such a scheme has no real effects when inflation is low.

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Appendix

Proof of Lemma 1. If the constraint is not binding then $\lambda_2 = 0$. Using (8) then $u'(q_{2b}) = 1$. Here trades are efficient. The buyer spends q^*/ϕ money, and we let $m^* = q^*/\phi$ denote money holdings such that the cash constraint does not bind. If the constraint is binding, $\lambda_2 > 0$, then (8) implies $u'(q_{2b}) = 1 + \frac{\lambda_2}{\phi}$ and $p_2 q_{2b} = m_2$. Here trades are inefficient. The buyer spends all his money, $p_2 q_{2b} = m_2 < m^*$, and consumes $q_{2b} = m_2/p_2 < q^*$.

To examine concavity of V_2 differentiate (6) with respect to m_2 to get

$$\begin{aligned} V_2'(m_2) = \; & \tfrac{1}{2}\left[u'(q_{2b})\tfrac{\partial q_{2b}}{\partial m_2} + V_3'(m_2 - p_2 q_{2b})\left(1 - p_2 \tfrac{\partial q_{2b}}{\partial m_2}\right)\right] \\ & + \tfrac{1}{2}\left[-\tfrac{\partial q_{2s}}{\partial m_2} + V_3'(m_2 + p_2 q_{2s})\left(1 + p_2 \tfrac{\partial q_{2s}}{\partial m_2}\right)\right] \end{aligned}$$

Then (7), (8) and $\phi = 1/p_2$ imply that

$$V_2'(m_2) = \frac{1}{2p_2}\left[u'(q_{2b}) + 1\right] + \frac{1}{2}\lambda_2\left[1 - p_2\frac{\partial q_{2b}}{\partial m_2}\right] \tag{16}$$

If $\lambda_2 = 0$, then $u'(q_{2b}) = 1$ and $V_2'(m_2) = \phi$, so $V_2(m_2)$ is linear in m_2 for $m \geq m^*$. If $\lambda_2 > 0$, then $p_2 q_{2b} = m_2$, which implies that $1 - p_2 \frac{\partial q_{2b}}{\partial m_2} = 0$. Hence, $V_2'(m_2) = \phi\left[\frac{u'(q_{2b})+1}{2}\right] > \phi$ since $u'(q_{2b}) > 1$. Note that $V_2''(m_2) < 0$ because $\frac{\partial q_{2b}}{\partial m_2} > 0$, so that $V_2(m_2)$ is concave $\forall m_2 < m^*$.■

Proof of Lemma 2. First prove that $\lambda_1 = 0$ always. Suppose $\lambda_1 > 0$. Then $m_2 = 0$ and $q_{2b} = 0$ implying $u'(0) = 1 + \lambda_2/\phi$, which is not possible since $u'(0) = \infty$. Thus $\lambda_1 = 0$, in which case (12)-(13) yield

$$u'(q_{1b}) = \frac{EV_2'(m_1 - p_1 q_{1b} + \tau)}{EV_2'(m_1 + p_1 q_{1s} + \tau)}. \tag{17}$$

If $m_1 - p_1 q_{1b} < m^*$ then $EV_2(m_1 - p_1 q_{1b} + \tau)$ is concave, hence $u'(q_{1b}) > 1$ and $q_{1b} < q^*$. If $m_1 - p_1 q_{1b} \geq m^*$ then both numerator and denominator are linear, hence $u'(q_{1b}) = 1$ and $q_{1b} = q^*$. Hence, $q_{1b} \leq q^*$.

Differentiating (11) with respect to m_1

$$\begin{aligned} V_1'(m_1) = \; & \tfrac{1}{2}\left[u'(q_{1b})\tfrac{\partial q_{1b}}{\partial m_1} + EV_2'(m_1 - p_1 q_{1b} + \tau)\left(1 - p_1 \tfrac{\partial q_{1b}}{\partial m_1}\right)\right] \\ & + \tfrac{1}{2}\left[-\tfrac{\partial q_{1s}}{\partial m_1} + EV_2'(m_1 + p_1 q_{1s} + \tau)\left(1 + p_1 \tfrac{\partial q_{1s}}{\partial m_1}\right)\right] \end{aligned}$$

Using (12) and (13) (for $\lambda_1 = 0$) yields

$$V_1'(m_1) = \frac{1}{2p_1}\left[u'(q_{1b}) + 1\right]$$

Thus, if $q_{1b} < q^*$ then $V_1(m_1)$ is strictly increasing and concave.

Since everyone enters the first market with identical money balances, and there is an identical number of buyers and sellers, in equilibrium, $q_{1b} = q_{1s} = q_1$.■

Proof of Proposition 1. The shock to the money supply is realized before the second market opens. Thus, p_2 adjusts instantly and proportionately to the change in the money stock, and so does the expected value of ϕ. Then (2) implies that

$$\frac{\phi^H}{E\phi} = k^H \equiv \frac{1-\varepsilon^L}{1+\varepsilon^H-\varepsilon^L} < 1 \tag{18}$$

$$\frac{\phi^L}{E\phi} = k^L \equiv \frac{1+\varepsilon^H}{1+\varepsilon^H-\varepsilon^L} > 1 \tag{19}$$

where k^i do not depend on μ and $k^H < k^L$. Note that k^i is the state price of money relative to the expected price. So when money is high the price of money is relatively low.

Suppose first that $\lambda_2^p = \lambda_2^r = 0$ for all states. Then $q_{2b} = q^*$ for all agents in all states. Therefore as shown in Lemma 1 $V_2'(m_2) = \phi$ and therefore $EV_2'(m_2) = E\phi$. Then from the first-order condition of the buyer in market 1 (13) we have $u'(q_1)/p_1 = E\phi$ and from the first-order condition of the seller in market 1 (12) we have $p_1 = 1/E\phi$. Finally, (14) implies that the marginal value of money at the beginning of a period is equal to the expected value at the end of the period

$$V_1'(m_1) = E\phi$$

This condition says that if agents take a unit of money into the first market but do not intend to spend it in either the first or second markets, then the value of this extra unit of money is the goods it buys in the last market. Substituting this expression into (4), and backdating it, gives

$$\phi_{-1}\left[\beta E\left(\frac{\phi}{\phi_{-1}}\right) - 1\right] = \phi_{-1}(\beta/\gamma - 1) \leq 0. \tag{20}$$

For $\beta/\gamma < 1$ this expression is negative implying $m_1 = 0$ which cannot be an equilibrium. For $\beta/\gamma > 1$ agents want to hold an infinite amount of money, since its rate of return is greater than the discount rate. This also cannot be an equilibrium. For $\beta/\gamma = 1$, there is an infinity of monetary equilibria, one for each value of ϕ_{-1}.

Suppose $\lambda_2^r > \lambda_2^p = 0$ in one or both states. From (8) this is a contradiction since m_2 is larger for rich agents.

Now consider the remaining possibilities.

Equilibrium 1: $\lambda_2^{pL} > 0$ **and** $\lambda_2^{pH} = \lambda_2^{rL} = \lambda_2^{rH} = 0$. In this case, $q_{2b}^{pH} = q_{2b}^{rH} = q_{2b}^{rL} = q^*$. First, we determine q_1. As shown in Lemma 1, $V_2'(m_1 + p_1 q_{1s} + \tau) = \phi$, thus $EV_2'(m_1 + p_1 q_{1s} + \tau) = E\phi$. Using (12), the first-order condition of the seller in market 1, we have $p_1 = 1/E\phi$. Next, (14) implies that the marginal value of money at the beginning of a period satisfies

$$2V_1'(m_1) = E\phi\left[u'(q_1) + 1\right]$$

Finally, the first-order condition in market 3 for the choice of money holdings (4) can be backdated to get

$$\frac{2\phi_{-1}}{\beta} = E\phi\left[u'(q_1) + 1\right].$$

Since $1/\gamma = E\phi/\phi_{-1}$then

$$u'(q_1) = 1 + 2\left(\frac{\gamma - \beta}{\beta}\right) \tag{21}$$

Because of strict concavity of $u(q)$ there is a unique value q_1 that solves (21), and for $\beta < \gamma$, $q_1 < q^*$. As $\gamma \to \beta$, $u'(q_1) \to 1$ and $q_1 \to q^*$.

Next we determine the real money balances Ω. Using (17) and noting that $q_{2b}^{pL} = \phi^L M^L - k^L q_1$ where $\phi^L M^L = \Omega$ we get

$$2u'(q_1) = (1-\pi)\,k^L\left[u'\left(\Omega - k^L q_1\right) + 1\right] + 2\pi k^H \tag{22}$$

For a given value of q_1, it is straightforward to show that a unique value of Ω exists.

It then follows that since the poor buyer spends all of his money in markets one and two, when the state is $i = L$, with $M^L = z^L M_{-1}$ we have $q_{2b}^{pL} = \Omega - k^L q_1$. Then,

$$\phi^L = \frac{\Omega}{M^L},\ \phi^H = \frac{\Omega}{M^H} \text{ and } 1/p_1 = (1-\pi)\frac{\Omega}{M^L} + \pi\frac{\Omega}{M^H}.$$

Finally, for this equilibrium to exist it must be the case that

$$q_{2b}^{pL} = \Omega - k^L q_1 < q^* \ \text{ and } q_{2b}^{pH} = q^* \leq \Omega - k^H q_1$$

which implies

$$q^* + k^H q_1 < \Omega < q^* + k^L q_1. \tag{23}$$

Since $q_{2b}^{pL} = \Omega - k^L q_1$ and $\Omega < q^* + k^L q_1$, then it follows that the poor agent can never buy the efficient quantity in the low state, i.e. $q_{2b}^{pL} < q^*$. As $\gamma \to \beta$, $q_1 \to q^*$ and $\Omega \to q^* \left(1 + k^L\right)$. Since (22) yields

$$\frac{d\Omega}{dq_1} = k^L + \frac{2u''(q_1)}{(1-\pi)\, u''\left(q_{2b}^{pL}\right) k^L} > k^L$$

as γ increases from β, Ω falls faster than the right-hand inequality in (23). For a sufficiently high value of γ, call it γ_1, the left-hand inequality will bind and beyond that will be violated. Hence, for $\gamma \in (\beta, \gamma_1]$ this equilibrium exists. Note, if $\varepsilon^L = \varepsilon^H = 0$ this equilibrium cannot exist for any γ.

Equilibrium 2: $\lambda_2^p > 0$ and $\lambda_2^r = 0$ for both states. In this case, $q_{2b}^{rH} = q_{2b}^{rL} = q^*$. Consequently, Lemma 1 implies that $EV_2'(m_1 + p_1 q_{1s} + \tau) = E\phi$ and the first-order condition of the seller in market 1 implies that $1/p_1 = E\phi$.

By using the same procedure as before one can show that the solution for q_1 is once again given by (21).

To find the real money balances Ω we once again use (17) to get

$$2u'(q_1) = (1-\pi)\, k^L \left[u'\left(\Omega - k^L q_1\right) + 1\right] + \pi k^H \left[u'\left(\Omega - k^H q_1\right) + 1\right]$$

Again a unique value of Ω exists. Using the solutions for Ω and q_1 we obtain

$$\begin{aligned} q_{2b}^{pH} &= \Omega - k^H q_1 > q_{2b}^{pL} = \Omega - k^L q_1 \\ \phi^L &= \frac{\Omega}{M^L},\ \phi^H = \frac{\Omega}{M^H} \text{ and } 1/p_1 = (1-\pi)\frac{\Omega}{M^L} + \pi \frac{\Omega}{M^H} \end{aligned}$$

For this equilibrium to exist we need that the poor buyers' money balances satisfy

$$q_{2b}^{pL} = \Omega - k^L q_1 < q^* \text{ and } q_{2b}^{pH} = \Omega - k^H q_1 < q^*$$

while the rich buyers' money balances satisfy

$$\Omega + k^L q_1 > q^* \text{ and } \Omega + k^H q_1 > q^*$$

Combining these two sets of inequalities, the sufficient condition for this equilibrium is

$$q^* - k^H q_1 < \Omega < q^* + k^H q_1$$

At γ_1, the right-hand inequality binds. As γ increases above γ_1 once again Ω falls faster than q_1. Finally at some $\gamma_2 > \gamma_1$ the left-hand inequality binds. Thus for $\gamma \in (\gamma_1, \gamma_2]$ this equilibrium exists.

Equilibrium 3: $\lambda_2^{pH}, \lambda_2^{pL}, \lambda_2^{rH} > 0$ **and** $\lambda_2^{rL} = 0$**.** In this case, $q_{2b}^{rL} = q^*$ and $q_{2b}^{rH}, q_{2b}^{pL}, q_{2b}^{pH} < q^*$. Consequently, Lemma 1 implies that

$$V_2'(m_2) = \frac{1}{2p_2}\left[u'(q_{2b}^r) + 1\right]$$

Then the first order condition of the seller in market 2 implies that $\frac{1}{p_2} = \phi$. Furthermore, the seller's first-order condition from market 1, $\frac{1}{p_1} = EV_2'(m_1 + p_1 q_{1s} + \tau)$, gives us

$$\frac{1}{p_1} = \frac{\phi^H}{2k^H}\left\{\pi k^H\left[u'\left(q_{2b}^{rH}\right) + 1\right] + 2(1-\pi)k^L\right\} \tag{24}$$

Then, (14) implies that the marginal value of money at the beginning of a period satisfies

$$V_1'(m_1) = \frac{\phi^H}{4k^H}\left\{\pi k^H\left[u'\left(q_{2b}^{rH}\right) + 1\right] + 2(1-\pi)k^L\right\}\left[u'(q_1) + 1\right]$$

Finally, the first-order condition in market 3 for the choice of money holdings (4) can be backdated to get

$$\frac{4\gamma}{\beta} = \left\{\pi k^H\left[u'\left(\Omega + \phi^H p_1 q_1\right) + 1\right] + 2(1-\pi)k^L\right\}\left[u'(q_1) + 1\right]. \tag{25}$$

where $q_{2b}^{rH} = \Omega + \phi^H p_1 q_1$.

We once again use (17) to get

$$2u'(q_1) = \pi k^H\left[u'\left(\Omega - \phi^H p_1 q_1\right) + 1\right] + (1-\pi)k^L\left[u'\left(\Omega - \frac{k^L}{k^H}\phi^H p_1 q_1\right) + 1\right] \tag{26}$$

Then solving (24), (25) and (26) yields $\phi^H p_1$, q_1 and Ω.

Using the solutions for Ω and q_1 we obtain

$$\begin{aligned}
q_{2b}^{rH} &= \Omega + \phi^H p_1 q_1 > q_{2b}^{pH} = \Omega - \phi^H p_1 q_1 > q_{2b}^{pL} = \Omega - \frac{k^L}{k^H}\phi^H p_1 q_1 \\
\phi^L &= \frac{\Omega}{M^L}, \phi^H = \frac{\Omega}{M^H}
\end{aligned}$$

For this equilibrium to exist we need that the poor buyers' money balances satisfy

$$q_{2b}^{pL} = \Omega - \frac{z^H}{z^L}\phi^H p_1 < q^* \text{ and } q_{2b}^{pH} = \Omega - \phi^H p_1 q_1 < q^*$$

while the rich buyers' money balances satisfy

$$\Omega + \phi^H p_1 q_1 < q^* < \Omega + \frac{k^L}{k^H}\phi^H p_1 q_1$$

Combining these two sets of inequalities, the sufficient condition for this equilibrium is

$$q^* - \frac{k^L}{k^H}\phi^H p_1 q_1 < \Omega < q^* - \phi^H p_1 q_1$$

At γ_2, the right-hand inequality binds. As γ increases above γ_2 once again Ω falls faster than $\phi^H p_1 q_1$. Finally at some $\gamma_3 > \gamma_2$ the left-hand inequality binds. Thus for $\gamma \in (\gamma_2, \gamma_3]$ this equilibrium exists if a solution to (24), (25) and (26) exists. Note, if $\varepsilon^L = \varepsilon^H = 0$ this equilibrium cannot exist for any $\gamma > \beta$.

Equilibrium 4: $\lambda_2^{pH}, \lambda_2^{pL}, \lambda_2^{rH}, \lambda_2^{rL} > 0$**.** In this case, $q_{2b}^{pH}, q_{2b}^{pL}, q_{2b}^{rH}, q_{2b}^{rL} < q^*$. Consequently, Lemma 1 implies that

$$V_2'(m_2) = \frac{1}{2p_2}\left[u'(q_{2b}^r) + 1\right]$$

Then the first order condition of the seller in market 2 implies that $\frac{1}{p_2} = \phi$. Furthermore, the seller's first-order condition from market 1, $\frac{1}{p_1} = EV_2'(m_1 + p_1 q_{1s} + \tau)$, gives us

$$\frac{1}{p_1} = \frac{\phi^H}{2k^H}\left\{\pi k^H\left[u'\left(q_{2b}^{rH}\right) + 1\right] + (1-\pi)\,k^L\left[u'\left(q_{2b}^{rL}\right) + 1\right]\right\} \tag{27}$$

Then, (14) implies that the marginal value of money at the beginning of a period satisfies

$$V_1'(m_1) = \frac{\phi^H}{4k^H}\left\{\pi k^H\left[u'\left(q_{2b}^{rH}\right) + 1\right] + (1-\pi)\,k^L\left[u'\left(q_{2b}^{rL}\right) + 1\right]\right\}\left[u'(q_1) + 1\right]$$

Finally, the first-order condition in market 3 for the choice of money holdings (4) can be backdated to get

$$\frac{4\gamma}{\beta} = \left\{\pi k^H\left[u'\left(q_{2b}^{rH}\right) + 1\right] + (1-\pi)\,k^L\left[u'\left(q_{2b}^{rL}\right) + 1\right]\right\}\left[u'(q_1) + 1\right]. \tag{28}$$

where $q_{2b}^{rH} = \Omega + \phi^H p_1 q_1$ and $q_{2b}^{rL} = \Omega + \frac{k^L}{k^H}\phi^H p_1 q_1$.

We once again use (17) to get

$$u'(q_1) = \frac{\pi k^H}{2}\left[u'\left(\Omega - \phi^H p_1 q_1\right) + 1\right] + \frac{(1-\pi)\,k^L}{2}\left[u'\left(\Omega - \frac{k^L}{k^H}\phi^H p_1 q_1\right) + 1\right] \tag{29}$$

Then solving (27), (28) and (29) yields $\phi^H p_1, q_1$ and Ω.

Using the solutions for Ω and q_1 we obtain

$$\begin{aligned}
q_{2b}^{rL} &= \Omega + \frac{k^L}{k^H}\phi^H p_1 q_1, \quad q_{2b}^{rH} = \Omega + \phi^H p_1 q_1, \\
q_{2b}^{pH} &= \Omega - \phi^H p_1 q_1, \quad q_{2b}^{pL} = \Omega - \frac{k^L}{k^H}\phi^H p_1 q_1 \\
\phi^L &= \frac{\Omega}{M^L}, \quad \phi^H = \frac{\Omega}{M^H}
\end{aligned}$$

From theses solutions we get $q_{2b}^{rL} > q_{2b}^{rH} > q_{2b}^{pH} > q_{2b}^{pL}$.

For this equilibrium to exist we need that the rich buyers' money balances satisfy

$$q_{2b}^{rL} = \Omega + \frac{k^L}{k^H}\phi^H p_1 q_1 < q^*$$

respectively

$$\Omega < q^* - \frac{k^L}{k^H}\phi^H p_1 q_1$$

As indicated above this inequality binds at γ_3. As γ increases above γ_3 once again Ω falls faster than $\phi^H p_1 q_1$. Thus for $\gamma \geq \gamma_3$ this equilibrium exists if a solution to (27), (28) and (29) exists.

Finally, it is straightforward to show that aggregate hours worked in market are q_3^* in all states and in all periods. To ensure that the richest agents have non-negative hours we need to impose that $q_3^* \geq 2q^*$. This requires scaling of $U(q)$ such that this condition holds.■

Proof of Proposition 4. Average aggregate output in market 2 is

$$\pi\left(q_{2b}^{pL} + q_{2b}^{rL}\right) + (1-\pi)\left(q_{2b}^{pH} + q_{2b}^{rH}\right)$$

Consider all possible equilibria.

In case (i) from Proposition 1 we know that $q_{2b}^{pL} < q_{2b}^{rL} = q_{2b}^{pH} = q_{2b}^{rH} = q^*$. If an unanticipated increase in money takes place aggregate output is realized as $q_{2b}^{pH} + q_{2b}^{rH} = 2q^*$, which is higher than average aggregate output since

$$\pi\left(q_{2b}^{pL} + q^*\right) + (1-\pi)2q^* < 2q^*.$$

In case (ii) from Proposition 1 we know that $q_{2b}^{pL} < q_{2b}^{pH} < q_{2b}^{rL} = q_{2b}^{rH} = q^*$. In state $i = H$ aggregate output is $q_{2b}^{pH} + q^* > q_{2b}^{pL} + q^*$. Thus

$$\pi\left(q_{2b}^{pL} + q^*\right) + (1-\pi)(q_{2b}^{pH} + q^*) < q_{2b}^{pH} + q^*$$

i.e. average aggregate output is smaller than aggregate output if the $i = H$ state is realized.

In case (iii) from Proposition 1 we know that $q_{2b}^{pL} < q_{2b}^{pH} < q_{2b}^{rH} < q_{2b}^{rL} = q^*$. We also know that

$$q_{2b}^{rH} = \Omega + \phi^H p_1 q_1 > q_{2b}^{pH} = \Omega - \phi^H p_1 q_1 > q_{2b}^{pL} = \Omega - \frac{z^H}{z^L}\phi^H p_1 q_1$$

Thus, in state $i = H$ aggregate output is $q_{2b}^{pH} + q_{2b}^{rH} = 2\Omega$. In state $i = L$ we have

$$q_{2b}^{pL} + q_{2b}^{rL} \quad = \Omega - \tfrac{k^L}{k^H}\phi^H p_1 q_1 + q^* < \Omega - \tfrac{k^L}{k^H}\phi^H p_1 q_1 + \Omega + \tfrac{k^L}{k^H}\phi^H p_1 q_1 = 2\Omega$$

since $q^* < \Omega + \frac{k^L}{k^H}\phi^H p_1 q_1$ in this equilibrium. Thus average aggregate output is

$$\pi\left(q_{2b}^{pL} + q_{2b}^{rL}\right) + (1-\pi)2\Omega < 2\Omega$$

i.e. average aggregate output is smaller than aggregate output if the $i = H$ state is realized.

In case (iv) from Proposition 1 we know that $q_{2b}^{pL} < q_{2b}^{pH} < q_{2b}^{rH} < q_{2b}^{rL} < q^*$. We also know that

$$q_{2b}^{rL} = \Omega + \frac{k^L}{k^H}\phi^H p_1 q_1 > q_{2b}^{rH} = \Omega + \phi^H p_1 q_1 > q_{2b}^{pH} = \Omega - \phi^H p_1 q_1 > q_{2b}^{pL} = \Omega - \frac{k^L}{k^H}\phi^H p_1 q_1$$

Thus, in state $i = H$ aggregate output is $q_{2b}^{pH} + q_{2b}^{rH} = 2\Omega$. In state $i = L$ aggregate output is $q_{2b}^{pL} + q_{2b}^{rL} = 2\Omega$. Thus aggregate output is independent across states and it is equal to average aggregate output.■

Proof of Proposition 5. In the low inflation economy, $p_1 = p_2 = 1/\phi$ and the following equations determine q_1 and q_{2b}^p :

$$u'(q_1) = 1 + 2\frac{\gamma - \beta}{\beta} \quad \text{and } u'(q_1) = \frac{u'\left(q_{2b}^p\right) + 1}{2}$$

Since neither of these expressions depend on x, the quantities $q_1$and q_{2b}^p are unaffected by a change in x. For a poor buyer we have $p_2 q_{2b}^p = M_{-1} - p_1 q_1 + \tau^p$, respectively, $q_{2b}^p + q_1 = \phi\left(M_{-1} + \tau^p\right)$. If the quantities bought by a poor buyer in markets one and two do not change then $\phi\left(M_{-1} + \tau^p\right)$ must remain the same. Thus,

$$\frac{\partial \phi}{\partial x} = -\frac{\phi\left(\mu - 1\right)}{1 + \left(\mu - 1\right)x} < 0 \quad \text{and} \quad \frac{\partial \phi M}{\partial x} < 0.$$

The proof for the high inflation economy is by contradiction. In this equilibrium both buyers spend all of their money in market two. This implies the following budget constraints

$$
\begin{aligned}
M_{-1} + x(\gamma - 1) M_{-1} &= p_1 q_1 + p_2 q_{2b}^p \\
M_{-1} + (2 - x)(\gamma - 1) M_{-1} + p_1 q_1 &= p_2 q_{2b}^r
\end{aligned}
$$

Noting that $\phi = 1/p_2$ these expressions can be written as

$$
\begin{aligned}
\phi\left[M_{-1} + x(\gamma - 1) M_{-1}\right] &= \phi p_1 q_1 + q_{2b}^p \\
\phi\left[M_{-1} + (2 - x)(\gamma - 1) M_{-1}\right] &= -\phi p_1 q_1 + q_{2b}^r
\end{aligned}
$$

Now add and subtract $\phi(1 - x)(\gamma - 1) M_{-1}$ on the left hand side of the first constraint and rewrite the second to get

$$
\phi\left[M - (1 - x)(\gamma - 1) M_{-1}\right] = \phi p_1 q_1 + q_{2b}^p \tag{30}
$$

$$
\phi\left[M + (1 - x)(\gamma - 1) M_{-1}\right] = -\phi p_1 q_1 + q_{2b}^r \tag{31}
$$

Now conjecture that a change in x leaves the quantities unchanged. Then it must also leave ϕp_1 unaffected since

$$
\frac{1}{\phi p_1} = \frac{V_2'(m_1 + p_1 q_{1s} + \tau)}{\phi} = \frac{1}{2}\left[u'(q_{2b}^r) + 1\right]
$$

Totally differentiate (30) and (31) holding the right hand sides constant to get

$$
\begin{aligned}
\frac{d\phi}{dx} &= -\frac{\phi(\gamma - 1) M_{-1}}{\left[M - (1 - x)(\gamma - 1) M_{-1}\right]} = -\frac{\phi(\gamma - 1)}{\left[\gamma - (1 - x)(\gamma - 1)\right]} \\
\frac{d\phi}{dx} &= \frac{\phi(\gamma - 1) M_{-1}}{\left[M + (1 - x)(\gamma - 1) M_{-1}\right]} = \frac{\phi(\gamma - 1)}{\left[\gamma + (1 - x)(\gamma - 1)\right]}
\end{aligned}
$$

Clearly both of these derivatives cannot be true for $\gamma \neq 1$. As a result, the quantities must change. ■

Proof of Proposition 6. From (27), (28), and (29) with $k^H = k^L$ and quantities constant across states, the following equations have to hold

$$
\begin{aligned}
2 &= p_1 \phi \left(1 + u'\left(\phi\left[M + p_1 q_1 + M(\gamma - 1)(2 - x)\right]\right)\right) \\
\frac{4\gamma}{\beta} &= \left(1 + u'(q_1)\right)\left(1 + u'\left(\phi\left[M + p_1 q_1 + M(\gamma - 1)(2 - x)\right]\right)\right) \\
2u'(q_1) &= p_1 \phi \left(1 + u'\left(\phi\left[M - p_1 q_1 + M(\gamma - 1) x\right]\right)\right)
\end{aligned}
$$

Totally differentiate these equations, evaluate the derivatives at $x = 1$ and solve for $\frac{dq_1}{dx}$, $\frac{d\phi}{dx}$, and $\frac{dp_1}{dx}$ to get

$$\frac{dq_1}{dx} = \frac{(\gamma - 1) p_1 \phi^2 [1 + u'(q_1)] u''(q_{2b}^r) u''(q_{2b}^p)}{u''(q_1) u''(q_{2b}^p) + u''(q_1) u''(q_{2b}^r) + p_1^2 \phi^2 u''(q_{2b}^r) u''(q_{2b}^p) [1 + u'(q_1) + q_1 u''(q_1)]}$$

where we have set $M = 1$ for simplicity. The numerator is positive if $\gamma > 1$. A sufficient condition for the denominator to be positive is that $1 + u'(q_1) + q_1 u''(q_1) \geq 0$ which can be expressed as

$$\frac{1 + u'(q_1)}{u'(q_1)} > 1 \geq -\frac{q_1 u''(q_1)}{u'(q_1)}$$

where $-\frac{q_1 u''(q_1)}{u'(q_1)}$ is the coefficient of relative risk aversion. This condition is satisfied for any CRRA utility function if the degree of risk aversion is less or equal to 1. For $\gamma < 1$ decreasing x increases q_1 because money is taken out of the economy. Furthermore, once we know that $\frac{dq_1}{dx} > 0$ one can show that $\frac{dq_{2b}^r}{dx} < 0$ and $\frac{dq_{2b}^p}{dx} > 0$.■

Proof of Corollary 7. Life-time expected utility of the representative agent equals

$$\begin{aligned} \mathcal{W}(1-\beta) &= \frac{1}{2}[u(q_1) - q_1] + \frac{1}{4}[u(q_{2b}^r) - q_{2b}^r] + \frac{1}{4}\left[u\left(q_{2b}^p\right) - q_{2b}^p\right] \\ &+ U(q_3^*) - q_3^* \end{aligned} \tag{32}$$

Differentiating with respect to x yields

$$\frac{\partial \mathcal{W}(1-\beta)}{\partial x} = \frac{1}{2}\left[u'(q_1) - 1\right]\frac{\partial q_1}{\partial x} + \frac{1}{4}\left[u'(q_{2b}^r) - 1\right]\frac{\partial q_{2b}^r}{\partial x} + \frac{1}{4}\left[u'\left(q_{2b}^p\right) - 1\right]\frac{\partial q_{2b}^p}{\partial x}$$

In this equilibrium all of the quantities are less than q^* so the bracketed terms are all positive. We also know $q_{2b}^p + q_{2b}^r = \phi M$ so

$$\frac{\partial q_{2b}^r}{\partial x} = M\frac{\partial \phi}{\partial x} - \frac{\partial q_{2b}^p}{\partial x}$$

Substitute in to obtain

$$\begin{aligned} \frac{\partial \mathcal{W}(1-\beta)}{\partial x} &= \frac{1}{2}\left[u'(q_1) - 1\right]\frac{\partial q_1}{\partial x} + \frac{1}{4}\left[u'\left(q_{2b}^p\right) - u'(q_{2b}^r)\right]\frac{\partial q_{2b}^p}{\partial x} \\ &+ \frac{1}{4}\left[u'(q_{2b}^r) - 1\right] M\frac{\partial \phi}{\partial x} \end{aligned}$$

The first two terms are strictly positive. From the Proof of Proposition 6 we have that

$$\frac{\partial \phi}{\partial x} = \frac{(1 - 1/\gamma)\phi u''(q_1)\left[u''\left(q_{2b}^p\right) - u''(q_{2b}^r)\right]}{u''(q_1) u''(q_{2b}^p) + u''(q_1) u''(q_{2b}^r) + p_1^2 \phi^2 u''(q_{2b}^r) u''(q_{2b}^p) [1 + u'(q_1) + q_1 u''(q_1)]}$$

Under our restriction on the coefficient of relative risk aversion, the denominator is positive. Since $u'''(.) \geq 0$ $\left[u''\left(q_{2b}^{p}\right) - u''\left(q_{2b}^{r}\right)\right] \geq 0$, the numerator is also positive. So $\frac{\partial \phi}{\partial x} > 0$. Consequently, $\frac{\partial \mathcal{W}(1-\beta)}{\partial x} > 0$.■