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A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily

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ABSTRACT

Often, researchers wish to analyze nonlinear dynamic discrete-time stochastic models. This paper provides a toolkit for solving such models easily, building on log-linearizing the necessary equations characterizing the equilibrium and solving for the recursive equilibrium law of motion with the method of undetermined coefficients. The innovation here is to demonstrate that log-linearizing the nonlinear equations can usually be done without the need for explicit differentiation, to extend the method of undetermined coefficients to models with more endogenous state variables than expectational equations, to provide a general solution and to provide frequency-domain techniques to calculate the second order properties of the model in its HP-filtered version without resorting to simulations. Since the method is an Euler-equation based approach rather than an approach based on solving a social planners problem, models with externalities or distortionary taxation do not pose additional problems. MATLAB programs to carry out the calculations in this paper are made available. This paper should be useful for researchers and Ph.D. students alike.

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1 Introduction

Often, researchers wish to analyze nonlinear dynamic discrete-time stochastic models. This paper provides a toolkit for solving such models easily, building on log-linearizing the necessary equations characterizing the equilibrium and solving for the recursive equilibrium law of motion with the method of undetermined coefficients. While these methods have been pioneered by McCallum (1983), King, Plosser and Rebelo (1987) as well as Campbell (1994), the new contribution here is to demonstrate that log-linearizing the nonlinear equations can usually be done without the need for explicit differentiation, to extend the method of undetermined coefficients to models with more endogenous state variables than expectational equations, to provide a general solution and to provide frequency-domain techniques, building on results in King and Rebelo (1993), to calculate the second-order moments of the model in its HP-filtered version without resorting to simulations. Since the method is an Euler-equation based approach rather than an approach based on solving a social planners problem, solving models with externalities or distortionary taxation does not pose additional problems. Since the (nonlinear) Euler equations usually need to be calculated in any case in order to find the steady state, applying the method described in this paper requires little in terms of additional manipulations by hand, given some preprogrammed routines to carry out the matrix calculations of section 5. MATLAB programs to carry out these calculations, given the log-linearized system, are available at the Federal Reserve Bank of Minneapolis web site[1]. The method in this paper therefore allows to solve nonlinear dynamic stochastic models easily.

Numerical solution methods for solving nonlinear stochastic dynamic models have been studied extensively in the literature, see in particular Kydland and Prescott (1982), the comparison by Taylor and Uhlig (1990) and the methods proposed by various authors in the same issue, Judd (1991), Hansen and Prescott (1995) and Danthine and Donaldson (1995). The literature on solving linear-quadratic dynamic stochastic models is even larger, see e.g. the textbook treatment in Sargent (1987), Chapters IX and XI, as well as, say, Blanchard and Kahn (1980), McGrattan (1994) or Hansen, McGrattan and Sargent (1994). Subject to applicability, all the method relying on a loglinear approximation to the steady state have in common that they will find the same recursive equilibrium law of motion as the method described in this paper, since the linear space approximating a nonlinear differentiable function is unique and "immune" to differentiable transformations of the parameter space. But while McGrattan (1994) and Hansen, McGrattan and Sargent (1994) focus on solving models

[1]The address of that web site is http://res.mpls.frb.fed.us/research/res.html. Go to the"Archive of Paper" and to the "Discussion Papers from the Institute for Empirical Macroeconomics". This discussion paper as well as the MATLAB programs are found under the header with the same title as this paper.

via maximizing a quadratic objective function, and while Blanchard and Kahn (1980) solve linear systems by searching for the stable manifold in the entire system of necessary equations describing the equilibrium relationships, this paper by contrast solves directly for the desired recursive equilibrium law of motion. This approach is very natural. The stability condition is imposed at the point, where a certain matrix quadratic equation is solved. It is shown how this matrix quadratic equation can be reduced to a standard eigenvalue problem of another matrix with twice as many dimensions.

Two related contributions are McCallum (1983), which is the key reference for the method of undetermined coefficients, and Ceria and Rios-Rull (1992). While these two contributions also derive the equilibrium law of motion with the method of undetermined coefficients for a multidimensional endogenous state space, their methods do not easily allow for the possibility that there are more endogenous state variables than there are expectational equations. McCallum (1983) reduces the coefficient-finding problem to a problem solvable with the methods in Blanchard and Kahn (1980), whereas Ceria and Rios-Rull (1992) reduce the problem to one of solving a matrix-quadratic equation as do we, but do not reduce the matrix-quadratic equation problem to a standard eigenvalue problem. Both contributions do not distinguish between endogenous variables which have to be part of the state vector, and other endogenous variables. Thus applying these models in somewhat larger system can either result in unnecessary large and computationally demanding eigenvalue problems in which "bubble solutions" have to be removed in a painstaking fashion, or one is always forced to reduce the system beforehand to make it fit their description. Furthermore, McCallum (1983) uses eigenvalue methods also to solve some other equations in his method, which are solved here by a simple linear-equation-solution techniques, compare his solution for equation (A.6) in his paper to equation (5.8).

Rather than just describing the new contributions, this paper describes the entire method as a "cookbook recipe", which should be of great practical use to Ph.D. students and researchers alike who are interested in analyzing nonlinear dynamic stochastic models in discrete time. Since the focus here is entirely on the computational aspect of studying these models, some issues are left aside entirely. In particular, the issue of existence or multiplicity of equilibria as well as the reasons for concentrating on stable solutions is not discussed. The methods in this paper should therefore not be applied blindly, but only in light of, say, McCallum (1983), Stokey, Lucas with Prescott (1989) and the related literature.

The outline of the paper will be evident from the description of the general procedure in the next section.

2 The general procedure

The general procedure to solve and analyze nonlinear dynamic stochastic models takes the following steps.

1. Find the necessary equations characterizing the equilibrium, i.e. constraints, first-order conditions, etc., see section 4.
2. Pick parameters and find the steady state(s), see section 4.
3. Log-linearize the necessary equations characterizing the equilibrium of the system to make the equations approximately linear in the log-deviations from the steady state, see section 3 and section 4.
4. Solve for the recursive equilibrium law of motion via the method of undetermined coefficients, employing the formulas of section 5.
5. Analyze the solution via impulse-response analysis and second-order-properties, possibly taking account of, say, the Hodrick-Prescott-Filter. This can be done without having to simulate the model, see section 6.

The next section skips directly to step 3 of the procedure outlined above and describes how to log-linearize nonlinear equations without explicit differentiation. Section 4 then provides two prototype examples, in which calculating the Euler equations, the steady state and the log-linearization is carried out to see how this method works. Once, a linearized system has been obtained, the methods in section 5 provide the desired recursive equilibrium law of motion.

3 Rules of log-linearization

Loglinearizing the necessary equations characterizing the equilibrium has been proposed in particular by King, Plosser and Rebelo (1987) and Campbell (1994). The idea is to use a Taylor approximation around the steady state to replace all equations by approximations, which are linear functions in the log-deviations of the variables.

Formally, let X_t be the vector of variables, $\bar{X}$ their steady state and

$$x_t = \log X_t - \log \bar{X}$$

the vector of log-deviations. The vector $100 \cdot x_t$ tells us, by how much the variables

differ from their steady state levels in period t in per cent. The necessary equations characterizing the equilibrium can be written as

$$1 = f(x_t, x_{t-1}) \tag{3.1}$$
$$1 = E_t[g(x_{t+1}, x_t)] \tag{3.2}$$

where $f(0,0) = 1$ and $g(0,0) = 1$, i.e. the left-hand side of (3.1) and (3.2). Taking first-order approximations around $(x_t, x_{t-1}) = (0,0)$ yields[2]

$$\begin{aligned} 0 &\approx f_1 \cdot x_t + f_2 \cdot x_{t-1} \\ 0 &\approx E_t[g_1 \cdot x_{t+1} + g_2 \cdot x_t] \end{aligned}$$

One obtains a linear system in x_t and x_{t-1} in the deterministic equations and x_{t+1} and x_t in the expectational equations. This linear system can be solved with the method of undetermined coefficients, described in section 5.

In the large majority of cases, there is no need to differentiate the functions f and g explicitely. Instead, the loglinearized system can usually be obtained as follows. Replace a variable X_t with $X_t = \bar{X}e^{x_t}$, where x_t is a real number close to zero. Let likewise y_t be a real number close to zero. Take logarithms, where both sides of an equation only involve products, or use the following three building blocks, where a is some constant:

$$\begin{aligned} e^{x_t + ay_t} &\approx 1 + x_t + ay_t \\ x_t y_t &\approx 0 \\ E_t[ae^{x_{t+1}}] &\approx E_t[ax_{t+1}] \text{ up to a constant .} \end{aligned}$$

For example, these building blocks yield

$$\begin{aligned} aX_t &\approx a\bar{X}x_t \text{ up to a constant} \\ (X_t + a)Y_t &\approx \bar{X}\bar{Y}x_t + (\bar{X} + a)\bar{Y}y_t \text{ up to a constant} \end{aligned}$$

[2] An alternative to approximate (3.2) rewrites it as

$$0 = \log(E_t[\exp(\tilde{g}(x_{t+1}, x_t))])$$

where $\tilde{g} = \log g$. Assuming x_t and x_{t+1} to be (approximately) conditionally jointly normally distributed with an (approximately) constant conditional variance-covariance matrix, and assuming that

$$\log g(0,0) \approx \frac{1}{2}\mathrm{Var}_t[\tilde{g}_1 \cdot x_{t+1} + \tilde{g}_2 \cdot x_t], \tag{3.3}$$

independent of t (rather than $\log g(0,0) = 0$) yields

$$\begin{aligned} 0 &\approx \log E_t[\exp(\tilde{g}(0,0) + \tilde{g}_1 \cdot x_{t+1} + \tilde{g}_2 \cdot x_t)] \\ &\approx E_t[\tilde{g}_1 \cdot x_{t+1} + \tilde{g}_2 \cdot x_t], \end{aligned}$$

using $E[e^X] = e^{E[X] + \mathrm{Var}[X]/2}$ for normally distributed variables. The two ways of approximating (3.2) differ essentially only in their choice for $g(0,0)$, since $g_1 = \tilde{g}_1$, if $g(0,0) = 1$.

Constants drop out of each equation in the end, since they satisfy steady state relationships, but they are important in intermediate steps: compare for example the two equations above. Rather than describing the general principles further, it is fruitful to consider two specific examples instead: this will be done in the next section.

4 Two Examples

The following two examples not only serve to explain how to perform the first three steps of the general procedure, but also as the two prototypes for the general solution in section 5: in the first model, the number of endogenous state variables equals the number of expectational equations, whereas in the second model, the number of endogenous state variables exceeds the number of expectational equations, as will become clear below. To avoid confusion in the application of the method in section 5, it is important to stick to the following *dating convention.* A new date starts with the arrival of new information. If a variable is chosen and/or (eventually) known at date t, it will be indexed with t. Use only variables dated t and $t-1$ in deterministic equations and variables dated $t+1$, t and $t-1$ in equations involving expectations $E_t[\cdot]$. The state variables are then those variables which are the linear combinations of variables dated $t-1$ in these equations. Occasionally the state variables need to include variables chosen at a date earlier than $t-1$ as well. One can recast this into the desired format as follows. Thelist of state variables might consist out of lagged values of the capital stock, k_{t-1} and k_{t-2}. This can and should be rewritten as $k_{1,t-1}$ and $k_{2,t-1}$ with $k_{1,t-1}$ replacing k_{t-1} and where the additional equation $k_{2,t} = k_{1,t-1}$ needs to be added to the system. With that notation, $k_{2,t}$ is "chosen" at date t, satisfying the dating convention.

4.1 Example 1: a real business cycle model

The following benchmark real business cycle model is due to Hansen (1985) and explained there in detail. Here, the mathematical description shall suffice.

The social planner solves the problem of the representative agent

$$\max E \sum_{t=1}^{\infty} \beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - AN_t \right)$$

s.t.

$$\begin{aligned} C_t + I_t &= Y_t \\ K_t &= I_t + (1-\delta)K_{t-1} \\ Y_t &= Z_t K_{t-1}^{\rho} N_t^{1-\rho} \end{aligned} \tag{4.1}$$

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.\mathcal{N}(0;\sigma^2),$$

where C_t is consumption, N_t is labor,I_t is investment, Y_t is production, K_t is capital Z_t is the total factor productivity and $A, \beta, \eta, \delta, \rho, \bar{Z}, \psi$ and σ^2 are parameters. Hansen only considered the case $\eta = 1$, so that the objective function is

$$E\sum_{t=0}^{\infty}\beta^t(\log C_t - AN_t)$$

As in Campbell (1994), there is no difficulty in considering arbitrary η, since no growth trend is assumed.

The first order conditions are

$$\begin{aligned} A &= C_t^{-\eta}(1-\rho)\frac{Y_t}{N_t} \\ 1 &= \beta E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right], \qquad (4.2) \\ R_t &= \rho\frac{Y_t}{K_{t-1}} + 1 - \delta. \qquad (4.3) \end{aligned}$$

Equation (4.2) is the Lucas asset pricing equations, see Lucas (1978), which typically arises in these models.

The steady state is obtained by dropping the time subscripts and stochastic shocks in the equations above, characterizing the equilibrium. Formally, this amounts to finding steady state values such that $f(0,0) = 1$ and $g(0,0) = 1$ in the notation of the previous section[3]. For example, equations (4.2) and (4.3) result in

$$\begin{aligned} 1 &= \beta\bar{R} \\ \bar{R} &= \rho\frac{\bar{Y}}{\bar{K}} + 1 - \delta, \end{aligned}$$

where bars over variables denote steady state values. One needs to decide what one wants to solve for. If one fixes β and δ, these two equations will imply values for $\bar{R}$ and $\bar{Y}/\bar{K}$. Conversely, one can fix $\bar{R}$ and $\bar{Y}/\bar{K}$ and then these two equations yield values for β and δ. The latter procedure maps observable characteristics of the economy into "deep parameters," and is the essence of calibration, see Kydland and Prescott (1991).

Introduce small letters to denote log-deviations, i.e. write

$$C_t = \bar{C}e^{c_t}$$

[3]Alternatively, find the steady state so that (3.3) is satisfied. This is, however, rarely done.

for example. The resource constraint (4.1) then reads

$$\bar{C}e^{c_t} + \bar{I}e^{i_t} = \bar{Y}e^{y_t}$$

This can be written approximately as

$$\bar{C}(1 + c_t) + \bar{I}(1 + i_t) = \bar{Y}(1 + y_t)$$

Since $\bar{C} + \bar{I} = \bar{Y}$ due to the definition of the steady state, the constant terms drop out[4] and one obtains

$$\bar{C}c_t + \bar{I}i_t = \bar{Y}y_t$$

The resource constraint is now stated in terms of percentage deviations: the steady state levels in this equation rescale the percentage deviations to make them comparable. Note that no explicit differentiation is required to obtain the log-linearized version of the resource constraint: log-linearization is obtained just by using the building blocks described in the previous section.

Similarly log-linearizating the other equations yields

$$\begin{aligned}
\bar{K}k_t &= \bar{I}i_t + (1-\delta)\bar{K}k_{t-1} \\
y_t &= z_t + \rho k_{t-1} + (1-\rho)n_t \\
z_t &= \psi z_{t-1} + \epsilon_t \\
0 &= -\eta c_t + y_t - n_t \\
0 &= E_t[\eta(c_t - c_{t+1}) + r_{t+1}] \\
\bar{R}r_t &= \rho\frac{\bar{Y}}{\bar{K}}(y_t - k_{t-1}).
\end{aligned}$$

To find the state variables, one needs to find all (linear combinations of) variables dated $t-1$ in these equations: the endogenous state variable is capital, k_{t-1} whereas the exogenous state variable is the technology parameter z_{t-1}. Note that there are as many expectational equations as there are endogenous state variables. The coefficients of the equations above need to be collected in the appropriate matrices to restate these equations in the form required for section 5: this is a straightforward exercise.

4.2 Example 2: a model with external habit formation

Using the external habit formulation of Campbell and Cochrane (1994), Lettau and Uhlig (1995) analyze the following business cycle model. Note that this equilibrium

[4]Another way to see that constants can in the end be dropped is to note that the steady state is characterized by $c_t = k_t = y_t = k_{t-1} = 0$. If one replaces all log-deviations with zero, only the constant terms remain, and that equation can be subtracted from the equation for general c_t, k_t, y_t and k_{t-1} above.

does not easily permit a social planners representation due to the presence of the external habit.

Let C_t be the consumption of the representative agent at date t, and X_t the habit, which the agent takes as given. The representative agent solves

$$\max_{C_t,K_t,N_t} E\sum_{t=1}^{\infty}\beta^t\left[\frac{(C_t-X_t)^{1-\eta}-1}{1-\eta}-AN_t\right]$$

s.t.

$$C_t+K_t=(D_t+1-\delta)K_{t-1}+W_tN_t+\pi_t$$

where K_t is capital, D_t are dividends per unit of capital, W_t is the wage and N_t is labor. The representative firm maximizes profit,

$$\pi_t=\max_{K_{t-1},N_t} Y_t-D_tK_{t-1}-W_tN_t,$$

s.t.

$$Y_t=Z_tK_{t-1}^{\rho}N_t^{1-\rho}$$

Z_t evolves according to

$$\log Z_t=\log\bar{Z}+\psi\log Z_{t-1}+\epsilon_t;\quad \epsilon_t\sim i.i.d.\mathcal{N}(0;\sigma^2)$$

Define the surplus consumption ratio

$$S_t=\frac{C_t-X_t}{C_t} \tag{4.4}$$

Let $\tilde{c}_t=\log C_t$ and $\tilde{s}_t=\log S_t$. X_t or equivalently $\tilde{s}_t$ evolves according[5] to

$$\tilde{s}_t=(1-\phi)\bar{s}+\phi\tilde{s}_{t-1}+\lambda(\tilde{c}_t-\tilde{c}_{t-1}). \tag{4.5}$$

Markets clear. Here, Y_t is output, Z_t s total factor productivity, and $\beta,\eta,\ A,\delta,\ \rho,\bar{Z},$ $\psi,\sigma^2,\ \phi,\bar{s}$ and λ are parameters.[6] For the usual reasons, one obtains

$$\begin{aligned} D_tK_{t-1} &= \rho Y_t \\ W_tN_t &= (1-\rho)Y_t \\ \pi_t &= 0 \end{aligned}$$

[5]It may be more natural to formulate the law of motion directly on X_t. This can be done by replacing s_t and s_{t-1} in (4.5) with the logarithm of the right hand side of (4.4) and solving for X_t as a function of X_{t-1}, C_t and C_{t-1}. The reason to equivalently use (4.5) instead is to make sure, that one always gets $S_t>0$ by construction.

[6]Campbell and Cochrane (1994) actually specify a rather elaborate function $\lambda(s_t)$ rather than a single number. For the purposes here, a single number suffices, see the discussion in Lettau and Uhlig (1995).

Keeping in mind that X_t and not S_t is external, the first order conditions for the agent can be written as

$$\begin{aligned} A &= (S_t C_t)^{-\eta}(1-\rho)\frac{Y_t}{N_t} \\ 1 &= \beta E_t\left[\left(\frac{S_t}{S_{t+1}}\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right], \\ R_t &= \rho\frac{Y_t}{K_{t-1}} + 1 - \delta. \end{aligned}$$

The steady state (denoted by bars over the variables) is obtained from these equations by dropping the time subscript as in the previous subsection. Similarly, the log-linearized version of the system in terms of log-deviations, denoted by small letters, is obtained by applying the building blocks described in the previous section. For expositional clarity, we do not reduce the number of equations by, say, substituting out dividends etc., although one would typically do so in practice. Using the full system, one obtains

$$\begin{aligned} \bar{C}c_t + \bar{K}k_t &= \bar{D}\bar{K}d_t + (\bar{D}+1-\delta)\bar{K}k_{t-1} + \bar{W}\bar{N}(w_t+n_t) \\ y_t &= z_t + \rho k_{t-1} + (1-\rho)n_t \\ z_t &= \psi z_{t-1} + \epsilon_t \\ s_t &= \phi s_{t-1} + \lambda(c_t - c_{t-1}) \\ d_t + k_{t-1} &= y_t \\ w_t + n_t &= y_t \\ 0 &= -\eta(s_t + c_t) + y_t - n_t \\ 0 &= E_t[\eta(s_t - s_{t+1} + c_t - c_{t+1}) + r_{t+1}] \\ \bar{R}r_t &= \rho\frac{\bar{Y}}{\bar{K}}(y_t - k_{t-1}). \end{aligned}$$

To find the state variables, one needs to find all linear combinations of variables dated $t-1$, i.e. k_{t-1} and $q_{t-1} = \phi s_{t-1} - \lambda c_{t-1}$. Thus, the equation

$$s_t = \phi s_{t-1} + \lambda(c_t - c_{t-1})$$

turns into

$$s_t = \lambda c_t + q_{t-1}$$

and one needs to add the equation

$$q_t = \phi s_t - \lambda c_t$$

to the system. Alternatively, use capital, k_{t-1}, the relative habit s_{t-1} and consumption c_{t-1} as endogenous state variables. Then, the state space is no longer of minimal size and the method of the next section may deliver more than the desired solution. In particular, the coefficient matrix P calculated in the next section will be singular at the desired solution. One therefore needs to be a bit more careful in applying the formulas below. In either case, there are are fewer expectational equations than there are endogenous state variables. The coefficients of the equations above need to be collected in the appropriate matrices to restate these equations in the form required for section 5: this is a straightforward exercise.

5 Solving recursive stochastic linear systems with the method of undetermined coefficients

This section describes how to find the solution to the recursive equilibrium law of motion in general, using the method of undetermined coefficients. MATLAB programs performing the calculations in this section are available at the Federal Reserve Bank of Minneapolis web site[7].

It is assumed that the system takes the following form[8]. There is an endogenous state vector x_t, size $m \times 1$, a list of other endogenous variables y_t, size $n \times 1$, and a list of exogenous stochastic processes z_t, size $k \times 1$. The equilibrium relationships between these variables are

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t \tag{5.1}$$

$$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \tag{5.2}$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0, \tag{5.3}$$

where it is assumed that C is of size $l \times n$, $l \geq n$ and[9] of rank n, that F is of size $(m + n - l) \times n$, and that N has only stable eigenvalues.

What one is looking for is the recursive equilibrium law of motion

$$x_t = Px_{t-1} + Qz_t \tag{5.4}$$

$$y_t = Rx_{t-1} + Sz_t, \tag{5.5}$$

[7]The address of that web site is http://res.mpls.frb.fed.us/research/res.html. Go to the"Archive of Paper" and to the "Discussion Papers from the Institute for Empirical Macroeconomics". This discussion paper as well as the MATLAB programs are found under the header with the same title as this paper.

[8]Note that the notation differs from the notation in section 3

[9]The case $l < n$ can be treated as well: in that case one needs to solve a matrix quadratic equation in R, where R is defined below. Since that case is rare in practice and involves considerable additional costs, it is excluded here.

i.e. matrices P, Q, R and S, so that the equilibrium described by these rules is stable. The solution is given in the next theorem. The important special case $l = n$ is treated in corrolary 1. In order to obtain P, a matrix quadratic equation needs to be solved (see also Ceria and Rios-Rull, 1992). The solution can be obtained via a standard eigenvalue problem for some $2n \times 2n$-matrix as stated in theorem 2.

Theorem 1 *If there is a recursive equilibrium law of motion solving equations (5.1), (5.2) and (5.3), then the coefficient matrices can be found as follows. Let C^+ be the pseudo-inverse*[10] *of C. Let C^0 be an $(l-n) \times l$ matrix, whose rows form a basis of the null space*[11] *of C'.*

1. *P satisfies the (matrix) quadratic equations*

$$0 = \qquad C^0AP + C^0B \tag{5.6}$$

$$0 = (F - JC^+A)P^2 \quad -(JC^+B - G + KC^+A)P - KC^+B + H \tag{5.7}$$

The equilibrium described by the recursive equilibrium law of motion (5.4), (5.5) and by (5.3) is stable iff all eigenvalues of P are smaller than unity in absolute value.

2. *R is given by*

$$R = -C^+(AP + B)$$

3. *Given P and R, let V be the matrix*

$$V = \begin{bmatrix} I_k \otimes A, & I_k \otimes C \\ N' \otimes F + I_k \otimes (FP + JR + G), & N' \otimes J + I_k \otimes K \end{bmatrix}.$$

where I_k is the identity matrix of size $k \times k$. If the matrix V is invertible, then

$$\begin{bmatrix} vec(Q) \\ vec(S) \end{bmatrix} = -V^{-1} \begin{bmatrix} vec(D) \\ vec(LN + M) \end{bmatrix}, \tag{5.8}$$

where $vec(\cdot)$ denotes columnwise vectorization.

Proof: *Plug the recursive equilibrium law of motion into equation (5.1). This yields*

$$(AP + CR + B)x_{t-1} + (AQ + CS + D)z_t = 0, \tag{5.9}$$

[10]The pseudo-inverse of the matrix C is the $n \times l$ matrix C^+ satisfying $C^+CC^+ = C^+$ and $CC^+C = C$. Since it is assumed that $\text{rank}(C) \geq n$, one gets $C^+ = (C'C)^{-1}C'$, see Strang (1980), p. 138. The MATLAB command to compute the pseudo-inverse is Cplus = pinv(C).

[11]C^0 can be found via the singular value decomposition of C', see Strang (1980), p. 142. The MATLAB command for computing C^0 is C0 = (null(C'))'.

which has to hold for arbitrary x_{t-1} and z_t. Thus, the coefficient matrices on x_{t-1} and z_t in (5.9) are zero. Plugging the recursive equilibrium law of motion into equation (5.2) twice and using (5.3) yields

$$\begin{aligned} 0 &= ((FP + JR + G)P + KR + H)x_{t-1} + \\ &\quad ((FQ + JS + L)N + \\ &\quad (FP + JR + G)Q + KS + M)z_t \end{aligned} \tag{5.10}$$

Again, the coefficient matrices on x_{t-1} and z_t need to be zero. Taking the column-wise vectorization of the coefficient matrices of z_t in equations (5.9) and (5.10) and collecting terms in vec(Q) and vec(S) yields the formula for Q and S.

To find P and thus R, rewrite the coefficient matrix on x_{t-1} in equation (5.9) as

$$\begin{aligned} R &= -C^{+}(AP + B) \\ 0 &= C^{0}AP + C^{0}B, \end{aligned} \tag{5.11}$$

noting that the matrix $[(C^{+})', (C^{0})']$ is nonsingular and that $C^{0}C = 0$, see Strang (1980), p. 88. Use (5.11) to replace R in the coefficient matrix on x_{t-1} in (5.10), yielding (5.7). Note finally that the stability of the equilibrium is determined by the stability of P, since N has stable roots by assumption. •

Corollary 1 *Suppose that $l = n$, i.e. that there are as many expectational equations as there are endogenous state variables. If there is a recursive equilibrium law of motion solving equations (5.1), (5.2) and (5.3), then their coefficient matrices can be found as follows.*

1. *P satisfies the (matrix) quadratic equation*[12]

$$(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0. \tag{5.12}$$

The equilibrium described by the recursive equilibrium law of motion (5.4), (5.5) and by (5.3) is stable iff all eigenvalues of P are smaller than unity in absolute value.

2. *R is given by*

$$R = -C^{-1}(AP + B)$$

[12]The solvability of equation (5.12) is discussed in Theorem 2 and in the text following it.

3. Q satisfies

$$\begin{aligned} vec(Q) &= (N' \otimes (F - JC^{-1}A) + I_k \otimes (JR + FP + G - KC^{-1}A))^{-1} \cdot \\ &\quad vec((JC^{-1}D - L)N + KC^{-1}D - M), \end{aligned} \tag{5.13}$$

where I_k is the identity matrix of size $k \times k$, provided the matrix which needs to be inverted in this formula is indeed invertible.

4. S is given by

$$S = -C^{-1}(AQ + D)$$

Proof: *This corollary can be obtained directly by inspecting the formulas of theorem 1 above for the special case $l = n$. In particular, C^+ is just the inverse of C. Alternatively, a direct proof can be obtained as follows by following the same proof strategy as above. Obtain (5.9) and (5.10) as before. Equation (5.9) implies*

$$\begin{aligned} R &= -C^{-1}(AP + B) \\ S &= -C^{-1}(AQ + D) \end{aligned}$$

Replacing S in the coefficient on z_t in equation (5.10) and collecting terms in Q yields

$$\begin{aligned} (F - JC^{-1}A)QN + (FP + JR + G - KC^{-1}A)Q &= \\ (JC^{-1}D - L)N + KC^{-1}D - M, \end{aligned} \tag{5.14}$$

which needs to be solved for Q. Take the columnwise vectorization operator $vec(\cdot)$ to find (5.13) stated above. Note that these formulas can be evaluated as soon as P and thus R is known. To find P and thus R, proceed as in the previous proof. •

The formulas in these theorems become simpler yet, if $m = 1$ or $k = 1$. If $m = 1$, there is just one endogenous state variable and the matrix quadratic equation above becomes a quadratic equation in the real number P, which can be solved using standard formulas. If $k = 1$, there is just one exogenous state variables, in which case the Kronecker product (i.e. "$\otimes$") in the formulas above becomes multiplication, and in which case $\text{vec}(Q) = Q$ and $\text{vec}(S) = S$, since Q and S are already vectors rather than matrices.

To generally solve the matrix quadratic equations (5.6), (5.7) for P, one needs to rewrite the equations to make the next theorem applicable. Rewrite (5.6) as

$$0 = C^0AP^2 + C^0BP. \tag{5.15}$$

This is in general a weaker condition than (5.6), but it is equivalent to it, if either $l = n$ or the desired solution P is invertible. P is invertible, if the state space is of minimal dimensionality. Let

$$\Psi = \begin{bmatrix} C^0 A \\ F - JC^+ A \end{bmatrix}.$$

Provided that Ψ is invertible, let

$$\begin{aligned} \Gamma &= \Psi^{-1} \begin{bmatrix} -C^0 B \\ JC^+ B - G + KC^+ A \end{bmatrix} \\ \Theta &= \Psi^{-1} \begin{bmatrix} 0_{l-n,m} \\ KC^+ B - H, \end{bmatrix} \end{aligned}$$

where $0_{l-n,m}$ is a $(l-n) \times m$ matrix with only zero entries. With that, equations (5.15) and (5.7) can be written as

$$P^2 - \Gamma P - \Theta = 0. \tag{5.16}$$

In the special case $l = n$, the formulas for Γ and Θ become slightly simpler:

$$\begin{aligned} \Gamma &= (F - JC^{-1}A)^{-1}(JC^{-1}B - G + KC^{-1}A) \\ \Theta &= (F - JC^{-1}A)^{-1}(KC^{-1}B - H). \end{aligned}$$

Equation (5.16) can now be solved by turning it into a standard eigenvalue and eigenvector problem:

Theorem 2 *To solve the quadratic matrix equation*

$$P^2 - \Gamma P - \Theta = 0, \tag{5.17}$$

for the $m \times m$ matrix P, given $m \times m$ matrices Γ and Θ, define the $2m \times 2m$ matrix Ξ via

$$\Xi = \begin{bmatrix} \Gamma & \Theta \\ I_m & O_{m,m} \end{bmatrix},$$

where I_m is the identity matrix of size m, and where $0_{m,m}$ is the $m \times m$ matrix with only zero entries.

1. *If there are m eigenvalues $\lambda, \ldots, \lambda_m$ together with eigenvectors $s_1, \ldots, s_m$ of Ξ, written as $s_i = [\lambda_i x_i', x_i']$ for some $x_i \in \mathbf{R}^m$, so that $(x_1, \ldots, x_m)$ is linearly independent, then*

$$P = \Omega \Lambda \Omega^{-1}$$

where $\Omega = [x_1, \ldots, x_m]$ *and* $\Lambda = diag(\lambda, \ldots, \lambda_m)$ *is a solution to the matrix quadratic equation (5.17). The solution* P *is stable if* $\mid \lambda_i \mid < 1$ *for all* $i = 1, \ldots, m$.

Conversely, any diagonalizable solution P *to (5.17) can be written in this way.*

2. *If* $m = 1$, *then the solutions* P *to equation (5.17) are given by*

$$P_{1,2} = \frac{1}{2}(\Gamma \pm \sqrt{\Gamma^2 + 4\Theta}).$$

Proof: *First note that any eigenvector* s *for some eigenvalue* λ *of the matrix* Ξ *can indeed be written as*

$$s = \begin{bmatrix} \lambda x \\ x \end{bmatrix}$$

for some $x \in \mathbf{R}^m$ *because of the special form of* Ξ. *Examining the first* m *rows of the relationship* $\lambda s = \Xi s$ *then shows that*

$$\lambda^2 x - \lambda \Gamma x - \Theta x = 0 \tag{5.18}$$

It follows that

$$\Omega\Lambda^2 - \Gamma\Omega\Lambda - \Theta\Omega = 0$$

and hence

$$P^2 - \Gamma P - \Theta = 0$$

as claimed, after multiplying with Ω^{-1} *from the right.*

Reversing the steps shows that any diagonalizable solution P *to (5.17) can be written in this way.* •

Solving the eigenvalue and eigenvector problem for a $2m \times 2m$ matrix is a well-studied problem and many numerical packages have preprogrammed solutions for this. Furthermore, this theorem links the approach to Blanchard and Kahn (1980). Consider solving the second order difference equation

$$x_{t+2} - \Gamma x_{t+1} - \Theta = 0. \tag{5.19}$$

The approach in Blanchard and Kahn (1980) amounts to finding the stable roots of Ξ. The approach here solves for the matrix P in the recursive equilibrium law of motion $x_{t+1} = Px_t$. The Theorem above states that both approaches amount to the same problem. The advantage of the method here is that it is easily applied to the entire system (5.1), (5.2) and (5.3), reducing it to (5.19) eventually, while finding the

stable roots in the entire system given by these equations and at the same time taking care of the expectation operators, using the Blanchard-Kahn (1980) procedure, can be quite daunting.

To apply this theorem, one needs to select m out of $2m$ possible eigenvalues. Note that P has only nonzero eigenvalues if the state space was chosen to be of minimal size: thus attention can be restricted to the roots $\mid \lambda_i \mid > 0$ in that case. This is important in particular for the case $l > n$ and $K = 0$, since rewriting (5.6) as (5.15) introduces spurious zero roots in that case. In general, there may be quite a bit of choice left. In practice, however, there will often be exactly m stable eigenvalues remaining so that the stable solution is unique[13]. The literature on solving linear rational expectations equilibria typically assumes this condition to hold or shows it to hold in social planning problems under reasonable conditions, see Blanchard and Kahn (1980) and Hansen, McGrattan and Sargent (1994). If there are fewer stable eigenvalues than endogenous state variables, the equilibrium might be inherently unstable. The method above then still permits calculation of an equilibrium which satisfies the nonlinear equilibrium conditions at least locally. In particular, in models involving more than one agent (or one country), one may find as many unit roots as there are more agents (countries) than one since shocks may affect the relative wealth of any two agents (countries) and thus may result in permanent changes in their consumption paths: in these cases, the method above allowing for unit roots still gives useful results, which obviously should then be used with some care. If there are more stable eigenvalues than endogenous state variables, enlarging the number of endogenous state variables by including further lagged values might help. Redoing the calculations might then result in a unique solution.

If not all eigenvalues of Ξ are distinct, P in turn might have repeated eigenvalues. Since the eigenspace for a repeated eigenvalue is (usually) multidimensional, there will be infinitely many choices for the eigenvectors and hence infinitely many choices for P in that case. Note, for example, that for any given λ and any three real numbers a, b, c satisfying $a^2 + bc = \lambda^2$, all matrices

$$P = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

[13]Another approach to select a unique solution is in McCallum (1983), who suggests to use those roots that can be obtained continuously from the zero roots of the equation $P^2 - \Gamma P - \alpha\Theta$ for $\alpha = 0$, as α changes from 0 to 1. However, not only is following these roots as functions of α computationally very demanding, it is also the case that uniqueness gets lost once two or more such paths cross each other. If these paths do not cross in a particular application, and if additionally all roots for all α are positive real numbers, say, then the McCallum proposal simply amounts to using the roots of minimal value. The MATLAB programs supplied by the author use the roots of minimal absolute value subject to eliminating spurious zero roots as described above and tries to use complex roots in conjugate pairs, as described below.

solve

$$P^2 - \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix} = 0.$$

These cases are rare in practice, since Ξ is diagonalizable with distinct eigenvalues generically in the coefficients of the system (5.1), (5.2) and (5.3).

More disconcerting is the possibility that some of the roots may be complex rather than real. Consider, for example, $\Gamma = -I_{2,2}$ and

$$\Theta = \begin{bmatrix} 0.23 & 0.64 \\ -0.64 & 0.23 \end{bmatrix}.$$

Using the theorem above, one obtains exactly two stable roots, which happen to be complex, $\lambda_{1,2} = 0.3 \pm 0.4i = 0.5e^{\pm \alpha i}$, where where $\alpha \approx 0.9273$. Their associated eigenvectors are complex, too. Calculating P results in a matrix with only real entries, however, given by

$$P = \begin{bmatrix} 0.3 & 0.4 \\ -0.4 & 0.3 \end{bmatrix} = 0.5 \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

Since Ξ is a real-valued matrix, complex eigenvalues only arise in complex-conjugate pairs. When using both roots of a complex-conjugate pair to calculate Λ and thus P, the resulting solution should be a real-valued matrix. The complex roots then give rise to endogenous damped cycles of frequency α.

Some additional properties of a solution P to (5.16) are stated in the following theorem[14]:

Theorem 3 *1. The eigenvalues λ of Ξ are the solutions to the equation*

$$det\left(\lambda^2 I_m - \lambda\Gamma - \Theta\right) = 0.$$

The lower half x of the eigenvector s to λ satisfies

$$\left(\lambda^2 I_m - \lambda\Gamma - \Theta\right) x = 0$$

2. If P is a real-valued solution to the matrix-quadratic equation (5.12), then

$$tr(4\Theta + \Gamma^2) \geq 0.$$

[14] I am grateful to Jan Magnus for pointing these out to me. Furthermore, Ceria and Rios-Rull, 1992, point to additional literature on this subject, which found and concentrated on part 1 of theorem 3, but apparently did not find the more useful theorem 2.

Proof: *The claim about λ follows from*

$$det\left(\begin{bmatrix} (\Gamma - \lambda I_m) & \Theta \\ I_m & -\lambda I_m \end{bmatrix}\right) = det\left(-\lambda(\Gamma - \lambda I_m) - \Theta\right),$$

which follows from inspecting the formula for the determinant. The claim about the eigenvector piece x is just (5.18). For the last claim, calculate that

$$0 = tr(P^2 - \Gamma P - \Theta) = tr((P - \frac{1}{2}\Gamma)^2 - (\Theta + \frac{1}{4}\Gamma^2)).$$

The conclusion follows since $tr((P - \frac{1}{2}\Gamma)^2) \geq 0$. •

6 Interpreting the results

The results obtained, i.e. the recursive equilibrium law of motion

$$x_t = Px_{t-1} + Qz_t$$

$$y_t = Rx_{t-1} + Sz_t$$

$$z_t = Nz_{t-1} + \epsilon_t$$

can be used to examine model implications. Since x_t, y_t and z_t are log-deviations, the entries in P, Q, R, S and N can be understood as elasticities and interpreted accordingly as in e.g. Campbell (1994).

Impulse responses to a particular shock ϵ_1 can be calculated by setting $x_0 = 0, y_0 = 0$ and $z_0 = 0$, as well as $\epsilon_t = 0$ for $t \geq 2$, and recursively calculating z_t and then x_t and y_t, given $x_{t-1}, y_{t-1}, z_{t-1}$ and ϵ_t for $t = 1, \ldots, T$ with the recursive equilibrium law of motion and the law of motion for z_t.

To find the second moment properties of the model such as variances and autocorrelations of certain variables as well as the small sample properties of their estimators, simulation methods are often used. Before calculating these moments, the Hodrick-Prescott filter is typically applied (short: HP-Filter). This section demonstrates a frequency-domain technique to obtain these moments (albeit without the small sample properties of their estimators) without the need for any simulations, see also Uhlig and Xu (1994). The matrix-valued spectral density for $[x_t', z_t']'$ is given by

$$\begin{aligned} f(\omega) &= \frac{1}{2\pi}\begin{bmatrix} (I_m - Pe^{-i\omega})^{-1}Q \\ I_k \end{bmatrix}(I_k - Ne^{-i\omega})^{-1}\Sigma \\ &\quad (I_k - N'e^{i\omega})^{-1}\left[Q'(I_m - P'e^{i\omega})^{-1}, I_k\right] \end{aligned}$$

where I_k and I_m are the identity matrices of dimension k and m, see Hamilton (1994), formula (10.4.43). Two ways to calculate the matrix-valued spectral density for the entire vector of variables $s_t = [x_t', y_t', z_t']'$ are calculate

$$\begin{aligned} g(\omega) &= \begin{bmatrix} I_m & 0_{m,k} \\ Re^{-i\omega} & S \\ 0_{k,m} & I_k \end{bmatrix} f(\omega) \begin{bmatrix} I_m & R'e^{i\omega} & 0_{m,k} \\ 0_{k,m} & S' & I_k \end{bmatrix} \\ &= Wf(\omega)W' \text{ , where } W = \begin{bmatrix} I_m, & 0_{m,k} \\ RP^+, & S - RP^+Q \\ 0_{k,m}, & I_k \end{bmatrix}, \end{aligned}$$

where P^+ is the pseudo-inverse of P and where the last equality exploits $s_t = W[x_t', z_t']'$, replacing x_{t-1} with $P^+x_t - P^+Qz_t$ in the recursive equilibrium law of motion for y_t. The HP filter aims at removing a smooth trend τ_t from some given data s_t by solving

$$\min_{\tau_t} \sum_{t=1}^{T} \left((s_t - \tau_t)^2 + \lambda \left((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right)^2 \right)$$

The solution is a linear lag polynomial $r_t = s_t - \tau_t = h(L)s_t$ which has the transfer function

$$\tilde{h}(\omega) = \frac{4\lambda(1 - \cos(\omega))^2}{1 + 4\lambda(1 - \cos(\omega))^2},$$

see King and Rebelo (1993). Thus, the matrix spectral density of the HP-filtered vector is simply

$$g_{HP}(\omega) = \tilde{h}^2(\omega)g(\omega),$$

from which one can obtain the autocorrelations of r_t in time domain via an inverse Fourier transformation,

$$\int_{-\pi}^{\pi} g_{HP}(\omega)e^{i\omega k}d\omega = E[r_t r_{t-k}'],$$

see formula (10.4.4) in Hamilton (1994). Inverse Fourier transformations are part of many numerical packages.

7 Conclusions

We have provided a toolkit to analyze nonlinear dynamic stochastic models easily. The main contribution of this paper is to show that loglinearzing the necessary equations characterizing the equilibrium typically does not require explicit differentiation,

to provide a general solution to a linearized system using the method of undetermined coefficients, allowing in particular for several endogenous states, and to provide simulation free frequency-domain based method to calculate the the model implications in its HP-filtered version. These methods are easy to use if a numerical package such as MATLAB or GAUSS is available. This paper should therefore be useful for anybody interested in analyzing stochastic dynamic models.

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