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The One-Sector Growth Model With Idiosyncratic Shocks

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ABSTRACT

This paper investigates the one-sector growth model where agents experience idiosyncratic endowment shocks and face a borrowing constraint. It is shown that a steady-state capital level lies strictly above the steady state in the model without shocks. In addition, the capital stock increases monotonically when it is sufficiently far below a steady state. However, near a steady state there can be interesting (nonmonotonic) economic dynamics.

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1. Introduction

This paper characterizes the steady state and dynamic properties of the one-sector growth model with two non-standard features. The first feature is that there are a continuum of agents in the economy experiencing idiosyncratic labor endowment shocks. The endowment uncertainty is such that there is uncertainty for individual agents but no uncertainty over the aggregate labor endowment. The second feature is that there are financial market imperfections. One of these imperfections is that agents face a borrowing constraint in that asset holding cannot be negative. The other imperfection is that there are by assumption no markets to insure against endowment uncertainty. Thus, individuals can only self-insure by holding non-negative quantities of a single asset- physical capital.

Becker and Foias (1987) and Hernandez (1991) have investigated the steady state and dynamic properties of this model when agents face a borrowing constraint but do not experience idiosyncratic endowment shocks. They show that the steady-state capital stock in the borrowing constrained economy is unique and coincides with that in the complete market economy. In addition, Hernandez (1991) shows that if capital income is increasing in the level of capital, then the turnpike property holds (i.e. the capital stock converges monotonically to the steady state) regardless of the distribution of capital across agents.[1] This turnpike property is a well-known property of the one-sector model with complete markets.

Much less is known about the theoretical properties of models with idiosyncratic shocks. This is not because of a lack of importance of models of this type. Economists have been interested in such models at least since the work of Friedman (1957). At a partial equilibrium level the theoretical properties of this model have been developed by Schechtman and Escudero (1977), Bewley (1977), Sotomayor (1984) and others. At a general equilibrium level the existing theoretical work has concentrated on existence of steady states and the properties of steady states. Contributors include Laitner (1979), Bewley (1984, 198?) and Clarida (1990). Recently there has been substantial interest in addressing quantitative questions within general equilibrium models of this type.[2] This

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[1] Becker and Foias (1987) examine the case in which agents discount at different rates instead of the equal discount factor case considered by Hernandez (1991). They prove that the capital stock also converges to the unique steady state and that this convergence is eventually monotonic.

[2] The questions have been in the areas of asset pricing, savings, optimal debt and tax structure, business cycles, as well as income and wealth distribution. See Huggett (1993), Aiyagari (1994), Krusell and Smith (1994), Castaneda, Diaz-Gimenez and Rios-Rull (1994), Rios-Rull (1995) and the references cited therein for a guide to work in these areas.

work provides additional motivation for understanding the theoretical properties of the underlying model.

The key findings of this paper are as follows. First, in steady state the capital stock is strictly greater than the steady-state capital stock in the complete markets economy. Thus, *in steady state the capital stock in the economy with idiosyncratic shocks is always strictly greater than the steady state capital stock in the economy without idiosyncratic shocks.* Second, the capital stock increases monotonically when the capital stock is sufficiently far below a steady state. Finally, an example is provided in which near a steady state there can be interesting non-monotonic economic dynamics. Thus, the turnpike property does not hold. The example shows that distribution can matter for the nature of economic dynamics even in one-sector models.

This paper is organized in five sections. Section 2 describes the economy. Section 3 characterizes steady states. Section 4 characterizes economic dynamics. Section 5 concludes.

2. The Economy

2.1 The Environment

The environment is composed of a continuum of infinitely-lived agents. The total mass of agents is normalized to equal 1. Each agent's preferences over consumption are given by a utility function:

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right], \qquad \text{where } 0 < \beta < 1$$

All agents have identical period utility functions u. In the theorems proved in this paper the period utility function satisfies some combination of the following assumptions:

A1 $u : R_+ \to R$ is bounded and continuous.

A2 $u' > 0$, $u'' < 0$ and $\lim_{c \to 0} u'(c) = \infty$.

A3 u' is convex.

Each agent receives a random labor endowment each time period that is independent and identically distributed. The endowment lies in a finite set of possible labor endowment shocks $E = \{e_1, e_2, ..., e_n\}$, where $0 < e_1 < ... < e_n$. Probabilities are given by a probability measure π, where $\pi(e) > 0$ for all $e \in E$. Endowments are independent across agents.

Thus, there is endowment uncertainty at the individual level but there is no uncertainty over the aggregate labor endowment.

The production technology is given by a total output function f(K). The technology maps the capital stock (equivalently the capital-labor ratio) in a time period into an output level for that time period. The technology can be related to a standard constant returns to scale production function F(K, L) as follows: $f(K) = F(K, 1) + (1 - \delta)K$. In this specification δ is the depreciation rate of capital. The technology will satisfy some combination of assumptions T1-2 below. Assumption T1 is the usual concavity assumption considered in neoclassical growth theory. Assumption T2 is sufficient to imply that there is a unique capital stock K satisfying $\beta f'(K) = 1$. The capital stock satisfying this condition is the steady-state capital stock in the one-sector growth model with complete markets.

T1 $f(0) = 0$, $f' > 0$ and $f'' < 0$

T2 $\lim_{K \to 0} f'(K) = \infty$ and $\lim_{K \to \infty} f'(K) < 1$

2.2 An Agent's Decision Problem

Each agent solves a version of the standard "income fluctuation problem" (Schechtman and Escudero (1977)). In this problem an agent faces a deterministic sequence $\{w_t, r_t\}$ of wage rates and gross interest rates. An agent then supplies labor inelastically and chooses capital holdings over time to maximize utility. Agents face a borrowing constraint in that each period capital holdings cannot be negative. By assumption there are no other assets available to insure against endowment uncertainty.

This decision problem is now described in the language of dynamic programming. An agent's position at a point in time is described by an individual state x. The individual state x = (k, e) is the agent's current capital holding and endowment. The individual state x lies in the individual state space $X = [0,\infty)\times E$. An agent in state x = (k, e) receives in period t a payment of $ew_t + kr_t$. The agent then chooses capital and consumption to solve the following dynamic programming problem. The expectation in the dynamic programming problem is with respect to the probability measure π defined on the agent's idiosyncratic shocks.

$$v(x,t) = \sup_{k'} \; u(ew_t + kr_t - k') + \beta E[\, v(k', e', t+1)\,] \quad \text{subject to } 0 \leq k' \leq ew_t + kr_t \tag{1}$$

If the period utility function u is bounded, then the contraction mapping theorem implies that a unique, bounded solution v to equation (1) exists. Furthermore, Theorem 3 in Denardo (1967) says that the solution v is the optimal value function. If the utility function is also continuous, then Corollary 2 in Denardo (1967) guarantees that optimal decision rules k(x,t) and c(x,t) exist that achieve the optimal value function. Thus, we have that assumption A1 is sufficient for the existence of optimal decision rules to problem (1). Fact 1-4 below lists some properties of optimal decision rules that will be useful later.

In the proof of Fact 1-4, I use the following standard results for this type of problem. First, the value function is increasing, strictly concave and differentiable in k and $v_1(k,e,t) = u'(c(k,e,t))r_t$. Second, the following Euler equation is a necessary condition for utility maximization:

$$u'(c(x,t)) \geq \beta r_{t+1} E[u'(c(k(x,t),e',t+1))]; \text{ where equality holds if } k(x,t) > 0.$$

Fact : Assume A1-2 and w_t, $r_t > 0$ for all t, then

1. c(x,t) and k(x,t) are continuous in x.
2. c(k,e,t) is strictly increasing in k and c(x,t) > 0 for all x.
   k(k,e,t) is increasing in k.
3. $c(k,e_1,t) < c(k,e_2,t) < \ldots < c(k,e_n,t)$.
4. If $(w_t, r_t) = (w,r)$ for all t and $\beta r \leq 1$, then $k(k,e_1,t) < k$ for all k > 0.

proof: See the Appendix. ◊

2.3 The Firm

There is a single firm that operates the technology f(K). The firm buys capital K each period to maximize profit. Profit maximization implies that in period t the payment to capital r_t satisfies condition (2) below. The wage rate w_t is also defined in condition (2).

$$r_t = f'(K) \tag{2}$$
$$w_t = f(K) - f'(K)K$$

2.3 Equilibrium

To state the equilibrium concept, some way of describing and keeping track of the heterogeneity in the economy is needed.[3] At time t the distribution of individual states across agents is described by the aggregate state y_t. The aggregate state is a probability measure defined on $\mathcal{H}$, where $\mathcal{H}$ denotes the Borel subsets of X. Thus, for all $B \in \mathcal{H}$, $y_t(B)$ is the mass of agents whose individual states lie in B at time t. Since y_t is a probability measure the total mass of agents is normalized to equal 1. The aggregate state y lies in the aggregate state space $Y = \{y : \int_X k\, dy < \infty\}$

To describe how the aggregate state and the capital stock evolve over time it is useful to define functions P(x, t, B) and K(y). The function P(x, t, B) is a transition function that gives the probability that an agent in state x at time t will have an individual state that lies in the set B next time period. The function K(y) simply gives the aggregate capital stock as a function of the aggregate state y. These functions are defined below.

$$P(x,t,B) = \pi(\{e' \in E: (k(x,t), e') \in B\})$$

$$K(y) = \int_X k\, dy$$

Definition: An equilibrium is a pair of functions c(x,t), k(x,t) and sequences $\{w_t, r_t, y_t\}$ satisfying:

1) c(x,t) and k(x,t) are optimal decision rules.
2) $\{w_t, r_t\}$ satisfy equation (2) for all $t \geq 0$.
3) Markets Clear: For all $t \geq 0$
   (i) $\int_X (c(x,t) + k(x,t))\, dy_t = f(K(y_t))$
   (ii) $\int_X e\, dy_t = 1$
4) Law of Motion for the Aggregate State: For all $t \geq 0$ and all $B \in \mathcal{H}$

$$y_{t+1}(B) = \int_X P(x,t,B)\, dy_t$$

[3] The way in which this paper handles agent heterogeneity is similar to the treatment in Lucas and Prescott (1974), Foley and Hellwig (1975), Lucas (1980) and others.

3. Steady States

This section considers the properties of a steady state. A steady-state equilibrium is an equilibrium where in all time periods $(w_t, r_t, y_t) = (w, r, y)$ and where $c(x,t)$ and $k(x,t)$ are time invariant functions of the state x. Theorem 1 states that in a positive capital steady state the capital stock satisfies the condition $\beta f'(K(y)) < 1$. This means that the capital stock is strictly larger than the steady-state capital stock in the economy without endowment uncertainty. It also means that the rate of return on capital will be strictly lower than the return without uncertainty. Recall that the capital level K solving $\beta f'(K) = 1$ is the steady-state in the economy without endowment uncertainty.

The proof of the theorem is based upon the fact that in steady state any statistic of the state of the economy must remain invariant over time. However, marginal utility averaged over the population shrinks over time when $\beta f'(K(y)) \geq 1$. The proof is in two steps. First, it is argued that $\beta f'(K(y)) > 1$ is not possible. If this were the case, then the Euler equation implies that marginal utility would shrink over time when averaged across agents. Second, it is argued that $\beta f'(K(y)) = 1$ is not possible. The argument is nearly identical except that it is shown that in this case there would be a positive mass of agents for whom the borrowing constraint would bind. Therefore, marginal utility shrinks over time.

Theorem 1: Assume A1-2 and T1. In a positive capital steady state $\beta f'(K(y)) < 1$.

proof: First show that $\beta f'(K(y)) \leq 1$. Suppose by way of contradiction that $\beta f'(K(y)) > 1$. The weak inequality below is a necessary condition for consumer maximization. The strict inequality below follows because $\beta r = \beta f'(K(y)) > 1$.

$$u'(c(x,t)) \geq \beta r \, E[u'(c(k(x,t),e',t+1))] > E[u'(c(k(x,t),e',t+1))]$$

Integrate both sides of the above equation using the stationary probability measure y. The integrals are finite as factor prices are strictly positive with a positive capital stock (assumption T1) and therefore Fact 2-3 imply that $c(x,t) \geq c(0,e_1,t) > 0$, for all values of x and t.

$$\int_X u'(c(x,t))\, y(dx) > \int_X E[u'(c(k(x,t),e',t+1))]\, y(dx)$$

The inequality below follows by rearrangement. The equality follows by Stokey and Lucas (1989, Theorem 8.3) and by the fact that the aggregate state is time invariant in steady state.

$$\int_X u'(c(x,t))\, y(dx) > \int_X \int_X u'(c(x',t+1))\, P(x,t,dx')\, y(dx) = \int_X u'(c(x',t+1))\, y(dx')$$

The contradiction follows as c(x,t) is time invariant. Lemma 1 below completes the proof.

Lemma 1: Assume A1-2 and T1. In a positive capital steady state $\beta f'(K(y)) \neq 1$.
proof: See the Appendix. ◊

Comments

1. Laitner (1979, 1992), Bewley (1984) and Clarida (1990) prove that there exist stationary equilibria where $\beta r < 1$. The basic logic in these proofs are similar. One of the key steps in these proofs is to argue that the capital stock in steady state is a continuous function of the value of the gross interest rate r. Then one shows that as the interest rate approaches the level $\beta r = 1$ from below that steady-state capital becomes arbitrarily large. One merit of the proof given here is its simplicity relative to the arguments given in the papers mentioned above. The simplicity is gained by not addressing the question of existence and thus approaching the question using a different logic.
2. Theorem 1 does not depend upon the sign of the third derivative of the period utility function as some people familiar with the precautionary savings literature might have conjectured. The result here implies that any idiosyncratic uncertainty at all will reduce the interest rate relative to the no uncertainty steady state, regardless of the sign of the third derivative of the period utility function. The natural conjecture from the precautionary savings literature would be that adding more idiosyncratic uncertainty would lead to an even lower interest rate, given appropriate restrictions on the utility function. I will not pursue this conjecture.
3. Theorem 1 can be extended to the case with more than one agent type. This is true as the reasoning in the theorem will hold for each agent type considered separately. Theorem 1 will also hold when discount factors differ across agent types for exactly the same reason.
4. Theorem 1 can also be extended to the case where agents are allowed to borrow. As long as the minimum earnings are sufficient to maintain asset holdings at the credit limit and allow strictly positive consumption, then the reasoning described above will work.

5. The first part of the argument (showing $\beta f'(K(y)) \leq 1$) can be extended without change to the case where the endowment shocks are Markov. However, the second part of the argument (showing $\beta f'(K(y)) \neq 1$) seems to require some extra structure on the Markov process. See Huggett (1993, Theorem 2) for some restrictions that work.

4. Dynamics of Capital Accumulation Paths

This section investigates the dynamics of capital paths. First, I briefly review existing arguments for why capital paths are monotone in the economy with borrowing constraints but without idiosyncratic shocks. I extend one of these arguments to apply to economies with idiosyncratic shocks and show that capital paths are monotone increasing when the capital stock is sufficiently far below a steady state. Second, I provide an example showing that capital paths need not be monotonic near a steady state.

4.1 A Partial Result on Monotonicity

The monotonicity result in the one-sector model with borrowing constraints but without idiosyncratic shocks is based on Euler equation arguments. The Euler equation is a necessary condition for consumer maximization which states that $u'(c_t) \geq \beta r_{t+1} u'(c_{t+1})$, where equality holds when an agent holds strictly positive quantities of capital. The equation implies that consumption grows when capital in period t+1 is below the steady state and falls when capital is above the steady state. The fact that one can unambiguously "back out" what happens to consumption allows one to prove that capital must be monotone increasing when capital is below steady state and that the opposite occurs when capital is above steady state. Actually, to prove that capital is monotone decreasing above the steady state Hernandez (1991) assumes that the production technology is such that capital income is increasing in the level of capital.

Matters are not as simple in economies with idiosyncratic shocks. The Euler equation is then $u'(c_t) \geq \beta r_{t+1} E[u'(c_{t+1})]$, where equality holds when an agent holds strictly positive quantities of capital. The expectation operator makes it unclear what will happen to either realized or expected consumption growth when βr_{t+1} is either above or below 1. Some progress could still be made with stronger assumptions on preferences. For example, when βr_{t+1} is greater than 1 it could still be argued that expected consumption for an individual grows if u' is a convex function. This is just Jensen's inequality. This allows one to argue that capital increases monotonically when $\beta r_{t+1} > 1$. The example in the next subsection suggests that it is unlikely that there is any general result on capital paths when $\beta r_{t+1} < 1$.

The next theorem formalizes the logic stated above. The theorem states that the capital stock is strictly increasing when the capital stock is below the steady state of the complete market economy. The theorem also rules out aggregate fluctuations where the capital stock starts above this level and then falls below this level. Thus, any non-monotonic fluctuations in the capital stock must occur at capital levels at or above the level of capital solving $\beta f'(K) = 1$.

Theorem 2: Assume A1-3 and T1-2.

If $\beta f'(K(y_t)) > 1$, then $K(y_{t-1}) < K(y_t) < K(y_{t+1})$.

proof: See the Appendix. ◊

4.2 An Example Where Capital Paths Are Not Monotone

This section provides an example where capital paths are not monotone. The intuition driving the example is as follows. Take an economy that is in steady state. Redistribute capital across agents without changing the aggregate capital stock. If the marginal savings propensities differ across agents in different states then the implied aggregate capital stock should not remain constant over time at the unchanged steady state prices.[4] There are then a couple of possibilities. One possibility is that the capital stock could monotonically converge to a different steady-state capital stock. This could happen if steady states are not unique. However, I have computed that for the example considered here there is only one steady state capital stock. Thus, the remaining possibility is that the capital path is not monotone.

Example:	Preferences:	$u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$, $(\beta, \sigma) = (.96, 1.5)$
	Endowments:	$E = \{e_1, e_2\} = \{.8, 1.2\}$, $\pi(e_1) = \pi(e_2) = .5$
	Technology:	$f(K) = AK^{\alpha} - (1 - \delta)K$, $(A, \alpha, \delta) = (1, .36, .1)$

The economy has preferences and a technology of the parametric classes commonly used in applied work. The utility functions of agents are identical and homothetic. This

[4] There are some redistributions that don't make a difference. For example, there are some redistributions that are symmetric about the average capital holdings that leave the distribution unchanged.

means that with complete markets the distribution of capital across agents does not affect the dynamics of prices or economic aggregates. The technology is Cobb-Douglas and therefore satisfies the conditions in Hernandez (1991) that guarantee that capital paths are monotone in the absense of endowment shocks. There are two possible values of the endowment that occur with equal probability. The economy is completed by describing the initial distribution of capital across agents. The initial distribution puts 20 percent of the agents exactly at zero asset holdings and equal numbers of agents at all capital levels between 0 and 10.8104. Thus, the aggregate capital stock is 4.3242 which is the steady-state capital stock that I calculate for this economy. The steady-state distribution looks roughly lognormal. Therefore, the initial distribution puts less agents at high capital values and more agents near zero as compared to the steady-state distribution.

The intuition described previously for why capital paths are not monotone was based on the non-linearity of the optimal decision rule for capital. Figure 1 graphs the steady-state optimal decision rule on a 45 degree line diagram for each of the two endowment shocks. The decision rule is non-linear near the borrowing constraint. The fact that the non-linearities are very slight in this example suggests that the magnitude of the departures from the steady-state path will also be small.[5]

I will now discuss the properties of equilibrium paths and postpone until the next section a description of the computational methods employed to approximate equilibria. Figure 2 describes the equilibrium path for the capital stock and for comparison purposes the steady-state path. Both paths are consistent with the results described in Theorems 1 and 2. Thus, the steady-state capital level lies strictly above the steady-state level in the complete market economy which for this example is 4.30. Also, both paths stay above the steady state in the complete market economy as Theorem 2 requires.

In Figure 2 the capital stock at first rises above and then below the steady state and thereafter slowly converges to the steady state. The dynamics of the capital stock can be thought of as depending on the initial distribution of capital and on the changes over time in the optimal decison rule for capital. For the economy considered here it turns out that the shape of the path is largely determined by the initial distribution of capital and not by changes in the decision rule over time. When the economy is simulated using the steady-state decision rule instead of the equilibrium decision rule then one generates a capital path that is qualitatively similar to that in Figure 2. Changes in the optimal decison rule over time serve to dampen these fluctuations considerably without changing the qualitative nature of

[5] More dramatic results could easily be obtained by (1) putting a larger mass of agents at the corner of their borrowing constraint, (2) considering utility functions that are not homothetic, (3) allowing for multiple types of agents with different preferences or (4) changing the endowment process.

the path. Using this intuition, capital at first increases in Figure 2 as the marginal dissavings of agents moved to zero asset holdings are less than the marginal savings of agents experiencing an increase in asset holdings.

To close this section I compare the present example to other examples of aggregate fluctuations that are based on financial market imperfections within the one-sector growth model. See Boldrin and Woodford (1990) for a much broader review of models of aggregate fluctuations. First, Bewley (1986) provides an example where the capital stock cycles. His example is based on there being two types of agents whose labor endowments cycle in opposite directions every other period. Second, Becker and Foias (1987) construct an example where the capital stock cycles. Their example relies on the low substitutability of capital for labor in the production technology. In their example the total payment to capital depends sensitively on the level of capital. Hernandez (1991) proves for the case when all agents discount future utility at the same rate that such examples cannot occur when the total payment to capital is increasing in the capital stock. In the example of this paper the production technology is of the type that rules out these fluctuations when there is no idiosyncratic variation in labor endowment. Thus, the example presented here is based on endowment fluctuation as is the example in Bewley (1986). The key difference is that the role of distribution is emphasized here rather than multiple types of agents with periodically fluctuating endowments. The present example also has the appeal of using the functional forms and parameter values commonly used in applied work.

4.3 Computational Details

The equilibrium discussed in the previous section are computed as follows. First, calculate the capital stock in a stationary equilibrium. This can be determined by guessing different values K of the steady-state capital and then backing out factor prices from marginal products. The capital stock implied by these prices in a stationary equilibrium can then be determined by computing time-independent decision rules for capital and then integrating these decision rules with respect to the stationary distribution y implied by the guess, K. The results of Hopenhayn and Prescott (1992, Theorem 2) can be applied to this problem to show that this stationary distribution is unique and that the integral can be computed up to arbitrary accuracy. Details of how to apply this theorem to the present context are provided in Huggett (1993). Steady-state capital levels are then determined by finding a fixed point of the mapping from the conjectured to the implied capital stock. When I computed this mapping, there was a unique point where the map crossed the 45 degree line. Thus, steady-state equilibria appear to be unique in this example. This completes the argument that there is only one steady state. The precise details of how to carry out these computations are provided in Huggett (1993).

The final step in the construction of the equilibrium is to determine the transition path from the initial capital stock level back to this steady-state level. The procedure is to assume that the economy is back in a steady state after a large number of periods, which in these calculations is 1000 periods. Next, a guess is made for the transition path for the economy. With the guess, the decision rules k(x,t) can be computed by backward recursion on Bellman's equation (1). The transition path implied by the guess can then be computed. If the implied path coincides with the guessed path and the path converges to the steady state then one has computed an equilibrium. If the guessed and implied paths do not agree then one updates the guess and repeats the process.

5. Conclusion

This paper shows that the steady-state and dynamic properties of the one-sector growth model do not survive the addition of idiosyncratic shocks and a borrowing constraint. In particular, there is more capital in steady state in the model investigated here than in similar models with complete markets and there can be interesting economic dynamics around a steady state. In the future it would be interesting to investigate the nature and the quantitative importance of these economic dynamics. The example provided here suggests that the economic fluctuations produced by "shocking" the higher moments of the distribution of wealth occur at much lower frequencies than business-cycle fluctuations.

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Appendix

Fact : Assume A1-2 and w_t, $r_t > 0$ for all t, then

1. $c(x,t)$ and $k(x,t)$ are continuous in x.
2. $c(k,e,t)$ is strictly increasing in k and $c(x,t) > 0$ for all x.
   $k(k,e,t)$ is increasing in k.
3. $c(k,e_1,t) < c(k,e_2,t) < \ldots < c(k,e_n,t)$.
4. If $(w_t, r_t) = (w, r)$ for all t and $\beta r \leq 1$, then $k(k,e_1,t) < k$ for all $k > 0$.

proof: (1.) The theorem of the maximum together with the strict concavity of u and v generates the result.

(2.) $c(k,e,t)$ is strictly increasing in k because $v_1(k,e,t) = u'(c(k,e,t))r_t$ and v is increasing and strictly concave in k. Assumption A2 guarantees that consumption is always strictly positive. Suppose by way of contradiction that there were values k_1, k_2, e and t such that $k_2 > k_1$ and $k(k_2,e,t) < k(k_1,e,t)$. Since $c(k,e,t)$ is increasing in k it is true that

$$\beta r_{t+1} \, E[\, u'(c(k(k_2,e,t),e',t+1))\,] > \beta r_{t+1} \, E[\, u'(c(k(k_1,e,t),e',t+1))\,]$$

The Euler equation then implies that $u'(c(k_2,e,t)) > u'(c(k_1,e,t))$, a contradiction.

(3.) Suppose by way of contradiction that there were values e_i, e_j, k and t such that $e_j > e_i$ and $c(k,e_i,t) \geq c(k,e_j,t)$. Then it is true that $k(k,e_j,t) > k(k,e_i,t)$. Since $c(x,t)$ is increasing in k, it is true that

$$\beta r_{t+1} \, E[\, u'(c(k(k,e_i,t),e',t+1))\,] > \beta r_{t+1} \, E[\, u'(c(k(k,e_j,t),e',t+1))\,]$$

The Euler equation then implies that $u'(c(k,e_i,t)) > u'(c(k,e_j,t))$, which contradicts the fact that u' is a decreasing function.

(4.) The inequality in the first line below holds by Fact 3 and $\beta r_{t+1} \leq 1$. The equality holds since $(w_t, r_t) = (w, r)$ for all t implies that optimal policy functions are time invariant. The second line below is the Euler equation.

$$u'(c(k,e_1,t)) > \beta r_{t+1} \, E[u'(c(k,e',t))] = \beta r_{t+1} \, E[u'(c(k,e',t+1))]$$

$$u'(c(k,e_1,t)) \geq \beta r_{t+1} \, E[u'(c(k(k,e_1,t),e',t+1))]; \quad = \text{if } \; k(k,e_1,t) > 0$$

For the Euler equation to hold there are two possibilities. If $k(k,e_1,t) > 0$, then $k(k,e_1,t) < k$ since $c(k,e',t+1)$ is increasing in k by Fact 2 and since u' is decreasing. The other possibility is that $k(k,e_1,t) = 0$. In either case Fact 4 holds. ◊

Lemma 1: Assume A1-2 and T1. In a positive capital steady state $\beta f'(K(y)) \neq 1$.

proof: Suppose by way of contradiction that $\beta f'(K(y)) = 1$. It will then be shown later that there is a set A such that $y(A) > 0$ and on this set the Euler equation holds with strict inequality. It then follows from the arguments in the first part of the proof of Theorem 1 that the following equation holds. To simplify notation the time index is dropped in this equation and throughout the proof as decision rules are time invariant in steady state.

$$\int_X u'(c(x))\, dy > \int_X E[u'(c(k(x),e'))]\, dy = \int_X u'(c(x))\, dy$$

Therefore, the lemma follows if such a set A exists. To prove that such a set A exists note that Fact 2-3 and $\beta r = 1$ imply that the following inequality holds for $k = 0$.

$$u'(c(k,e_1)) > \beta r\, E[u'(c(k(k,e_1),e'))\,]$$

The continuity of u' and continuity of the optimal decision rules imply that there is a constant $a > 0$ and a set of states $A = [0,a)\times\{e_1\}$ where the strict inequality above also holds. Next, argue that in steady state $y(A) > 0$. For x in X define a sequence $\{x_n(x)\}$ by

$$x_1 = (k(x),e_1) \text{ and } x_n = (k(x_{n-1}),e_1) \text{ for } n = 2, 3,...$$

Fact 4 implies that $x_n(x)$ converges to $(0,e_1)$. So it is possible to go from x to A in N or greater steps, for some value N which depends on x. Let $P^n(x,A)$ denote such n-step probabilities. Therefore, $P^n(x,A) \geq \pi(e_1)^n > 0$ for all $n \geq N$. Select $B = [0,k]\times E$ such that $y(B) > 0$. Fact 2 then implies that there is an N such that $P^n(x,A) \geq \pi(e_1)^n > 0$ for all $n \geq N$ and for all x in B. Therefore, the following is true.

$$y(A) = \int_X P(x,A)\, dy = \int_X P^n(x,A)\, dy \geq \pi(e_1)^n\, y(B) > 0. \qquad \Diamond$$

Theorem 2: Assume A1-3 and T1-2.

If $\beta f'(K(y_t)) > 1$, then $K(y_{t-1}) < K(y_t) < K(y_{t+1})$.

proof: Suppose by way of contradiction that $K(y_{t-1}) \geq K(y_t)$. Apply Lemma 2 below repeatedly to get that $\{K(y_{t-1}), K(y_t), K(y_{t+1}),...\}$ is decreasing. A necessary condition for maximization is

$$u'(c(x,t-1)) \geq \beta r_t \, E[u'(c(k(x,t-1),e',t))\,] = \beta r_t \int_X u'(c(x_1,t))\, P(x,t-1,dx_1)$$

Integrate this expression with respect to the measure y_{t-1} and apply Stokey and Lucas (1989, Theorem 8.3) to get the inequality and the equality in the first line below. Repeat the argument to get the second line below.

$$\int_X u'(c(x,t-1))dy_{t-1} \geq \beta r_t \int_X E[u'(c(k(x,t-1),e',t))]dy_{t-1} = \beta r_t \int_X u'(c(x,t))\, dy_t$$

$$\int_X u'(c(x,t-1))dy_{t-1} \geq \beta^{n+1} r_t \ldots r_{t+n} \int_X u'(c(x,t+n))\, dy_{t+n}$$

This implies that $\int u'(c(x,t+n))\, dy_{t+n}$ tends to zero as n increases because the term on the right-hand side preceeding the integral grows without bound. Given any $\varepsilon > 0$, let n be such that $\int u'(c(x,t+n))\, dy_{t+n} < \varepsilon$. Since marginal utility is a convex function, Jensen's inequality implies the line below.

$$u'(\int c(x,t+n)\, dy_{t+n}) \leq \int u'(c(x,t+n))\, dy_{t+n} < \varepsilon$$

This then implies that aggregate consumption becomes arbitrarily large which contradicts the fact that output is bounded. Therefore, it is the case that $K(y_{t-1}) < K(y_t)$. Repeat the same argument to obtain the second inequality in the Theorem. ◊

Lemma 2: Assume A1-3 and T1-2.

If $\beta f'(K(y_t)) \geq 1$ and $K(y_{t-1}) \geq K(y_t)$, then $K(y_t) \geq K(y_{t+1})$.

proof: The first inequality below is a necessary condition for consumer maximization. The second inequality follows because $\beta r_t \geq 1$.

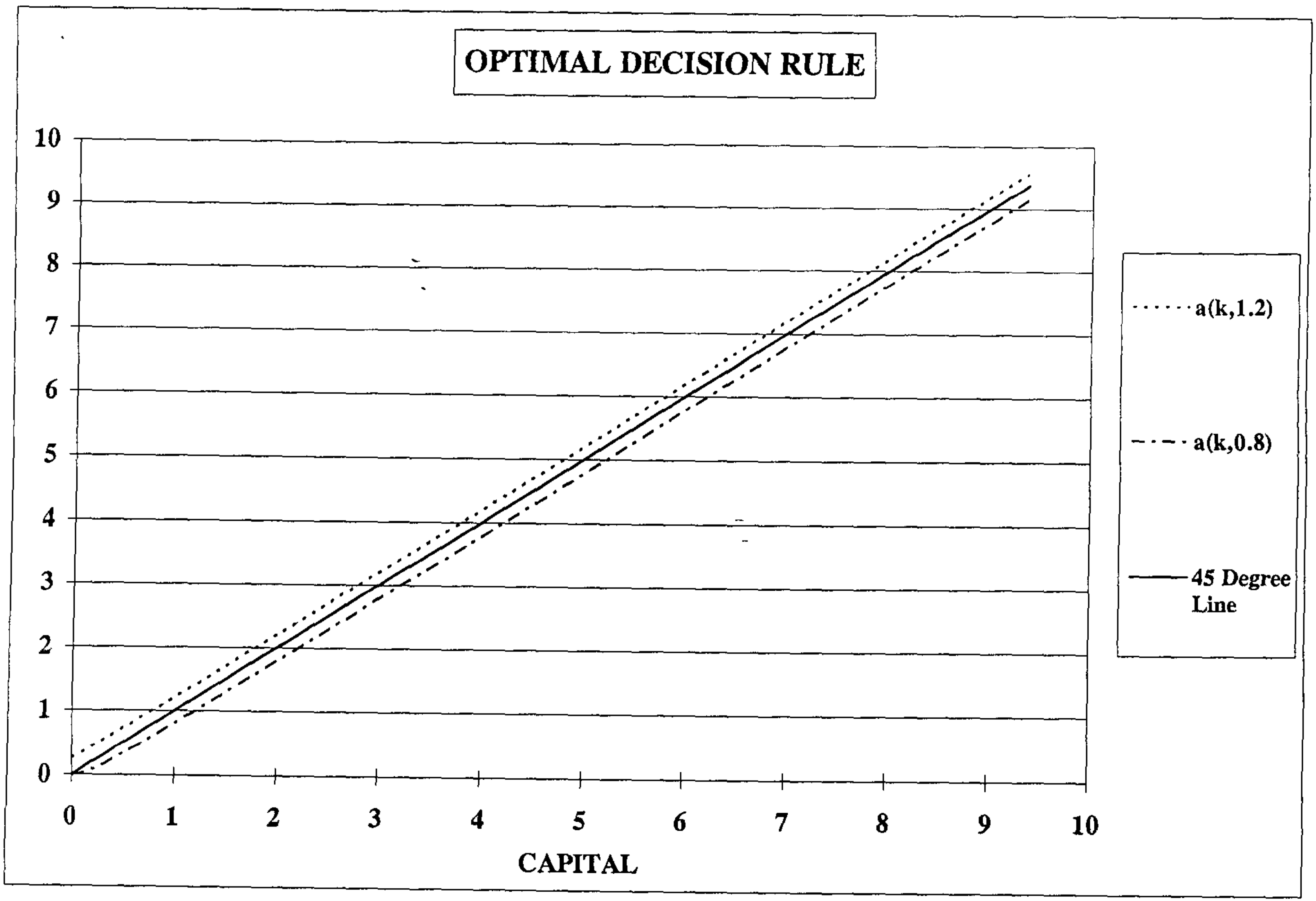
$$u'(c(x,t-1)) \geq \beta r_t \; E[u'(c(k(x,t-1),e',t))] \; \geq E[u'(c(k(x,t-1),e',t)) \;]$$

The inequality below follows by applying Jensen's inequality to the equation above and then integrating. The equality follows by Stokey and Lucas (1989, Theorem 8.3).

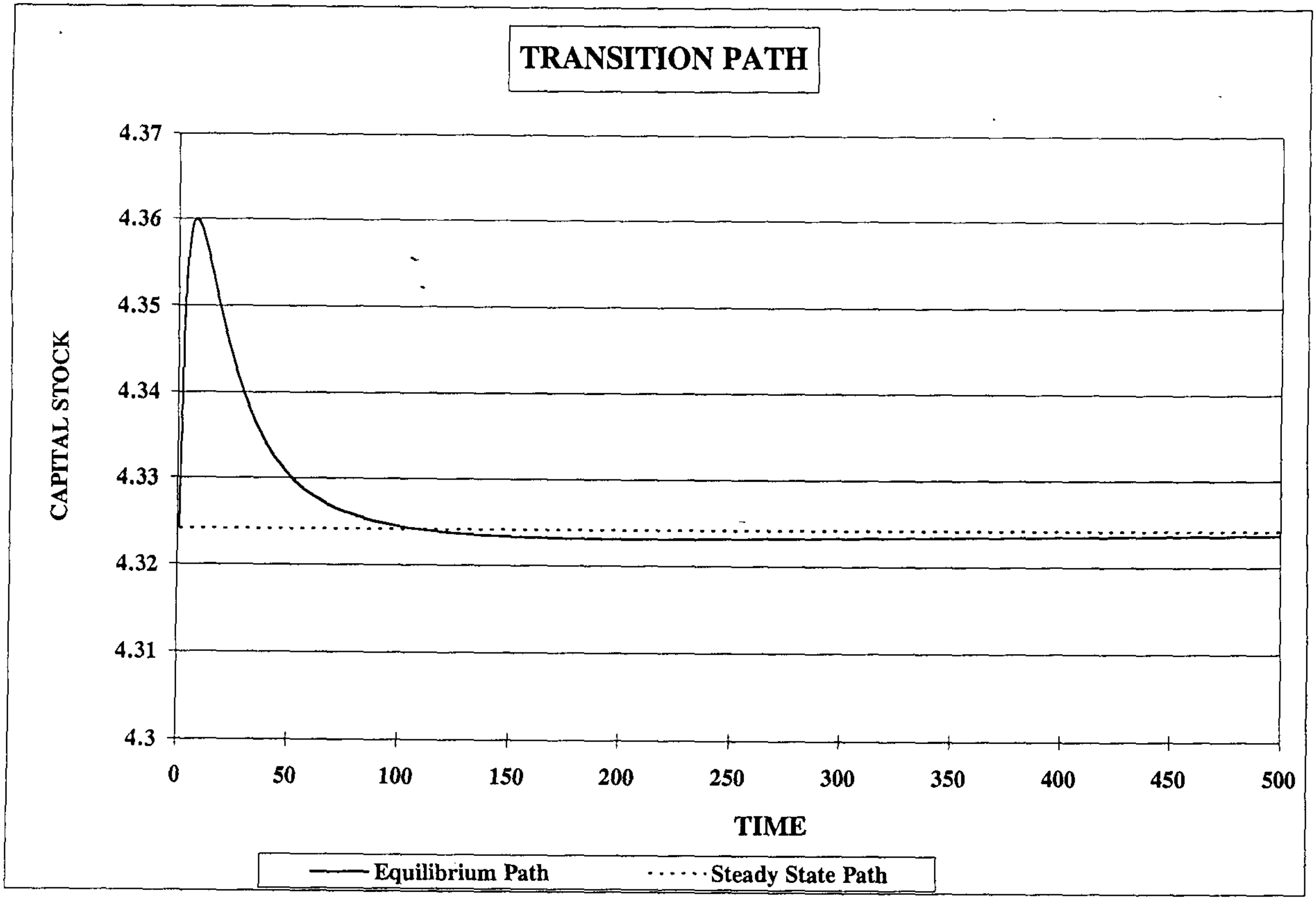
$$\int_X c(x,t-1) \, y_{t-1}(dx) \; \leq \; \int_X E[c(k(x,t-1),e',t)] \, y_{t-1}(dx) = \; \int_X c(x,t) \, y_t \, (dx)$$

Finally, $f(K(y_{t-1})) \geq f(K(y_t))$ and the previous step yield

$$K(y_t) = f(K(y_{t-1})) - \int_X c(x,t-1) \, y_{t-1}(dx) \; \geq \; f(K(y_t)) - \int_X c(x,y_t) \, y_t(dx) = K(y_{t+1}) \; . \; \Diamond$$



OPTIMAL DECISION RULE
10
9
8
7
6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8
9
10
CAPITAL
a(k,1.2)
a(k,0.8)
45 Degree Line


TRANSITION PATH
CAPITAL STOCK
4.37
4.36
4.35
4.34
4.33
4.32
4.31
4.3
0
50
100
150
200
250
300
350
400
450
500
TIME
Equilibrium Path
Steady State Path