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A Note on Dynamic Programming With Homogeneous Functions

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ABSTRACT

This note shows that the basic theorems of dynamic programming hold when the return function is homogenous of degree $\theta \leq 1$.

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A Note on Dynamic Programming with Homogeneous Functions

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Abstract

This note shows that the basic theorems of dynamic programming hold when the return function is homogeneous of degree $\theta \leq 1$.

A major limitation in applying the tools of dynamic programming to many economic problems has been the lack of a general theory in the case where returns are unbounded. Except for a few special cases where closed form solutions are available (linear-quadratic models and the log-Cobb-Douglas growth model being the leading examples), recursive methods have not been useful. This note establishes that the basic fixed point argument of dynamic programming, which is fundamental for both theoretical and computational purposes, can be applied to a limited but useful family of unbounded problems, those that are homogeneous.

Many problems in economics are conveniently modeled with return functions that are homogeneous of degree $\theta \leq 1$ and constraints that are homogeneous of degree one. For example, in much of the endogenous growth literature (Lucas (1988), Jones and Manuelli (1990), Rebelo (1991), and many others), a utility function with a constant elasticity of intertemporal substitution is used,

$$U(c) = \frac{c^\theta}{\theta}, \quad \theta < 1,$$

together with technologies that display constant returns to scale. The case $\theta = 0$ is interpreted as $U(c) = \ln(c)$. Preferences in this class are used because homogeneous functions are the only ones consistent with balanced growth.

This note shows that the basic contraction mapping argument of dynamic programming holds for problems of this type. The main steps are to find appropriate restrictions on the state space and feasibility constraints and an appropriate space of functions. With this done, standard arguments apply.

1. Preliminaries

Consider the problem

$$\sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{1}$$

$$\text{s.t.} \quad x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, ...,$$

$$x_0 \in X \text{ given,}$$

and the corresponding Bellman equation

$$v(x) = \sup_{y \in \Gamma(x)} \left[F(x, y) + \beta v(y)\right], \quad \text{all } x_0 \in X. \tag{2}$$

Let $v^*(x_0)$ denote the value of the supremum in (1). It is often convenient, for both theoretical and computational purposes, to study (2) instead of (1). To do so, however, it must be established that there is a one-to-one relationship between solutions to the two problems. Moreover, the convenience comes only if it can be shown that standard arguments establishing the existence and uniqueness of a solution apply to the latter.

Both can be accomplished when Γ is homogeneous of degree one and F is homogeneous of degree $\theta \leq 1$. Slightly different assumptions and arguments are needed for the case $\theta \in (0, 1]$, the case $\theta < 0$ and the logarithmic case ($\theta = 0$).

The following space of functions is suitable for analyzing Bellman equations when $\theta \neq 0$. Let $X \subseteq \mathbf{R}^{\ell}$ be a cone, excluding the origin if $\theta < 0$, and let $H(X, \theta)$ be the linear space of functions f: $X \to \mathbf{R}$ that are continuous, homogeneous of degree θ, and bounded in the norm

$$\|f\| = \sup_{\|x\|=1, x \in X} |f(x)|. \tag{3}$$

It is straightforward to show that (3) defines a norm and that with it, $H(X, \theta)$ is a complete metric space. To see this, notice that there is a one-to-one relationship between elements of $H(X, \theta)$ and elements of the set of bounded, continuous functions defined on the intersection of X with the unit circle. If $\theta < 0$, then functions in $H(X, \theta)$ diverge as $\|x\| \to 0$, so the origin must be excluded.

Note that for any $f \in H(X, \theta)$, homogeneity of degree θ implies that

$$f(x) = \|x\|^{\theta} f\left(\frac{x}{\|x\|}\right) \leq \|x\|^{\theta} \|f\|. \tag{4}$$

It follows that for any $f, g \in H(X, \theta)$,

$$\begin{aligned} f(x) &= g(x) + [f(x) - g(x)] \\ &\leq g(x) + \|x\|^{\theta} \|f - g\|, \quad \text{all } x \in X. \end{aligned} \tag{5}$$

In addition, for any $f \in H$, let $f + a$ denote the function

$$(f + a)(x) = f(x) + a \|x\|^{\theta},$$

so $f + a$ is also continuous and homogeneous of degree θ. That is, $(f + a) \in H(X, \theta)$. Lemma 1 provides an analogue of Blackwell's sufficient conditions for a contraction. The lemma applies to $H(X, \theta)$, but to other spaces as well, and others arise in applications. For example, in some settings it is useful to apply the result to the subset of $H(X, \theta)$ consisting of functions that are weakly quasi-concave.

Lemma 1. Let $\theta \in \mathbf{R}$, with $\theta \neq 0$; let $X \subseteq \mathbf{R}^{\ell}$ be a cone, excluding the origin if $\theta < 0$; and let $J(X,\theta)$ be a space of functions that are homogeneous of degree θ and bounded in the norm in (3). Let T: $J(X,\theta) \rightarrow J(X,\theta)$ be an operator satisfying

a. (monotonicity) $f, g \in J(X,\theta)$ and $f \leq g$, implies $Tf \leq Tg$;

b. (discounting) there exists some $\beta \in (0,1)$ such that

$$T(f+a) \leq (Tf) + \beta a, \text{ all } f \in J(X,\theta),\ a \geq 0.$$

Then T is a contraction of modulus β.

Proof. Let $f, g \in J(X,\theta)$. Using (5), monotonicity and discounting imply that,

$$\begin{aligned}(Tf)(x) &\leq T\left(g(x) + \|x\|^{\theta}\|f-g\|\right)\\ &\leq Tg(x) + \beta\|x\|^{\theta}\|f-g\|, \text{ all } x \in X.\end{aligned}$$

The same argument applies with the roles of f and g reversed. Using these facts and confining attention to the subset of X where $\|x\| = 1$, one finds that $\|Tf - Tg\| \leq \beta\|f-g\|$. □

2. Homogeneous functions: $0 < \theta \leq 1$

Next we must find suitable restrictions on the state space, feasibility constraints, and return function. For the case $\theta \in (0,1]$, the following assumption is useful.

Assumption 1. a. $\theta \in (0,1]$ and $X \subseteq \mathbf{R}^{\ell}$ is a cone;

b. the correspondence Γ: $X \rightarrow X$ is nonempty, compact-valued, and continuous, and the graph of Γ, call it A, is a cone: $\Gamma(0) = 0$, and

$$y \in \Gamma(x) \Rightarrow \lambda y \in \Gamma(\lambda x), \quad \text{all } \lambda > 0, \text{ all } x \in X;$$

c. $\beta \in (0,1)$, and there exists $\alpha > 0$ with $\gamma \equiv \alpha^\theta \beta < 1$, such that

$$\|y\| \leq \alpha \|x\|, \quad \text{all } (x,y) \in A;$$

d. $F: A \to \mathbf{R}$ is continuous and homogeneous of degree θ, and for some $0 < B < \infty$,

$$|F(x,y)| \leq B(\|x\| + \|y\|)^\theta, \text{ all } (x,y) \in A.$$

Assumption 1c, which is the Brock-Gale (1969) condition for existence of optimal paths, bounds the rate of growth of feasible sequencesby $\beta^{-1/\theta}$, and 1d imposes a uniform bound on the ratio of F to the norm of its arguments. The latter is equivalent to assuming that $F(x,y)$ is bounded for $\|x\| = 1$ and $y \in \Gamma(x)$, and implies that $F(0,0) = 0$.

Given any $x_0 \in X$, define

$$\Pi(x_0) = \{\{x_t\}_{t=0}^\infty : x_{t+1} \in \Gamma(x_t), \quad t = 0,1,...\}$$

to be the set of all sequences in X that are feasible from x_0. As above, let v^* denote the supremum function for (1).

Lemma 2. Let $(\theta, X, \Gamma, \beta, F)$ satisfy Assumption 1. Then

a. $v^* \in H(X,\theta)$;

b. v^* satisfies (2);

c. if a feasible plan $\{x_t^*\} \in \Pi(x_0)$ attains the supremum in (1) for initial state x_0, then

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*), \quad t = 0,1,2,...; \tag{6}$$

d. if $v^* \in H(X,\theta)$ and $\{x_t^*\} \in \Pi(x_0)$ satisfies (6) for initial state x_0, then $\{x_t^*\}$ attains the supremum in (1).

Proof. To show that (b) - (d) hold, it suffices to show that $\Pi(x_0)$ is nonempty, for all $x_0 \in X$; that $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists, for all $x_0 \in X$ and all $\{x_t\} \in \Pi(x_0)$, (although it may be plus or minus infinity); and that

$$\lim_{t\to\infty} \sup \beta^t v^*(x_t) \le 0, \quad \text{all } \{x_t\} \in \Pi(x_0), \text{ all } x_0 \in X. \tag{7}$$

The claims then follow from Theorems 4.2, 4.4, and 4.5 in Stokey, Lucas, and Prescott (1989). Part (a) will be proved directly.

It follows immediately from Assumption 1b that $\Pi(x_0)$ is nonempty, for all $x_0 \in X$.

Next, choose any $x_0 \in X$ and any $\{x_t\} \in \Pi(x_0)$. Assumption 1c implies that

$$\|x_t\| \le \alpha^t \|x_0\|, \quad \text{all } t,$$

so

$$\|x_t\| + \|x_{t+1}\| \le \alpha^t \|x_0\| (1+\alpha), \quad \text{all } t,$$

and

$$(\|x_t\| + \|x_{t+1}\|)^\theta \le \alpha^{\theta t} \|x_0\|^\theta (1+\alpha)^\theta, \quad \text{all } t.$$

Then Assumption 1d implies that

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t |F(x_t, x_{t+1})| &\le B \|x_0\|^\theta (1+\alpha)^\theta \sum_{t=0}^{\infty} \left(\beta\alpha^\theta\right)^t \\ &= \|x_0\|^\theta \frac{B(1+\alpha)^\theta}{1-\gamma},\end{aligned} \tag{8}$$

where $\gamma = \alpha^\theta \beta < 1$. Hence $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists.

To show that (a) holds, we must show that v^* is bounded in the norm in (3), continuous, and homogeneous of degree θ. For boundedness, note that (8) implies that if $\|x_0\| = 1$, then $|v^*(x_0)| \le B(1+\alpha)^\theta / (1-\gamma)$. Continuity follows from the fact that Γ and F are continuous. For homogeneity, note that since Γ is homogeneous of degree one, it follows that for any $x_0 \in X$ and $\lambda > 0$, if $\{x_t\} \in \Pi(x_0)$, then

$\{\lambda x_t\} \in \Pi(\lambda x_0)$. Since F is homogeneous of degree θ, the discounted returns from these paths satisfy

$$u(\lambda \underline{x}) = \sum_{t=0}^{\infty} \beta^t F(\lambda x_t, \lambda x_{t+1}) = \sum_{t=0}^{\infty} \beta^t \lambda^\theta F(x_t, x_{t+1}) = \lambda^\theta u(\underline{x}),$$

where $u(\underline{x})$ denotes the total discounted returns from any feasible path. Hence

$$v^*(\lambda x_0) = \sup_{\{x_t\}\in\Pi(\lambda x_0)} u(\underline{x}) = \sup_{\{x_t\}\in\Pi(x_0)} u(\lambda \underline{x}) = \sup_{\{x_t\}\in\Pi(x_0)} \lambda^\theta u(\underline{x}) = \lambda^\theta v^*(x_0),$$

so v^* is homogeneous of degree θ.

Finally, since $v^* \in H(X, \theta)$, it follows from (4) and Assumption 1c that

$$\lim_{t\to\infty} \beta^t |v^*(x_t)| \leq \lim_{t\to\infty} \beta^t \|x_t\|^\theta \|v^*\| \leq \lim_{t\to\infty} \gamma^t \|x_0\|^\theta \|v^*\| = 0,$$

$$\text{all } \{x_t\} \in \Pi(x_0), \text{ all } x_0 \in X,$$

so (7) holds. □

Theorem 1. Let $(\theta, X, \Gamma, \beta, F)$ satisfy Assumption 1, define $H(X, \theta)$ as above, and define the operator T on $H(X, \theta)$ by

$$(Tf)(x) = \sup_{y\in\Gamma(x)} [F(x,y) + \beta f(y)]. \tag{9}$$

Then T: $H(X,\theta) \to H(X,\theta)$, T is a contraction of modulus β, and the supremum function v^* for (1) is the unique fixed point of T. If a feasible plan $\{x_t^*\} \in \Pi(x_0)$ satisfies (7), then it attains the supremum in (1) for initial state x_0. The policy correspondence G is nonempty, compact-valued, u.h.c., and homogeneous of degree one:

$$y \in G(x) \quad \text{implies} \quad \lambda y \in G(\lambda x), \quad \text{all } \lambda > 0.$$

Proof. Assumptions 1c and 1d imply that for $(x, y) \in A$ with $\|x\| = 1$, $F(x, y)$ is bounded and $\|y\| \leq \alpha$, so the right side of (9) is bounded by $B(1+\alpha)^\theta + \alpha \|f\|$. Hence $\|Tf\|$ is bounded. Clearly T preserves homogeneity:

$$y^* \in \arg\max_{y\in\Gamma(x)} [F(x,y) + \beta f(y)] \quad \text{implies} \tag{10}$$

$$\lambda y^* \in \arg\max_{y \in \Gamma(\lambda x)} [F(\lambda x, y) + \beta f(y)], \quad \text{all } \lambda > 0.$$

That Tf is continuous follows from Assumption 1 and the Theorem of the Maximum.

Obviously T satisfies the hypotheses of Lemma 1, so T is a contraction of modulus β. Since $H(X, \theta)$ is complete, T has a unique fixed point. This fixed point is the unique element of $H(X, \theta)$ that satisfies (2), and so by Lemma 2 is v^*. The homogeneity of the policy function is clear from (10), and the other properties are standard. □

3. Homogeneous functions: $\theta < 0$

The space of functions $H(X, \theta)$ defined above applies for this case as well. A somewhat different set of assumptions on the state space are useful, however,

If $\theta > 0$, the value of a homogeneous function $f(x)$ grows (in absolute value) as $\|x\|$ grows. Thus, to ensure that total discounted returns did not diverge along any feasible path, Assumption 1 put an upper bound on the growth rate of $\|x_t\|$ along every feasible path. If $\theta < 0$, the value of a homogeneous function grows (in absolute value) as $\|x\|$ shrinks. Thus, one possibility for this case would be to put a *lower* bound on the growth rate of $\|x_t\|$ along every feasible path. Most applications do not fit this restriction, however.

Instead, in most applications the return function takes nonpositive values, $F \leq 0$, so returns are bounded above by zero but are potentially unbounded below. Then, since total returns are being maximized, it is enough to assume that from every initial condition there is at least *one* feasible path along which returns do not diverge to minus infinity. Hence it suffices to assume that from every initial condition $x_0 \in X$, there is at least one feasible path $\{x_t\} \in \Pi(x_0)$ along which $\|x_t\|$ does not shrink too quickly. This in turn implies that $(\|x_t\| + \|x_{t+1}\|)^\theta$ does not grow too quickly, so total discounted returns are bounded below.

Assumption 2. a. $\theta < 0$ and $X \subseteq \mathbf{R}^\ell$ is a cone, minus the origin;

b. the correspondence Γ: $X \to X$ is nonempty, compact-valued, and continuous, and the graph of Γ, call it A, is a cone, minus the origin:

$$y \in \Gamma(x) \Rightarrow \lambda y \in \Gamma(\lambda x), \quad \text{all } \lambda > 0, \text{ all } x \in X.$$

c. $\beta \in (0,1)$, and there exists $\zeta > 0$ with $\gamma \equiv \beta\zeta^\theta < 1$, such that for every $x \in X$,

$$\|y\| \geq \zeta \|x\|, \quad \text{for some } y \in \Gamma(x);$$

d. $F: A \to \mathbf{R}_-$ is continuous and homogeneous of degree θ, and for some $0 < B < \infty$,

$$|F(x,y)| \leq B(\|x\| + \|y\|)^\theta, \text{ all } (x,y) \in A.$$

As before, Assumption 2d is equivalent to assuming that $|F(x,y)|$ is bounded for $\|x\| = 1$ and $y \in \Gamma(x)$.

Under Assumption 2 there may be many feasible paths long which returns diverge to $-\infty$. For any initial state, however, there exists at least one feasible path along which the growth rate of $\|x_{t+1}\| / \|x_t\|$ is bounded below by ζ. For any such path, the bound in part 2d and the assumption that $\beta\zeta^\theta < 1$ together imply that total returns are bounded. Thus, under Assumption 2, we have the following analogue of Lemma 2.

Lemma 3. If $(\theta, X, \Gamma, \beta, F)$ satisfy Assumption 2, then (a) - (d) of Lemma 2 hold.

Proof. Much of the proof parallels the proof of Lemma 2.

It follows immediately from Assumption 2b that $\Pi(x_0)$ is nonempty, for all $x_0 \in X$. Also, since $F \leq 0$, the limit exists, although it may be $-\infty$, and $v^* \leq 0$, so (7) holds. Hence claims (b) - (d) hold.

To show that $v^* \in H(X,\theta)$, we must show that v^* is bounded in the norm in (3), continuous, and homogeneou of degree θ. Since $F \leq 0$, v^* is bounded above by zero. Assumption 2c implies that for any $x_0 \in X$, there exists $\{x_t\} \in \Pi(x_0)$ such that

$$\|x_t\| \geq \zeta^t \|x_0\|, \quad \text{all } t \geq 0,$$

so

$$(\|x_t\| + \|x_{t+1}\|) \geq \zeta^t \|x_0\| (1+\zeta), \quad \text{all } t \geq 0,$$

so

$$(\|x_t\| + \|x_{t+1}\|)^\theta \leq \zeta^{\theta t} \|x_0\|^\theta (1+\zeta)^\theta, \quad \text{all } t \geq 0.$$

Then Assumption 1d implies that

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t |F(x_t, x_{t+1})| &\leq B \|x_0\|^\theta \, (1+\zeta)^\theta \sum_{t=0}^{\infty} \left(\beta\zeta^\theta\right)^t \\ &= \|x\|^\theta \frac{B(1+\zeta)^\theta}{1-\gamma}, \end{aligned}$$

where $\gamma = \beta\zeta^\theta < 1$. In particular, if $\|x_0\| = 1$, then $-B(1+\zeta)^\theta / (1-\gamma) \leq v^*(x_0) \leq 0$. Hence v^* is bounded in the norm in (3). The arguments showing that v^* is continuous and homogeneous of degree θ are the same as before. □

The conclusions of Theorem 1 then go through without change.

Theorem 2. Let $(\theta, X, \Gamma, \beta, F)$ satisfy Assumption 2, define $H(X,\theta)$ as above, and define the operator T on $H(X,\theta)$ by (9). Then the conclusions of Theorem 1 hold.

Proof. Unchanged. □

4. Logarithmic functions: $\theta = 0$

For the logarithmic case, $\theta = 0$, a similar argument applies. A somewhat unusual space of functions must be used, however. Let $X \subset \mathbf{R}^\ell$ be a cone, minus the origin, and for any $\beta \in (0,1)$ consider the space of continuous functions f: $X \to \mathbf{R}$ that satisfy

$$f(x) = f\left(\frac{x}{\|x\|}\right) + (1-\beta)^{-1} \ln \|x\|, \quad \text{all } x \in X. \tag{11}$$

Define the "zero" function

$$o(x) = (1-\beta)^{-1} \ln \|x\|;$$

define addition by

$$(f+g)(x) = f\left(\frac{x}{\|x\|}\right) + g\left(\frac{x}{\|x\|}\right) + (1-\beta)^{-1} \ln \|x\|;$$

and define scalar multiplication by

$$\alpha f(x) = \alpha f\left(\frac{x}{\|x\|}\right) + (1-\beta)^{-1} \ln \|x\|.$$

It is straightforward to show that

$$\|f\| = \sup_{\|x\|=1, x\in X} |f(x)| \tag{12}$$

defines a norm on this space. Let $H_\beta(X,0)$ denote the space of continuous functions satisfying (11) that are bounded in the norm in (12). It is straightforward to show that $H_\beta(X,0)$ is complete. As before, the key idea is that functions in $H_\beta(X,0)$ are characterized by their behavior on the intersection of X with the unit circle.

Also note that for any $f \in H_\beta(X,0)$,

$$f(x) \leq \|f\| + (1-\beta)^{-1} \ln \|x\|. \tag{13}$$

It follows that for any $f, g \in H_\beta(X,\theta)$,

$$\begin{aligned} f(x) &= g(x) + [f(x) - g(x)] \\ &= g(x) + \left[f\left(\frac{x}{\|x\|}\right) - g\left(\frac{x}{\|x\|}\right)\right] \end{aligned}$$

$$= g(x) + \|f - g\|, \quad \text{all } x \in X. \tag{14}$$

The next lemma provides an analogue of Blackwell's sufficient conditions for a contraction. As before, the lemma holds for $H_\beta(X,0)$, but also more generally.

Lemma 4. Let $X \subseteq \mathbf{R}^\ell$ be a cone, excluding the origin; let $\beta \in (0,1)$; let $J_\beta(X,0)$ be a space of functions that satisfy (11) and are bounded in the norm in (12). Let T: $J_\beta(X,0) \to J_\beta(X,0)$ be an operator satisfying (a) and (b) (monotonicity and discounting) of Lemma 1. Then T is a contraction of modulus β.

Proof. Let $f, g \in J_\beta(X,0)$. Using (14), monotonicity and discounting imply that,

$$(Tf)(x) \leq T\left(g(x) + \|f - g\|\right)$$

$$\leq Tg(x) + \beta\,\|f - g\|, \text{ all } x \in X.$$

Reversing the roles of f and g, one finds that $\|Tf - Tg\| \leq \beta\,\|f - g\|$. □

When the return function is logarithmic, it is unbounded both above and below. Hence the growth rate of $\|x_t\|$ must be bounded above for all feasible sequences, and for every initial condition $x_0 \in X$, there must be at least one feasible sequence along which the growth rate of $\|x_t\|$ is bounded below. In addition, the ratio of the return function to the norm of its arguments must be bounded from above and below. The analogue of Assumptions 1 and 2 is the following.

Assumption 3. a. $X \subseteq \mathbf{R}^\ell$ is a cone, minus the origin.

b. The correspondence Γ: $X \to X$ is nonempty, compact-valued, and continuous, and the graph of Γ, call it A, is a cone, minus the origin.

c. $\beta \in (0,1)$, there exist $0 < \zeta < 1$ and $1 < \alpha < +\infty$ such that for any $x \in X$,

$$\|y\| \geq \zeta\,\|x\|, \quad \text{some } y \in \Gamma(x),$$

$$\|y\| \leq \alpha \|x\|, \quad \text{all } y \in \Gamma(x);$$

d. F: $A \to \mathbf{R}$ has the property that $F(x,y) = \ln \phi(x,y)$, where ϕ: $A \to \mathbf{R}_+$ is continuous and homogeneous of degree one. In addition, there exist $0 < b < 1$ and $1 < B < +\infty$ such that

$$b(\|x\| + \|y\|) \leq \phi(x,y) \leq B(\|x\| + \|y\|), \text{ all } (x,y) \in A.$$

Part d is equivalent to assuming that $\phi(x,y)$ is bounded, above and below, for $\|x\| = 1$, and $y \in \Gamma(x)$.

Lemma 5. If (X, Γ, β, F) satisfy Assumption 3, then $v^* \in H_\beta(X, 0)$, and (b) - (d) of Lemma 2 hold.

Proof. Much of the proof parallels the proof of Lemma 2.

Assumption 3b implies that $\Pi(x_0)$ is nonempty, for all $x_0 \in X$.

To show that $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists, it suffices to show that $\sum_{t=0}^{\infty} \beta^t F^+(x_t, x_{t+1})$ is bounded, where $F^+(x,y) = \max\{F(x,y), 0\}$. Assumption 3c implies that for all $\{x_t\} \in \Pi(x_0)$ and all $x_0 \in X$,

$$\ln \|x_t\| \leq \ln\left(\alpha^t \|x_0\|\right) = t \ln \alpha + \ln \|x_0\|, \quad t = 1, 2, ... \tag{15}$$

Since ϕ is homogeneous of degree one,

$$\begin{aligned} F(x_t, x_{t+1}) &= \ln(\phi(x_t, x_{t+1})) \\ &= \ln\left(\|x_t\| \ \phi\left(\frac{x_t}{\|x_t\|}, \frac{x_{t+1}}{\|x_t\|}\right)\right) \\ &= \ln(\|x_t\| \ ) + \ln\left(\ \phi\left(\frac{x_t}{\|x_t\|}, \frac{x_{t+1}}{\|x_t\|}\right)\right). \end{aligned}$$

It then follows from (15) and Assumption 3d that

$$\beta^t F(x_t, x_{t+1}) \leq \beta^t \left[t \ln \alpha + \ln \|x_0\| \ + \ln(B(1+\alpha))\right].$$

Therefore, since $\alpha, B > 1$ and $\beta < 1$,

$$\sum_{t=0}^{\infty} \beta^t F^+(x_t, x_{t+1}) \leq \sum_{t=0}^{\infty} \beta^t \left[t \ln \alpha + \max\{\ln \|x_0\|, 0\} + \ln (B(1+\alpha))\right]$$

$$= \ln \alpha \sum_{t=0}^{\infty} \beta^t t + \frac{\max\{\ln \|x_0\|, 0\} + \ln (B(1+\alpha))}{1-\beta},$$

which is finite. Hence $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists, although it may be $-\infty$ for some paths.

To show that $v^* \in H_\beta(X, 0)$, we must show that v^* is bounded and continuous and satisfies (11). Choose any $x_0 \in X$ with $\|x_0\| = 1$. It follows from the argument above that for any $\{x_t\} \in \Pi(x_0)$,

$$\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \leq \ln \alpha \sum_{t=0}^{\infty} \beta^t t + \frac{\ln (B(1+\alpha))}{1-\beta}.$$

In addition, it follows from Assumption 1c that there exists $\{x_t\} \in \Pi(x_0)$, such that

$$\ln \|x_t\| \geq \ln \left(\zeta^t \|x_0\|\right) = t \ln \zeta + \ln \|x_0\| = t \ln \zeta, \quad t = 1, 2, \ldots.$$

Then Assumption 1d implies that

$$\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \geq \ln \zeta \sum_{t=0}^{\infty} \beta^t t + \frac{\ln (b(1+\zeta))}{1-\beta}.$$

Hence $v^*(x_0)$ is uniformly bounded above and below for $\|x_0\| = 1$, so v^* is bounded in the norm in (12).

The continuity of v^* follows from the continuity of Γ and ϕ.

For any $x_0 \in X$ and any $\lambda > 0$, if $\{x_t\} \in \Pi(x_0)$, then $\{\lambda x_t\} \in \Pi(\lambda x_0)$. Let $u(x)$ denote the total discounted returns along any feasible path. Then

$$u(\lambda x) = \sum_{t=0}^{\infty} \beta^t \ln (\phi(\lambda x_t, \lambda x_{t+1}))$$

$$= \sum_{t=0}^{\infty} \beta^t \ln (\phi(x_t, x_{t+1})) + \sum_{t=0}^{\infty} \beta^t \ln \lambda$$

$$= u(x) + \frac{\ln \lambda}{1-\beta}.$$

Hence v^* satisfies (11).

Finally, using (13) and (15), one finds that for any $f \in H_\beta(X,0)$, any $x_0 \in X$, and any $\{x_t\} \in \Pi(x_0)$,

$$\lim_{t\to\infty} \sup \beta^t f(x_t) \le \lim_{t\to\infty} \sup \beta^t \left[\|f\| + \frac{\ln \|x_t\|}{1-\beta}\right]$$
$$\le \lim_{t\to\infty} \beta^t \left[\|f\| + \frac{t \ln \alpha + \ln \|x_0\|}{1-\beta}\right] = 0,$$

so (7) holds. □

Again, the conclusions of Theorem 1 go through without change.

Theorem 3. Let (X, Γ, β, F) satisfy Assumption 3, define $H_\beta(X,0)$ as above, and define the operator T on $H_\beta(X,0)$ by (9). Then the conclusions of Theorem 1 hold.
Proof. Unchanged. □

5. Conclusion

Arguments analogous to those in section 4.3 of Stokey, Lucas, and Prescott (1989) can be used to establish (weak or strict) concavity of the value function. Several additional assumptions are needed: that X is convex, that $\Gamma(x)$ is convex-valued, for all $x \in X$, and that F or ϕ is (weakly or strictly) quasi-concave. The arguments above can then be applied to the closed subset of $H(X,\theta)$ consisting of functions that are weakly quasi-concave.

To simplify the exposition, the deterministic case was discussed here. The same arguments apply to stochastic models, however. In this case the bounds on the growth rate of $\|x_t\|$ and on the ratio of F or ϕ to the norm of its arguments must hold contingent on every realization of the exogenous stochastic shock.

Dolmas (1993) shows that return functions that are recursive and homogeneous are also consistent with balanced growth, even if they are not additively separable over time. The extension of the arguments above to return functions in this broader class is a topic for future research.

Homogeneous models have the attractive feature that for computational purposes, the dimensionality of the problem is essentially reduced by one. Since both the value and policy functions are homogeneous, it suffices to compute their values on the unit circle. Thus, for a problem with state space $X \subset \mathbf{R}^\ell$, it is enough to compute the value and policy function on a manifold of dimension $\ell - 1$. Hence the "curse of dimensionality" operates a little more slowly.

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