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On Convergence in Endogenous Growth Models

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ABSTRACT

In this paper we analyze the rate of convergence to a balanced path in a class of endogenous growth models with physical and human capital. We show that such rate depends locally on the technological parameters of the model, but does not depend on those parameters related to preferences. This result stands in sharp contrast with that of the one-sector neoclassical growth model, where both preferences and technologies determine the speed of convergence to a steady-state growth path.

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1. Introduction

In this paper we analyze the speed of convergence to a balanced path in an endogenous growth framework. The speed of convergence provides important information in testing a model on the relative emphasis that should be placed on the steady-state behavior and transitional dynamics. If the speed of convergence to a steady state or balanced path is high, then the long-run behavior of the model should be determined by its predictions at the steady state. However, if such rate of convergence is low, then transitional dynamics may play a relevant role in ascertaining the predictive power of a model even if long-run considerations are called into the analysis.

Our study is restricted to a class of endogenous growth models with physical and human capital. This class of models —initiated in the work of Uzawa (1965) and Lucas (1988)— has been the focus of some current research on growth theory, since they generate processes of permanent growth propelled by a human capital technology. Our basic result shows that the speed of convergence depends in a quasilinear fashion on the technological parameters of the model, but does not depend on those parameters characteristic of preferences. Roughly, this rate increases with the productivity of the human capital technology and decreases with the share of physical capital in the goods sector. These findings stand in sharp contrast with corresponding results for the neoclassical growth model, where both preferences and technological parameters determine the speed of convergence.

The fact that preferences are irrelevant in the determination of the speed of convergence is linked to the multi-sectoral structure of our economies. The intuition behind this result can be explained from the logic underlying our method of analysis. We first note that along the stable path all variables in the economy must converge at the same speed to the steady-state equilibrium; hence, in order to determine the speed of convergence (or to devise an appropriate test of convergence) we may just focus on a single, chosen variable. In particular, let us concentrate on the optimal quantity of labor devoted to goods production; in our model, such quantity is determined by the relative price and marginal productivities in both sectors. Moreover, the equilibrium law of motion of the relative price depends only on the wedge between those marginal productivities. From these facts, and after some simple algebraic manipulations, we show below that only technological parameters determine the rate of convergence.

An extensive empirical literature has analyzed the problem of convergence, broadly defined as a general tendency for poor countries or regions to exhibit higher growth rates. Barro (1991), Barro and Sala-i-Martin (1992) and Mankiw *et al.* (1992) study several cross-section data sets and document slow rates of convergence (annual rates of the order of 2%). These results conflict with simulations of the standard growth model, where for calibrated parameter values the rate of convergence is of the order of 7% (e.g., see below). The contrast between empirical results and model simulations is also manifested in temporal analyses of individual countries [cf., Christiano (1989)], in which the exogenous growth model seems to display much faster convergence than the reported evidence.

The cross-section analyses by Barro (1991), Barro and Sala-i-Martin (1992) and Mankiw *et al.* (1992) have nevertheless been challenged in several respects. Various authors [e.g., Bernard and Durlauf (1995), Durlauf and Johnson (1994), and Quah (1993a,b)] have argued that cross-section regressions cannot capture certain dynamic properties of the evolving income distribution, and that the negative coefficient associated with the speed of convergence in those analyses may be compatible with the absence of convergence or with the existence of multiple steady-state growth patterns. More recently, Canova and Marcet (1995) have contended that the relatively low estimates for the speed of convergence reported in those cross-section studies may be subject to a fixed-effect bias arising from pooling heterogeneous units with different data generating processes. Also, den Haan (1995) highlights a possible bias and inconsistency of these estimates in the presence of productivity shocks.

Although further research is needed to clarify some of these issues, one conclusion that appears to emerge from this line of research —and that would be one of the premises of the present study —is that reasonably calibrated deteministic growth models display higher rates of convergence than those supported by the empirical evidence. This becomes transparent in the dynamic analysis of selected units [as in Christiano (1989)] or small groups of countries or regions [cf., Barro and Sala-i-Martin (1992)]. Hence, it seems that important features of the convergence problem are not well captured by standard growth models. Yet, den Haan (1995) has observed that small rates of convergence are compatible with plausible calibrations of stochastic growth models, as long as those shocks are sufficiently persistent. Likewise, we shall illustrate below that the presence of a simple adjustment costs technology for the accumulation of physical capital can reasonably reduce the rate of convergence without altering substantially some other predictions of the model.

The paper is organized as follows. We begin in Section 2 with a review of some basic facts on convergence in the one-sector model. Then in Section 3 we present our main results on convergence for a family of two-sector growth models that include qualified leisure. Under the assumption of constant returns to scale in both sectors, we show that our results hold true for general concave utility and production functions; likewise, these results are robust to several types of flat-rate taxes. In Section 4, the scope of these theoretical findings is further qualified with the aid of some numerical computations. Although our theoretical analysis applies locally, these numerical exercises will illustrate that our results trace reasonably well the global, non-linear behavior of the economy over a considerable portion of the transition. Also, certain departures from our basic assumptions will only lead to marginal deviations of our benchmark results. We conclude in Section 5 with a summary of our main contributions.

2. Convergence in the Neoclassical Growth Model

In this section we present a simple version of the neoclassical growth model, and summarize some well known results on convergence. This setting will also prove useful to facilitate the exposition in the subsequent development.

We consider an economy where at each time $t \geq 0$ the production of the single homogeneous good is represented by the production process

$$y(t) = Ak(t)^{\beta}$$

where both variables $y(t)$ and $k(t)$ are measured in per capita units, and $A > 0$ and $0 < \beta < 1$ are technological parameters. Output, $y(t)$, is devoted either to consumption, $c(t)$, or to investment, $i(t)$. Physical capital, $k(t)$, depreciates at a fixed rate, $\pi \geq 0$. The instantaneous utility derived from consumption is represented by a CES function

$$U(c(t)) = \frac{c(t)^{1-\sigma} - 1}{1-\sigma}$$

with $\sigma > 0$. Future utilities are discounted at a given rate, $\rho > 0$, and population grows at an exogenous rate, $n \geq 0$, with $\rho > n$.

Under these assumptions, the planning problem can be written as

$$\max \quad \int_0^\infty e^{-(\rho-n)t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} dt \qquad (P)$$

subject to

$$\dot{k}(t) = Ak(t)^\beta - (\pi + n)k(t) - c(t) \qquad (2.1)$$

$$c(t) \geq 0, \quad k(t) \geq 0$$

$$k(0) = k_0 \text{ given,}$$

where $\dot{k}(t)$ is the time derivative. It is well known that problem (P) has a unique, differentiable solution $\{(c(t), k(t))\}_{t\geq 0}$, which must satisfy at every $t \geq 0$ the following system of first-order conditions

$$c(t)^{-\sigma} = \eta_1(t) \qquad (2.2)$$

$$\dot{\eta}_1(t) = \left[\rho + \pi - \beta Ak(t)^{\beta-1}\right] \eta_1(t) \qquad (2.3)$$

Here $\eta_1(t)$ denotes the co-state variable associated with $k(t)$. The optimal solution is characterized by (2.1)-(2.3) and the transversality condition,

$$\lim_{t\to\infty} e^{-(\rho-n)t} \eta_1(t) k(t) = 0 \qquad (2.4)$$

From (2.1)-(2.3) we then obtain that the following two-dimensional dynamical system determines the evolution of consumption and investment,

$$\dot{c}(t) = -\frac{c(t)}{\sigma} \left[\rho + \pi - \beta Ak(t)^{\beta-1}\right] \qquad (2.5)$$

$$\dot{k}(t) = Ak(t)^\beta - (\pi + n)k(t) - c(t) \qquad (2.6)$$

The system reaches a steady state if $\dot{c}(t) = \dot{k}(t) = 0$. It is easy to see from these equations that there is a unique positive solution that conforms a steady state (c^*, k^*).

In order to study the stability properties of the system, we linearize (2.5) and (2.6) at the steady-state values (c^*, k^*). The linearized dynamical system is thus given by

$$\begin{pmatrix} \dot{c}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{c^*}{\sigma}(1-\beta)\beta A k^{*^{\beta-2}} \\ -1 & \beta A k^{*^{\beta-1}} - (\pi + n) \end{pmatrix} \begin{pmatrix} c(t) - c^* \\ k(t) - k^* \end{pmatrix} \tag{2.7}$$

The characteristic equation corresponding to this linear system is then

$$\lambda^2 - (\rho - n)\lambda - \frac{(1-\beta)(\rho+\pi)}{\sigma}\left[\left(\frac{\rho+\pi}{\beta}\right) - (\pi + n)\right] = 0 \tag{2.8}$$

It follows that there is a negative root λ_1, which can be computed as

$$\lambda_1 = \frac{\rho - n - \left((\rho - n)^2 + \frac{4(1-\beta)(\rho+\pi)}{\sigma}\left[\left(\frac{\rho+\pi}{\beta}\right) - (\pi + n)\right]\right)^{1/2}}{2} \tag{2.9}$$

Consequently, equations (2.5)-(2.6) contain a one-dimensional stable manifold. This manifold satisfies (2.4)-(2.6), and hence it is made up of optimal solutions. Moreover, by the strict concavity of our optimization problem these solutions are the only optimal ones.

Let $\hat{\lambda} = -\lambda_1$. We call $\hat{\lambda}$ the *rate or speed of convergence* to the steady state (c^*, k^*). The relatively complex form in (2.9) suggests that further extensions of the model — such as leisure in the utility function or many types of goods— may render the rate of convergence $\hat{\lambda}$ hard to compute analytically. In the above simple case, however, we can see from (2.9) how different parameters related to preferences and technologies affect the value $\hat{\lambda}$. In the economic growth literature [e.g., Barro and Sala-i-Martin (1992)] it has been stressed the role of parameter β; such parameter affects negatively the value $\hat{\lambda}$. Also, preferences parameters σ and ρ have a non-negligible influence on $\hat{\lambda}$. Indeed, one easily sees from (2.9) that $\hat{\lambda}$ becomes unbounded as σ approaches zero.

3. Convergence in an Endogenous Growth Model

We now study the rate of convergence in a parameterized family of growth models with physical and human capital. In contrast to the previous section, we show that the rate of convergence is only determined by technological parameters. Moreover, this result is robust to several extensions of the basic framework, including general convex techonologies with constant returns to scale and various types of flat-rate taxes.

3.1. The basic model

With respect to the exogenous growth framework, the present model features an added educational sector and a choice of a time variable allocated to three margins: production of the aggregate good, schooling, and leisure.

At every time $t \geq 0$, production of the single, homogeneous good is represented by the production process

$$y(t) = Ak(t)^{\beta}(u(t)h(t))^{1-\beta}$$

where $u(t)$ connotes the relative amount of effort devoted to the production of the good, and $h(t)$ is the level of education or human capital. All variables are measured in per capita units. In the educational sector, the law of motion of $h(t)$ is given by a linear technology

$$\dot{h}(t) = \delta(1 - u(t) - l(t))h(t) - \theta h(t)$$

where $(1 - u(t) - l(t))$ is the fraction of time devoted to education, and $l(t)$ is the fraction of time spent in leisure activities. Parameter $\theta \geq 0$ is the rate of depreciation of $h(t)$, and parameter $\delta > 0$ is the constant marginal productivity.

The instantaneous utility derived from consumption and leisure is represented by a CES function

$$U(c, lh) = \frac{\left(c^{\gamma}(lh)^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma} \qquad \text{for } \sigma > 0,\ \sigma \neq 1,\ 0 < \gamma \leq 1$$

and

$$U(c, lh) = \gamma \ln c + (1-\gamma)\ln(lh) \qquad \text{for} \quad \sigma = 1, \; 0 < \gamma \leq 1$$

Observe that if $\gamma = 1$ this formulation reduces to the utility function postulated in Lucas (1988) with no leisure. A discussion of alternative modelizations of consumption and leisure will be provided below.

The planning problem to be considered is written as

$$V(k(0), h(0)) \;=\; \max \int_0^\infty e^{-(\rho-n)t} \frac{\left(c(t)^\gamma (l(t)h(t))^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma} dt \qquad (P')$$

subject to

$$\dot{k}(t) = Ak(t)^\beta (u(t)h(t))^{1-\beta} - (\pi + n)k(t) - c(t) \qquad (3.1)$$

$$\dot{h}(t) = \delta(1 - u(t) - l(t))h(t) - \theta h(t) \qquad (3.2)$$

$$0 \leq u(t) \leq 1, \; 0 \leq l(t) \leq 1, \; 0 \leq u(t) + l(t) \leq 1$$

$$c(t) \geq 0, \; k(t) \geq 0, \; h(t) \geq 0$$

$$k(0), h(0) \text{ given}, \; \rho - n > (\delta - \theta)(1-\sigma)$$

Under these conditions, problem (P') has a unique optimal solution $\{(c(t), k(t), h(t), l(t), u(t))\}_{t\geq 0}$, which in the interior case must satisfy the following system of first-order equations

$$\gamma c(t)^{(1-\sigma)\gamma - 1} (l(t)h(t))^{(1-\sigma)(1-\gamma)} = \eta_1(t) \qquad (3.3)$$

$$(1-\gamma)c(t)^{(1-\sigma)\gamma} (l(t)h(t))^{-\sigma-(1-\sigma)\gamma} = \eta_2(t)\delta \qquad (3.4)$$

$$\eta_1(t)(1-\beta)Ak(t)^\beta (u(t)h(t))^{-\beta} = \eta_2(t)\delta \qquad (3.5)$$

$$\dot{\eta}_1(t) = [\rho + \pi - \beta Ak(t)^{\beta-1}(u(t)h(t))^{1-\beta}]\eta_1(t) \qquad (3.6)$$

$$\dot{\eta}_2(t) = [\rho - n - \delta + \theta]\eta_2(t) \qquad (3.7)$$

where $\eta_1(t)$ and $\eta_2(t)$ denote the co-state variables associated with $k(t)$ and $h(t)$, respectively. The optimal solution must also fulfil the transversality condition,

$$\lim_{t\to\infty} e^{-(\rho-n)t} (\eta_1(t)k(t) + \eta_2(t)h(t)) = 0 \qquad (3.8)$$

A *balanced path* is an optimal solution $\{(c(t)^*, k(t)^*, h(t)^*, l(t)^*, u(t)^*)\}_{t\geq 0}$ to (P') such that $c(t)^*, k(t)^*$ and $h(t)^*$ grow at constant rates, and $l(t)^*$ and $u(t)^*$ remain constant. It follows from Caballé and Santos (1993) and Ortigueira (1994) that there is a unique ray of balanced paths, which is globally stable. Along this ray, consumption, $c(t)^*$, and both types of capital, $k(t)^*$ and $h(t)^*$, grow at a constant rate, say ν.

Let $z(t) = \frac{c(t)}{k(t)}$ and $x(t) = \frac{h(t)}{k(t)}$. Then the ray of balanced paths can be parameterized by the vector (z^*, x^*, l^*, u^*). Moreover, under this new set of variables convergence to the balanced path is determined by a unidimensional stable manifold corresponding to the unique negative eigenvalue of the system [cf., *op. cit.*].

We now present our basic result on the rate of convergence.

Theorem 3.1: *Assume that (z^*, x^*, l^*, u^*) is an interior stationary solution to problem (P'). Then every other optimal solution $(z(t), x(t), l(t), u(t))$ approaches (z^*, x^*, l^*, u^*) at the rate of convergence,*

$$\hat{\lambda} = \left(\frac{1-\beta}{\beta}\right)(\pi + n + \delta - \theta) \tag{3.9}$$

It can be shown that the existence of an interior steady state requires $\hat{\lambda} > 0$. The theorem is proved in the appendix. As pointed out in the introductory section, the intuition behind this result can be explained from the strategy underlying our method of proof. After rewriting system (3.3)-(3.7) in the variables $z(t) = \frac{c(t)}{k(t)}$, $x(t) = \frac{h(t)}{k(t)}$ and $\eta(t) = \frac{\eta_1(t)}{\eta_2(t)}$, the linearization of these equations contains a unique negative eigenvalue that defines a one-dimensional manifold for the non-linear system [cf., Caballé and Santos (1993)]. Assuming that this manifold is in general position, all variables must then converge at the same speed to the ray of balanced paths, say $\hat{\lambda}$. Hence, we only need to detemine the speed of convergence for a given chosen variable.[1] In particular, we may focus on the variable working time, $u(t)$. From equation (3.5), we can see that $u(t)$ can be defined in terms of the ratio of the two capitals, $x = \frac{h}{k}$, and the relative price, $\eta = \frac{\eta_1}{\eta_2}$. Moreover, it follows from (3.6)-(3.7) that the law of motion of η is not dependent upon preferences

[1]This procedure suggests that in order to design the most effective tests of convergence one should focus on that variable that displays more variability in the data, or that is easiest to measure and manipulate.

parameters. Now, exploiting again the fact that variables x and η must converge at the same rate, $\hat{\lambda}$, and after some simple algebraic computations, we show in the appendix that in fact, $\hat{\lambda}$, can be fully determined from (3.5)-(3.7). Consequently, preferences parameters do not affect the rate of convergence.

We should note from (3.9) that the productivity of the human capital technology, $(\delta - \theta)$, increases the rate of convergence, whereas the weight of physical capital in the production sector, β, has the opposite effect. Moreover, physical capital depreciation, π, and population growth, n, exert a positive influence on the rate of convergence to the extent that they increase the rate of replacement of average physical capital. This result is to be contrasted with a related extension of the one-sector growth model in which investments in physical and human capital are perfect substitutes [cf., Barro *et al.* (1995)]. In this latter case, the ratio $x = \frac{h}{k}$ remains constant along the transitional dynamics to the steady-state path, and as a result both types of capital exert a symmetric effect on the rate of convergence. Our analysis illustrates that this symmetry is lost in the presence of a specific sector for human capital accumulation. Since we are concerned with the ratio $x = \frac{h}{k}$, it is to be expected that the productivities of these factors should have opposite effects on the speed of convergence of such variable. Thus, if there is a sudden increase in our economy in the quantity of physical capital from a given steady-state solution, then a more productive human capital technology will counteract this imbalance more quickly. As expressed in (3.9), the effectiveness of this channel is, however, dependent upon the relative weight of human capital in the production of physical capital investment; that is, on the ratio $\frac{1-\beta}{\beta}$. Likewise, a higher physical capital depreciation rate, π, or a higher population growth, n, will help restore a physical capital imbalance, and such effect is more pronounced the smaller the weight of physical capital in the goods sector.

Another interpretational issue is whether the computed rate of convergence in (3.9) is comparable to that derived in the preceding section [see (2.9)]. The answer is in the affirmative. The above exogenous growth model could be reinterpreted as having a production function of the form, $Ak^{\beta}h^{1-\beta}$, and where variable h is growing exogenously. Then, considering variables $z(t) = \frac{c(t)}{k(t)}$ and $x(t) = \frac{h(t)}{k(t)}$ in optimization problem (P), the rate of convergence is again given by (2.9). For such redefined variables, this is precisely the computed rate of convergence in (3.9).[2] Moreover, as shown in Caballé and Santos (1993) and Mulligan and Sala-i-Martin (1993), if we temporarily abstract from leisure considerations, then for $\sigma = \beta$ the endogenous growth framework reduces to an exogenous growth model with a deterministic law of motion for the human capital variable. For this particular case, (2.9) and (3.9) must yield the same value.

It follows then from this analysis that if $\sigma = \beta$ both models display the same speed of convergence. Now, standard calibrations assign a much higher value for σ (e.g., $\sigma = 1.5$ and $\beta = 0.4$). Also, parameter σ affects negatively the rate of convergence in the exogenous growth model but not in our endogenous growth framework [see equations (2.9) and (3.9)]. Therefore, for standard calibrated values convergence should be faster in the endogenous growth model. Indeed, we show below that for our baseline parameters values, $\hat{\lambda} = 0.0694$, for the exogenous growth model, whereas $\hat{\lambda} = 0.165$ for the endogenous growth model. This is over a two-fold increase in the rate of convergence.

3.2. Extensions of the basic framework

Our previous results have been derived for a parameterized family of utility and production functional forms. This basic framework has been chosen for convenience of the analysis. We should note, however, that our arguments extend to more general settings. In particular, under the assumption of constant returns to scale in both production sectors, our results hold for general concave utility and production functions, and several types of flat-rate taxes. Moreover, for some other extensions of the model preferences parameters have a marginal effect on the speed of convergence.

Concavity of utility and production functions

It should be observed that the existence of a balanced path imposes certain restrictions on the utility and production functional forms [cf., King, Plosser and Rebelo (1988)]. In particular, the utility function must display a constant intertemporal elasticity of substitution with respect to consumption, and substitution and income effects associated with

[2]Note that if the economy features a positive rate of growth, then (2.9) must be suitably modified.

increases in productivity along the balanced growth path must leave unaltered the optimal quantity of leisure.

The above utility function, $U(c, lh) = \frac{\left(c^{\gamma}(lh)^{1-\gamma}\right)^{1-\sigma}-1}{1-\sigma}$, is compatible with the existence of a balanced path. Under this type of utility, human capital affects the productivity of leisure, and such formulation seems consistent with certain theories of allocation of time and household production [e.g., Becker (1965)]. There is, however, a diammetrically opposite modelization of leisure, $U(c, l) = \frac{\left(c^{\gamma}l^{1-\gamma}\right)^{1-\sigma}-1}{1-\sigma}$, which is also compatible with the existence of a balanced path [cf., Lucas (1990)]. Although further microeconometric work is needed to elucidate the empirical plausibility of these formulations, it seems that in the context of macroeconomic theory the utility form $U(c, l)$ is more appropriate. The problem, however, is that under this latter functional form optimization problem (P') is not jointly concave in the control and state variables [see Ladrón de Guevara *et al.* (1994)], since the stock of human capital affects asymmetrically the three given activities. [3] This asymmetry brings about an additional effect of preferences on the rate of convergence. Indeed, under the utility function $U(c, l) = \frac{\left(c^{\gamma}l^{1-\gamma}\right)^{1-\sigma}-1}{1-\sigma}$, equation (3.7) becomes

$$\dot{\eta}_2(t) = [\rho - n - \delta(1 - l(t)) + \theta]\eta_2 \tag{3.7'}$$

In this case only the fraction $(1 - l(t))h(t)$ is productive, and as specified in the new law of motion (3.7′), such fraction must account for all human capital rents. Imbalances in the ratio of the two capitals may change the time devoted to leisure, l, and consequently preferences are going to affect the law of motion of the relative price. Thus, the rate of convergence must be influenced by preferences parameters. We shall nevertheless illustrate below with the aid of some numerical exercises that this possible effect of preferences is fairly small.

[3] To illustrate this technical point on concavity, consider for instance the utility $U(c, l) = \gamma \ln c + (1 - \gamma)\ln l$. Then $U(c, l) = U(c, lh) - \ln h$, and so the term $-\ln h$ acts as a negative externality. In other words, under the utility form $U(c, lh)$ the above dynamic optimization problem is concave, and the stock h can be distributed symmetrically among the three alternative activities. Under the utility form $U(c, l)$, however, the global concavity of the optimization problem may break down. Of course, for a utility function of the form $U(c)$, problem (P') is again globally concave and equation (3.7) remains unaffected. Hence, in this latter case preferences do not affect the convergence rate.

Regarding goods production, our results are not dependent upon the specific functional forms adopted above. Thus, it is shown in the appendix [see (6.8) below] that an alternative expression for the rate of convergence is

$$\hat{\lambda} = F_L(1, u^*x^*)u^*x^* \tag{3.10}$$

where $F_L(1, u^*x^*)$ denotes the derivative of the production fuction $F(k, uh)$ with respect to the second argument, $L = uh$, at the steady-state value $(1, u^*x^*)$, for $x = \frac{h}{k}$. This expression remains valid in the more general case in which $F(k, L)$ is merely a linearly homogeneous function in k and L.

It is worth emphasizing that these results are also heavily dependent on the homogeneity of degree one of the human capital technology in the amount of qualified labor employed in that sector, $(1 - u - l)h$. If, for instance, the human capital technology is characterized by a production function of the form $G(1 - u - l)h$, where $G(\cdot)$ is an increasing concave function with second order derivative $G''(\cdot) < 0$, then the above results are no longer valid, since in such case production function $G(1 - u - l)h$ is not linearly homogeneous in $(1 - u - l)h$. Also, if technological externalities are present [as in Lucas (1988) and Benhabib and Perli (1994)] then preferences parameters may affect the speed of convergence. Indeed, for some preferences parameters the equilibrium steady-state may contain two negative eigenvalues, and consequently equilibria are indeterminate.

Finally, we could consider a more general class of models in which physical capital also enters into the production of human capital, as in Mulligan and Sala-i-Martin (1993). Again, in a previous version of this paper we have shown that, under the assumptions of concavity and constant returns to scale in both production sectors, preferences parameters do not affect the speed of convergence. This rate depends upon the concavity of the instantaneous production possibility frontier derived from the technologies of both sectors. Mulligan and Sala-i-Martin (1993) interpret the curvature of this frontier as point-in-time adjustment costs.

Installment or adjustment costs

Theories of investment often introduce installment or adjustment costs as a way to insure a determinate demand flow for capital goods [cf., Lucas (1967), Hayashi (1982), Mendoza (1993)]. The role of adjustment costs, however, has not been emphasized in the context of growth theory [some notable exceptions include Abel and Blanchard (1983), Barro and Sala-i-Martin (1995) and King and Rebelo (1993)]. Of particular interest to us is how these costs affect the speed of convergence.

The introduction of adjustment costs seems especially attractive regarding human capital accumulation; unfortunately, relatively little is known concerning the modelization and quantification of these costs. For illustrative purposes, we shall consider here a particular form of adjustment costs embedded in the physical capital sector. In order to account for such costs, the technological law of motion (3.1) is now replaced by the following two equations

$$c(t) + i(t)\left[1 + h\left(\frac{i(t)}{k(t)}\right)\right] = Ak(t)^{\beta}(u(t)h(t))^{1-\beta} \tag{3.11}$$

$$\dot{k}(t) = i(t) - (\pi + n)k(t), \tag{3.12}$$

where $h(\cdot)$ is an increasing, non-negative function, and $i[1+h(i/k)]$ represents gross expenditures on investment. This formulation gives rise to a linearly homogeneous technology, and as shown by Hayashi (1982) in such case Tobin's marginal q is equal to the average q.

One may assume that the function $ih(i/k)$ is jointly convex in both arguments [c.f., Lucas and Prescott (1971) and Uzawa (1969)]. However, in order to draw some comparisons our numerical computations will be restricted to the functional form $h(i/k) = \frac{\frac{b}{2}\left(\frac{i}{k}-a\right)^2}{\frac{i}{k}}$, where, $b > 0$, $a \geq 0$. This technology is adopted from King and Rebelo (1993) and Summers (1981), and it is compatible with the existence of a balanced path. Observe that under this formulation the global concavity of the optimization problem is no longer guaranteed. Nevertheless, it can be shown that an interior optimal path can still be characterized from the first-order conditions.

Following our previous analysis, it is then readily deduced that the optimal quantity of labor, u, can be determined by the value of the marginal productivities in both sectors [cf., equation (3.5)]. However, the marginal productivity of labor in the physical sector should now be evaluated by the price of the consumption good, and such value is different from the shadow price of installing new capital units. (Tobin's q is simply the ratio of the shadow price of installing new capital over the price of consumption.)

As one can infer from our benchmark result (3.9), variations in the technological possibilities of the economy should influence the speed of convergence. Consequently, adjustment costs would have a certain impact on such rate (albeit in this case there is not a clean analytical solution). Moreover, the dynamics are now driven by the wedge between the price of installing new capital and the price of the consumption good. This wedge is a function of the amount of consumption and investment, and so it is dependent upon preferences parameters. Hence, one should expect that preferences will also affect the speed of convergence. As our computations in the next section show, this effect from preferences is nevertheless small.

Fiscal policy and convergence

One major conclusion from the endogenous growth literature is that distortionary taxation may have important effects on welfare [e.g., King and Rebelo (1990), Jones, Manuelli and Rossi (1993), and Lucas (1990)]. However, as we presently show the presence of flat-rate taxes in our model will not affect the rate of convergence. In order to introduce taxation in our framework, we now reformulate our analysis in the context of a competitive economy.

We consider that consumption, investment, and income from capital and labor are subject to *ad valorem* taxes, $\tau_c, \tau_i, \tau_k, \tau_l$, respectively. Total revenues from taxation are rebated in lump-sum form to the representative consumer. For given initial values k_0 and h_0, the consumer is confronted with maximizing the objective in (P') subject to the instantaneous budget balance

$$(1+\tau_c)c(t) + (1+\tau_i)i(t) = (1-\tau_k)r(t)k(t) + (1-\tau_l)\omega(t)u(t)h(t) + T(t)$$

and the laws of accumulation for the capital stocks

$$\begin{aligned}\dot{k}(t) &= i(t) - (\pi + n)k(t) \\ \dot{h}(t) &= \delta(1 - u(t) - l(t))h(t) - \theta h(t)\end{aligned}$$

for all $t \geq 0$. Here, $T(t)$ is the lump-sum transfer to the agent, and $r(t)$ and $\omega(t)$ denote the rental prices for capital and labor, respectively.

The representative consumer rents capital and labor to the firm and these factors are paid according to the values of their marginal productivities. Also, in a competitive equilibrium, the goods and labor markets must clear. Therefore,

$$\begin{aligned} \omega(t) &= (1-\beta)k(t)^{\beta}(u(t)h(t))^{-\beta} \\ r(t) &= \beta k(t)^{\beta-1}(u(t)h(t))^{1-\beta} \end{aligned}$$

and

$$y(t) = c(t) + i(t)$$

for all $t \geq 0$.

For this reformulated problem, our previous optimization analysis applies, considering now gross prices for consumption and investment, and net returns for physical capital and labor. It follows then from the first-order conditions that an interior steady state must satisfy the following equations system

$$\frac{1-\gamma}{\gamma} = \frac{l}{zu}\left(\frac{1-\tau_l}{1+\tau_c}\right)(1-\beta)Ax^{1-\beta}u^{1-\beta} \tag{3.13}$$

$$z = x^{1-\beta}u^{1-\beta} - \nu - n - \pi \tag{3.14}$$

$$\rho + \sigma\nu + \pi = \left(\frac{1-\tau_k}{1+\tau_i}\right)\beta Ax^{1-\beta}u^{1-\beta} \tag{3.15}$$

$$\rho - n + \sigma\nu + \theta = \delta \tag{3.16}$$

$$\nu = \delta(1-u-l) - \theta \tag{3.17}$$

The tax structure may affect the steady-state values z^*, x^*, l^*, and u^*, but in our simple model one can see from (3.16) that it will not affect the growth rate ν. This invariance of the growth rate implies that the steady-state, net rate of return stays unchanged [cf., equation (3.15)]. As the net rate of return determines the equilibrium law of motion of the relative price, the speed of convergence should remain unaffected. Indeed, following the same arguments as in Theorem 3.1 the corresponding formulation of (3.10) is now

$$\hat{\lambda} = (1-\beta)\left(\frac{1-\tau_k}{1+\tau_i}\right)A{x^*}^{1-\beta}{u^*}^{1-\beta}$$

Making then use of equations (3.15) and (3.16) it is readily shown that

$$\hat{\lambda} = \left(\frac{1-\beta}{\beta}\right)(\pi + n + \delta - \theta)$$

This is, of course, expression (3.9) obtained in the preceding section.

This result again differentiates our endogenous growth framework from the neoclassical model where under the same functional forms flat-rate taxes on capital rents have a significant impact on the rate of convergence. Therefore, our analysis highlights the testable restriction that the rate of convergence is independent of the fiscal regime.

4. Numerical Experiments

Our purpose in this section is to present a set of numerical computations in order to shed light on some of the issues raised above. Specifically, in the context of some benchmark cases we show that the endogenous growth model displays a much faster speed of convergence, and that in all cases studied our local, analytical results apply for sizable portions of the transitional dynamics. Also, we illustrate that, in the presence of unqualified leisure and adjustment costs, preferences parameters have a marginal effect on such rate.

The neoclassical growth model

We now calculate the rate of convergence $\hat{\lambda}$ in two examples for the model of Section 2. These cases will serve as reference in our later study. We first consider our benchmark economy with parameters values

$$\sigma = 1.5, \;\; \rho = 0.05, \;\; n = 0.01, \;\; A = 1, \;\; \beta = 0.4, \;\; \pi = 0.05,$$

These calibrations agree with those generally invoked in the economic literature [cf., Lucas (1988) and Prescott (1986)]. In this case, we obtain the following steady-state values, $c^* = 1.9151$ and $k^* = 10.0794$. The rate of convergence is $\hat{\lambda} = 0.0691$.

The second example involves a simple variation of the preceding values, in which $\sigma = 0.4$ and the remaining parameters stay unchanged. In this situation, $\hat{\lambda} = 0.15$. One then sees that in the benchmark economy the speed of convergence $\hat{\lambda}$ is relatively high, and increases substantially with decrements in σ.

In order to investigate how accurately this local analytical result approximates the global converging behavior, we shall calculate numerically the stable manifold of the system in both examples. For such purpose, we just follow a simple numerical technique[4] where the stable manifold is extended backwards from an arbitrarily small neighborhood of the steady state, k^*. (In such small neighborhood, the stable manifold is approximated by the linearized stable system.) For each case, Figure 1 displays the laws of motion of the linear and non-linear systems. It can be observed that the linearized system mimics well the non-linear dynamic behavior over a significative range of the capital domain. Therefore, in both examples the local speed of convergence, $\hat{\lambda}$, is a good estimate of the global converging behavior.

The endogenous growth model with qualified leisure

In order to compare the speed of convergence with that of the exogenous growth model we shall also consider two alternative examples for the basic model of Section 3. Our benchmark economy has the following parameterization

$$\gamma = 0.45, \sigma = 1.5, \ \rho = 0.05, \ n = 0.01, \ A = 3, \ \beta = 0.4, \ \pi = 0.05, \ \delta = 0.07, \ \theta = 0$$

These parameters give rise to the following steady-state values: $z^* = 0.2450$, $x^* = 0.0874$, $l^* = 0.4326$, $u^* = 0.2817$, $\nu = 0.02$. The speed of convergence is $\hat{\lambda} = 0.1950$.

The presence of both leisure and a positive growth rate precludes a close comparison with our numerical computations of the exogenous growth framework. Thus, in our next example, we abstract from leisure considerations in the utility function, and let $\gamma = 1$. [5] As already pointed out, if in this case $\sigma = \beta$ then the time devoted to education remains locally constant over the optimal solution, and so the transitional dynamics are

[4] Our computations are effected by a standard Euler method [see, e.g., Gerald and Wheatley (1990, Ch. 5)].

[5] Observe that this parameterization results in the utility function set forth in Lucas (1988), which is also included under our general formulation.

qualitatively the same as those of the neoclassical growth model. We then fix $\gamma = 1$, $\sigma = \beta = 0.4$, $\delta = 0.05$, and leave unchanged the remaining parameter values of our benchmark economy. (Parameter δ has been adjusted in order to obtain an interior steady state with a more realistic time allocation for work and education.) This calibration yields a steady-state growth rate $\nu = 0.025$, and a speed of convergence $\hat{\lambda} = 0.165$. Notice that the small difference in the rate of convergence with respect to the similar calibration in the exogenous growth model (where $\hat{\lambda} = 0.15$) is due to the fact that for the postulated parameters values the rate of growth is now $\nu = 0.0250$, whereas in the preceding model, $\nu = 0$.

From these numerical exercises we can see that the speed of convergence differs substantially over the two benchmark cases (in the preceding model the benchmark value, $\hat{\lambda} = 0.0694$, whereas in this section, $\hat{\lambda} = 0.1950$). Furthermore, as it is evident from the foregoing calculations, in order to equate both rates of convergence parameter σ must take values near 0.4 in the exogenous growth model.

Following our earlier numerical procedures, Figure 2 portrays the global dynamics of the linear and non-linear systems for our benchmark economies with endogenous growth. We again see that in both cases the linear system mimics reasonably well the non-linear converging behavior of the optimal solution over a sizable domain of state variable $\frac{k}{h}$. Similar results were obtained for various alternative parameterizations of the model. Hence, it appears that the same dynamic forces prevail over substantial phases of the transition with the effect that expressions (3.9)-(3.10) are good global estimates of the speed at which an economy will reach the steady-state behavior.

The endogenous growth model with pure leisure

We consider now the aforementioned polar modelization of leisure, where human capital does not affect its marginal utility. The utility function is here written as follows

$$\begin{aligned} U(c,l) &= \frac{(c^{\gamma}l^{1-\gamma})^{1-\sigma}-1}{1-\sigma} \quad \text{for } \sigma > 0,\ \sigma \neq 1 \text{ and } 0 < \gamma \leq 1 \\ U(c,l) &= \gamma \ln c + (1-\gamma)\ln l \quad \text{for } \sigma = 1 \text{ and } 0 < \gamma \leq 1 \end{aligned}$$

This model is studied in Lucas (1990) and Ladrón-de-Guevara *et al.* (1994). The rationale behind this formulation is that the utility obtained from certain leisure activities —such as resting or spending time with the family— does not depend upon the attained level of education. Ladrón-de-Guevara *et al.* (1994) show that for certain parameterizations this model may contain multiple steady-state equilibria. For our present illustrative purposes, we shall focus on a calibrated economy with a unique balanced path and verify that changes in preferences parameters γ and σ will lead to marginal variations in the speed of convergence. At this stage, it is worth pointing out that for $\gamma = 1$, the utility function is encompassed in the analysis of Section 3, and accordingly in such a case preferences parameters have no effect in the convergence rate.

In this new setting, we now assign the following parameters values

$$\gamma = 0.4,\ \ \sigma = 1.5,\ \ \rho = 0.05,\ \ n = 0.01,\ \ A = 3,\ \ \beta = 0.4,\ \ \pi = 0.05,\ \ \delta = 0.2,\ \ \theta = 0.05$$

This model gives rise to the following steady-state values: $z^* = 0.2359, x^* = 0.1070, l^* = 0.4124$, $u^* = 0.2229$, $\nu = 0.0229$. The speed of convergence about such stationary solution is $\hat{\lambda} = 0.1908$. Starting from this reference model, we now vary separately parameters values γ and σ. Table 1 displays the computed rate of convergence in the present model for different calibrations of γ against the estimated values of the rate of convergence from analytic expression (3.10) of Section 3. Table 2 reproduces the same results for several values of σ, letting $\gamma = 0.4$ fixed. It is seen from this exercise that these changes result in marginal variations from the estimated rate of convergence.

The endogenous growth model with adjustment costs

We finally consider the effects of adjustment costs for physical capital accumulation on the speed of convergence. Our numerical computations show that the presence of adjustment costs may reduce substantially the speed of convergence. In this setting preferences parameters also have an influence on such rate, but this effect is nonetheless fairly small.

Following the analysis of Section 3, the technological law of motion (3.1) is now replaced by the following equations

$$c(t) + i(t)\left[1 + h\left(\frac{i(t)}{k(t)}\right)\right] = Ak(t)^{\beta}(u(t)h(t))^{1-\beta} \tag{4.1}$$

$$\dot{k}(t) = i(t) - (\pi + n)k(t) \tag{4.2}$$

where $h(\cdot) = \frac{\frac{b}{2}\left(\frac{i}{k}-a\right)^2}{\frac{i}{k}}$, for $b > 0$, and $a \geq 0$. This technological law of motion is taken from King and Rebelo (1993) and Summers (1981). The remaining elements of the model are as defined in optimization problem (P').

Our baseline economy considers the following parameters values

$$\gamma = 0.45,\ \sigma = 1.5,\ \rho = 0.05,\ n = 0.01,\ A = 3,\ \beta = 0.4,\ \pi = 0.05,\ \delta = 0.07,\ \theta = 0,$$

$$b = 16, a = 0.06$$

Parameter a is set equal to $\pi + n$, so that only net investment undergoes adjustment costs. As we will see, it is crucial for our results whether gross or net investments are subject to such costs. The value of the adjustment cost coefficient, $b = 16$, is chosen to get a plausible value for Tobin's q.

The steady-state values of our baseline economy are reported simultaneously in the first row of Tables 3 and 4. From this benchmark calibration, Table 3 replicates the same numerical experiment for alternative values for b, whereas Table 4 focusses on variations of parameter a.

The first observation to be made from these computations is that in our baseline economy the speed of convergence $\hat{\lambda} = 0.0338$ is obtained at the cost of a value for $q = 1.32$. In the present context, these values seem quite reasonable. (Empirical estimates of q [e.g., Hayashi (1982) and Summers (1981)] are not directly comparable, since such estimates include taxes and allowances for depreciation and investment.) Also, we can see that the presence of adjustment costs does not change the steady-state growth rate, ν, and that it leads to marginal changes in l^* and u^*. As compared to our benchmark endogenous growth model without adjustment costs (last row in Table 3), the main changes are in the steady-state values z^* and x^*, but nevertheless such variations are not substantial.

As already pointed out, it matters for these results whether gross or net investments are affected by adjustment costs. This is illustrated in Table 4, where we consider alternative calibrations for parameter a. It is seen that parameter a has a minor effect on the speed of convergence, and a larger impact on Tobin's q. Hence, if gross investment is subjected to adjustment costs, then low speeds of convergence require high values for q.[6] The empirical estimates reported in Summers (1981) seem to suggest, however, that only net investment is affected by adjustment costs. Under this latter assumption, we have experimented with some other alternative calibrations for endogenous and exogenous growth models and obtained relatively low rates of convergence for plausible q-values. Hence, adjustment costs on net investment seem to be an effective device to bridge the gap between models predictions and the reported evidence.

We lastly consider the effect of preferences parameters on the speed of convergence. This is illustrated in Tables 5 and 6, where we analyze individual variations of parameters values γ and σ. In all these computations, the remaining parameters stay unchanged, as specified in our baseline model with adjustment costs (first row in Tables 3 and 4). In order to interpret these results, note that our previous benchmark values are $\gamma = 0.45$ and $\sigma = 1.5$.

Table 5 illustrates that increases in σ result in small decrements in the rate of convergence. In our view, this effect is small due to the existence of some countervailing forces. A higher value for σ lowers the elasticity of intertemporal substitution, and hence should exert a negative influence on the speed of convergence. At the same time, a higher value for σ lowers the growth rate, and consequently the steady-state value for q goes down. A lower value for q drives up the speed of convergence.

In contrast, Table 6 shows that variations in parameter γ have a negligible effect on the steady-state growth rate. The reported results, however, conflict with conventional intuition: a lower γ (i.e., a larger weight of leisure in the utility function) leaves more

[6] Barro and Sala-i-Martin (1995, Ch. 3) consider an exogenous growth model in which adjustment costs depend on gross investment, and not on net investment as studied here. They also conclude that low rates of convergence require high values for q.

scope for substitution and leads to an increased speed of convergence. But in the presence of adjustment costs in the goods sector, it seems that the economy may be willing to bear these costs more quickly the greater the weight of consumption in the utility function (i.e., the greater is γ). Again, these countervailing forces result in a marginal effect of parameter γ on the speed of convergence.

5. Concluding Remarks

In this paper we have considered a family of endogenous growth models with physical and human capital. This framework seems potentially attractive to growth theorists, since it allows for the possibility of convergence in growth rates while preserving differences in income levels. Moreover, in response to imbalances in the ratio of physical to human capital this class of models features a rich dynamic behavior in the process of convergence to the steady-state path.

Our study has focussed on the speed of convergence to the steady-state path. A few testable propositions have emerged from this analysis. Our principal finding in Section 3 is a neat characterization of the local rate of convergence. The result asserts that the productivity of the human capital technology has a positive influence on the speed of convergence, while the income share of physical capital exhibits the opposite effect. Preferences parameters and distortionary taxes exert no influence on the rate at which an economy will approach the steady-state path. Although this analytical result holds in a neighborhood of the balanced path, our numerical computations attest that this local behavior extrapolates over sizable phases of the transition.

These findings should be contrasted with an analogous analysis of the exogenous growth model, where preference data as well as flat-rate taxes on capital rentals play a significant role in assessing the speed of convergence. Furthermore, for standard calibrated parameter values the endogenous growth framework displays a higher rate of convergence. In order for the exogenous growth model to exhibit the same rate of convergence, the inverse of the elasticity of intertemporal substitution, σ, must roughly match the value of the elasticity of labor with respect to physical capital, β. Standard calibrations, however, attach a much higher value to σ. A higher value for σ lowers the speed of convergence in the exogenous growth model, but not in our endogenous growth framework where preferences parameters are ineffective. For our baseline economies, convergence is at least twice as fast in the endogenous growth model.

Some empirical studies [e.g., Barro and Sala-i-Martin (1992)] report annual rates of convergence of the order of 2%, whereas for standard calibrations the models considered here yield rates between 6% and 20%. One typical way to reconcile model predictions with the reported evidence is to assume values of β around 0.8, interpreting such share of the capital variable as a broad measure of physical and human capital. Barro *et al.* (1995) provide theoretical foundations for such an aggregate share in a generalized version of the exogenous growth model with physical and human capital. This conclusion, however, does not extend to our endogenous growth framework, where there is a specific sector for the production of human capital. Indeed, in our case physical and human capital variables exert opposite effects on the speed of convergence. The productivity of the human capital technology as well as the income share of human capital in the goods sector accelerate the process of convergence to the balanced path, whereas the share of physical capital in the goods sector has the reverse effect. Therefore, the inclusion of a human capital variable in standard growth settings does not necessarily resolve the puzzling difference between models predictions and reported evidence concerning the speed of convergence.

There are other possible extensions of the model worthwhile exploring that may yield lower rates of convergence. Our analysis suggests, however, that these studies should focus on specifications of the technological possibilities of the economy rather than on changes in the preference structure, since in all cases considered preferences have at most a marginal influence on the rate of convergence. Of course, the most attractive extensions would be those that can lower the rate of convergence and leave practically unmodified some other relevant predictions. As illustrated above, some adjustment costs technologies share this property.

6. Appendix

Throughout this appendix we shall assume that the one-dimensional stable manifold lies in a general position, so that it can be parameterized by each of the coordinates (z, x, l, u, η), where $z = \frac{c}{k}, x = \frac{h}{k}$ and $\eta = \frac{\eta_1}{\eta_2}$. One can show from the analysis of Caballé and Santos (1993) and Ortigueira (1994) that this property must hold generically. Moreover, if a property on the value of a characteristic root holds true for a generic subset of matrices, by continuity such property must hold true for the whole set of matrices.

In the proof of the theorem, we make use of the following lemma.

Lemma 1: *Let $(z^*, x^*, l^*, u^*, \eta^*)$ be a vector of steady-states values, and let $\frac{dx}{du}(u^*)$ be the derivative of x with respect to u at the steady-state value u^*. Then the elasticity*

$$-\frac{u^*}{x^*}\frac{dx}{du}(u^*) \neq 1.$$

Proof: Taking *logs* in (3.5) and differentiating with respect to t, we have

$$\frac{\dot{\eta}(t)}{\eta(t)} - \beta\frac{\dot{x}(t)}{x(t)} - \beta\frac{\dot{u}(t)}{u(t)} = 0$$

Now, write $\dot{\eta}(t)$, $\dot{x}(t)$ and $\dot{u}(t)$ as functions of their own variable. That is, $\dot{\eta}(t) = g(\eta(t))$, $\dot{x}(t) = f(x(t))$ and $\dot{u}(t) = h(u(t))$. This is possible if the stable manifold lies in a general position. Then

$$\frac{g(\eta(t))}{\eta(t)} - \beta\frac{f(x(t))}{x(t)} - \beta\frac{h(u(t))}{u(t)} = 0 \tag{6.1}$$

Considering that derivatives $g'(\eta^*) = f'(x^*) = h'(u^*) = \lambda_1 = -\hat{\lambda}$, we differentiate expression (6.1) with respect to $x(t)$, and evaluate the derivatives at their steady-state values. We then obtain

$$\frac{d\eta}{dx}(x^*) = \left[\frac{\beta}{x^*} + \frac{\beta\frac{du}{dx}(x^*)}{u^*}\right]\eta^*$$

Assume now that

$$\frac{u^*}{x^*}\frac{dx}{du}(u^*) = -1$$

This implies

$$\frac{d\eta}{dx}(x^*) = \left[\frac{\beta}{x^*} - \frac{\beta}{x^*}\right]\eta^* = 0 \tag{6.2}$$

However, (6.2) is in contradiction with the fact that the value function $V(k,h)$ for problem (P') is a smooth, strictly concave function, homogeneous of degree $1-\sigma$. This contradiction is easily derived from the expression

$$\eta(t) = \frac{V_k(1,x(t))}{V_h(1,x(t))} \tag{6.3}$$

where $V_j(\cdot,\cdot)$ is the partial derivative of $V(\cdot,\cdot)$ with respect to j, for $j = k, h$. The lemma is thus established.[7] Q.E.D.

Proof of Theorem 3.1: By the definition of $\hat{\lambda}$, for every arbitrarily small $\epsilon > 0$, there is T such that

$$\left|\frac{\dot{u}(t)}{u(t) - u^*} - \lambda_1\right| < \epsilon \tag{6.4}$$

for all $t \geq T$.

Taking *logs* in (3.5), and differentiating with respect to t, we get

$$\frac{\dot{\eta}(t)}{\eta(t)} - \beta\frac{\dot{x}(t)}{x(t)} - \beta\frac{\dot{u}(t)}{u(t)} = 0 \tag{6.5}$$

Using the fact that $\frac{\dot{\eta}(t)}{\eta(t)} = \frac{\dot{\eta}_1(t)}{\eta_1(t)} - \frac{\dot{\eta}_2(t)}{\eta_2(t)}$, and substituting out (3.6)-(3.7) in (6.5) it follows that

$$\dot{u}(t) = \left[\delta + n + \pi - \theta - \beta A x(t)^{1-\beta}u(t)^{1-\beta} - \beta\frac{\dot{x}(t)}{x(t)}\right]\frac{u(t)}{\beta}$$

Now, linearizing this latter equation with respect to u, and considering steady-state values, we get

$$\dot{u}(t) = \left[-(1-\beta)Ax^{*^{1-\beta}}u^{*^{1-\beta}} - (1-\beta)Ax^{*^{-\beta}}u^{*^{2-\beta}}\frac{dx}{du}(u^*) - u^*\frac{d}{du}\left(\frac{\dot{x}(u)}{x(u)}\right)_{u=u^*}\right](u(t)-u^*)$$

Then write $\dot{x}(t) = f(x(t))$, and observe that $f'(x^*) = \lambda_1$. This implies

$$\dot{u}(t) = \left[-(1-\beta)Ax^{*^{1-\beta}}u^{*^{1-\beta}} - (1-\beta)Ax^{*^{-\beta}}u^{*^{2-\beta}}\frac{dx}{du}(u^*) - u^*\left(\frac{\lambda_1}{x^*}\frac{dx}{du}(u^*)\right)\right](u(t) - u^*) \tag{6.6}$$

[7] There is an alternative proof of the lemma, which does not involve the value function and which is relevant in the presence of flat-rate taxes. The proof is available upon request from the authors.

Moreover, from (6.4) and (6.6), it follows that

$$\lambda_1 = -(1-\beta)Ax^{*^{1-\beta}}u^{*^{1-\beta}} - \left((1-\beta)Ax^{*^{-\beta}}u^{*^{2-\beta}} + \frac{\lambda_1}{x^*}u^*\right)\frac{dx}{du}(u^*)$$

Hence,

$$\left[\lambda_1 + (1-\beta)Ax^{*^{1-\beta}}u^{*^{1-\beta}}\right] = \left[-\frac{u^*}{x^*}\frac{dx}{du}(u^*)\right]\left[\lambda_1 + (1-\beta)Ax^{*^{1-\beta}}u^{*^{1-\beta}}\right] \tag{6.7}$$

Therefore, by Lemma 1,

$$\lambda_1 = -(1-\beta)Ax^{*^{1-\beta}}u^{*^{1-\beta}} \tag{6.8}$$

Finally, evaluating conditions (3.6)-(3.7) at their steady-state values [cf., (3.15)-(3.16)], and rearranging terms, we obtain from (6.8) that

$$\lambda_1 = -\left(\frac{1-\beta}{\beta}\right)(\pi + n + \delta - \theta)$$

The theorem now follows from our notational convention, $\hat{\lambda} = -\lambda_1$. Q.E.D.

Observe that (6.4) and all subsequent steps apply to any arbitrary eigenvalue of the non-linear system; however, as it clear from (6.7)-(6.8), our computations pick only the negative eigenvalue, λ_1. This seems to be due to the condition $-\frac{u^*}{x^*}\frac{dx}{du}(u^*) \neq 1$ in the above lemma.

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	true value $\hat{\lambda}$	estimated value $F_L(1, u^*x^*)u^*x^*$
$\gamma = 1$	.3150	.3150
$\gamma = .6$	.2500	.2477
$\gamma = .55$	.2384	.2364
$\gamma = .5$	.2251	.2236
$\gamma = .45$	.2096	.2089
$\gamma = .4$	.1908	.1913
$\gamma = .35$	.1668	.1693

Table 1. True and estimated values for the rate of convergence.

	true value $\hat{\lambda}$	estimated value $F_L(1, u^*x^*)u^*x^*$
$\sigma = 4$	.1889	.1743
$\sigma = 3$	.1887	.1774
$\sigma = 2$	.1892	.1840
$\sigma = 1.85$	.1895	.1857
$\sigma = 1.75$	.1897	.1871
$\sigma = 1.65$	.1901	.1886
$\sigma = 1.5$	.1908	.1913
$\sigma = 1.25$	.1931	.1976
$\sigma = 1$	.1980	.2077

Table 2. True and estimated values for the rate of convergence.

b	rate of convergence	Tobin's q	z	x	l	u	ν
16	0.0338	1.32	0.3458	0.1446	0.4139	0.2704	0.0200
32	0.0235	1.64	0.4466	0.2128	0.4504	0.2639	0.0200
20	0.0299	1.40	0.3710	0.1607	0.4159	0.2684	0.0200
10	0.0443	1.20	0.3080	0.1218	0.4405	0.2738	0.0200
5	0.0659	1.10	0.2765	0.1041	0.4369	0.2773	0.0200
0	0.1950	1	0.2450	0.0874	0.4326	0.2817	0.0200

Table 3. Effect of parameter b on steady-state values.

a	rate of convergence	Tobin's q	z	x	l	u	ν
0.06	0.0338	1.32	0.3458	0.1446	0.4139	0.2701	0.0200
0.08	0.0262	1	0.2450	0.0874	0.4326	0.2817	0.0200
0.04	0.0403	1.64	0.4402	0.2109	0.4180	0.2663	0.0200
0.02	0.0459	1.96	0.5282	0.2845	0.4187	0.2656	0.0200
0	0.0508	2.28	0.6098	0.3643	0.4474	0.2669	0.0200

Table 4. Effect of parameter a on steady-state values.

σ	rate of convergence	Tobin's q	z	x	l	u	ν
4	0.0318	1.12	0.2960	0.0885	0.5568	0.3361	0.0075
3	0.0322	1.16	0.3062	0.0976	0.5342	0.3229	0.0100
2	0.0331	1.24	0.3262	0.1188	0.4891	0.2966	0.0150
1.5	0.0338	1.32	0.3458	0.1446	0.4139	0.2704	0.0200
1	0.0353	1.48	0.3838	0.2174	0.3538	0.2177	0.0300

Table 5. Effect of parameter σ on steady-state values.

γ	rate of convergence	Tobin's q	z	x	l	u	ν
0.85	0.0352	1.32	0.3458	0.0677	0.1369	0.5774	0.0200
0.75	0.0349	1.32	0.3458	0.0793	0.2209	0.4934	0.0200
0.65	0.0345	1.32	0.3458	0.0943	0.2998	0.4145	0.0200
0.55	0.0342	1.32	0.3458	0.1149	0.3740	0.3403	0.0200
0.45	0.0338	1.32	0.3458	0.1446	0.4439	0.2701	0.0200
0.35	0.0335	1.32	0.3458	0.1913	0.5099	0.2044	0.0200

Table 6. Effect of parameter γ on steady-state values.

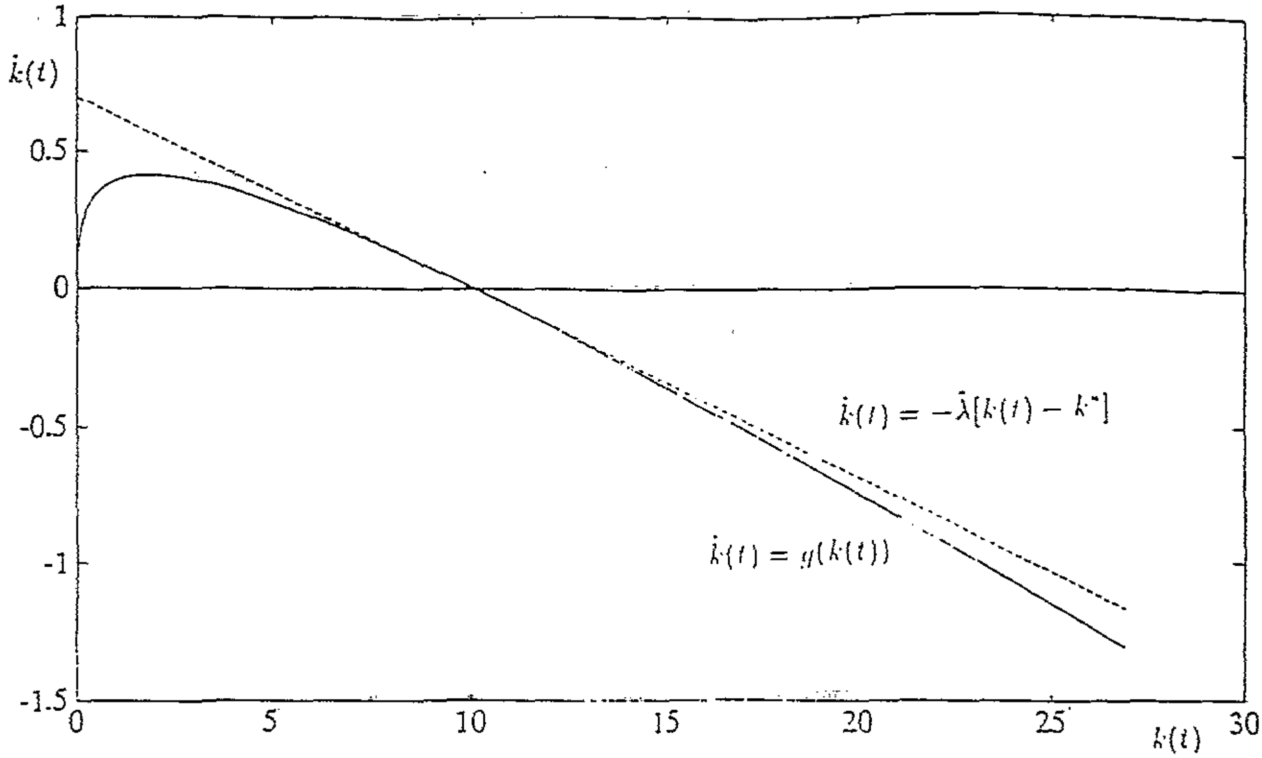

Figure 1(a). *Dynamics in the Neoclassical Growth Model: Stable manifolds of the linear and non-linear systems for stable variable* k. *Parameters values,* $\sigma = 1.5$, $\rho = 0.05$, $n = 0.01$, $A = 1$, $\beta = 0.4$, $\pi = 0.05$. *Steady state* $k^* = 10.0794$.

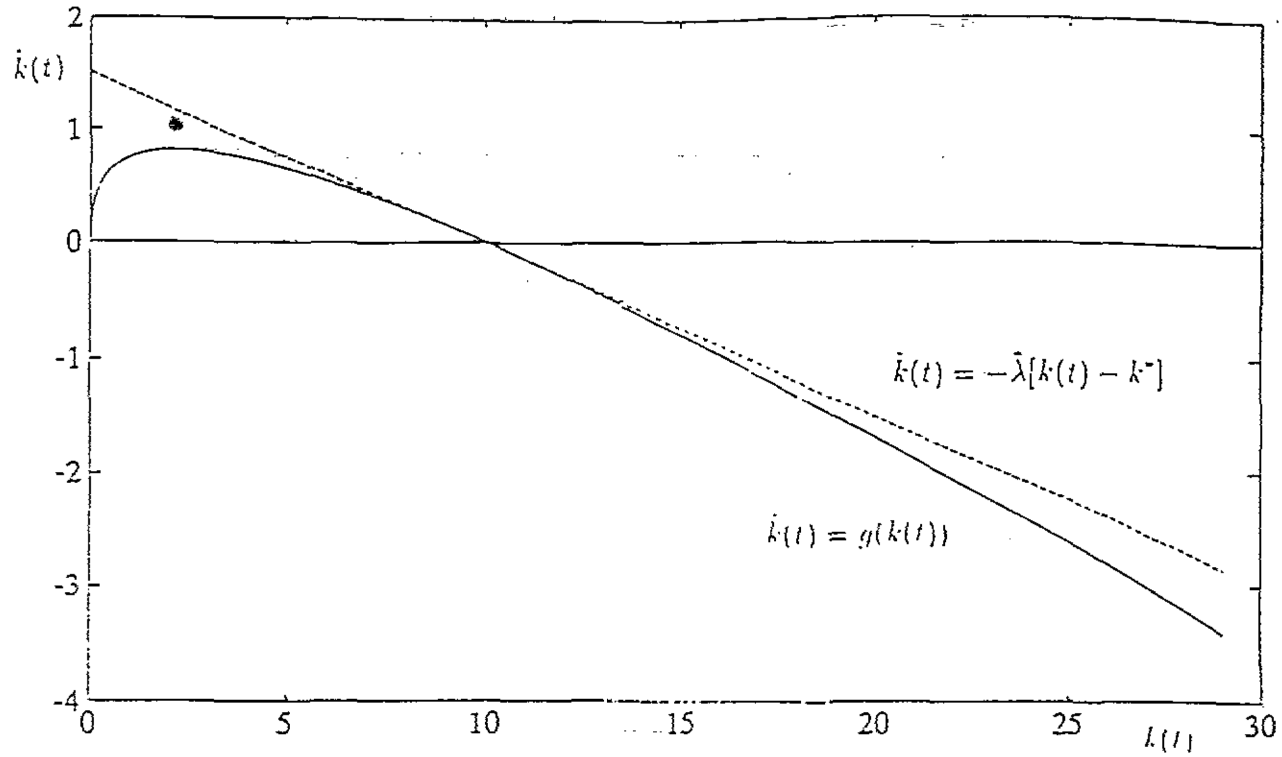

Figure 1(b). *Dynamics in the Neoclassical Growth Model: Stable manifolds of the linear and non-linear systems for stable variable k. Parameters values,* $\sigma = 0.1$, $\rho = 0.05$, $n = 0.01$, $A = 1$, $\beta = 0.4$, $\pi = 0.05$. *Steady state* $k^* = 10.0791$.

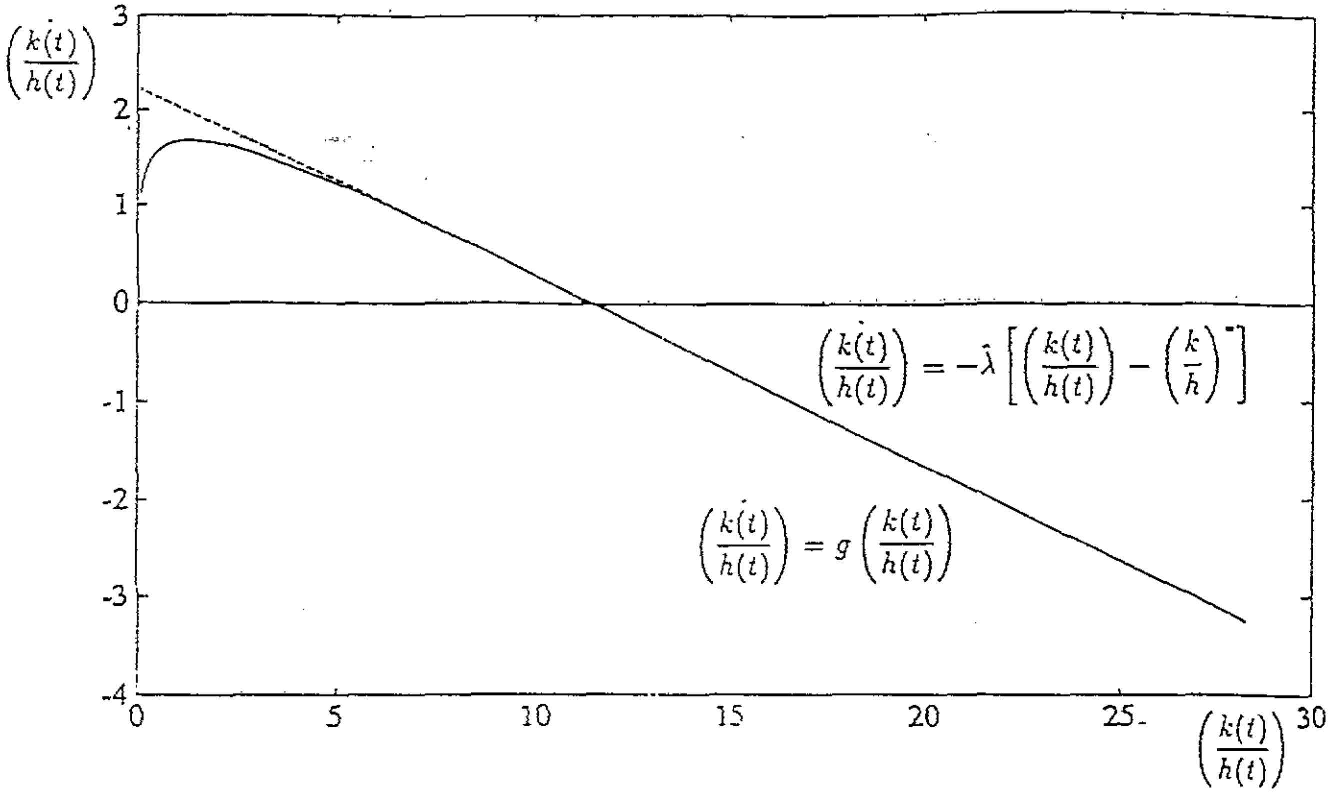

Figure 2(a). *Dynamics in the Endogenous Growth Model: Stable manifolds of the linear and non-linear systems for stable variable* $\left(\frac{k}{h}\right)$. *Parameters values,* $\gamma = 0.45$, $\sigma = 1.5$, $\rho = 0.05$, $n = 0.01$, $A = 3$, $\beta = 0.4$, $\pi = 0.05$, $\delta = 0.07$, $\theta = 0$. *Steady state* $\left(\frac{k}{h}\right) = 11.4424$.

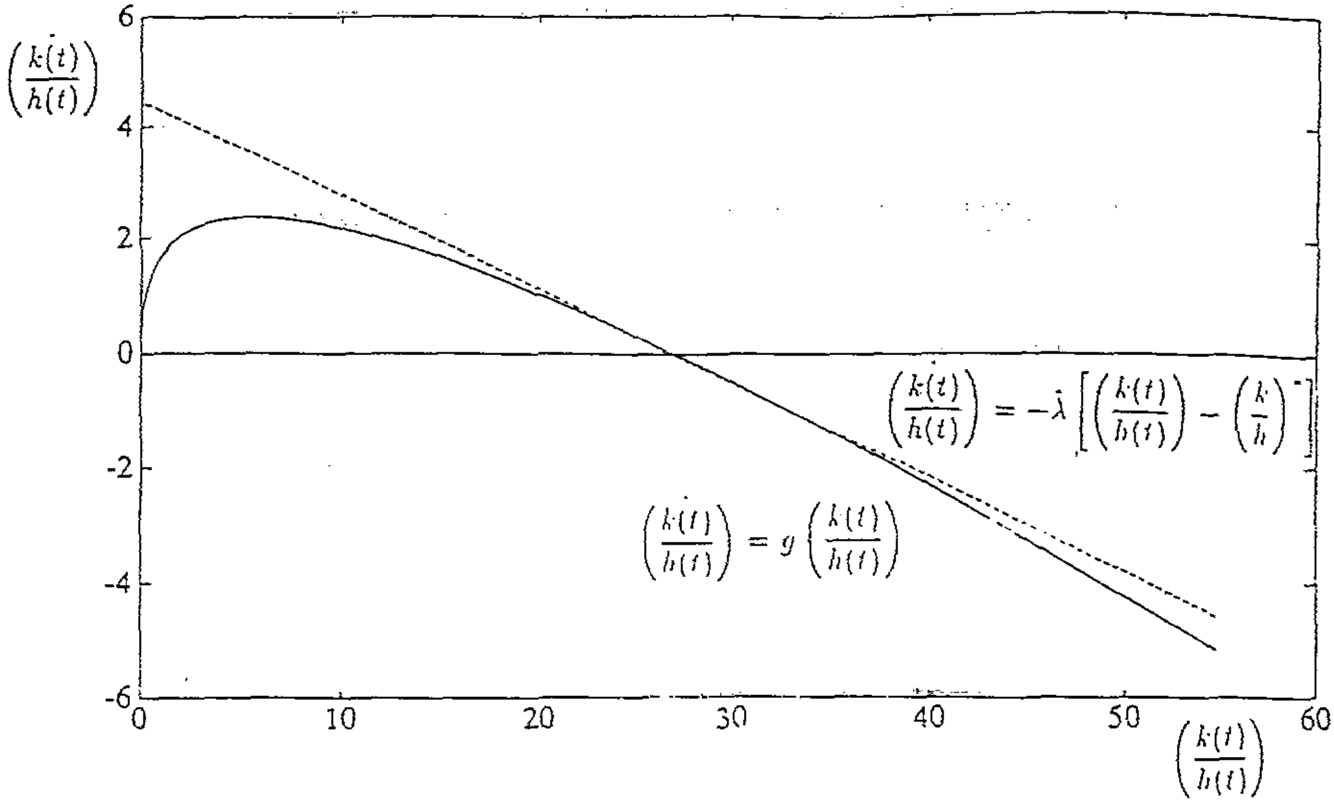

Figure 2(b). *Dynamics in the Endogenous Growth Model: Stable manifolds of the linear and non-linear systems for stable variable* $\left(\frac{k}{h}\right)$. *Parameters values,* $\gamma = 1$, $\sigma = 0.4$, $\rho = 0.05$, $n = 0.01$, $A = 3$, $\beta = 0.4$, $\pi = 0.05$, $\delta = 0.07$, $\theta = 0$. *Steady state* $\left(\frac{k}{h}\right) = 26.8097$.