Discussion Paper 12

Institute for Empirical Macroeconomics
Federal Reserve Bank of Minneapolis
250 Marquette Avenue
Minneapolis, Minnesota 55480

February 1989

STOCHASTIC INFLATION AND THE EQUITY PREMIUM

Pamela Labadie
Graduate School of Business
Columbia University
New York

ABSTRACT

The effects of stochastic inflation on equity prices and the equity premium are studied in a pure-endowment asset-pricing model with a cash-in-advance constraint. Stochastic inflation affects the equity premium through two channels: the assessment of an inflation tax and the presence of an inflation premium. Real and monetary versions of the model are simulated and the comparative dynamic results corroborate the conclusion that inflation has quantitatively important effects.

The other important result is that the equity premium in the real version of the model--a continuous state-space generalization of Mehra and Prescott (1985)--and the monetary model is very sensitive to the conditional variance of endowment growth. When the standard deviation of endowment growth is increased from 3.49 percent (the estimated value) to 5.59 percent, the real model can generate an equity premium of 2.8 percent in the range of the risk aversion parameters considered by Mehra and Prescott. The monetary model displays similar sensitivity and can generate an equity premium of 5.81 percent.

JEL Numbers: 023, 132, 311

This material is based on work supported by the National Science Foundation under Grant No. SES-8722451. The Government has certain rights to this material.

The effects of stochastic inflation on the equity premium are studied in a pure endowment representative agent economy with a cash-in-advance constraint. The random endowment is growing over time and the nondistortionary monetary transfers are stochastic. Inflation affects the equity premium through two channels: the assessment of the entire time path of the future inflation tax and the presence of an inflation premium.

The equilibrium equity price is a function of the discounted present value of the real dividend stream. Dividends are denominated in units of currency so that their real value depends on the realization of endowment and the inflation tax. The effect of the inflation tax on the equity price depends on the conditional covariance of the tax with the intertemporal marginal rate of substitution (MRS). This covariance is positive or negative depending on the sign of the conditional covariance of the endowment shock with the monetary transfer. The intuition is this: Whether the inflation tax tends to high in good times (higher than anticipated endowment) or high in bad times (lower than anticipated endowment) affects the riskiness of the equity. The riskiness of an equity is measured by the conditional covariance of its return with the MRS.

Measuring the equity premium -- the difference between the conditional expected equity return and the return to an indexed bond -- may be difficult in the presence of stochastic inflation. An indexed bond, which has the important property of zero correlation of its return with the MRS, is generally not traded so data on its return are unavailable. An alternative is to use the real return to a short-term nominal bond. This return displays nonzero correlation with the MRS if the conditional covariance of inflation with the MRS is nonzero. The difference between the real return to the nominal bond and the return to the indexed bond is the inflation premium. The measured equity premium -- the difference between the expected equity return and the real return to the nominal bond -- differs from the equity premium by the inflation premium.

Are the effects of stochastic inflation on the equity premium quantitatively important? To answer this question, I simulate real and monetary versions of the model. The model is a parameterized version of the models devised by Lucas [1978, 1980, 1982]. The cash-in-advance constraint is assumed to be binding in all states. This assumption has some empirical support; see the paper by Hodrick, Kocharlakota and Lucas [1989]. A covariance stationary system with gaussian disturbances that is bivariate and autoregressive is specified for the growth rates of endowment and money. With this specification and the assumption of isoelastic preferences, the equity price is a geometric distributed lead of log-normally distributed random variables; an iterative solution method to evaluate the equity price is described.

The real model is a continuous state space generalization of Mehra and Prescott [1985]. Simulations reveal that the equity premium is very sensitive to the conditional covariance of the endowment process. When the standard deviation is increased from 3.49% (the estimated value) to 5.59%, the equity premium increases from 1.17% to 2.81% in the range of the risk aversion parameter considered by Mehra and Prescott.[1] Moreover, the equity return is 7.16% and the indexed bond return is 4.36%. While this parameter sensitivity diminishes somewhat the extent of the equity premium puzzle, an equity premium of 2.81% is still less than half of the observed premium.

The quantitative effects of stochastic inflation appear to be important. When the standard deviation of inflation and the endowment growth are increased from their estimated values of (4.47%, 3.45%) to (6.32%, 5.48%), the equity premium increases from 2.23% to 5.76%.

The model and the effects of inflation are described in section 1. The model is parameterized and the solution algorithm is described in section 2; the results of the simulations are reported in section 3.

1. A model of equity prices that incorporates inflation

A version of the asset-pricing model devised by Lucas [1978, 1980, 1982] is used to study the effects of inflation on equity prices and the equity premium. Since most of the theoretical properties are established, my discussion focuses at the outset on a specific parametric version of the model which is simulated to derive comparative dynamic results. Incorporating inflation into the model reveals two channels through which stochastic inflation affects the equity premium; the two channels are analyzed after the basic model is described.

The economy experiences monetary instability and stochastic endowment shocks. The per capita endowment is nonstorable, exogenous and growing over time. Before trading starts, stochastic monetary transfers are made at the beginning of each time period to currency holders. Each member of the identical and fixed population maximizes an isoelastic utility function over an infinite planning horizon. The representative agent holds wealth carried over from the previous period in the form of currency M_{t-1} and equity shares z_{t-1}. The equity share is a claim to the dollar-denominated current and future dividend stream. All variables are expressed as per capita.

At the beginning of the period and prior to any trading, the stochastic monetary shock and endowment shock are realized and observed by all. Currency holdings are augmented by a lump-sum transfer $w_t M_{t-1}$ so that an agent's post-transfer currency holdings (before any trade occurs) are

$$M_t = (1 + w_t)M_{t-1}. \tag{1.1}$$

The endowment good, denoted as y_t at time t, evolves over time according to

$$y_{t+1} = \lambda_{t+1}\, y_t. \tag{1.2}$$

The motion over time of the growth rate of endowment ($\ell n\ \lambda$) and the growth rate of currency ($\ell n\ (1+\omega)$) is described by a bivariate system that is covariance stationary; this is made explicit in section 2 and motivated by the properties of the data described in section 3.

The exchange of equities, currency and goods takes place in two stages. In the first phase of trading, the agent divides his post-transfer wealth between equity claims z_t -- with each claim purchased at a real price q_t -- and currency holdings M_t^D -- with each dollar valued at $(p_t)^{-1}$ units of the endowment good.

Goods trading and dividend collection occurs during the second phase of trading. The agent must finance consumption purchases with currency accumulated previously so that

$$p_t c_t \leq M_t^D. \tag{1.3}$$

During the second trading phase, the agent collects nominal dividends $p_t z_t y_t$; this income is unavailable for spending until the next period.

At the beginning of period t+1, the pre-transfer, dollar-denominated wealth available for spending is

$$z_t[p_t y_t + p_{t+1} q_{t+1}]. \tag{1.4}$$

The maximization problem solved by the representative agent is

$$\max_{\{z_t, c_t, M_t^D\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\gamma}\right) c_t^{1-\gamma} \right\}, \ \gamma > 0 \tag{1.5}$$

subject to the cash-in-advance constraint (1.3) and the wealth constraint (1.4). In (1.5) E_0 denotes the expectations operator conditioned on information available at time 0 and $0 < \beta < 1$. The first order condition for equity holdings is

$$c_t^{-\gamma} q_t = \beta E_t\{c_{t+1}^{-\gamma}(z_t(q_{t+1} + y_t \pi_{t+1}^{-1}))\} \tag{1.6}$$

where π_{t+1} is the gross inflation rate $(p_{t+1}(p_t)^{-1})$.

Markets are cleared when all equity shares are held ($z_t = 1$), all currency is held ($M_t = M_t^D$), and the endowment is consumed ($c_t = y_t$). Let $q(\cdot)$ and $p(\cdot)$ denote the equilibrium equity price function and equilibrium goods price function and let s_t denote the state vector at time t. Finally, let q_t and p_t denote the value of the functions q and p evaluated at the current state s_t. The constraint (1.3) is assumed to be binding so that the equilibrium price function is

$$p(s_t) = M_t(y_t)^{-1}. \tag{1.7}$$

The <u>equilibrium</u> first order condition is

$$\begin{aligned} y_t^{-\gamma} q(s_t) &= \beta E_t\{(y_{t+1})^{-\gamma}((q(s_{t+1}) + y_t \pi_{t+1}^{-1}))\} \\ &= \beta E_t\{(y_{t+1})^{-\gamma}((q(s_{t+1}) + y_{t+1}\phi_{t+1})\} \end{aligned} \tag{1.8}$$

where ϕ_{t+1} denotes $(1 + w_{t+1})^{-1}$ for notational convenience. The second equality in (1.8) follows from (1.1) and (1.7).

The equilibrium first-order condition (1.8) reveals the dependence of the real equity price on the stochastic properties of both the endowment process and the monetary shock. The stochastic monetary transfer can either detract from or enhance the real value of the equity through the inflation tax. Risk averse agents assess the real impact of the inflation tax to determine their optimal equity holdings; in a representative agent model this assessment affects the equilibrium equity price and creates a nontrivial link between the inflation tax and the real equity return. The exact nature of this link is now explored more carefully.

The real return to the equity from period t to (t+1) is

$$R^q_{t+1} = (q_{t+1} + \pi^{-1}_{t+1} y_t)(q_t)^{-1} - 1. \tag{1.9}$$

Substituting (1.9) into the equilibrium first order condition (1.8) and rewriting results in

$$1 = E_t\{\beta(\lambda_{t+1})^{-\gamma}[1 + R^q_{t+1}]\}. \tag{1.10}$$

Let S_{t+1} denote the MRS $(\beta\lambda^{-\gamma}_{t+1})$. The conditional equity return is then expressed as[2]

$$1 + E_t R^q_{t+1} = (E_t S_{t+1})^{-1}[1 - \mathrm{Cov}_t(S_{t+1}, R^q_{t+1})] \tag{1.11}$$

where Cov_t denotes the conditional covariance. One measure of the riskiness of an asset is the correlation of the asset's return with the MRS. An asset is risky if the covariance in (1.11) is negative. One implication is this: If stochastic inflation affects the covariance in

(1.11), stochastic inflation affects the risk characteristics (defined as the equity premium and the correlation of consumption with the asset's return) of the asset.

How does stochastic inflation affect the conditional covariance of the asset's return with the MRS? Substituting (1.9) into the covariance reveals that

$$\begin{aligned}\mathrm{Cov}_t(S_{t+1},R^q_{t+1}) &= \mathrm{Cov}_t(S_{t+1},(q_{t+1} + \pi^{-1}_{t+1}y_t)(q_t)^{-1} - 1)\\ &= (q_t)^{-1}[\mathrm{Cov}_t(S_{t+1},q_{t+1}) + y_t\ \mathrm{Cov}_t(\pi^{-1}_{t+1},S_{t+1})]. \qquad (1.12)\end{aligned}$$

The first covariance, $\mathrm{Cov}_t(S_{t+1},q_{t+1})$, is affected by inflation through the equilibrium price process. To illustrate this, notice that (1.8) is linear in the function $h(s_t)$ where

$$h(s_t) = y_t^{-\gamma}q(s_t)$$

so that (1.8) is

$$h(s_t) = \beta E_t\{h(s_{t+1}) + (y_{t+1})^{-\gamma}y_t\pi^{-1}_{t+1}\}. \qquad (1.13)$$

Under certain conditions,[3] (1.13) can be solved forward to result in

$$h(s_t) = E_t \sum_{j=1}^{\infty} \beta^j (y_{t+j})^{-\gamma}\pi^{-1}_{t+j}y_{t+j-1}$$

so that

$$q(s_t) = h(s_t)y_t^{\gamma}.$$

The equilibrium equity price is a function of the discounted present value of the marginal utility of the future dividend stream. This present value depends on the entire path of the future inflation tax.

The other covariance, $\mathrm{cov}_t(\pi_{t+1}^{-1}, S_{t+1})$, is the conditional covariance of the MRS with the appreciation in the purchasing power of money. In my model,

$$\mathrm{Cov}_t(\pi_{t+1}^{-1}, S_{t+1}) = \beta\ \mathrm{Cov}_t(\lambda_{t+1}\phi_{t+1}, \lambda_{t+1}^{-\gamma}). \qquad (1.14)$$

The sign of (1.14) can be positive or negative depending on how λ and ϕ co-vary contemporaneously. If the endowment shock λ and the monetary transfer $((1 + w) = \phi^{-1})$ co-vary negatively, the covariance (1.13) is negative.

Stochastic inflation affects the risk characteristics of the equity's return through the assessment of the entire path of the inflation tax measured by $\mathrm{Cov}_t(S_{t+1}, q_{t+1})$ -- and the correlation of the MRS with the rate of appreciation in the purchasing power of money -- measured by $\mathrm{Cov}_t(S_{t+1}, \pi_{t+1}^{-1})$. This is the first channel through which inflation affects the equity premium.

The equity premium is defined as the difference between the real return to the equity (R^q) and the real return to an indexed bond -- a bond that pays with certainty one unit of the endowment good one period hence. The indexed bond has a real price that, in equilibrium, satisfies

$$q_t^0 = E_t S_{t+1}$$

and a real return

$$1 + R_t^0 = (E_t S_{t+1})^{-1}. \qquad (1.15)$$

The indexed bond has, by definition, the property that the conditional covariance of its return with the MRS is zero.

Efforts to measure the equity premium are confounded because, in general, an indexed asset of this type -- zero covariance with the MRS --

is not traded so that data on its return are unavailable. One approach, used by Mehra and Prescott [1985], is to measure the indexed bond's return by the real return to a nominal bond (computed by subtracting realized inflation from the nominal interest rate). If the conditional covariance of the MRS with inflation is nonzero, the real return to a nominal bond will display a nonzero correlation with the marginal rate of substitution. As a result, the measured equity premium computed with the real return of a nominal bond may differ systematically from the equity premium computed with the indexed bond. The difference between the real return to the nominal bond and the indexed bond return is defined as the inflation premium.

The inflation premium, Π_{t+1}, is defined as

$$\Pi_{t+1} = E_t R^i_{t+1} - R^0_t \tag{1.16}$$

where R^i_{t+1} is the real return to the nominal bond. A nominal bond is a claim to one unit of currency one period hence. The nominal price at time t (B_t) of a one period bond that is a claim to $(p_{t+1})^{-1}$ units of (t+1)-goods satisfies

$$B_t = E_t S_{t+1} (\pi_{t+1})^{-1}$$

and the nominal interest rate is

$$i_t = (B_t)^{-1} - 1. \tag{1.17}$$

The expected real return to the nominal bond is

$$\begin{aligned} 1 + E_t R^i_{t+1} &= (1 + i_t) E_t \pi^{-1}_{t+1} \\ &= E_t \pi^{-1}_{t+1} [E_t S_{t+1} \pi^{-1}_{t+1}]^{-1}. \end{aligned} \tag{1.18}$$

A property of the covariance is

$$E_t[S_{t+1}\pi_{t+1}^{-1}] = Cov_t(S_{t+1},\pi_{t+1}^{-1}) + E_tS_{t+1}E_t\pi_{t+1}^{-1}$$

so that (1.18) may be expressed as

$$1 + E_tR_{t+1}^{i} = \frac{E_t\pi_{t+1}^{-1}}{Cov_t(S_{t+1},\pi_{t+1}^{-1}) + E_tS_{t+1}E_t\pi_{t+1}^{-1}} . \qquad (1.19)$$

Whether an inflation premium is present depends on the sign of $Cov_t(S_{t+1},\pi_{t+1}^{-1})$.[4] There are three cases.

Case 1 No inflation premium

If $Cov_t(S_{t+1},\pi_{t+1}^{-1})$ equals zero,

$$1 + E_tR_{t+1}^{i} = (E_tS_{t+1})^{-1} = 1 + R_t^0.$$

Using the real return to the nominal bond to measure the equity premium results in no systematic measurement error.

Case 2 Negative inflation premium

A negative inflation premium occurs when $Cov_t(S_{t+1},\pi_{t+1}^{-1})$ is positive (endowment shocks and monetary transfer shocks are positively correlated) because

$$E_t\pi_{t+1}^{-1}[Cov_t(S_{t+1},\pi_{t+1}^{-1}) + E_t\pi_{t+1}^{-1}E_tS_{t+1}]^{-1} < (E_tS_{t+1})^{-1} = 1 + R_t^0.$$

The measured equity premium exceeds the equity premium because of the negative inflation premium.

Case 3 Positive inflation premium

If $Cov_t(S_{t+1},\pi_{t+1}^{-1})$ is negative,

$$E_t R^i_{t+1} > R^o_t,$$

and the measured equity premium will understate the equity premium.

To conclude, the two channels through which stochastic inflation can affect the equity premium are: assessing the current and future inflation tax which alters both the equity return and its correlation with consumption; and measuring the indexed bond's return in the presence of an inflation premium.

2. Computing the equilibrium price

While stochastic inflation affects the risk characteristics of the asset and may create an inflation premium, the question remains: Are these effects of sufficient magnitude that their omission leads to a serious misinterpretation of the data? To answer this question, the equilibrium equity price function $q(\cdot)$ must be expressed as an explicit function of the current state s_t.

The equilibrium equity price satisfies

$$y_t^{-\gamma} q(s_t) = \beta E_t \{ y_{t+1}^{-\gamma} [q(s_{t+1}) + y_{t+1}\phi_{t+1}] \}. \tag{2.1}$$

As illustrated in (1.13), the equilibrium first order condition (2.1) becomes linear in the function $h(\cdot)$ -- where $h(s_t)$ equals $y_t^{-\gamma} q(s_t)$ -- so that

$$h(s_t) = \beta E_t [h(s_{t+1}) + y_{t+1}^{\rho} \phi_{t+1}] \tag{2.2}$$

where ρ equals $1-\gamma$ for notational convenience. This is a linear stochastic equation that can be solved forward, if the sum converges, as

$$h(s_t) = E_t\{\sum_{j=1}^{\infty} \beta^j y_{t+j}^{\rho} \phi_{t+j}\}. \tag{2.3}$$

To evaluate the conditional expectation in (2.3), the joint distribution of the endowment and monetary processes must be specified.

To this end, I make

Assumption A. The distribution of (λ,ϕ) is a stationary bivariate process that is log-normal:

$$\begin{bmatrix} \ell n\ \lambda_{t+1} \\ \ell n\ \phi_{t+1} \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} \delta_1 & \eta \\ \psi & \theta_1 \end{bmatrix} \begin{bmatrix} \ell n\ \lambda_t \\ \ell n\ \phi_t \end{bmatrix} + \begin{bmatrix} v_{t+1} \\ u_{t+1} \end{bmatrix} \tag{2.4}$$

where (v,u) are jointly normally distributed with zero mean and variance-covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_v^2 & \sigma_{vu} \\ \sigma_{vu} & \sigma_u^2 \end{bmatrix} . \tag{2.5}$$

The (v,u) process also satisfies: $Eu_t u_s = Ev_t v_s = Eu_t v_s = 0$ for $s \neq t$.

This assumption is motivated by the properties of the data which are described in section 3. Conditions that result in stationarity are detailed in the appendix. The state at time t is a vector $s_t = (y_t, \lambda_t, M_t, \phi_t)$.

Assumption B. The long-run average growth rate of marginal utility times the real dividend $(y^{\rho}\phi)$ is of exponential order less than β^{-1}.

Under assumptions A and B, (2.3) is a geometric distributed lead of log-normally distributed random variables. Evaluation of this sum results in

Theorem 1. Under assumptions A and B, the equilibrium equity price satisfies

$$q(s_t) = y_t \sum_{j=1}^{\infty} A_j \lambda_t^{a_j} \phi_t^{b_j}$$

where

$$A_1 = \beta \exp[\rho(\delta_0 + .5\rho\sigma_v^2) + \theta_0 + .5\sigma_u^2 + \sigma_{vu}]$$

$$a_1 = \rho\delta_1 + \psi$$

$$b_1 = \theta_1 + \rho\eta \tag{2.6}$$

and, for j > 1,

$$A_{j+1} = A_j\beta \exp[(a_j + \rho)(\delta_0 + .5(a_j + \rho)\sigma_v^2) + b_j(\theta_0 + .5b_j\sigma_u^2) + (a_j + \rho)b_j\sigma_{vu}]$$

$$a_{j+1} = \delta_1(a_j + \rho) + b_j\psi$$

$$b_{j+1} = \theta_1 b_j + \eta(a_j + \rho) \tag{2.7}$$

Proof. The proof is in the appendix.[5]

There are five steps in the algorithm to compute the equity price.

Algorithm

1. Choose values for the parameters

$$(\sigma_v^2, \sigma_u^2, \sigma_{vu}, \theta_0, \theta_1, \delta_0, \delta_1, \psi, \eta, \beta, \gamma)$$

and values for the initial conditions (y_0, λ_0, ϕ_0). Generate a sample realization of the bivariate normal process $\{u_t, v_t\}_{t=1}^{T}$. Use the realization, the parameter values and the initial conditions to construct $\{y_t, \lambda_t, \phi_t\}_{t=1}^{T}$.

2. Evaluate the system

$$A_1 = \beta \exp\{\rho\delta_0 + .5\rho^2\sigma_v^2 + .5\sigma_u^2 + \theta_0 + \rho\sigma_{uv}\}$$

$$a_1 = \rho\delta_1 + \psi$$

$$b_1 = \theta_1 + \rho\eta$$

and compute

$$H_t^1 = y_t^\rho A_1 \lambda_t^{a_1} \phi_t^{b_1}.$$

3. For $j \geqq 2$ evaluate (A_j, a_j, b_j) in (2.7) and compute

$$H_t^j = H_t^{j-1} + y_t^\rho A_j \lambda_t^{a_j} \phi_t^{b_j}.$$

4. Repeat step 3 until

$$\max_t \mid H_t^{N+1} - H_t^N \mid < \text{epsilon} \tag{2.8}$$

where epsilon is a small and positive number.

Let N denote the iteration number at which (2.8) is satisfied; then

$$q_t^N = y_t^\gamma H_t^N$$

and q_t^N is an arbitrarily good approximation to q_t. The approximation error is an increasing function of epsilon.

Evaluated and summarized in Table 1 are the other important variables of the model: the real return to the indexed bond; the real return to the nominal bond; the return to the nominal bond; and the conditional covariance of the MRS with the rate of change in the purchasing power of money.

1. Indexed bond return

$$1 + R_t^0 = (E_t S_{t+1})^{-1} = [\beta \exp(-\gamma\delta_0 + .5\gamma^2\sigma_v^2)]^{-1}[\lambda_t^{\delta_1\gamma}\phi_t^{\eta\gamma}]$$

2. Nominal interest rate

$$\begin{aligned} 1 + i_t &= (E_t S_{t+1}\pi_{t+1}^{-1})^{-1} \\ &= [\beta \exp(\rho\delta_0 + .5\rho^2\sigma_v^2 + \theta_0 + .5\sigma_u^2 + \rho\sigma_{uv}]^{-1}(\lambda_t^{\rho\delta_1+\psi}\phi_t^{\rho\eta+\theta_1})^{-1} \end{aligned}$$

3. Expected real return to the nominal bond

$$\begin{aligned} 1 + E_t r_{t+1} &= [E_t S_{t+1}\pi_{t+1}^{-1}]^{-1}E_t\pi_{t+1}^{-1} \\ &= [\beta \exp(-\gamma\delta_0 + .5(\gamma^2 - 2\gamma)\sigma_v^2 - \gamma\sigma_{uv})]^{-1}[\lambda_t^{\delta_1}\phi_t^{\eta}]^{-\gamma} \end{aligned}$$

4. Conditional covariance of MRS and the inverse of inflation

$$\begin{aligned} \mathrm{Cov}_t(S_{t+1},\pi_{t+1}^{-1}) &= \beta\lambda_t^{\delta_1\rho+\psi}\phi_t^{\eta\rho+\theta_1} \\ &\exp(\rho\delta_0 + \theta_0 + .5\sigma_u^2 + (1+\gamma^2)\sigma_v^2 + \sigma_{vu})[\exp(-\gamma\sigma_v^2 - \gamma\sigma_{vu}) - 1] \end{aligned}$$

Table 1: Important variables of the monetary model expressed as explicit functions of the current state and parameters of the model.

When the cash-in-advance constraint is binding, assumption (A) imposes conditions on the behavior over time of the inflation process. To see this, recall that gross inflation is

$$\pi_{t+1} = p_{t+1}(p_t)^{-1} = \left(\frac{M_{t+1}}{y_{t+1}}\right)\left(\frac{y_t}{M_t}\right)$$

so that, from assumption A,

$$\begin{aligned}\ln \pi_{t+1} &= -(\ln \phi_{t+1} + \ln \lambda_{t+1}) \\ &= -(\delta_0 + \theta_0) - ((\delta_1 + \psi)\ln \lambda_t + (\theta_1 + \eta)\ln \phi_t) - (v_{t+1} + u_{t+1}) \\ &= -(\delta_0 + \theta_0) - ((\delta_1 + \psi)\ln \lambda_t + (\theta_1 + \eta)(\ln \pi_t + \ln \lambda_t)) \\ &\quad - (v_{t+1} + u_{t+1}) \\ &= \alpha_0 + \alpha_1 \ln \lambda_t + \alpha_2 \ln \pi_t + e_{t+1}. \end{aligned} \tag{2.9}$$

The conditional variance of the inflation process is

$$\sigma_e^2 = E(-(v + u)^2) = \sigma_v^2 + \sigma_u^2 + 2\sigma_{uv}, \tag{2.10}$$

and the conditional covariances of inflation with the endowment shock and the monetary process are

$$\sigma_{ev} = \mathrm{Cov}(e,v) = E[-(v + u)v] = -\sigma_v^2 - \sigma_{uv}$$

and (2.11)

$$\sigma_{eu} = \mathrm{Cov}(e,u) = E[-(v + u)u] = -\sigma_u^2 - \sigma_{uv}.$$

If σ_{uv} is negative, the conditional covariance of inflation with the endowment may be near zero. As long as σ_{uv} is negative and

$$|\sigma_{uv}| - \sigma_v^2 \neq 0$$

the conditional covariance of the MRS with the inverse of inflation (see Table 1, number 4) will be positive. This covariance (σ_{uv}) is critical to the sign and magnitude of the inflation premium. The magnitude of the covariance also depends on the risk parameter γ and the current realization (λ_t,ϕ_t).

The binding cash-in-advance constraint has another implication: The monetary transfer $(1 + w)$ equals the growth of nominal consumption $(\pi\lambda)$ since

$$\pi_{t+1} = p_{t+1}(p_t)^{-1} = (\frac{M_{t+1}}{y_{t+1}})(\frac{y_t}{M_t})$$

$$= (1 + w_{t+1})\lambda_{t+1}^{-1} \tag{2.12}$$

or

$$(1 + w_{t+1}) = \pi_{t+1}\lambda_{t+1}.$$

Assumption A and the binding cash-in-advance constraint imply strong cross-equation restrictions on the rate of inflation and the growth rates of endowment, money, and nominal consumption.

3. Comparative dynamics and the equity premium

Mehra and Prescott [1985] observe that the equity premium for a particular sample averages about 6%. They pose the question: Can a simple asset-pricing model with random and growing endowment generate an average equity premium that matches the sample average? They find that, in a simple two-state Markov model with constant relative risk aversion, the answer is no if the risk aversion parameter is restricted to a range that seems reasonable in light of microeconomic data. When the risk aversion parameter is varied between zero and ten, Mehra and Prescott find that the largest equity premium generated by their model is .38%.

Mehra and Prescott use the Grossman and Shiller [1981] data set to construct a measure of the equity premium. The data are:[5]

i. Series C: Kuznets-Kendrick-USNIA real, per capita annual U.S. consumption of nondurables and services 1889-1979, measured in 1972 dollars.

ii. Series PC: Consumption deflator -- computed by dividing nominal consumption by real consumption.

iii. Series PSN: Average Annual Standard and Poor's Composite Stock Price Index, 1889-1979.

iv. Series DS: Annual nominal dividend for the Standard and Poor's Stock Price Index. DS is the dividend that accrues over time period t and is paid at the beginning of t + 1.

v. Series RF: Nominal yield on relatively short-term securities -- this series is described and plotted in Mehra and Prescott (1985, p. 149, fig. 3). Ninety-day government Treasury Bills are used for 1931-78, Treasury Certificates for 1920-30, and 60-90 day Prime Commercial paper prior to 1920.

Mehra and Prescott measure the real equity return in the data by

$$R_t^D = \left(\frac{PSN_{t+1}}{PC_{t+1}} + \frac{DS_t}{PC_t}\right)\left(\frac{PSN_t}{PC_t}\right)^{-1} - 1. \tag{3.1}$$

They compute the real return on the indexed bond by

$$r_{t+1}^D = RF_t - \left[\frac{PC_{t+1} - PC_t}{PC_t}\right] \tag{3.2}$$

where the last term on the right-hand-side of (3.2) is the realized inflation. Finally, they measure the expost equity premium by

$$\Pi_t^D = R_t^D - r_t^D. \tag{3.3}$$

Some sample statistics for these variables are reported in Tables 2a and 2b.

	(1) Return on S&P 500 (%)	(2) Nominal yield Short-term asset (%)	(3) Inflation (%)	(4) Measured Equity Premium (%)
	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)
1889-1978	6.98 (16.54)	3.36 (2.065)	2.40 (5.10)	6.18 (16.66)
1889-1898	7.58 (10.01)	4.58 (1.32)	-1.26 (2.65)	1.78 (11.57)
1899-1908	7.71 (17.21)	4.73 (7.87)	2.06 (2.38)	5.08 (16.86)
1909-1918	-.14 (12.81)	4.58 (8.26)	5.75 (7.64)	1.49 (9.18)
1919-1928	18.94 (16.18)	3.92 (9.85)	-.58 (6.47)	14.64 (15.94)
1929-1938	2.56 (27.90)	9.78 (1.39)	-1.58 (5.93)	.18 (31.63)
1939-1948	3.08 (14.66)	3.02 (2.43)	5.88 (3.73)	8.90 (14.22)
1949-1958	17.49 (13.08)	1.69 (.67)	2.45 (1.75)	18.29 (13.20)
1959-1968	5.58 (10.59)	3.52 (.99)	2.41 (1.22)	4.50 (10.17)
1969-1978	.04 (14.02)	5.95 (1.40)	6.43 (2.36)	.76 (11.64)

Table 2a: The real return to the Standard and Poor 500 (col. 1) is computed as $R_t^D = [PSN_{t+1}(pc_{t+1})^{-1} + D_t(pc_t)^{-1}][PSN_t(pc_t)^{-1}]^{-1} - 1$. Inflation (col. 3) is computed as $(pc_{t+1} - pc_t)(pc_t)^{-1}$. The real return to the bond with nominal return RF_t (col. 2) is $RF_t - (pc_{t+1} - pc_t)(pc_t)^{-1}$ and the measured equity premium (col. 4) is $R_t^D - [RF_t - (pc_{t+1} - pc_t)(pc_t)^{-1}]$. (S.D.) is the standard deviation.

	(1) % Growth Rate per capita real consumption	(2) % Growth Rate of per capita nominal consumption	(3) Correlation of growth rates of real consumption and inflation	(4) Correlation of growth rates real and nominal consumption
	Mean (S.D.)	Mean (S.D.)		
1889-1978	1.75 (3.53)	4.15 (6.47)	.09	.62
1889-1898	2.18 (4.77)	.92 (6.58)	.54	.94
1899-1908	2.40 (5.20)	4.47 (5.04)	-.30	.89
1909-1918	.39 (3.06)	6.14 (7.72)	-.17	.22
1919-1928	2.88 (3.95)	2.30 (6.70)	-.25	.35
1929-1938	-.37 (5.33)	-1.95 (10.53)	.75	.93
1939-1948	2.14 (2.47)	8.02 (4.32)	-.07	.51
1949-1958	1.46 (.98)	3.91 (1.58)	-.44	.13
1959-1968	2.34 (.98)	4.75 (1.76)	.27	.74
1969-1978	2.37 (1.45)	8.80 (1.86)	-.61	.01

Table 2b: The compounded growth rate (col. 1) is computed as $\ell n(c_t(c_{t-1})^{-1})$. Nominal consumption is $(pc_t)c_t$ and its compounded growth (col. 2) is $\ell n(((pc_{t+1})c_{t+1})((pc_t)c_t)^{-1}$. (S.D.) is the standard deviation.

Mehra and Prescott assume a two-state Markov process for the growth rate of the endowment. A continuous state-space generalization of their growth process is

$$\ln \lambda_{t+1} = \Omega_0 + \Omega_1 \ln \lambda_t + \varepsilon_{t+1} \tag{3.4}$$

where $\ln \lambda$ is the continuously compounded rate of growth and ε is an error process that is independent and identically distributed with mean zero and finite variance σ_ε^2. This specification (3.4) allows for positive average growth and serial correlation. By setting λ_{t+1} equal to $c_{t+1}(c_t)^{-1}$ -- where c is real per capita consumption of nondurables over 1889-1979 -- the parameters of the process can be estimated. This estimation results in: $\hat{\Omega}_0 = .01765$, $\hat{\Omega}_1 = .018145$, and $\sigma_\varepsilon^2 = .0012167$.[6]

In the real version of the model, the equity is a claim to the future dividend stream so that the purchase today of a claim entitles the holder to tomorrow's endowment y_{t+1} (in equilibrium) and its resale price q_{t+1}. If the error process $\{\varepsilon_t\}$ is normally distributed, the equilibrium equity price and its return can be expressed as explicit functions of the current state by following the steps of the algorithm described in section 2.[7] To compute the equity premium, the return of the indexed bond must be evaluated; its price is

$$\bar{q}_t^f = \beta E_t \lambda_{t+1}^{-\gamma} = \beta \exp(-\gamma(\Omega_0 + .5\gamma^2\sigma_\varepsilon^2))\lambda_t^{-\gamma\Omega_1}.$$

The model can be simulated by generating a realization of the error process $\{\varepsilon_t\}$ according to its distribution (a normal distribution with mean zero and variance σ_ε^2). The endowment process is computed by setting $y_0 = c_{1889}$, $\lambda_1 = (c_{1890})(c_{1889})^{-1}$, using (3.4) to generate λ_j $(j > 1)$, and setting, for $t > 1$,

$$y_{t+1} = \lambda_{t+1} y_t .$$

The final step is to choose values for (γ,β) to use in the solution algorithm. The realized equity return, indexed bond return, and equity premium can then be evaluated. The results are reported in Table 3.

In Table 3, the value of β is fixed at .98. The risk premium parameter (γ) is varied from .75 to 9.75. While several values of the conditional variance σ_ε^2 are examined, the parameters (ϕ_0,ϕ_1) are fixed at their point estimates. The sample size of each simulation is 200. For every set of parameters values considered, each statistic reported -- the equity return, the indexed bond return and the equity premium -- is an average over fifty drawings of length 200. Specifically, a sample $\{\varepsilon_t\}_{t=1}^{200}$ is generated, sample paths $\{\lambda_t,\phi_t,y_t\}$ evaluated, and computations made of the equity return, indexed bond return and the premium. The sample average of the returns and the premium are calculated; this process is repeated fifty times. The average of the fifty sample averages is compiled and reported in Table 3.

The first column in the table reports the sample averages of the ex post equity return, the indexed bond return and the premium using the point estimates of the parameters in (3.4). Column 2 reports the results when the conditional variance is increased. The standard deviation of endowment growth in column 1 is 3.49% whereas the standard deviation in column 2 is 5.59%. In the decade sample averages for the data summarized in Table 2b, the standard deviation varies from 1% to 5.33% so that a standard deviation of 5.59% is not extraordinary.

Value of Risk Aversion Parameter γ	Case (1) $\sigma_\varepsilon^2 = .0012167$ Equity Return (%)	Indexed Bond Return (%)	Equity Premium (%)	Case (2) $\sigma_\varepsilon^2 = .00312167$ Equity Return (%)	Indexed Bond Return (%)	Equity Premium (%)
.75	3.48	3.39	.09	3.58	3.33	.25
1.75	5.32	5.11	.21	5.37	4.79	.58
2.75	7.06	6.72	.34	6.83	5.12	.91
3.75	8.69	8.23	.46	7.95	6.72	1.23
4.75	10.21	9.62	.57	8.72	7.18	1.54
5.75	11.61	10.90	.71	9.13	7.30	1.84
6.75	12.88	12.06	.83	9.18	7.07	2.11
7.75	14.03	13.09	.94	8.87	6.50	2.37
8.75	15.05	13.99	1.06	8.20	5.59	2.60
9.75	15.93	14.76	1.17	7.16	4.36	2.81

Table 3: Simulation of the real model assuming that endowment growth follows $\ell n\ \lambda_{t+1} = \Omega_0 + \Omega_1\ \ell n\ \lambda_t + \varepsilon_{t+1}$ with $\Omega_0 = .01765$ and $\Omega_1 = .018145$ (the point estimates) and the initial endowment and growth rate λ set equal to their historical values. β is fixed at .98. The endowment variance in column 1 is the point estimate; its standard deviation is 3.49% whereas the standard deviation in column 2 is 5.59%. Each statistic is the average over 50 draws of length 200.

The equity premiums in Table 3 are surprisingly large: when σ_ε equals 3.49% and γ equals 9.75 the equity premium is 1.17%; when σ_ε equals 5.59% and γ equals 9.75 the equity premium is 2.81%. Moreover, the equity return is 7.16% and the indexed bond return is 4.36% in the second case ($\sigma_\varepsilon = 5.59\%$). The largest equity premium reported by Mehra and Prescott in this range for γ is .38%. The statistics reported in column one are directly comparable to Mehra and Prescott; the key difference is that I use a continuous state space version of their 2-state model. The statistics in the second column suggest that the Mehra and Prescott [1985] equity premium puzzle is very sensitive to the conditional variance of endowment. Despite this sensitivity, an equity premium of 2.81% is less than half of the observed premium so this feature of the data is unexplained by the real model.

Incorporating Inflation

In section 1, I show that stochastic inflation affects the risk characteristics of the equity and the measurement of the equity premium. The question now is: Are the effects of inflation on the equity premium quantitatively significant?

To study the effects of stochastic inflation, I use the Mehra-Prescott data set to estimate a bivariate system describing the motion of the growth of endowment and inflation. The binding cash-in-advance constraint imposes strong cross-equation restrictions on inflation and the growth rates of endowment, nominal consumption, and the monetary transfer; these restrictions are described in (2.9)-(2.11). The model is matched to the data by setting λ_t equal to $(c_{t+1}(c_t)^{-1})$ and π_{t+1} equal to $(pc_{t+1}(pc_t)^{-1})$ where c is per capita real consumption of nondurables and pc is the consumption deflator. The bivariate system I estimate is

$$\begin{bmatrix} \ln \lambda_{t+1} \\ \ln \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \alpha_0 \end{bmatrix} + \begin{bmatrix} \delta_1 & \delta_2 \\ \alpha_2 & \alpha_1 \end{bmatrix} \begin{bmatrix} \ln \lambda_t \\ \ln \pi_t \end{bmatrix} + \begin{bmatrix} v_{t+1} \\ e_{t+1} \end{bmatrix} \tag{3.5}$$

The results are reported in Table 4.

Equation 1: $\ell n\ \lambda_t = \delta_0 + \bar{\delta}_1\ \ell n\ \lambda_{t-1} + \delta_2\ \ell n\ \pi_{t-1} + v_t$

Parameter	Estimate	Standard Error
δ_0	.01835	.105013
$\bar{\delta}_1$	-.15196	.073321
δ_2	.10197	.0043976

Equation 2: $\ell n\ \pi_t = \alpha_0 + \alpha_1\ \ell n\ \pi_{t-1} + \alpha_2\ \ell n\ \lambda_{t-1} + e_t$

Parameter	Estimate	Standard Error
α_0	.01034	.0056949
α_1	.46456	.094951
α_2	.18608	.1359928

Table 4: Estimates of the bivariate system where $\ell n\ \lambda_t = \ell n(c_t) - \ell n(c_{t-1})$ and c_t is annual per capita consumption of nondurables in year t, and $\ell n\ \pi_t = \ell n(pc_t) - \ell n(pc_{t-1})$ where pc_t is the consumption deflator in year t. The sample period is 1891-1978 (89 observations).

In the simulation results contained in Table 5, the parameters $(\alpha_0, \delta_0, \bar{\delta}_1, \delta_2, \alpha_1)$ are set equal to their point estimates reported in Table 4. The implied values of the parameters of the monetary transfer process are: $\delta_1 = -.2539$, $\psi = .5314$, $\eta = -.1019$, $\theta_0 = -.02869$, and $\theta_1 = .5655$.

By specifying values of $(\sigma_v^2, \sigma_e^2, \sigma_{ev})$ (which determines the values of σ_u^2 and σ_{uv}), a realization $\{v_t, u_t\}$ can be generated; the procedure is described in Appendix A. The initial values selected are: $y_0 = c_{1889}$, $\lambda_1 = (c_{1890})(c_{1889})^{-1}$, $\pi_1 = (pc_{1890})(pc_{1889})$, and $\phi_1 = (\pi_1\lambda_1)^{-1}$.

Values of the parameters (γ,β) are chosen and the equity price and its return are computed with the solution algorithm described in section 2. The other variables of interest -- the real and nominal returns to the bond and the indexed bond return -- are calculated according to Table 1. The results of this exercise are reported in Table 5.

In Table 5, the value of β is fixed at .98. Just as in Table 3, the statistics reported are the sample averages over fifty realizations of $\{v_t, u_t\}$ of length 200.

Values of γ	Case (1) $\sigma_e^2 = .001994$		Case (2) $\sigma_e^2 = .004$	
	(1a) $\sigma_v^2 = .001189$ $\sigma_{ev} = .000083$	(1b) $\sigma_v^2 = .003189$ $\sigma_{ev} = .00021$	(2a) $\sigma_v^2 = .001189$ $\sigma_{ev} = .000083$	(2b) $\sigma_v^2 = .003189$ $\sigma_{ev} = .00021$
$\gamma = .75$				
R (%)	3.49	3.61	3.52	3.60
R^o	3.41	3.36	3.41	3.35
R^i	3.40	3.34	3.40	3.32
$\gamma = 1.75$				
R (%)	5.38	5.47	5.41	5.46
R^o	5.16	4.88	5.16	4.83
R^i	5.14	4.82	5.14	4.78
$\gamma = 2.75$				
R (%)	7.21	7.09	7.23	7.07
R^o	6.82	6.09	6.82	6.02
R^i	6.79	6.01	6.80	5.93
$\gamma = 3.75$				
R (%)	8.97	8.45	9.00	8.34
R^o	8.39	6.99	8.39	6.89
R^i	8.36	6.87	8.36	6.77
$\gamma = 4.75$				
R (%)	10.66	9.56	10.68	9.54
R^o	9.86	7.56	9.87	7.44
R^i	9.82	7.42	9.82	7.29

Values of γ	Case (1) (cont.) $\sigma^2_e = .001994$		Case (2) (cont.) $\sigma^2_e = .004$	
	(1a) $\sigma^2_v = .001189$ $\sigma_{ev} = .000083$	(1b) $\sigma^2_v = .003189$ $\sigma_{ev} = .00021$	(2a) $\sigma^2_v = .001189$ $\sigma_{ev} = .000083$	(2b) $\sigma^2_v = .003189$ $\sigma_{ev} = .00021$
$\gamma = 5.75$				
R (%)	12.26	10.41	12.29	10.37
R^o	11.24	7.82	11.24	7.66
R^i	11.18	7.64	11.19	7.49
$\gamma = 6.75$				
R (%)	13.79	10.99	13.82	10.94
R^o	12.50	7.74	12.51	7.56
R^i	12.44	7.54	12.44	7.36
$\gamma = 7.75$				
R (%)	15.24	11.30	15.27	11.25
R^o	13.66	7.34	13.66	7.13
R^i	13.59	7.10	13.59	6.90
$\gamma = 8.75$				
R (%)	16.60	11.34	16.62	11.29
R^o	14.70	6.61	14.71	6.38
R^i	14.62	6.35	14.63	6.12
$\gamma = 9.75$				
R (%)	17.86	11.13	17.89	11.07
R^o	15.63	5.57	15.64	5.31
R^i	15.54	5.28	15.54	5.02

Table 5: Reported are simulation results for the monetary model. Each statistic is an average over 50 draws of length 200. Reported are: the ex post equity return (R), the indexed bond return R^o, the real return to the nominal bond (R^i). The conditional variance of inflation is σ^2_e, the conditional variance of endowment is σ^2_e and their conditional covariance is σ_{ev}.

Incorporating inflation results in higher average returns for the equity. In case (1a), the equity premium ranges from a low of .08% when γ equals .75 to 2.32% when γ equals 9.75. In Case (1a), the parameter values are set equal to their point estimates (reported in Table 4). The standard deviation of the inflation disturbance in column 1 is 4.47%; in column 2 its standard deviation is 6.32% -- this is not unusually high since the decade sample standard deviations of inflation range from 1.22% to 7.64%. For each value of the conditional variance of inflation considered (of which there are two), two sets of values for the endowment variance and the inflation-endowment covariance are examined. In cases (1a) and (2a), the parameters σ_v^2 and σ_{ev} are set at their point estimates; the standard deviation of endowment is 3.45%. In cases (1b) and (2b), the standard deviation of endowment is 5.47%. Comparing columns 1 and 2 reveals that the sample statistics for the monetary model, just as in the real model, are sensitive to the parameter specification of the conditional variances and covariance. The positive covariance σ_{ev} implies that the covariance of the endowment shock and monetary transfer is positive (the inflation tax tends to be high when endowment is high). This corresponds to the case of a negative inflation premium. The inflation premium appears to be small (in absolute value) and fairly constant -- see Table 6 where the inflation premium can be calculated as $R^i - R^o$.

Conclusion

Currency is introduced into a pure exchange asset-pricing model via a cash-in-advance constraint. The timing of information acquisition and trading is designed to leave the real side of the economy as unchanged as possible. Despite this construction, stochastic inflation affects the risk characteristics of the equity's return and its premium. The equity is a claim to a currency-denominated dividend stream whose real value depends on the realizations of the endowment and the inflation tax. The riskiness of the equity, measured by the conditional covariance of its return with the MRS, depends on two factors: the assessment of the entire time path of the future inflation tax and the conditional covariance of the MRS with the rate of appreciation in the purchasing power of money.

Are the effects of stochastic inflation quantitatively important? To determine the answer, the model is simulated. The growth rates of the endowment and monetary transfer are modelled as a bivariate autoregressive system with gaussian disturbances. Preferences are isoelastic. As a result of this specification, the equity price is a geometric distributed lead of log-normally distributed random variables; an iterative solution method is described in section 2. The parameters of the bivariate system are estimated for the Mehra and Prescott [1985] data set.

When the real model -- a continuous state-space version of the Mehra and Prescott model -- is simulated, I find that the equity premium is very sensitive to the conditional variance of endowment growth. For the range of risk aversion parameters considered by Mehra and Prescott, the equity premium varies from .09% to 1.17%. If the standard deviation of endowment growth is increased from its estimated value 3.49% to 5.59%, the equity premium varies from .25% to 2.81%. If the endowment data (annual real per capita consumption of nondurables for 1889-1978) contain measurement error that causes the series to appear smoother than it actually is, this increase in the standard deviation does not seem unreasonable. Even so, the model-generated equity premium of 2.81% is less than half of the observed equity premium.

Stochastic inflation does have quantitatively significant effects especially as the risk aversion parameter is increased. This seems quite natural since agents are confronted with two sources of risk (endowment shocks and random inflation tax). The monetary model can generate an equity premium of 5.81% in the relevant range of the risk aversion parameter if the conditional variances of inflation and endowment are increased. The inflation premium seems to be quite small for the set of parameter values examined here; elsewhere (Labadie [1988]) I have found larger and more volatile inflation premiums. Most of the impact of inflation on equity prices results from the inflation tax assessment.

Footnotes

1. The endowment series is real per capita consumption of nondurables for the sample period 1889-1978 (see section 3). There is a strong possibility that the data contain measurement error which could make the series appear smoother (lower variance) than it actually is.
2. The relationship between the riskiness of an asset and the covariance of its return with the MRS is discussed in Grossman and Shiller [1981] and Donaldson and Mehra [1984].
3. This is the data set used by Grossman and Shiller [1981] and Mehra and Prescott [1985]. Mehra and Prescott provide a description of the properties of the data.
3. The conditions are described in detail in section 2 and the appendix.
4. LeRoy [1984a,b] and Svenssen [1985] have made this point.
5. The key property is, for X a log-normally distributed random variable and k a scalar,

$$\ell n\, E_t\, X_{t+1}^k = E_t(k\, \ell n\, X_{t+1}) + .5k^2\, \mathrm{var}_t(\ell n\, X_{t+1}).$$

6. Estimation of the endowment process results in

$$\ell n\, \hat{\lambda}_{t+1} = \underset{(.003719)}{.01765} + \underset{(.016618)}{.018145}\, \ell n\, \lambda_t$$

with the sum of squared residuals .013365. The sample size is 90. The F-statistic is 1.192 and t-statistics for the estimates are 4.74 (Ω_0) and 1.092 (Ω_1).

7. The equity price of the real model is

$$\bar{q}_t = y_t^{\gamma}\, E_t \sum_{j=1}^{\infty} \beta^j y_{t+j}^{\rho}$$

where $\rho = 1 - \gamma$. The first term is

$$\beta E_t \; y_{t+1}^{\rho} = \beta \; y_t^{\rho} \; E_t \; \lambda_{t+1}^{\rho}$$

$$= \beta \; y_t^{\rho} \; \exp(\rho\Omega_0 + .5\rho^2\sigma_\varepsilon^2)\lambda_t^{\rho\Omega_1}$$

$$= y_t^{\rho} \; A_1 \; \lambda_t^{a_1}.$$

The second term is

$$\beta^2 \; E_t \; y_{t+2}^{\rho} = \beta^2 \; y_t^{\rho} \; E_t \; \lambda_{t+2}^{\rho} \; \lambda_{t+1}^{\rho}$$

$$= \beta \; y_t^{\rho} \; A_1 \; E_t \; \lambda_{t+1}^{\rho+a_1}$$

$$= \beta \; y_t^{\rho} \; A_1 \; \exp((\rho + a_1)(\Omega_0 + .5(\rho + a_1)\sigma_\varepsilon^2)\lambda_t^{\Omega_1(\rho+a_1)}$$

$$= y_t^{\rho} \; A_2 \; \lambda_t^{a_2}.$$

This sets up a recursive scheme

$$A_{j+1} = \beta \; A_j \; \exp((\rho + a_j)(\Omega_0 + .5(\rho + a_j)\sigma_\varepsilon^2)$$

$$a_{j+1} = \Omega_1(\rho + a_j).$$

The equilibrium equity price is

$$\bar{q}_t = y_t^{\gamma} \sum_{j=1}^{\infty} A_j \; \lambda_t^{a_j}.$$

References

Donaldson, John B., and Mehra, Rajnish U. [1984]. "Comparative Dynamics of an Equilibrium Intertemporal Asset Pricing Model," Review of Economic Studies, LI, 491-508.

Grossman, S., and Shiller R. J. [1981]. "The Determinants of the Variability of Stock Market Prices," American Economic Review, 71, 222-27.

Hodrick, R. J., Kocharlakota, N., Lucas, D. [1989]. The variability of velocity in cash-in-advance models. NBER working paper.

Labadie, Pamela [1988]. The effects of stochastic inflation on asset prices. Discussion paper No. 5, Institute for Empirical Macroeconomics, Federal Reserve Bank of Minneapolis, MN.

LeRoy, Stephen F. [1984a]. Nominal prices and interest rates in general equilibrium: Endowment shocks. Journal of Business, 57, No. 2, 177-95.

LeRoy, Stephen F. [1984b]. Nominal prices and interest rates in general equilibrium: money shocks. Journal of Business, 57, No. 2, 197-213.

Lucas, Robert E., Jr. [1978]. "Asset Prices in an Exchange Economy," Econometrica, 46, 1429-45.

________ [1980]. "Equilibrium in a Pure Currency Market," Economic Inquiry, 18, 203-20.

________ [1982]. "Interest Rates and Currency Prices in a Two-Country World." Journal of Monetary Economics, 10, 336-60.

Mehra, R., and Prescott, E. [1985]. "The Equity Premium: A Puzzle," Journal of Monetary Economics, 15, 145-62.

Svensson, Lars E. O. [1985]. "Money and Asset Prices in a Cash-in-Advance Economy," Journal of Political Economy, Vol. 93, 919-94.

Appendix

Proof of Theorem 1

The equation to be solved (2.3) is

$$h(s_t) = E_t\{\sum_{j=1}^{\infty} \beta^j \; y_{t+j}^{\rho} \; \phi_{t+j}\}. \tag{A1}$$

The linearity of the expectations operator allows (A1) to be evaluated term-by-term. The variables y_{t+j} and ϕ_{t+j} are jointly log-normally distributed. A property used repeatedly is

$$\ell n(E_t \; X_{t+1}^{a}) = E_t(a \; \ell n \; X_{t+1}) + \frac{a^2}{2} \; \mathrm{var}_t(\ell n \; X_{t+1}).$$

Evaluating term-by-term:

1. $$\beta E_t[y_{t+1}^{\rho}\phi_{t+1}] = \beta y_t^{\rho} \; E_t \; \lambda_{t+1}^{\rho} \; \phi_{t+1}$$

$$= \beta y_t^{\rho} \; \lambda_t^{\rho\delta_1+\psi} \; \phi_t^{\rho\eta+\theta_1} \; E_t \; \exp(\rho\delta_0 + \theta_0 + \rho v_{t+1} + u_{t+1})$$

$$= \beta y_t^{\rho} \; \lambda_t^{\rho\delta_1+\psi} \; \phi_t^{\rho\eta+\theta_1} \; \exp[\rho\delta_0 + \theta_0 + .5\rho^2\sigma_v^2 + .5\sigma_u^2 + \rho\sigma_{uv})$$

$$= A_1 \; y_t^{\rho} \; \lambda_t^{a_1} \; \phi_t^{b_1}$$

where A_1, a_1 and b_1 are defined in (2.6).

2. $$\beta^2 E_t[y_{t+2}^{\rho}\phi_{t+2}] = \beta^2 y_t^{\rho} \; E_t \; \lambda_{t+2}^{\rho} \; \phi_{t+2} \; \lambda_{t+1}^{\rho}$$

$$= \beta^2 y_t^{\rho} \; E_t\{\exp(\rho\delta_0 + .5\rho^2\sigma_v^2 + \theta_0$$

$$+ .5\sigma_u^2 + \sigma_{uv})\lambda_{t+1}^{\rho\delta_1+\psi+\rho}\phi_{t+1}^{\rho\eta+\theta_1}\}$$

$$= \beta \; A_1 \; y_t^{\rho} \; E_t \; \lambda_{t+1}^{a_1+\rho} \; \phi_{t+1}^{b_1}$$

$$= \beta\ A_1\ y_t^{\rho}\ \exp((a_1 + \rho)(\delta_0 + .5(a_1 + \rho)\sigma_v^2 + b_1\theta_0$$

$$+ b^2\sigma_u^2 + (a_1 + \rho)b_1\sigma_{uv})\lambda_t^{\delta_1(a_1+\rho)+b_1\psi}\ \phi_t^{\theta_1 b_1+\eta(a_1+\rho)}$$

$$= A_2\ y_t^{\rho}\ \lambda_t^{a_2}\ \phi_t^{b_2}$$

where A_2, a_2 and b_2 are defined in (2.7).

Evaluating the third term results in

$$\beta^3 E_t\ y_{t+3}^{\rho}\ \phi_{t+3} = \beta^2\ A_1\ y_t^{\rho}\ E_t\ \lambda_{t+2}^{a_1+\rho}\ \phi_{t+2}^{b_1}\ \lambda_{t+1}^{\rho}$$

$$= \beta\ A_2\ y_t^{\rho}\ E_t\ \lambda_{t+1}^{a_2+\rho}\ \phi_{t+1}^{b_2}$$

$$= \beta\ A_2\ y_t^{\rho}\ \exp((a_2 + \rho)(\delta_0 + .5(a_2 + \rho)\sigma_v^2 + b_2\theta_0 + b^2\sigma_u^2$$

$$+ (a_2 + \rho)b_2\sigma_{uv})\lambda_t^{\delta_1(a_2+\rho)+b_2\psi}\ \phi_t^{\theta_1 b_2+\eta(a_2+\rho)}$$

$$= A_3\ y_t^{\rho}\ \lambda_t^{a_3}\phi_t^{b_3}.$$

There is nothing special about the second and third terms and evaluation of higher terms proceeds according to the recursive system established in (2.7). The equilibrium is computed as

$$q(s_t) = y_t^{\gamma}\ h(s_t)$$

Stationarity of the bivariate system in Assumption A

The unconditional mean of the bivariate system is

$$\ell n\ \bar{\lambda} = \frac{\delta_0}{1 - \delta_1}[1 - \frac{\psi}{(1 - \theta_1)(1 - \delta_1) - \psi\eta}] + \frac{\eta\theta_0}{(1 - \theta_1)(1 - \delta_1) - \psi\eta_1}$$

and

$$\ell n\ \bar{\phi} = \frac{(1 - \delta_1)\theta_0 + \psi\delta_0}{(1 - \theta_1)(1 - \delta_1) - \psi\eta}\ .$$

The unconditional mean is finite if

1. $|\delta_1| < 1$ and $|\theta_1| < 1$
2. $|\theta_1 + \delta_1 - \theta_1\delta_1 + \psi\eta| < 1$

For (2.4) to be invertible, the determinant must be nonzero or

$$\theta_1\delta_1 - \psi\eta + \psi + \eta - (\theta_1 + \delta_1) \neq 0. \tag{A2}$$

This condition is assumed to be satisfied.

Convergence of the sum $\Sigma A_j \lambda_t^{a_j} \phi_t^{b_j}$

The steady state of the system (2.7) is

$$a = \rho[\delta_1(1 - \theta_1) + \psi\eta][(1 - \delta_1)(1 - \theta_1) - \psi\eta]^{-1}$$

and

$$b = \rho\eta[(1 - \delta_1)(1 - \theta_1) - \psi\eta]^{-1}.$$

The earlier condition $|\theta_1 + \delta_1 - \theta_1\delta_1 + \psi\eta| < 1$, along with a finite and positive γ, ensures that a and b are finite. Define a constant k as

$$k = \exp[(a + \rho)(\delta_0 + .5(a + \rho)\sigma_v^2) + b(\theta_0 + .5b\sigma_u^2) + (a + \rho)b\sigma_{uv}].$$

If $\beta k < 1$,

$$\lim_{j\to\infty} A_{j+1} = \lim_{j\to\infty} A_j\beta k = 0$$

and, for some j large (say j = N where N is finite),

$$\sum_{j=N}^{\infty} A_j \lambda_t^{a_j} \phi_t^{b_j} = \lambda_t^a \phi_t^b \sum_{j=N}^{\infty} A_N(\beta k)^j$$

$$= \frac{\lambda_t^a \phi_t^b A_N(\beta k)^N}{1 - \beta k} < \infty$$

so that $h(s_t)$, written as

$$h(s_t) = y_t \sum_{j=1}^{\infty} A_j \lambda_t^{a_j} \phi_t^{b_j} = y_t \left[\sum_{j=1}^{N-1} A_j \lambda_j^{a_j} \phi_t^{b_j} + \frac{\lambda_t^a \phi_t^b A_N(\beta k)^N}{1 - \beta k} \right],$$

is the sum of two finite partial sums and hence $h(s_t)$ is well-defined.

Generating a realization of (u,v)

The system

$$\begin{bmatrix} \ln \lambda_{t+1} - \ln \bar{\lambda} \\ \ln \phi_{t+1} - \ln \bar{\phi} \end{bmatrix} = \begin{bmatrix} \delta_1 & \eta \\ \psi & \theta_1 \end{bmatrix} \begin{bmatrix} \ln \lambda_{t+1} - \ln \bar{\lambda} \\ \ln \phi_{t+1} - \ln \bar{\phi} \end{bmatrix} + \begin{bmatrix} v_{t+1} \\ u_{t+1} \end{bmatrix}$$

can be written in terms of the lag operator L ($LX_t = X_{t-1}$) as

$$C(L) \begin{bmatrix} \ln \lambda_{t+1} - \ln \bar{\lambda} \\ \ln \phi_{t+1} - \ln \bar{\phi} \end{bmatrix} = \begin{bmatrix} v_{t+1} \\ u_{t+1} \end{bmatrix}$$

and the bivariate process (expressed as deviation from mean) has a Wold moving average representation

$$\begin{bmatrix} \ln \lambda_{t+1} - \ln \bar{\lambda} \\ \ln \phi_{t+1} - \ln \bar{\phi} \end{bmatrix} = D(L) \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}$$

where (z_{1t}, z_{2t}) are jointly fundamental and

$$D(L) = C^{-1}(L) \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where (v_t, u_t) and (z_{1t}, z_{2t}) are related by

$$\ell n\ \lambda_t - P[\ell n\ \lambda_t \mid 1, \ell n\ \lambda_{t-1}, \ldots, \ell n\ \phi_{t-1}, \ldots] = v_t = c_{11} z_{1t} + c_{12} z_{2t}$$
$$\ell n\ \phi_t - P[\ell n\ \phi_t \mid 1, \ell n\ \lambda_{t-1}, \ldots, \ell n\ \phi_{t-1}, \ldots] = u_t = c_{21} z_{1t} + c_{22} z_{2t}$$

and P is the linear least squares projection operator.

In order for C(L) to be invertible the condition for the determinant (A2) must be satisfied. If the variances σ_1^2, σ_2^2 are set equal to one, the variance-covariance matrix of (v_t, u_t) is

$$\Sigma = \begin{bmatrix} \sigma_v^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_u^2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix}$$

Once the values in Σ are chosen, a Cholesky decomposition can be used to determine $(c_{11}, c_{12}, c_{21}, c_{22})$. A realization of (u, v) can be generated by drawing realizations (z_1, z_2) according to their joint normal distribution (i.i.d. with variances σ^2, σ_2^2 equal to one) to compute

$$v_t = c_{11} z_{1t} + c_{12} z_{2t}$$
$$u_t = c_{21} z_{1t} + c_{22} z_{2t}.$$