REASONABLE EXTREME BOUNDS ANALYSIS

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ABSTRACT
Leamer (1983) suggested to study the range of estimators $\beta_0$ in the model
$y = X_0 + \epsilon$ when imposing linear constraints of the form $M (C\beta - c) = 0$
where only $C$ and $c$ are fixed. However the extremes may come from models with a
bad $R^2$, say. In this paper we give the exact bounds when only considering models
with $R^2 \geq (1-\delta) R_{\max}^2 + \delta R_{\min}^2$. These exact bounds can be found from
calculating only two regressions. We apply our techniques to study the velocity of
money.

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1. Introduction

A modeller is faced with many different possible specifications for the model when there are several possible explanatory variables, each of which can enter with various lags. Leamer (1983) has suggested that certain essential features of the model can vary greatly between alternative specifications, thus making the interpretation of the model difficult or "fragile." An easily understood version of his argument has a single dependent variable $y$, a group of variables $X_F$ that "should" be used as explanatory variables in any model of $y$ when a particular question is under consideration, and a second group of variables $X_D$ that may or may not enter the model as explanatory variables. A basic or "restricted" model would be

\begin{equation}
(1.1) \quad y = \beta_F X_F + \text{residual}
\end{equation}

a complete model would be

\begin{equation}
(1.2) \quad y = \beta_F X_F + \beta_D X_D + \text{residual}
\end{equation}

with some linear constraints on $\beta_D$, such as requiring that certain variables be given a zero coefficient in (1.2). Thus, for example, if one was interested in the effect of an interest rate on velocity then the equation for velocity would certainly include this interest rate in $X_F$ plus possibly also velocity lagged once, money lagged once and a price index lagged. These explanatory variables might be thought of as a minimum set of variables necessary to explain velocity, according to some theory. This would
give the basic model. However, it may be thought necessary, by some modellers, to augment the basic model so that the model better explains the main features of the actual velocity series. This augmentation may include further lags of the variables already used plus other variables, such as variability of money base. As there are many possible ways to augment the basic model so there will be many different specifications. Suppose that we are most interested in the value of a particular coefficient, denoted \( \beta_0 \), such as the coefficients on interest rate in the velocity equation. The estimates of \( \beta_0 \) may vary considerably from one specification to another, and the extremes taken by the alternative estimates are called the "extreme bounds" by Leamer (1983). The extent of these bounds are viewed as measuring the fragility of the estimate of \( \beta_0 \) as alternative specifications are used. The value and interpretation of these bounds have been strongly criticized by McAleer, Pagan and Volker (1983, 1985) and also by Breusch (1985) and defended by Leamer (1985).

One criticism of the use of extreme bounds, which has some impact, in that the actual extremes may come from models that most economists would find unreasonable in some way, such as having low Durbin–Watson statistics, for example, in a time-series context. One way to express this problem is in terms of \( R^2 \) statistics. We are not defending \( R^2 \) as an ideal measure of the quality of a model, but it is possibly a relevant statistic and some exact results are achievable using it. Suppose that the maximum value achievable for \( R^2 \) is \( R^2_{\text{max}} \), which is certainly found by using all of the variables \( X_F, X_D \) in (1.2) with no exclusions, which might be called the "full" model. Of course, other specifications may also achieve this \( R^2_{\text{max}} \). The above worry about the virtues of using extreme bounds analysis is that the extreme may come from specifications that achieve \( R^2 \) values very much smaller than
$R^2_{\text{max}}$ and these specifications might be considered irrelevant because of their relatively low goodness of fit so that estimates of $\beta_0$ based on them would also be strongly discounted. It may be thought that specifications that achieve $R^2$ values not too far from $R^2_{\text{max}}$ would produce much narrower extreme bounds for $\beta_0$. It is this possibility that we consider in this paper. Suppose that $R^2_{\text{max}}$ is found from the full model and $R^2_{\text{min}}$ from the basic model (1.1). Consider model specification achieving $R^2$ values equal to or greater than

$$R^2_{\delta} = (1-\delta) R^2_{\text{max}} + \delta R^2_{\text{min}}$$

where $0 < \delta < 1$. For $\delta$ small these may be considered as being "reasonable" specifications as they are not far from the "best" model in terms of goodness of fit, as measured by $R^2$. In the next section of the paper an equation for the values of the extreme bounds of $\beta_0$ is presented for any given $\delta$. The proof is found in the appendix. A numerical example is presented in Section 3, concerning the modelling of velocity and using time series models. It is found that quite wide extreme bounds can occur using $\delta$ values as low as 0.1 or 0.2, relative to the extreme bounds found from the full set of possible specifications. This result strengthens Leamer's arguments about the difficulties that can arise when interpreting particular coefficients. Some further considerations are presented in the final section.

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\(^1\)See also Leamer (1981), in which similar ideas and results are obtained when constraining ridge estimates to achieve a given level of significance.
2. The Model and Results

The model being considered is

\[ y = X\beta + \epsilon \quad (2.1) \]

where \( y \) is the vector of observations on the variable \( y \) and \( X \) is the matrix of observations on a vector of explanatory variables. At this stage, no distinction is being made between time-series or cross-section situations. It will be assumed that \( \epsilon \) is \( N(0, \sigma^2 \Omega) \) where the covariance matrix \( \sigma^2 \Omega \) is assumed known for the time being. The object of primary interest is the "focus" coefficient

\[ \beta_o = \psi \beta \quad (2.2) \]

so that \( \beta_o \) can be any individual coefficient or a weighted linear combination of coefficients. There will be a set of prior linear constraints

\[ C\beta = c \quad (2.3) \]

It is convenient to use this general form for the constraints, but if the two types of variables \( X_F, X_D \) are considered, as in the first section, then the coefficients on \( X_F \) are free of restrictions (hence the notation \( \beta_F \)) and the coefficients on \( X_D \) are "doubtful" in that they may or may not appear in any particular specification. The restrictions would then be \( C = (0, I) \) with appropriate sizes of the zero and unit vectors, and \( c = 0 \). In a particular specification any linear combination of these
restrictions can be used, so the objective is to study the range of estimators for $\beta_0$
when imposing linear constraints of the form

\begin{equation}
M(C\beta - c) = 0
\end{equation}

for some matrix $M$, which is assumed without loss of generating to be of full row
rank. If no restrictions are placed on $M$, one gets the extreme bounds suggested by
Leamer (1983). The following notation is used:

The generalized least square (GLS) estimates of $\beta$ using the full model (2.1)
with no exclusions is

$$ b = (X' \Omega^{-1} X)^{-1} X' \Omega^{-1} y $$

which gives the estimate of $\beta_0$, $b_0 = \psi' b$.

Let

$$ D = \sigma^2(X' \Omega^{-1} X)^{-1} $$

$$ A = CDC' $$

and let $A^{-\frac{1}{2}}$ be the unique symmetric square root of $A^{-1}$.

For a given $M$ in (2.4) define

$$ W = A^\frac{1}{2} M' $$

5
noting that we get any full column rank \( W \) as \( M \) ranges over all full row rank matrices and vice versa.

Two important vectors are

\[
(2.5) \\
 \quad u = A^{-\frac{1}{2}} CD\psi
\]

and

\[
(2.6) \\
 \quad v = A^{-\frac{1}{2}} (Cb-c)
\]

The Euclidean norm of \( u \) is \( \|u\| = (u' u)^{\frac{1}{2}} \). It is convenient to define an angle \( \theta \in [0, \pi/2] \) by

\[
(2.7) \\
 \quad \cos 2\theta = \cos (u, v)
\]

\[
\equiv \frac{u' v}{\|u\| \|v\|}
\]

The GLSE \( \hat{\beta} \) of \( \beta_0 \) under the restriction (2.4) is

\[
\hat{\beta}_0 = b_0 - u' W(W' W)^{-1} W' v
\]

Breusch (1955) proved:
Theorem 1

The extreme values of $\hat{\beta}_0$ over all choices of (full column rank matrices) $W$ are

$$b_0 - \frac{1}{2} (\cos 2\theta + 1) \|u\| \|v\|$$

i.e. $$b_0 - \cos^2 \theta \|u\| \|v\|$$

and

$$b_0 + \sin^2 \theta \|u\| \|v\|$$

The bounds can be attained for some $W$.

Suppose now that a value for $\delta$ is chosen, with $0 \leq \delta \leq 1$ and models are considered having $R^2$ values greater or equal to

$$R^2_\delta = (1-\delta) R^2_{\text{max}} + \delta R^2_{\text{min}}$$

Define an angle $\lambda \in [0, \pi/2]$ such that

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2Alternatively, restrict $M$ to be of rank $m$ with $0 < m < \text{rank}(C)$. Consider all models where the $F$-statistic for testing the set of linear constraints $M(C\beta-c)$ is less or equal to $F_{\delta,m} = \delta F_{m,\text{max}} + (1-\delta) F_{m,\text{min}}$, where $F_{m,\text{max}}$ is the maximum and $F_{m,\text{min}}$ is the minimum among the achievable $F$-statistics when there are $m$ restrictions. Again, Theorem 2 gives the correct bounds.

Observe that the $F$-statistic is essentially $R^2$ except for the number of regressors. Furthermore, any $R^2$ can be achieved for any given number $m$, $0 < m < \text{rank}(C)$ of restrictions: for the maximal $R^2$, simply include the regressor that arises as the linear combination of regressors with coefficients taken from the unrestricted regression. For the minimal $R^2$, choose regressors orthogonal to that regressor. Therefore fixing the number $m$ is no restriction.
\[ \sin^2 \lambda = \delta. \]

The upper and lower bounds on \( \beta_0 \) will be given by

\[ b_o - \phi_L \|u\| \|v\| \]

and

\[ b_o + \phi_U \|u\| \|v\| \]

where \( \phi_L \) and \( \phi_U \) are always positive and depend on the chosen \( \delta \). The precise formula is given in:

**Theorem 2**

(i) if \( \lambda \leq \theta \) then \( \phi_U = \sin^2 \theta - \sin^2 (\lambda - \theta) \)

if \( \lambda \geq \theta \) then \( \phi_U = \sin^2 \theta \)

(ii) if \( \lambda \leq \pi/2 - \theta \) then \( \phi_L = \cos^2 \theta - \cos^2 (\lambda + \theta) \)

if \( \lambda \geq \pi/2 - \theta \) then \( \phi_L = \cos^2 \theta \).

The proof is in the appendix.

For given \( \delta \), and thus, \( \lambda \), the extreme bounds can be found directly from the
two regressions\(^3\), the basic regression involving just the free variables \(X_F\), which gives \(\hat{b}_o\) as the estimate of \(\beta_o\) and also provides \(R^2_{\min}\), and the complete regression in which all variables enter, which gives \(b\) as the estimate of \(\beta\), and \(b_o\) as the estimate of \(\beta_o\) and \(R^2_{\max}\). It follows that

\[
u'v = b_o - \hat{b}_o
\]

\[
\|v\| = (Cb - c)'A^{-1}(b - c)
\]

and

\[
\|u\| = \text{var}(b_o) - \text{var}(\hat{b}_o)
\]

From these quantities \(\theta\) is determined as \(\cos 2\theta = \cos (u, v)\). It may be suggested that it is good economic practice to report the values of \(u'v\), \(\|v\|\), \(\|u\|\) as well as the extreme bounds on \(\beta_o\) for various values of \(\delta\), say 0.1 and 0.05. Table 1, at the end of the paper makes this latter task easy by tabulating the values of \(\phi_L\) and \(\phi_U\) for various values of \(\cos 2\theta\).

3. The effect of interest rates and inflation on the velocity of money.

We want to analyze the effect of interest rates and inflation on the increase in the velocity of money. To this end, we consider two models in quarterly data and homoskedastic errors with unknown variance.

\(^3\)Observe that the bounds do not depend on \(\sigma^2\). Thus one might therefore set \(\sigma^2=1\) to simplify the calculation.
We used the following list of variables:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>veloc</td>
<td>velocity of money, computed as GNP/M1,</td>
</tr>
<tr>
<td>dveloc</td>
<td>first differences in the velocity of money</td>
</tr>
<tr>
<td>tbill</td>
<td>3-month treasury bill rate,</td>
</tr>
<tr>
<td>infla</td>
<td>inflation, computed from the consumer price index CPI,</td>
</tr>
<tr>
<td>gnp</td>
<td>gross national product,</td>
</tr>
<tr>
<td>mbvariab</td>
<td>variability of money⁴, computed as standard deviation within one year of the growth rate of the monetary base from its global mean and trend and 12 own lags, using monthly data. We multiply that number by 10000 for numerical reasons.</td>
</tr>
</tbody>
</table>

GNP, CPI, M1 and monetary base are seasonally adjusted. We use first differences in the velocity of money in our regression.⁵ The time index t counts quarters.

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⁴It has been argued that the recent decline in the velocity of money was caused by changes in the variability of money supply (see Friedman (1984) and Hall–Noble (1987)).

⁵It has been argued that the velocity of money follows a random walk, i.e. that there are unit roots in the corresponding regression equation in levels. Then, first differencing the velocity series is a reasonable procedure. But also in the level model the inference drawn from OLS (instead of using unit roots distribution theory) is valid, if the linear combination of regressors that we look at for our focus coefficient doesn’t lie in the eigenspace of the unit root in the joint VAR (see Sims–Stock–Watson). Another way to justify using levels is the Bayesian point of view, in which the posterior distribution of the coefficient given the data is still almost normal even in the presence of unit roots, see Sims (1987). In fact, the results don’t change much using levels instead of first differences except that the range of R² becomes much smaller. We chose a model in first differences because the results are more instructive and not because we believe the level model to yield incorrect results.
Model I:

a) unrestricted: ("equation 1")
\[
\begin{align*}
dveloc(t) &= \alpha + \beta_{0,0}t + \sum_{l=1}^{5} \beta_{1,1}dveloc(t-l) + \sum_{l=0}^{6} \beta_{1,2}tbill(t-l) + \sum_{l=0}^{5} \beta_{1,3}infla(t-l) \\
&\quad + \sum_{l=0}^{6} \beta_{1,4}gnp(t-l) + \sum_{l=0}^{5} \beta_{1,5}mbvariab(t-l) + \epsilon_t
\end{align*}
\]

b) restricted: ("equation 2")
\[
\begin{align*}
dveloc(t) &= \alpha + \beta_{0,0}t + \sum_{l=1}^{2} \beta_{1,1}dveloc(t-l) + \sum_{l=0}^{1} \beta_{1,2}tbill(t-l) + \epsilon_t.
\end{align*}
\]

Model II:

a) unrestricted: ("equation 1"): same as in model I

b) restricted: ("equation 2")
\[
\begin{align*}
dveloc(t) &= \alpha + \beta_{0,0}t + \sum_{l=1}^{2} \beta_{1,1}dveloc(t-l) + \sum_{l=0}^{2} \beta_{1,2}tbill(t-l) + \sum_{l=0}^{2} \beta_{1,3}infla(t-l) \\
&\quad + \sum_{l=0}^{1} \beta_{1,5}mbvariab(t-l) + \epsilon_t
\end{align*}
\]

We used ordinary least squares to estimate the models. The regression output is in appendix 2. Let us highlight a few results (the numbers for $\|u\|, \|v\|$ and $\cos(u,v)$ are calculated for $\sigma^2=1$, see remarks above):
Table 1:

Model I

<table>
<thead>
<tr>
<th>Focus Coeff.</th>
<th>$\delta^2 = 1.0$</th>
<th>$\delta^2 = 0.1$</th>
<th>$\delta^2 = 0.05$</th>
<th>$\delta^2 = 0.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{1,2}$ (tbill — first lag)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>upper</td>
<td>.08110</td>
<td>.07342</td>
<td>.06806</td>
<td>.05154</td>
</tr>
<tr>
<td>lower</td>
<td>-.00540</td>
<td>.02420</td>
<td>.03230</td>
<td>.05154</td>
</tr>
<tr>
<td>$\beta_{0,0}$ (trend)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>upper</td>
<td>.00300</td>
<td>.00161</td>
<td>.00107</td>
<td>-.00034</td>
</tr>
<tr>
<td>lower</td>
<td>-.00342</td>
<td>-.00224</td>
<td>-.00172</td>
<td>-.00034</td>
</tr>
</tbody>
</table>

$\sum_{1 = 0}^{6} \beta_{1,2}$ (effect of permanent increase in interest rates)

<table>
<thead>
<tr>
<th>Focus Coeff.</th>
<th>$\delta^2 = 1.0$</th>
<th>$\delta^2 = 0.1$</th>
<th>$\delta^2 = 0.05$</th>
<th>$\delta^2 = 0.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{1,2}$ (tbill — first lag)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>upper</td>
<td>.07635</td>
<td>.07044</td>
<td>.06591</td>
<td>.05155</td>
</tr>
<tr>
<td>lower</td>
<td>-.00108</td>
<td>.02709</td>
<td>.03441</td>
<td>.05155</td>
</tr>
<tr>
<td>$\beta_{2,2}$ (tbill — second lag)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>upper</td>
<td>.00725</td>
<td>-.02404</td>
<td>-.03436</td>
<td>-.06019</td>
</tr>
<tr>
<td>lower</td>
<td>-.10847</td>
<td>-.09251</td>
<td>-.08411</td>
<td>-.06019</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Focus Coeff.</th>
<th>$\delta^2 = 1.0$</th>
<th>$\delta^2 = 0.1$</th>
<th>$\delta^2 = 0.05$</th>
<th>$\delta^2 = 0.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{1,3}$ (inflation — first lag)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>upper</td>
<td>.01156</td>
<td>.00918</td>
<td>.00795</td>
<td>.00448</td>
</tr>
<tr>
<td>lower</td>
<td>-.00513</td>
<td>-.00072</td>
<td>.00076</td>
<td>.00448</td>
</tr>
</tbody>
</table>

These results contain several interesting aspects:
The range of possible estimates for the trend variable is bigger than the two coefficient estimates in the restricted and the unrestricted model suggest. This coincides with the uncertainty about the trend coefficient as indicated by the t-statistic (see McAleer et al (1985)).

The shape of the range of coefficient estimates for the first lag of tbills changes little in either model. In both cases it has a positive coefficient as long as we stay in the top 20% of $R^2$, say. Similarly, the second lag has a negative coefficient. Notice that the unrestricted EBA doesn't allow here for that conclusion (since coefficients of the opposite sign in these cases are included here when $R^2$ is not restricted). It is of course subject to debate whether it is reasonable to look at a fraction of the range of possible $R^2$. The point is, that we have to make the unrestricted model pretty much as bad as possible within the admissible range to arrive at coefficients of the opposite sign in either case. It is now up to the judgement of the individual researcher to decide whether (s)he wants to rule out these coefficients as unreasonable (because of the "bad" $R^2$) or not. This type of sensitivity analysis, putting the facts on the table, is of course the whole point of this approach.

We also see that not much can be said about the permanent effects of e.g a permanent rise in the nominal interest rate. Different models allow for different conclusions within the top 20%-range of $R^2$ and that might be all that can be said. The same applies to money base variability which we did not find to have a clear effect on velocity. Looking at a plot of real interest rates (computed as tbill(t)–inflation(t), where contemporaneous inflation is used as a crude substitute for
the inflation expectation of agents in our economy), it seems that the recent change in the behavior of the velocity of money coincides with the shift of real interest rates from negative values to positive values.

An analysis as shown here helps to understand better the possibilities for the outcomes from different models and the type of restrictions we impose when passing from a large "benchmark" model to a smaller model.

4. Conclusions:

Our result enables the researcher with a "continuum" of choices between classical econometrics and Leamer's "extreme" EBA: if one only wants the maximal $R^2$, the theorem will give the coefficient of the classical analysis. If one allows for any $R^2$, the theorem gives the extreme bounds as e.g. given by Breusch's theorem (which is contained in our theorem as a special result). It seems reasonable, as explained above to give the extreme bounds of the coefficient of interest subject to restricting $R^2$ to be in the top 5% or top 10% of the range of possible $R^2$: using ordinary regression output, this can be done using the formulas of the theorem or the tables at the end of this paper. Of course, one would like extensions of our result: how can we deal with a vector of coefficients of interest? Is there a similar version of the theorem that controls for the Durbin-Watson statistic? How can we incorporate uncertainty about $\Omega$ in our model? Are there similar results for other inference procedures (e.g. probit models) or in the presence of nonnormal distributions (as e.g. in the unit roots case). How do we include coefficient-uncertainty arising from any of the specific models
included in our range? How can we find bounds if we pose positivity constraints on certain coefficients? Are there interesting asymptotic results?

The theorem above allows us to get a feel of how much actually changes, if we proceed from a general "benchmark" model to a smaller, restricted model. The procedure described above is intended to append current practice in that way, not to replace it and can thus provide useful insights about the data.

Appendix 1:

Proof of the theorem:

Given a restriction matrix $M$ as in (2.4), find the corresponding $W$. Write

$$\beta_0 = b_0 - \varphi(W)||v||^2 \text{ and}$$

$$R^2 = 1 - \frac{e'e + \gamma(W)||v||^2 \sigma^2}{(y - \bar{y})'}(y - \bar{y}),$$

where $e$ is the vector of residuals under the unrestricted model, $\bar{y}$ is the mean of $y$ and $\gamma(W)$ is assumed to be positive.

Assume that $\{u, v\}$ is linear independent and that $C$ is not trivial (i.e. there is at least one nonzero constraint).

Note that (i) and (ii) are equivalent to the following formulation, using the usual addition theorems for sin and cos. It is this version that we are going to prove:

\[\text{We have of course } \bar{y} = 0, \text{ i.e. the model is already in deviation form.}\]
For a chosen \( \delta \in [0,1] \) and matrix \( W \), suppose that \( \gamma(W) \leq \delta \).

(i') If \( \delta \leq (1-\cos(u, v))/2 \),
then \( \varphi(W) \geq \phi_L = \delta^{1/2}\sin(\lambda - 2\theta) \)
else \( \varphi(W) \geq \phi_L = (\cos(u, v)-1)/2 \)

(ii') If \( \delta \leq (1+\cos(u, v))/2 \),
then \( \varphi(W) \leq \phi_U = \delta^{1/2}\sin(\lambda + 2\theta) \)
else \( \varphi(W) \leq \phi_U = (\cos(u, v)+1)/2 \).

These bounds can be achieved for properly chosen matrices \( W_L \) and \( W_U \), i.e. for these matrices we get the upper bound \( \phi_L = \varphi(W_L) \) and the lower bound \( \phi_U = \varphi(W_U) \).

The proof involves two steps. We consider a special class of \( W \)-matrices \( \{W(t) | t \in \mathbb{R}\} \).
First, we show, that any other \( W \) can do only "worse" than some \( W(t) \), i.e. at the same \( \mathbb{R}^2 \), \( \varphi(W) \) will be restricted to the range given by the corresponding \( \varphi(W(t)) \).
Secondly, we show that we get the bounds mentioned in the Theorem within that class of \( W(t) \)'s, proving the theorem.

Consider any (full column rank) matrix \( W \) of size \( k \times m \), say, where \( 1 \leq m \leq k \) and \( k \) is the number of possible restrictions, \( k \geq 1 \) by assumption. We find that

\[
\varphi(W) = u'W(W'W)W'v/(\|u\| \|v\|) \quad \text{and} \quad \gamma(W) = v'W(W'W)W'v/(\|v\| \|v\|),
\]
so assume w.l.o.g. (in order to avoid the norm–signs) that \( \|u\| = \|v\| = 1 \). The special class of \( W \) that we want to consider are all \( W \) of size \( k \times 1 \) which are linear combinations of \( u \) and \( v \).
Claim: For all $W$, there is a linear combination $\zeta$ of $x$ and $y$ and a real number $\eta$ with $0 \leq \eta \leq 1$ such that $\gamma(W) = \gamma(\zeta)$ and $\varphi(W) = \eta \varphi(\zeta)$.

Proof of the Claim:

Recall from linear algebra, that $W(W^TW)W'$ is a matrix that maps any $k$-dimensional vector in an orthogonal way on the range of $W$. Hence $\gamma(W) \leq 1$. Fix a $W$. Let $P$ be the orthogonal projector on the plane spanned by $u$ and $v$. Let $z_W = PW(W^TW)W'v$, we then have $\varphi(W) = u'z_W$ and similarly $\gamma(W) = v'z_W$. Assume $z_W \neq 0$ (otherwise we are done). Let $x$ be the orthogonal projection of $v$ on $z_W$, i.e. $x = \tau z_W$, where $\tau = (z_W'v)/(z_W'z_W)$. By Lemma 1 (below), we have $1 \leq \tau$. Now, find a linear combination $\zeta$ of $u$ and $v$ such that $\zeta'v = z_W'v$ and $\zeta$ is the orthogonal projection of $v$ on $\zeta$ itself, i.e. $\zeta = \zeta(\zeta'\zeta)^{-1}\zeta'v$. By Lemma 2 (below), $\zeta$ exists and satisfies $\|\zeta\| \geq \|z_W\|$. Write $\zeta' = (z_W'v)v + q$, where $q$ is orthogonal to $v$. Notice that $\tilde{\zeta}' = (z_W'v)v - q$ also satisfies $\zeta'v = z_W'v$, $\|\tilde{\zeta}\| \geq \|z_W\|$ and $\tilde{\zeta}$ is the orthogonal projection of $v$ on $\tilde{\zeta}$ itself. Therefore, we can assume w.l.o.g. that $(u'q)(u'z_W) \geq 0$.

Write $z_W = (z_W'v)v + \alpha q$, where $|\alpha| \leq 1$. Now we are done, since $\gamma(\zeta)' = v'\zeta = v'z_W = \gamma(W)$ and (for the case $u'z_W \neq 0$, otherwise the claim is now trivial) $\varphi(\zeta)' = u'\zeta = u'z_W + (1-\alpha)u'q = \varphi(W)/\eta$, where $1/\eta = 1 + (1-\alpha)(u'q)/(u'z_W) \geq 1$. This finishes the first part of the proof.

We now proceed to the second part of the proof.

Consider $W(t) = u + tv$ (recall that we assumed w.l.o.g. $\|u\| = \|v\| = 1$). Observe that we exclude the matrix $W=v$ among the class of all linear combinations of $u$ and $v$. However, since $W(W^TW)^{-1}W'$ is an orthogonal projector on the range of $W$, we can include $W=v$ by considering $t=0$. A look then at the algebraic results below shows
that \( W = v \) is "irrelevant" for the bounds given in the theorem (actually, \( W = v \) gives the worst possible \( R^2 \) among our class).

Let \( f(t) = \varphi(W(t)) \) and \( g^2(t) = \gamma(W(t)) \). Let \( \kappa = \cos(u,v) \). Calculate \( f \) and \( g^2 \) to be
\[
f(t) = (\kappa t + 1)(t + \kappa) / (t^2 + 2\kappa t + 1) \quad \text{and}
\]
\[
g^2(t) = \frac{(t + \kappa)^2}{(t^2 + 2\kappa t + 1)}
\]
so that we might set
\[
g(t) = \frac{(t + \kappa)}{(t^2 + 2\kappa t + 1)^{1/2}}.
\]
Observe that \( g \) vanishes only for \( t = -\kappa \) and that also \( f(-\kappa) = 0 \). Define the ratio \( r(t) \) by \( r(t) = f(t)/g(t) \), whenever \( g(t) \neq 0 \). We find
\[
r(t) = \frac{(\kappa t + 1)}{(t^2 + 2\kappa t + 1)^{1/2}},
\]
which is also defined for \( t = -\kappa \). (For \( W = v \), observe that \( |r(t)| \) converges to \( \kappa \), \( |g(t)| \to 1 \) and \( f(t) \to \kappa \) as \( t \to \infty \) and that \( |g(t)| < 1 \) for all \( t \). Observe that \( r(-\kappa) = (1 - \kappa^2)^{1/2} \). To get to the exact bounds, note that \( f \) has its absolute minimum at \( t = -1 \) and its absolute maximum at \( t = 1 \), \( f \) is a strictly increasing function in between. Using the relationship between \( (t - \kappa)^2 \) and \( g^2(t) \) as stated above, we find at \( g^2(t) = \gamma^2 \):
\[
r(t) = \kappa g(t) + (1 - \kappa^2)^{1/2} (1 - g^2(t))^{1/2}
\]
\[
= \text{sign}(g(t)) \cos(2\theta) \sin(\lambda) + \sin(2\theta) \cos(\lambda)
\]
\[
= \text{sign}(g(t)) \sin(\lambda + \text{sign}(g(t)) 2\theta).
\]
Using the fact that \( g \) is also a strictly increasing function on the whole real line with \( g^2(-1) = -f(-1) = (1 - \kappa)/2 = \sin^2(\theta) \) and \( g^2(1) = f(1) = (1 + \kappa)/2 = \cos^2(\theta) \) we obtain the sharp bounds as stated in (i') and (ii'). This concludes the proof of the Theorem.
In the proof of the Theorem we made use of the following two Lemmata:

**Lemma 1:**
Suppose for $u,v,w,x \in \mathbb{R}^k$, that $v=w+x$, that $w$ and $x$ are orthogonal to each other and that $\{u,v\}$ is linear independent. Project $w$ in the plane spanned by $u$ and $v$, call this projection $z_w$. Then $v'z_w \geq z_w'z_w$.

**Proof:**
Decompose $w$ in $w=\alpha v+p+q$, where $\alpha v+p=z_w$ and $p$ is orthogonal to $v$. Then $y=p+q$ is orthogonal to $v$ and we have

$$z_w'z_w \geq \alpha^2 v'v + y'y = w'w = w'v = z_w'v.$$

**Lemma 2:**
Let $v,x,y,z \in \mathbb{R}^k$, $k \geq 2$ and $1 \leq \tau$ be such that $v=x+y$, $x$ orthogonal to $y$ and $x=\tau z$. Then there exists a linear combination $\zeta$ of $x$ and $v$ such that $\zeta'v=\zeta'x$ and $v-\zeta$ orthogonal to $\zeta$ and we have $\|z\| \leq \|\zeta\|$.

**Proof:**
W.l.o.g. $\|v\|=1$ (rescale every vector). Assume w.l.o.g. that $\{x,v\}$ is linear independent, otherwise the claim is. Let $q$ be orthogonal to $v$ in the plane spanned by $x$ and $v$ and of unit length. Set $\zeta = (z'v)v + \alpha q$, where $\alpha$ is a solution to the equation $\alpha^2 = (z'v)(1-z'v)$. Check that $\zeta'v = \zeta'x = z'v$. Observe that with a similar argument, we can write $x$ as $x = (x'v)v + aq$, where $a$ now satisfies $a^2 = (x'v)(1-x'v)$. Hence $|a| \leq \tau |\alpha|$, proving $\|z\| \leq \|\zeta\|$.
Appendix IIa)

-------------------

Rats subroutine programs for performing the calculation of restricted extreme bounds.

*  *
* ********************************************
*  *
* RATS-procedures for the calculation of restricted extreme  *
* bounds acc. to Granger-Uhlig (1988)
*  *
* ********************************************
*  *
* Usage:  *
* increase BMA COMPILE 500  *
* Define two equations:  *
* Equation 1 - the "unrestricted regression". It is assumed that  *
* a constant is included in the regression and that the sum of  *
* the coefficients 1 to up1 (1 and up1 are options for  *
* EBASET) is the focus coefficient.
*  *
* Equation 2 - the "restricted regression". Again, a constant  *
* has to be included and the sum of the coefficients lo2 to  *
* up2 (lo2 and up2 are options for EBASET) is the focus  *
* coefficient.
*  *
* Equation 2 should define the regression equation for a  *
* subset of the regressors of equation 1 and the sum of the  *
* first coefficients lo2 to up2 should correspond to the sum  *
* of the coefficients 1 to up1 of equation 1 with  *
* (up2-lo2+1)-(up1-lo1+1) zero restrictions.
*  *
* Per default, lo1=up1=lo2=up2=1, i.e. the first coefficient is  *
* the focus coefficient.
*  *
* First execute the procedure EBASESTART to do the initial  *
* calculation of the regressions etc. Then decide on your focus  *
* coefficient and set the necessary data for the calculations  *
* of the bounds with the procedure EBASET. Then, with each run  *
* of EBACKCALC, the bounds are calculated according to the given  *
* restriction on Rsquared. EBASET and EBACKCALC can be executed  *
* several times after an initial EBASESTART.
*  *
* Reserved names used in these procedures are:
*  *
* Reserved names used in these procedures are:
*  *
* zzzru zzzrr zzzbu zzzbr zzzvare zzzvaren zzzulen zzzulen
* zzzuv zzzco_uv zzzg 101 up1 102 up2 zzzxzx zzzxux zzzbetau
* zzzbetar zzznobs zzzyvar
*  *
* Output includes the lengths of the vectors u and v and their  *
* inner product and covariances, assuming sigma=1.0 (recall that*  *
* the calculated bounds do not depend on sigma).
*  *
* Example: suppose, the effect of x1 on y is to be determined  *
*  *
* Other possible regressors are const, trend, x2,..,x5,*  *
* and we always want to include x2. Define  *
*  *
* SMPL firstper lastper
*  *** allow for all lags in equation 1 !
*  EQUATION 1 y
*  # x1 const trend x2 x3 x4 x5
*  *
* EQUATION 2 y
* # x1 const x2
* EXEC EBASTART y
* EXEC EBASET(lo1=1,up1=1,lo2=1,up2=1)
* (or simply EXEC EBASET)
* EVAL g2 = 0.1
* EXEC EBACALC lower upper g2
* EVAL g2 = 0.05
* EXEC EBACALC lower upper g2
*
******************************************************************************
* PROCEDURE EBASTART y
******************************************************************************
* Instructions: y is the dependent variable
******************************************************************************
TYPE EQV y
    DECLARE SYMM zzzxxu zzzxxr
    DECLARE VECTOR zzzbetau zzzbetar
    OLS(EQUATION=1)
    FETCH zzzru = RSQUARED
    MATRIX zzzxxu = XX
    MATRIX zzzbetau = BETA
    OLS(EQUATION=2)
    FETCH zzzrr = RSQUARED
    MATRIX zzzxxr = XX
    MATRIX zzzbetar = BETA
    STATISTIC y
    EVAL zzznobs = NOBS
    EVAL zzzyvar = VAR
END
*
*
PROCEDURE EBASET
OPTION LO1 INTEGER 1
OPTION UP1 INTEGER 1
OPTION LO2 INTEGER 1
OPTION UP2 INTEGER 1
******************************************************************************
* lo1, up1, lo2 and up2 are options.
* the sum of the coefficients lo1 to up1 is the focus
* coefficient in equation 1, the unrestricted equation.
* the sum of the coefficients lo2 to up2 is the focus
* coefficient in equation 2, the unrestricted equation.
******************************************************************************
* LOCAL INTEGER counta countb
*
    EVAL zzzbu = 0.0
    EVAL zzzvaru = 0.0
    DO counta = lo1,up1
        EVAL zzzbu = zzzbu + zzzbetau(counta)
        DO countb = lo1,up1
            EVAL zzzvaru = zzzvaru + zzzxxu(counta,countb)
        END
    END
    EVAL zzzbr = 0.0
    EVAL zzzvarr = 0.0
    DO counta = lo2,up2
        EVAL zzzbr = zzzbr + zzzbetar(counta)
DO countb = lo2,up2
   EVAL zzzvvar = zzzvvar + zzzxxr(counta,countb)
END

END

EVAL zzzvlen = SQRT( (zzznobs-1.0)*zzzyvar*(zzzru-zzzrr) )
EVAL zzzvul = SQRT( zzzvvar - zzzvvar )
EVAL zzzuv = zzzbu - zzzbr
EVAL zzzco_uv = zzzuv / (zzzulen*zzzvlen)
DISPLAY

DISPLAY 'REstricted EXTREME BOUNDS. DATA:'
DISPLAY 'FOCUS ON SUM OF ' lo1 ' TO ' up1 ' IN EQUATION 1'
DISPLAY 'FOCUS ON SUM OF ' lo2 ' TO ' up2 ' IN EQUATION 2'
DISPLAY 'LENGTH OF U-VECTOR : ' zzzulen
DISPLAY 'LENGTH OF V-VECTOR : ' zzzvlen
DISPLAY 'INNER PRODUCT : ' zzzuv  COS(U,V) = ' zzzco_uv
DISPLAY

DISPLAY

END

*
*

PROCEDURE EBACALC lower upper g2
TYPE REAL lower upper g2
************************************************************************************************************
* Instructions: lower is the variable that is going to be set  *
* to the lower extreme bound. upper is the variable that is  *
* going to be set to the upper extreme bound. g2 is the para- *
* meter gamma-squared, determining the fraction of the range of* *
* admissible R-squares, 0.0<=g2<=1.0  *
************************************************************************************************************
EVAL zzzg = SQRT(g2)
IF g2.LE.((1.0-zzzco_uv)/2.0)
   EVAL upper = zzzbu - zzzg*( zzzco_uv*zzzg -
      SQRT(1-zzzco_uv**2)*SQRT(1-g2) )*zzzulen*zzzvlen
ELSE
   EVAL upper = zzzbu - ((zzzco_uv - 1.0)/2.0)*zzzulen*zzzvlen
* IF g2.LE.((1.0+zzzco_uv)/2.0)
   EVAL lower = zzzbu - zzzg*( zzzco_uv*zzzg +
      SQRT(1-zzzco_uv**2)*SQRT(1-g2) )*zzzulen*zzzvlen
ELSE
   EVAL lower = zzzbu - ((zzzco_uv + 1.0)/2.0)*zzzulen*zzzvlen
END

*************************************************************************************************************
* End of EBA-Subroutines *
*************************************************************************************************************
*
*
Appendix IIb

The following two tables list the lower and upper bounds as given in the theorem for different values of delta.

### LOWER BOUND

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Appendix III

Application to the analysis of the velocity of money. (See main text for the description of the model.)

The following plots of restricted extreme bounds show the range of coefficient estimates as a function of the fraction delta of the possible range of correlation coefficients.

The graphs contain lines at the 5%, 10% and 20% fraction of that range as well as a horizontal line for the coefficient estimates from the restricted model.
Model I and II - velocity in first differences

EQUATION 1
DEPENDENT VARIABLE 13 DVELOC
FROM 49:0 UNTIL 87:4
OBSERVATIONS 154 DEGREES OF FREEDOM 113
R**2 .79616921 RBAR**2 .72401672
SSR .14906528 SEE .36320267E-01
DURBIN-WATSON 1.99182778
Q(36) = 32.4226 SIGNIFICANCE LEVEL .639492

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Model I - velocity in first differences

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OBSERVATIONS 154  DEGREES OF FREEDOM  147
R**2  .43596533  RBAR**2  .41294350
SSR  .41248914  SEE  .52972149E-01
DURBIN-WATSON 1.79940473  SIGNIFICANCE LEVEL  .743578
Q( 36)= 30.1241

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Model II - velocity in first differences

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OBSERVATIONS 154  DEGREES OF FREEDOM  142
R**2  .47591604  RBAR**2  .43531799
SSR  .38327243  SEE  .51952879E-01
DURBIN-WATSON 1.78195047  SIGNIFICANCE LEVEL  .300441
Q( 36)= 39.9104

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TREND
model II

INFLATION RATE (FIRST LAG)

TBILL (FIRST LAG)

TBILL (SUM OF COEFF.)

TBILL (SECOND LAG)
VELOCITY AND REAL TBILL

[Graph showing trends in velocity and real TBILL from 1948 to 1987]
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