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The Replacement Problem

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ABSTRACT

We construct a vintage capital model of economic growth in which the decision to replace old technologies with new ones is modeled explicitly. Depreciation in this environment is an economic, not a physical concept. We describe the balanced growth paths and the transitional dynamics of this economy. We illustrate the importance of vintage capital by analyzing the response of the economy to fiscal policies designed to stimulate investment in new technologies.

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1. Introduction

The neoclassical model of capital accumulation views changes in the capital stock as taking place smoothly over time. Microeconomic data, however, show that this view of investment is not the right one; investment at the plant level occurs infrequently and in bursts. A recent study by Doms and Dunne (1993) confirms this picture: their study of 12,000 plants over a 15 year period shows that, on average, 25% of a plant's investment was concentrated in a single year and more than 50% was concentrated over 3 contiguous years.[1] Moreover studies by Dunne, Roberts and Samuelson (1989) show that employment flows are driven primarily by the entry of new plants and the gradual contraction of older plants. Dunne (1994) finds that firms using the newest technology have more employees. In this paper we construct a vintage capital model which is consistent with these observations. In this model changes in the capital stock are lumpy, occuring all at once as old technologies are replaced by new ones. Our objective is to study the decision to replace old vintages of capital with new ones and explore how economic growth is tied to this decision. We also study the dynamic adjustment of an economy characterised by this more realistic view of the investment process.

It is now well documented [by Gordon (1990) and others] that the relative price of capital has declined fairly steadily and rapidly in the postwar U.S. economy. Further, the ratio of equipment to output has increased steadily. These two observations suggest that a large part of technological change has been specific to the investment goods sector. That is, over time, new capital goods must be ever more efficient in terms of forgone consumption.[2] There is microeconomic evidence that this investment-specific technological progress may be important for growth. Bahk and Gort (1993), using a cross section of more than 2000 firms from 41 industries, find that a one year change in the average age of capital is associated with a 2.5 to 3.5 % change in output. These observations suggest that a successful model of vintage capital should treat the investment and consumption goods sectors separately, and should link the process of growth with investment in new technologies.

To examine these issues we develop a vintage capital model of economic growth.

[1]Cooper, Haltiwanger and Power (1994) estimate hazard functions on the same data used by Doms and Dunne and find the same lumpy investment behavior.

[2]Greenwood, Hercowitz and Krusell (1994) construct a two-sector model of investment-specific technological change where the relative price of capital declines and the equipment-to-output ratio rises. They use this as the basis of their argument that as much as 60% of post-war U.S. growth can be accounted for by investment-specific technological progress.

Technological change is embodied in new capital goods. Firms in this economy must decide when to replace existing capital with new vintages. Here investment is a lumpy decision and depreciation is an economic concept, not a physical one. The firms produce consumption goods and new capital by using capital and two kinds of labor, designated as skilled and unskilled. A distinguishing feature of this environment is that growth results from the ability to produce ever more efficient capital goods. This occurs because the skilled agents in the economy make continuing investments in human capital.[3] In this setting, the age distribution of the capital stock, economic growth, and the distribution of income between skilled and unskilled workers are endogenously determined. Also, the relative price of capital declines, and the capital-to-income ratio increases, over time. In addition, the economy has a government which taxes factor incomes, offers tax credits for new investment and rebates it's net revenues to households.

The classic vintage capital models where technological change is embodied in new capital goods were developed by Robert Solow. In Solow (1960) new capital goods incorporate the latest technology. Capital can be combined with a variable amount of labor and depreciates at a geometric rate. At any point in time plants with new and old capital coexist, but Solow illustrated how this world with heterogeneity could be represented in terms of the standard growth model with a single aggregate stock of captal. In Solow (1962) capital has a fixed lifetime and the amount of labor allocated to a given unit of capital is fixed at the time it is introduced (the technology is "putty-clay"). The current analysis is different from previous vintage capital models in several important respects. First, the decision to replace old capital with new more efficient capital is modeled explicitly. In contrast, the typical vintage capital model treats depreciation as exogenous. Old capital never becomes obsolete, it either vanishes gradually due to the assumed fixed rate of capital consumption or it dies suddenly because of a fixed lifetime. In the environment described here capital only disappears because of replacement; depreciation is an economic, not a physical, concept. Second, consistent with observations at the microeconomic level, labor is allocated efficiently across vintages so that older technologies have less labor assigned to them. Finally, growth is modeled endogenously here rather than being treated as exogenous, as in the Solow models.

The model developed here bears some resemblance to the vintage human cap-

[3]In related work, Krusell (1991) develops a model where capital goods can be produced more efficiently over time. There, monopolisitically competitive firms can invest in R&D that will allow them to produce capital goods at a lower cost. This leads to sustained growth.

ital models of Chari and Hopenhayn (1991) and Parente (1991). Both of these models result in equilibrium distributions of knowledge or skills across agents that are similar to equilibrium age distribution of capital over plants produced by the current model. Additionally, in Parente (1992) an agent must decide to when to acquire new skills. This decision is related to adoption-replacement problem studied here. The current work here is also related to Campbell's (1994) model of the relationship between the adoption of new technologies, and the exit and entry decisions of plants. The exit decision of a plant has aspects that are similar to adoption-replacement choice modeled in the current analysis.

Clearly, the incentives to develop (through R&D) and to adopt (through replacement) more efficient capital goods will be integrally connected. Therefore, it seems worth exploring how an economy's long-run growth may be affected by the adoption-replacement decision. In fact, the notion that new technology is embodied in new investment and that the adoption of new technologies is an important factor in economic growth has been enshrined in U.S. fiscal policy since the early part of this century. With the exception of a few short term reversals, the tax treatment of capital, particularly policies toward depreciation, have been marked by increasingly generous tax treatment of the useful life of capital. The average age of the capital stock has declined for most of the postwar period. Figure 1 shows the average age of the capital stock from 1929 to the present.[4] This dramatic decline in the average age of the aggregate capital stock over the post-war period is consistent with this trend toward leniency in the tax treatment of capital income. In the later part of the paper we illustrate how our model economy responds to changes in the tax treatment of capital.

The remainder of the paper proceeds as follows: In Section 2 the economic environment is outlined. Here the adoption-replacement decision of firms is explicitly modeled. Economic obsolescence is the only reason for replacing old capital. The economy's balanced growth path and the calibration of the model are described in Sections 3 and 4. In Sections 5 and 6 some operating characteristics of the model under study are developed. This is done by analyzing the impact of various tax policies that are designed to stimulate new investment. The final section concludes.

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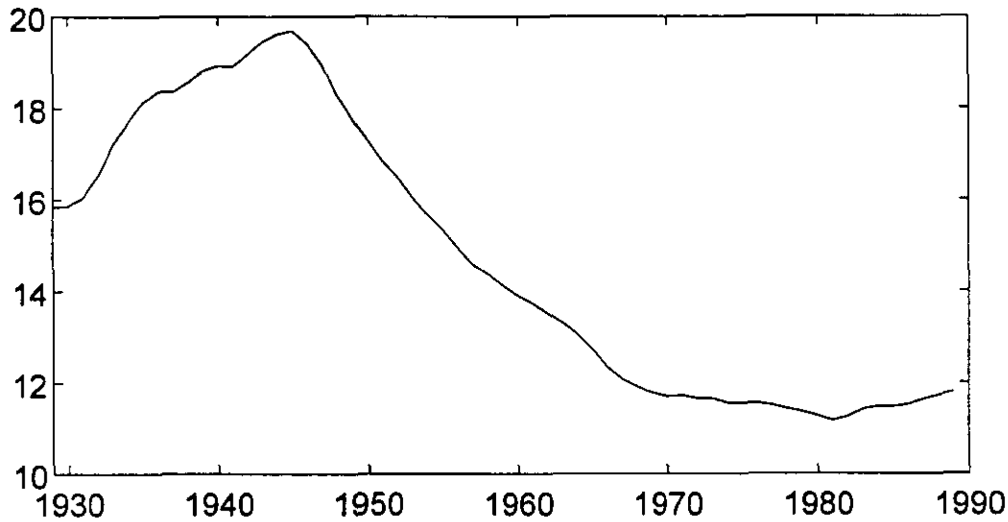
[4]The data are taken from Table A7 in *Fixed Reproducable Tangible Wealth in the United States, 1925-1990*, a publication of the U.S. Department of Commerce.

Figure 1 - Average Age of Capital -U.S.



2. The Economic Environment

Imagine an economy inhabited by two types of households, a firm, and a government. The firm produces consumption and investment goods using factor inputs, namely, capital and two types of labor. Households earn income by supplying labor to the firm, by lending funds to finance the firm's acquisition of capital, and from their ownership claim to the stream of profits on the firm's activity. There is a government in the economy that taxes both labor income, and interest and profits. This revenue is used to give households transfer payments and to provide firms with investment subsidies and capital consumption allowances.

2.1. The Firm

The firm in the economy owns a variety of plants, which are distributed uniformly over the unit interval. In any given period, a plant can produce one of two types of goods: consumption goods and capital goods. Production of these two goods requires the input of capital and labor. Each plant has associated with it a capital stock of a certain age or vintage. At each point in time, the operator of the firm must decide whether to replace the existing capital stock in each plant with the latest vintage. Since capital has a maximum life of N years, replacement is inevitable. The question to be addressed here is when? Let p_i represent the fraction of plants that are currently using capital of age i; clearly, then $\sum_{i=1}^{N} p_i = 1$.

Consider a representative plant of vintage i — i.e., a plant using capital of age i. Let this plant have k_i *efficiency* units of vintage-i capital at its disposal. This plant can be used to produce either consumption or capital goods. Consumption goods can be produced according to the technology

$$c_i = k_i^{\alpha} l_i^{\beta},\ 0 \leq \alpha, \beta, \alpha + \beta \leq 1, \tag{2.1}$$

where c_i is the output of consumption goods and k_i, l_i represent the inputs of capital and unskilled labor.[5]

Growth in the economy results from the ability to produce ever more efficient capital goods over time. The development of new capital goods requires the use of skilled, in addition to unskilled, labor . New capital goods are produced according to the technology

$$x_i = k_i^{\alpha} b_i^{\xi} (\eta h_i)^{\zeta},\ 0 \leq \alpha, \xi, \zeta, \alpha + \xi + \zeta \leq 1. \tag{2.2}$$

[5]One could easily allow a plant's total factor productivity to vary with the age of the capital stock to capture learning-by-doing and/or depreciation effects, as in Klenow (1993).

Here x_i represents the amount of new capital goods produced by plant i using k_i units of capital, while b_i and ηh_i denote the quantities used of unskilled and skilled labor.

At any point in time the firm maximizes the present-value of profits. Now, suppose that plant i produces consumption goods in the current period. It should hire unskilled labor to maximize plant profits, π_i. Specifically, it should solve the problem

$$P(k_i, w) \equiv \max_{l_i} \ \pi_i = k_i^{\alpha} l_i^{\beta} - w l_i, \tag{P(1)}$$

where w is the wage rate for unskilled labor. The first-order condition associated with the problem is:

$$\beta k_i^{\alpha} l_i^{\beta-1} = w. \tag{2.3}$$

By making use of (2.3) in P(1) it is straightforward to deduce that the profits accruing from this location can be expressed as $\pi_i = (1-\beta)k_i^{\alpha} l_i^{\beta}$. Alternatively, the plant could be assigned to the production of capital goods. Let q represent the price of capital goods in terms of consumption goods. Now, the maximization problem for the plant would be

$$\max_{b_i, \eta h_i} \ \pi_i = q k_i^{\alpha} b_i^{\xi} (\eta h_i)^{\zeta} - w b_i - v \eta h_i, \tag{P(2)}$$

where v is the wage rate for skilled labor. The first-order conditions tied to this problem are:

$$q \xi k_i^{\alpha} b_i^{\xi-1} (\eta h_i)^{\zeta} = w, \tag{2.4}$$

and

$$q \zeta k_i^{\alpha} b_i^{\xi} (\eta h_i)^{\zeta-1} = v. \tag{2.5}$$

The profits derived from this activity are $\pi_i = (1-\xi-\zeta) q k_i^{\alpha} b_i^{\xi} (\eta h_i)^{\zeta}$.

Observe that no plant has a *comparative* advantage in producing one type of good over the other.[6] Since each plant is free to choose which production activity to engage in, it must be true that there is indifference between these choices.

$$(1-\beta)k_i^{\alpha} l_i^{\beta} = (1-\xi-\zeta) q k_i^{\alpha} b_i^{\xi} (\eta h_i)^{\zeta}. \tag{2.6}$$

Without loss of generality, assume that the fraction f of each type of plants will produce consumption goods in the current period.

[6] If a plant of type i decided to produce consumption goods its profits would be $(1-\beta)[\beta^{\beta} k_i^{\alpha} w^{-\beta}]^{1/(1-\beta)}$. Alternatively, if it produced investment goods it would earn $(1-\xi-\zeta)[\xi^{-\xi}\zeta^{-\zeta} q k_i^{\alpha} w^{-\xi} v^{-\zeta}]^{1/(1-\xi-\zeta)}$. Provided that labor's share of income is the same across the two activities, or $\beta = \xi + \zeta$, the ratio of profits is the same for all i.

The manager of the firm must decide how many plants of each vintage to operate and how much new capital to place into the plants that are being modernized. New capital formation is subsidized by the government at rate τ_x. There is also a capital consumption allowance in place. In particular, the owners of the firm can write-off from their taxes, in equal installments over a Δ-period time horizon, any investment spending (net of the investment subsidy) that is undertaken. The manager undertakes these decisions in line with the dynamic programming problem shown below where:[7]

$$\begin{aligned} V(p_1, ..., k_1, ...; s) &= \max_{\{p'_i\}_{i=1}^{N}, k'_1} \{\textstyle\sum_{i=1}^{N-1} p_i(1-\tau_k)P(k_i, w) \\ &\quad -(1-\tau_x)(1-d)qp'_1k'_1 \qquad P(3) \\ &\quad + \tfrac{V(p'_1,...,k'_1,...;s')}{[1+(1-\tau_k)r']}\}, \end{aligned}$$

subject to

$$\sum_{i=1}^{N} p'_i \leq 1, \qquad (2.7)$$

$$p'_{i+1} \leq p_i, \qquad (2.8)$$

$$k'_{i+1} = k_i. \qquad (2.9)$$

In the above s denotes the aggregate state-of-the-world in the current period (a precise definition for s is given later) while r' represents the interest rate between today and tomorrow. The variable d proxies for the present-value of the capital consumption allowance on a unit of investment spending; its value in period t reads $d_t \equiv (\tau_k/\Delta)\{1 + \sum_{j=1}^{\Delta-1} 1/(\prod_{m=1}^{j}[1 + (1-\tau_k)r_{t+m}])\}$.[8] The first constraint (2.7) limits the number of plants that can be operated next period. Next, the number of plants using capital of age-$i+1$ next period must be no bigger than the number using age-i this period. This is what (2.8) states. Similarly, capital that is i periods old today will be $i+1$ periods old tomorrow, cf. (2.9).

[7]The manager of the firm maximizes it's present-value from the owner's perspective. This implies that after-tax profits should be discounted using the after-tax interest rate.

[8]Time subscripts are added in standard fashion, as needed. Thus, for instance, the amount of capital in an age-j plant in period t would be denoted by $k_{j,t}$. In the formulae for d_t, r_{t+m} denotes the interest rate bridging periods $t+m-1$ and $t+m$.

The upshot of this dynamic programming problem is the following set of efficiency conditions:

$$(1-\tau_x)(1-d)qk_1' - \frac{[V_1(\cdot') - V_i(\cdot')]}{[1+(1-\tau_k)r']} \begin{cases} \leq 0, & \text{if } p_i' = 0, \\ = 0, & \text{if } 0 < p_i' < p_{i-1}, \\ \geq 0, & \text{if } p_i' = p_{i-1}, \end{cases} \tag{2.10}$$

for $i = 2, ..., N$, with

$$\begin{aligned} V_i(\cdot') &= (1-\tau_k)P(k_i', w') + \\ & \max\{-(1-\tau_x)(1-d')q'k_1'' + \tfrac{V_1(\cdot'')}{[1+(1-\tau_k)r'']}, \tfrac{V_{i+1}(\cdot'')}{[1+(1-\tau_k)r'']}\}, \end{aligned} \tag{2.11}$$

and

$$(1-\tau_x)(1-d)p_1'q = V_{N+1}(\cdot')/[1+(1-\tau_k)r'], \tag{2.12}$$

with

$$V_{N+i}(\cdot') = (1-\tau_k)p_i'P_1(k_i', w') + V_{N+i+1}(\cdot'')/[1+(1-\tau_k)r''] \tag{2.13}$$

Equation (2.10) determines how many plants of vintage i should be operated in next period.[9] Suppose that the firm decides to replace the age i capital in a plant in period t+1 with new capital. There are two costs associated with doing this. First, is the direct cost, $(1-\tau_x)(1-d)qk_1'$, of buying the new capital. Second, is the opportunity cost associated with junking the old capital, $V_i(\cdot')/[1+(1-\tau_k)r']$. From equation (2.11) this can be seen to equal the after-tax present-value of the profits over the life of the plant that would obtain if this replacement decision is delayed a period. The benefit of replacing the age i capital is $V_1(\cdot')/[1+(1-\tau_k)r']$, or the after-tax present value of profits that would be derived from new capital. Equation (2.10) states that (a) if all vintage i plants are to be upgraded then these benefits exceed the costs, (b) if only some are renovated there must be indifference between these options at the margin, and (c) if none of this plants are to be refitted then the cost must exceed the benefits. Equation (2.12) determines the amount of new capital that will be placed in each plant that is getting modernized. The cost of an extra unit of capital is q while its benefit is $V_{N+1}(\cdot')/[1+(1-\tau_k)r']$,

[9] The notation $V_i(\cdot')$ is used to signify that the function V_i is being evaluated at next period's values for its arguments .

which from (2.13) is the present value of the marginal product of capital over its economic life.[10]

It is interesting to note that the firm's replacement decision is driven by the lure of earning increased rents at plants. In the absence of rents from modernization, the firm will never update the stock of capital in a plant before it is N years old. This is easy to see from equation (2.10). Consider a plant with capital of age $i < N$. Now, suppose there are no rents from modernization in the sense that the after-tax profits derived from updating a plant, $V_1(\cdot')$, exactly equal the direct renovation costs, $(1-\tau_x)(1-d)qk_1'$. The plant will not be updated, since the firm looses the forgone rents derived from the age-i capital, $V_i(\cdot')$, which exceed the (zero) net profits that will be realized from the new capital. This is always the case if the production technologies exhibits constant-returns-to-scale.

Lemma 1. *If $\alpha+\beta=\alpha+\xi+\zeta=1$ then $V_1(\cdot')=(1-\tau_x)(1-d)qk_1'$.*

Proof. The proof is by induction. Consider the T-period horizon version of problem P(3). Let $V^{T+1-t}(\cdot_t)$ represent the firm's value function for period t.[11] Clearly, $V^0(\cdot_{T+1}) = V_1^0(\cdot_{T+1}) = 0$. Now, suppose that $V_1^{T-t-j}(\cdot_{t+j+1}) - (1-\tau_x)(1-d_t)q_tk_{1,t+j+1} \leq 0$ for all $j \geq 1$. From the T-horizon analog to equation (2.10) this implies that $p_{j+1,t+j+1} = p_{1,t+1}$ for all $j \geq 1$. Also, the analog to equation (2.11) would give

$$V_1^{T-t}(\cdot_{t+1}) = (1-\tau_k)[P(\cdot_{t+1}) + \sum_{j=1}^{N-1} \frac{P(\cdot_{t+1+j})}{\Pi_{m=1}^{j}[1+(1-\tau_k)r_{t+1+m}]}]. \tag{2.14}$$

[10]Solving (2.13) forward yields $V_{N+1}(\cdot_{t+1}) = (1-\tau_k)\{p_{1,t+1}P_1(k_{1,t+1},w_{t+1}) + \sum_{j=1}^{N-1} p_{j+1,t+j+1}P_1(k_{j+1,t+j+1},w_{t+j+1})/[\Pi_{m=1}^{j}(1+(1-\tau_k)r_{t+1+m})]\}$. The notation $V_{N+1}(\cdot_{t+1})$ is used to signify that the function V_{N+1} is being evaluated at arguments as of date $t+1$.

[11]The period-t dynamic programming problem is

$$\begin{aligned} V^{T+1-t}(\cdot_t) = \max_{\{p_{i,t+1}\}_{i=1}^{N}, k_{1,t+1}} & \{\textstyle\sum_{i=1}^{N-1} p_{i,t}(1-\tau_k)P(\cdot_t) \\ & -(1-\tau_x)(1-d_t)q_tp_{1,t+1}k_{1,t+1} + \frac{V^{T-t}(\cdot_{t+1})}{[1+(1-\tau_k)r_{t+1}]}\}, \end{aligned}$$

subject to (2.7), (2.8) and (2.9).

Next, substitute (2.13) into (2.12) and then multiply both sides of resulting expression by $k_{1,t+1}$ to get

$$
\begin{aligned}
(1-\tau_x)(1-d_t)p_{1,t+1}q_t k_{1,t+1} &= (1-\tau_k)\{p_{1,t+1}P_1(\cdot_{t+1+j})k_{1,t+1}+ \\
&\quad \textstyle\sum_{j=1}^{N-1} p_{j+1,t+j+1}\frac{P_1(\cdot_{t+1+j})k_{1,t+1}}{\prod_{m=1}^{j}[1+(1-\tau_k)r_{t+1+m}]}\} \\
&= (1-\tau_k)\{p_{1,t+1}P(\cdot_{t+1+j})+ \\
&\quad \textstyle\sum_{j=1}^{N-1} p_{1,t+1}\frac{P(\cdot_{t+1+j})}{\prod_{m=1}^{j}[1+(1-\tau_k)r_{t+1+m}]}\},
\end{aligned}
\tag{2.15}
$$

where use has been made of the fact that $P_1(\cdot_{t+1+j})k_{1,t+1} = P_1(\cdot_{t+1+j})k_{j+1,t+j+1} = P(\cdot_{t+1+j})$. Substituting (2.14) into (2.15) then yields $V_1^{T-t}(\cdot_{t+1}) = (1-\tau_x)(1-d_t)q_t k_{1,t+1}$. The desired result obtains by letting $T \to \infty$. □

2.2. Households

There are two types of household in the economy, described as skilled and unskilled. There are M times more unskilled workers than skilled ones. Each period unskilled workers decide how much to consume, c, work, l, and save in the form of one period bonds, a'. These agents derive income from working, wl, saving (interest income, ra) and from government lump-sum transfer payments, τ. Labor and interest income are taxed at the rates, τ_l and τ_k. The dynamic programming problem for unskilled agents is:

$$
J(a;s) = \max_{c,l,a'} \{U(c,l;\lambda) + \rho J(a';s')\} \tag{P(4)}
$$

subject to

$$
c + a' = (1-\tau_l)wl + [1+(1-\tau_k)r]a + \tau. \tag{2.16}
$$

The momentary utility function $U(\cdot)$ is given by

$$
U(c,l;\lambda) = \ln(c - \lambda\Theta\frac{l^{1+\theta}}{1+\theta}),\ \theta,\Theta > 1, \tag{2.17}
$$

where the term λ represents the state of technological advance in the household sector. This form for the utility function can be justified by appealing to household production theory — see Greenwood, Rogerson and Wright (1994).[12] Its adoption

[12] It has also been successfully used in applied work; an example is Hercowitz and Sampson (1991).

simplifies the analysis since the economy's general equilibrium will not be affected by the distribution of income across skilled and unskilled workers. The first-order conditions associated with the unskilled household's problem are:

$$U_1(c,l;\lambda) = \rho[1+(1-\tau_k)r']U_1(c',l';\lambda'), \tag{2.18}$$

and

$$U_1(c,l;\lambda)(1-\tau_l)w = -U_2(c,l;\lambda). \tag{2.19}$$

Skilled agents in this economy own the firm. This means that decisions concerning R&D (human capital investment) and replacement of old technologies are made by the owner/operators of the firm. Assume that the firm's current indebtedness is b. The firm will then owe rb in interest. The firm's period-t profits after paying off this interest will be $\sum_{i=0}^{N} p_i P_i(k_i,w) - rb$. Additionally, recall that the firm is intending to spend $(1-\tau_x)qp_1'k_1'$ on new capital. This can be financed by issuing new debt, b'. Like unskilled agents, skilled agents must decide for each period t how much to consume, z, and how much to work, h. They must further allocate their effort, however, across two activities: working in plants, $h-e$, and human capital formation, e. The skilled agent's dynamic programming problem is:[13]

$$X(b,\eta;s) = \max_{z,h,e,b'} \{W(z,h;\lambda) + \rho X(b',\eta';s')\} \qquad P(5)$$

subject to the flow budget constraint,

$$\begin{aligned} z \;=\; & (1-\tau_l)v\eta(h-e) + (1-\tau_k)[\textstyle\sum_{i=1}^{N} p_i P_i(k_i,w) - rb] \\ & -(1-\tau_x)qp_1'k_1' + (\tau_k/\Delta)\{(1-\tau_x)[p_1'k_1' + \textstyle\sum_{i=0}^{\Delta-2} q_{-i}p_{1,-i}k_{1,-i}]\} \\ & +b' - b + \tau, \end{aligned} \tag{2.20}$$

and the law of motion for human capital,

$$\eta' = H(e)\eta. \tag{2.21}$$

The efficiency conditions associated with this problem are:

$$W_1(z,h;\lambda) = \rho[1+(1-\tau_k)r']W_1(z',h';\lambda'), \tag{2.22}$$

$$W_1(z,h;\lambda)(1-\tau_l)\eta v = -W_2(z,h;\lambda), \tag{2.23}$$

[13] Let x_{-i} denote the value that x had i periods ago.

and

$$\rho[W_1(z',h';\lambda')(1-\tau_l)v'(h'-e') - W_2(z',h';\lambda')\frac{H(e')}{H_1(e')\eta'}]H_1(e)\eta \\ = -W_2(z,h;\lambda). \tag{2.24}$$

In the subsequent analysis, the functions W and H will be restricted to have the forms

$$W(z,h;\lambda) = \ln(z - \lambda\Omega\frac{h}{1+\omega}^{1+\omega}),\ \omega,\Omega > 1, \tag{2.25}$$

and

$$H(e) = \chi e^{\phi},\ 0 < \phi < 1.$$

2.3. Government

The last actor in the economy is the government. As has been mentioned, it taxes labor income at rate τ_l, and interest and profits (net of the capital consumption) at τ_k. It uses the revenues raised from these taxes to provide lump-sum transfer payments,τ_t, to subsidize gross investment at the rate, τ_x and to give a capital consumption allowance. The government's budget constraint reads

$$\begin{aligned}(M+1)\tau + \tau_x q p_1' k_1' &= \tau_l[Mwl + v\eta(h-e)] + \tau_k \textstyle\sum_{i=1}^{N} p_i P_i(k_i, w) \\ &\quad -(\tau_k/\Delta)(1-\tau_x)[qp_1'k_1' + \textstyle\sum_{i=0}^{\Delta-2} q_{-i}p_{1,-i}k_{1,-i}].\end{aligned} \tag{2.26}$$

2.4. Competitive Equilibrium

We complete the description of the economy under study with a definition of the competitive equilibrium. First, the aggregate state-of-the-world is given by the vector $s = (p_1, ..., p_N, k_1, ..., k_N, \eta)$. Second, the equilibrium wage and interest rates, the price of capital goods, and individual transfer payments τ are expressed as functions of the aggregate state vector as follows: $w = \mathcal{W}(s)$, $v = \mathcal{V}(s)$, $r' = \mathcal{R}(s)$, $q = \mathcal{Q}(s)$ and $\tau = \mathcal{T}(s)$. Next, suppose that the aggregate state variables evolve according to $p_i' = \mathcal{P}_i(s)$, $k_i' = \mathcal{K}_i(s)$, and $\eta' = \mathcal{N}(s)$. Hence the law of motion for s is $s' = \mathcal{S}(s) \equiv (\mathcal{P}_1(s), ..., \mathcal{K}_1(s), ..., \mathcal{N}(s))$. Finally, it is easy to see that the above expression imply that d (the value of the capital consumption allowance) can be represented as $d = \mathcal{D}(s)$.

Definition: A competitive equilibrium is a set of allocation rules $l_i = \mathcal{L}_i(s)$, $b_i = \mathcal{B}_i(s)$, $h_i = \mathcal{H}_i(s)$, $p_i' = \mathcal{P}_i(s)$, $k_i' = \mathcal{K}_i(s)$, for $i = 1, ...N$, $c = \mathcal{C}(s)$, $l = \mathcal{L}(s)$, $z = \mathcal{Z}(s)$, $h = \mathcal{H}(s)$, $e = \mathcal{E}(s)$, and set of pricing and transfer payments $w = \mathcal{W}(s)$,

$v = \mathcal{V}(s)$, $q = \mathcal{Q}(s)$ $d = \mathcal{D}(s)$ and $\tau = \mathcal{T}(s)$, and an aggregate law of motion $s' = \mathcal{S}(s)$, such that

1. Consumption goods plants, capital goods plants, and firms solve problems (P1), (P2), and (P3), respectively, taking as given the aggregate state-of-the-world s and the form of the functions $\mathcal{W}(\cdot)$, $\mathcal{V}(\cdot)$, $\mathcal{R}(\cdot)$, $\mathcal{Q}(\cdot)$,$\mathcal{D}(\cdot)$, and $\mathcal{S}(\cdot)$, with the equilibrium solutions to these problems satisfying $l_i = \mathcal{L}_i(s)$, $b_i' = \mathcal{B}_i(s)$, $h_i = \mathcal{H}_i(s)$, $p_i' = \mathcal{P}_i(s)$, $k_i' = \mathcal{K}_i(s)$, for $i = 1, ..., N$.

2. Unskilled workers solve problem (P4), taking as given the aggregate state-of-the-world s and the form of the functions $\mathcal{W}(\cdot)$, $\mathcal{R}(\cdot)$, $\mathcal{T}(\cdot)$and $\mathcal{S}(\cdot)$, with the equilibrium solution to this problem satisfying $c = \mathcal{C}(s)$,and $l = \mathcal{L}(s)$.

3. Skilled workers solve problem (P5), taking as given the aggregate state-of-the-world s and the form of the functions $\mathcal{W}(\cdot)$, $\mathcal{V}(\cdot)$, $\mathcal{R}(\cdot)$, $\mathcal{Q}(\cdot)$, $\mathcal{D}(\cdot)$, $\mathcal{T}(\cdot)$ and $\mathcal{S}(\cdot)$, with the equilibrium solution to this problem satisfying $z = \mathcal{Z}(s)$, $h = \mathcal{H}(s), \eta' = \mathcal{N}(s)$, where $\mathcal{N}(s) = H(\mathcal{E}(s))\eta$.

4. All markets clear, implying

$$Mc + z = f \sum_{i=1}^{N} p_i k_i^{\alpha} l_i^{\beta}, \tag{2.27}$$

$$p_1' k_1' = (1 - f) \sum_{i=1}^{N} p_i k_i^{\alpha} b_i^{\xi} (\eta_i h_i)^{\zeta}, \tag{2.28}$$

$$\sum_{i=1}^{N} p_i [f l_i + (1 - f) b_i] = M l, \tag{2.29}$$

$$\sum_{i=1}^{N} p_i (1 - f) h_i = h - e, \tag{2.30}$$

in addition to the production arbitrage condition (2.6), the law of motion for capital (2.9), and the government's budget constraint (2.26) holding.

3. Balanced Growth

The balanced growth path for the economy can now be characterized. Clearly, along a balanced growth path some variables, such as consumption, will be growing at some fixed rate, while others, such as aggregate employment, will remain constant. Some basic properties of the economy's balanced path will now be derived in a heuristic fashion.

To begin with, it seems reasonable to conjecture that along a balanced growth the labor variables l_i, b_i, h_i, l, h and e will all be constant. Given this conjecture, equation (2.21) implies that the stock of human capital grows at some constant rate, say γ_η. Second, it seems likely that in balanced growth the age distribution of plants $\{p_i\}_{i=1}^{N}$ will be constant. Using (2.28) it is then straightforward to compute the rate, γ_k, at which the economy's distribution of capital shifts to the right over time. One finds

$$\gamma_k = (\gamma_\eta)^{\zeta/(1-\alpha)}.$$

Note that $k_i = \gamma_k^{1-i} k_1$. Next, from the above condition and (2.6) it follows that the relative price of capital must grow at the rate, γ_q, given by

$$\gamma_q = \gamma_\eta^{-\zeta} < 1.$$

Thus, the price of capital declines in balanced growth . Finally, let the γ_y represent constant rate at which aggregate consumption grows. Condition (2.27) restricts this rate of growth to be

$$\gamma_y = \gamma_\eta^{\alpha\zeta/(1-\alpha)} < \gamma_k.$$

Note that aggregate investment spending when measured in terms of consumption goods, or $qp_1'k_1'$, grows at this rate too, a fact that follows from the formulae for γ_k and γ_q. Therefore, γ_y is the rate at which aggregate output – or $c + z + qp_1'k_1'$ when taking consumption as the numeraire – grows. To take stock of the discussion so far, observe that in balanced growth the relative price of capital goods falls at the same time as capital-to-income ratio rises.

Moving on, it is easy to deduce how wages, the interest rate, and profits, etc., behave along a balanced growth path. From (2.3) and (2.5), it is transparent that the wage rates per unit of time worked by skilled and unskilled labor, or w and $v\eta$, grow at rate γ_y. The wage rate, v, for an *efficiency* unit of skilled labor, however, increases at the lower rate γ_y/γ_η. For leisure to remain constant in balanced growth, productivity in the household sector must grow at the same rate as in the consumption sector. This is readily apparent from the forms of (2.17), (2.19), (2.25) and (2.23). To insure that this is the case let $\lambda = \eta^{\alpha\zeta/(1-\alpha)}$. Note that it is the relentless rise in real wages that motivates capital replacement in economy. As wages increase the profits for a plant using old capital are continuously shrinking. To increase these dwindling profits the plant must invest in new capital.

As is readily observable from either (2.18) or (2.22), the after tax interest rate, $1 + (1 - \tau_k)r$, remains constant at γ_y/ρ. The profits, $P(k_i, \mu_i, w)$, made by a plant in the i-th period of its life rise at rate γ_y along a balanced growth path.[14] Equation (2.11) implies that the present-value of a vintage-i plant's profits, or V_i, are growing at this rate too. Last, the contribution that an extra unit of capital makes to a plant's profits, $P_1(k_i, \mu_i, w)$, increases at rate $\gamma_q < 1$, or declines over time. Thus, using (2.13) the marginal product in terms of the present-value of profits derived from a unit new capital, or V_{N+1}, declines at this rate also. Should not this decline in the productivity of new capital eventually choke off capital accumulation and hence growth? The answer is no: observe that while the marginal unit of new capital is becoming less productive over time, the cost of purchasing it is falling at the same rate.

The steady-state age distribution of capital across plants has a simple characterization. Two cases can obtain. In the first, *only* the plants with the oldest vintage of capital are modernized. All of these plants are updated. In the second case, *some* next-to-oldest vintage plants are renovated as well. The following lemma makes this characterization precise, where the oldest capital has an age of M.

Lemma 2. $p_1 = p_2 = ... = p_{M-1} \geq p_M > 0$.

Proof. There are two cases to consider: either $p_{M-1} = p_M$ or $p_{M-1} > p_M$. First, suppose that $p_{M-1} > p_M$. Here equation (2.10) holds with equality for p_M. Now, assume, for the moment, that $V'_{i+1} < V'_i$, for all $i \geq 1$. Then if the righthand of (2.10) is equal to zero for p_M it must exceed zero for p_{M-1}, p_{M-2}, ..., p_2. Consequently, $p_1 = p_2 = ... = p_{M-1} > p_M$. It remains to be established that $V'_{i+1} < V'_i$. This can be shown by induction. To begin with, recall that in balanced growth $V_i = V'_i/\gamma_y$. Next, suppose $V'_{i+2} < V'_{i+1}$. Then equation (2.11)

[14] See footnote 6.

implies that $V'_{i+1} < V'_i$. To start the induction hypothesis off note from (2.11) that $V'_{N+1} < V'_N$, since capital has a maximum physical life of N years. The first case where $p_{M-1} = p_M$ can be analyzed the same way. □

4. Calibration

The next step in the analysis is to choose values for the model's parameters. As is now conventional, as many parameter values as possible are chosen on the basis of either (i) *a priori* information or (ii), so that along the model's balanced growth path various endogenous variables assume the long-run values that are observed in the U.S. data — for a discussion of the calibration methodology see Cooley and Prescott (1994). The parameters in question are:

Utility:	$\theta, \Theta, \omega, \Omega, \rho$
Technology:	$\alpha, \beta, \xi, \zeta, \mu_1, ..., \mu_N, \chi, \phi,$
Government:	$\tau_l, \tau_k, \tau_x, \Delta.$

A time period in the model corresponds to one year.

Six parameters are determined on the basis of a priori information. Over the postwar period labor's share of income had an average of .65 in the U.S. economy. This dictates setting $\beta = \xi + \zeta = .65$. The tax rates on labor and capital income, τ_l and τ_k, were set at .30 and .30, respectively. This is the baseline tax rate structure used in Auerbach, Kotlikoff and Skinner (1983). The investment subsidy has changed considerably over the postwar period. A value of .10 was picked for the investment subsidy, τ_x, the rate reported by Fullerton and Gordon (1983) for after 1975. Finally, one of the most volatile elements of the tax treatment of capital over the post-war period has been the capital consumption allowance. An accounting life of 20 years was chosen for capital in the model; i.e., $\Delta = 20.0$. This is roughly in accord with the prevailing policy in 1980, as reported by Fullerton and Karayannis (1993).

The values of $1/\theta$ and $1/\omega$ correspond to the labor supply elasticities for unskilled and skilled labor. Following Greenwood, Hercowitz and Huffman (1988) a value of .6 was chosen for θ and ω. This implies a value of 1.7 for the labor supply elasticities, which is an average of low numbers found by researchers for males and the high ones for females.

The rest of the parameters were chosen so that the model's growth path shares certain characteristics with the long-run U.S. data. To begin with, the average growth rate of output per hour was 1.24 percent between 1954-90. Thus, the model should satisfy the restriction

$$\gamma_y = 1.0124. \tag{4.1}$$

The average ratio of hours worked to non-sleeping hours of the working-age population is .25. This implies that

$$l = h = .25. \tag{4.2}$$

Evidence on the amount of time devoted to human capital formation in R&D activities for the U.S. economy is scant. As a result, the following arbitrary restriction is imposed on the model:

$$e = 0.1. \tag{4.3}$$

This condition implies that approximately 0.4 percent of working is time spent on R&D activities. This is roughly in accord with Birdsall and Rhee's (1993) calculation that approximately 0.2 percent of the population are involved in R&D activity.

U.S. income distribution statistics indicate that the top 1 percent of the population earn approximately 8 times that of the bottom 99 percent.[15] Let skilled labor be identified as representing the top 1 percent of the income distribution. Then, $M = 99$. Next, assume that the top 1 percent of the population earn 8 times more labor income than the bottom 99 percent. This yields the condition

$$\frac{q\zeta k_1^{\alpha} b_1^{\xi} h_1^{\zeta-1}(h-e)}{\beta k_1^{\alpha} l_1^{\beta-1} l} = 8. \tag{4.4}$$

Further, let skilled workers have 8 times the wealth of unskilled workers implying

$$z/c = 8. \tag{4.5}$$

In 1989 the average age of capital in the U.S. was 11.9.[16] This implies that the model's balanced growth path should obey the restriction

$$\frac{\sum_{j=1}^{N+1} j p_j \gamma_k^{1-j} k_1}{\sum_{j=1}^{N+1} p_j \gamma_k^{1-j} k_1} = 11.9. \tag{4.6}$$

[15]This estimate is taken from Gomme and Greenwood (1993), Appendix B, who fit a Pareto distribution to the tail of the U.S. income distribution.

[16]This number is from Table A7 in *Fixed Reproducable Tangible Wealth in the United States, 1925-1990.*

Finally, the after-tax real interest rate is taken to be 7 percent. This is approximately the value estimated by Cooley and Prescott (1994) from the *National Income and Product Accounts*. Therefore,

$$\rho\gamma_y^{-1} = 1/1.07. \tag{4.7}$$

Now, the conjectured solution for the model's balanced growth path suggests deflating the nonstationary variables by functions of η to render them stationary. To this end, let $\widehat{c} = c/\eta^{\alpha\zeta/(1-\alpha)}$, $\widehat{z} = z/\eta^{\alpha\zeta/(1-\alpha)}$, $\widehat{k}_i = k_i/\eta^{\zeta/(1-\alpha)}$, for $i = 1, ..., N$, $\widehat{q} = q/\eta^{-\zeta}$, etc, where the circumflex over a variable denotes its transformed value. Note that (2.4)-(2.6), (2.7), (2.10)-(2.13), (2.19), (2.23),(2.24), and (2.26)-(2.30) represent a system of 5N+ 9 equations in the 5N+ 9 unknowns: the firm's variables l_j, b_j, h_j, p_j, $\widehat{V}_j$, for $j = 1, ...N$, and $\widehat{k}_1$ and $\widehat{V}_{N+1}$; the households' variables $M\widehat{c} + \widehat{z}$, l, h, and s; the market variables $\widehat{w}, \widehat{v}, \widehat{q}$, and f. (See Appendix A for more detail on the transformed system.) Observe that the balanced growth path is invariant to the distribution of income between skilled and unskilled agents in the sense that the solution to the above system of equations can be determined independently of the breakdown of aggregate consumption, $M\widehat{c}+\widehat{z}$, between $\widehat{c}$ and $\widehat{z}$. This breakdown depends upon the long-run distribution of wealth represented by $\widehat{a}$. By appending the eight long-run restrictions (4.1)-(4.7) to the above system the seven parameters α, Θ, Ω, ζ, ϕ, ρ, and χ can also be solved for simultaneously in addition to $\widehat{a}$. Doing this yielded the following parameter values: $\alpha = 0.2$, $\theta = 0.6$, $\Theta = 0.39$, $\omega = 0.6$, $\Omega = 5.26$, $\zeta = 0.504$ (implying $\xi = 0.146$), $\phi = 0.407$, $\rho = 0.9462$, and $\chi = 0.2623$.

5. Quantitative Properties of Balanced Growth Paths

In this section the results of several experiments that highlight the role of the adoption-replacement decision in the vintage capital economy are reported. We compare the balanced growth paths for economies characterized by various tax rates for labor income, and capital income and examine how these are affected by policies designed specifically to encourage innovation and the adoption of new technologies. Before reporting these results we first discuss the functional dependence of model's steady state on the tax policy in place.

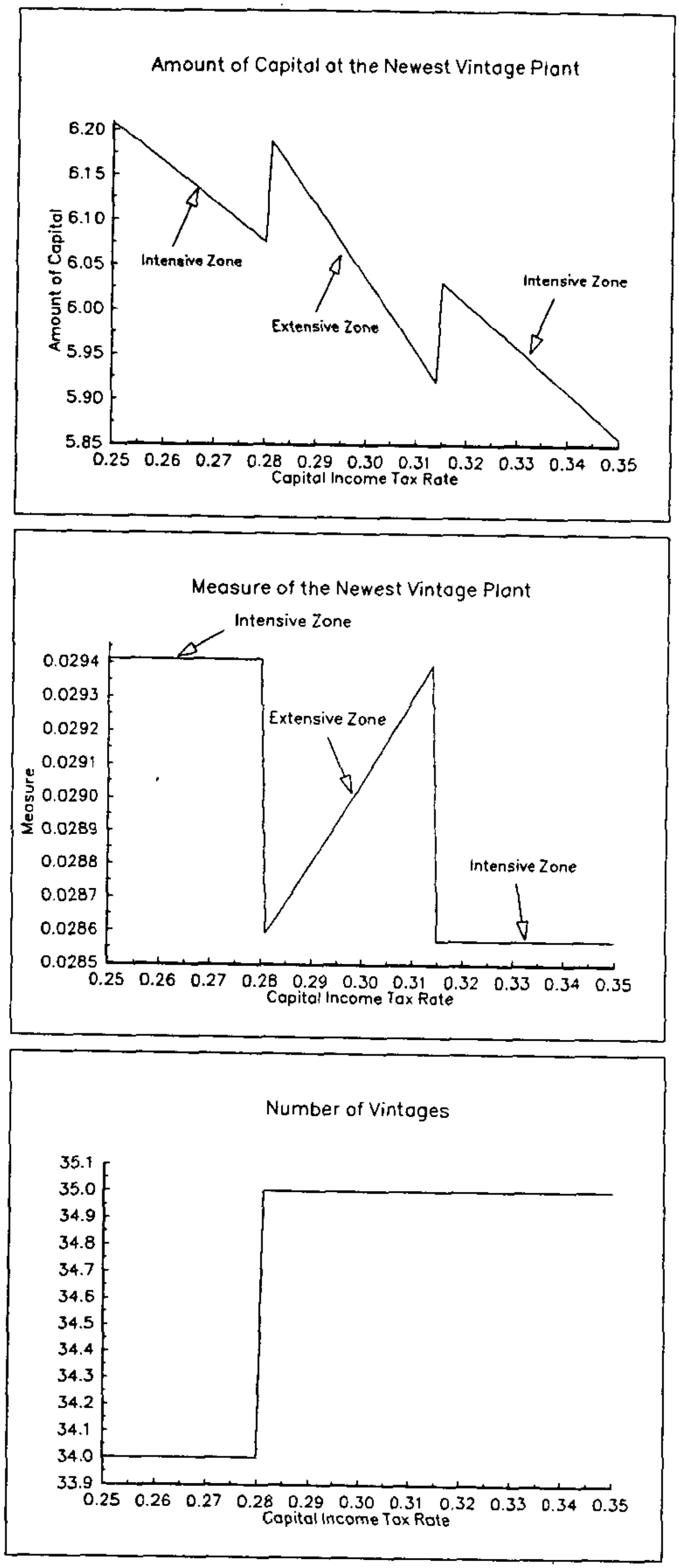
In line with Lemma 2, depending on the particular configuration of tax rates, the steady state can be characterized by one of two cases. In the first case *only* the plants with the oldest vintage of capital are modernized. All of these plants are updated. The plant renovation equation, (2.10), is slack in this situation. In the second case *some* next-to-oldest vintage plants are renovated as well. Equation (2.10) now holds with equality for the next-to-oldest vintage of plants. Figure 2 illustrates the two cases. In the zones marked "intensive" the first case transpires, while in the ones labeled "extensive" the second case occurs. This figure traces out the effect that the capital income tax has on the amount of investment in a plant, k_1, the number of plants renovated, p_1, and vintages of capital, M.[17]

Consider taxes in the $[0.25, 0.28]$ interval. Here the economy is in the first case. At any point of time there are 34 vintages of capital in existence and $p_1 = p_2 = ... = p_{34}$. At higher rates of capital income taxation the amount of investment in a new plant declines, as one would expect from equation (2.12) governing physical capital accumulation. All adjustment is along the intensive margin here, since p_1 remains fixed. Now in the zone under consideration the lefthand of equation (2.10) is strictly negative for p_{35}. As the rate of capital income taxation increases new plants become less profitable relative to old ones, causing $V_1 - V_{34}$ to fall, resulting in the lefthand side eventually becoming positive. Eventually, it pays to delay modernization by one period. This happens as tax rates move into the $[0.28, 0.35]$ range. Observe that at the point where modernization is postponed by a period investment in new plants takes an upward jump. This makes intuitive sense. Since renovation is more infrequent, investment should be larger since it makes do for an extra period — a fact transparent from (2.13).

Now, consider taxes in the interval $[0.28, 0.314]$. The economy is in the second case here. There are 35 vintages of capital in existence with $p_1 = p_2 = ... = p_{34} > p_{35}$. Equation (2.10) holds tightly here for p_{35}. In this zone k_1 and p_1 move in opposite directions. Again, as the capital income tax rate rises investment in a renovated plant, k_1, drops. This lowers the cost of renovating an old plant. Hence the value of a new plant increases vis-a-vis an old one. Consequently, p_1 increases implying adjustment along the extensive margin. The amount of equilibrium investment, $p_1 k_1$, and thus the relative price of capital, q, still decreases over this range. It is the fall in the relative price of capital that allows (2.10) to remain tight. There is a limit to how far this process can go since p_1 is bounded above by

[17]It is worth pointing out that a two sector model with fixed proportions (putty-clay technology) is more complicated. Suppose that a plant can switch between producing capital and consumption goods — so that there is only one age distribution for capital. It is easy to show that the newer plants will produce capital goods while the older ones will manufacture consumption goods. Firms must now decide both on when to replace old machines, and at what time to switch plants over from producing investment goods to consumption goods.

Figure 2


Amount of Capital at the Newest Vintage Plant
Amount of Capital
6.20
6.15
6.10
6.05
6.00
5.95
5.90
5.85
Intensive Zone
Extensive Zone
Intensive Zone
0.25 0.26 0.27 0.28 0.29 0.30 0.31 0.32 0.33 0.34 0.35
Capital Income Tax Rate
Measure of the Newest Vintage Plant
Intensive Zone
Measure
0.0294
0.0293
0.0292
0.0291
0.0290
0.0289
0.0288
0.0287
0.0286
0.0285
Extensive Zone
Intensive Zone
0.25 0.26 0.27 0.28 0.29 0.30 0.31 0.32 0.33 0.34 0.35
Capital Income Tax Rate
Number of Vintages
35.1
35.0
34.9
34.8
34.7
34.6
34.5
34.4
34.3
34.2
34.1
34.0
33.9
0.25 0.26 0.27 0.28 0.29 0.30 0.31 0.32 0.33 0.34 0.35
Capital Income Tax Rate

1/34. This limit is reached as τ_k approaches 0.314. At this point the number of plants being renovated takes a plunge. This is associated with an upward surge in the amount of new capital placed in a renovated plant. The economy is now in the first case, analyzed previously.

5.1. Economic Depreciation and The Tax Treatment of Capital

One of the advantages of the explicit treatment of the replacement problem is that it permits a more detailed analysis of the different elements that comprise the tax treatment of capital. In the standard user-cost-of-capital approach these different features of the tax system are condensed into a single effective tax rate on capital income. Policies governing the taxation of capital income, on the other hand, are considerably more diverse and detailed. Accordingly, it is of some interest to examine in detail these separate elements. We consider here the effect of investment tax credits, the accounting life of capital and the use of tax "holidays". We first illustrate how these policies affect the age distribution of the capital stock and economic growth. We then examine the same policies in the context of the standard two sector model to illustrate how much of the growth and other effects are due to the vintage structure of the model. Since the agents in these economies predicate their behavior on the certain belief that tax rates are constant over time, the findings should be viewed as a comparison across economies characterized by different fiscal policies.

The Capital Tax Rate and Accounting Life

The reference point for the first set of experiments is the baseline calibrated model presented in the previous section. This model assumes that 1% of the agents in the economy are skilled. Skilled agents have approximately 8 times the consumption of unskilled workers. The baseline fiscal policy assumes that capital income is taxed at the rate of 30%, labor income of both skilled and unskilled workers is taxed at the rate of 30%, and there is a 10% investment tax credit. In addition capital is assumed to have an accounting life for depreciation purposes of 20 years. This implies that capital tax revenues as a fraction of capital's share of income is about 21%, a number that is close to that found in U.S. data for the 1980's.

One of the most volatile elements of U.S. fiscal policy toward capital has been the variation in the tax treatment of depreciation and the useful life of capital.

Gravelle (1994) argues that the history of the tax treatment of depreciation is one of steadily more lenient policy toward depreciation up until 1980. The policy since 1980 has been more volatile as is documented by Fullerton and Karayannis (1993). The following Table shows the effect on this economy of varying the capital income tax rate and the accounting life of capital.

Table 1: Response to Tax and Depreciation Policies

Accounting Life	Average Age			Growth Rate			Welfare Costs		
	$\tau_k=$			$\tau_k=$			$\tau_k=$		
	.10	.30	.40	.10	.30	.40	.10	.30	.40
5 Years	10.98	10.83	10.65	1.35	1.34	1.33	-2.22	-1.88	-1.56
10 Years	11.47	11.38	11.70	1.34	1.29	1.26	-1.90	-1.10	-0.48
20 Years	11.47	11.90	12.57	1.32	1.24	1.16	-1.60	0	1.16
	Price Decline			**Excess Burden**			**Income Distribution**		
	$\tau_k=$			$\tau_k=$			$\tau_k=$		
	.10	.30	.40	.10	.30	.40	.10	.30	.40
5 Years	5.24	5.19	5.13	1.18	1.21	1.23	10.11	8.91	8.31
10 Years	5.17	5.00	4.88	1.30	1.29	1.32	9.91	8.50	7.64
20 Years	5.12	4.81	4.60	1.40	1.43	1.46	9.78	8.00	7.03

The age distribution of the capital stock is characterized here by the average age of capital and the number of vintages in existence. The upper left hand panel of Table 1, and Figures 3 and 4, show the effect of varying depreciation policy and the capital income tax rate on the age distribution of the capital stock. The response of the average age of capital to changes in the accounting life of capital is much greater at higher tax rates. Figures 5 and 6, and the corresponding panels of Table 1, show the effects of varying depreciation policy on the growth rate of output and the decline in the relative price of capital. The magnitude of the growth rate effects in this economy are small. This is a consequence of the law of motion for human capital accumulation that we used in the analysis in conjunction with the calibrated value for the amount of time devoted to human capital acquisition. This result is in accord with recent empirical work, however, which has not found the presence of a strong link between taxes and growth. The effect of the capital tax rate on growth rates is negligible when the accounting life

Figure 3- Number of Vintages

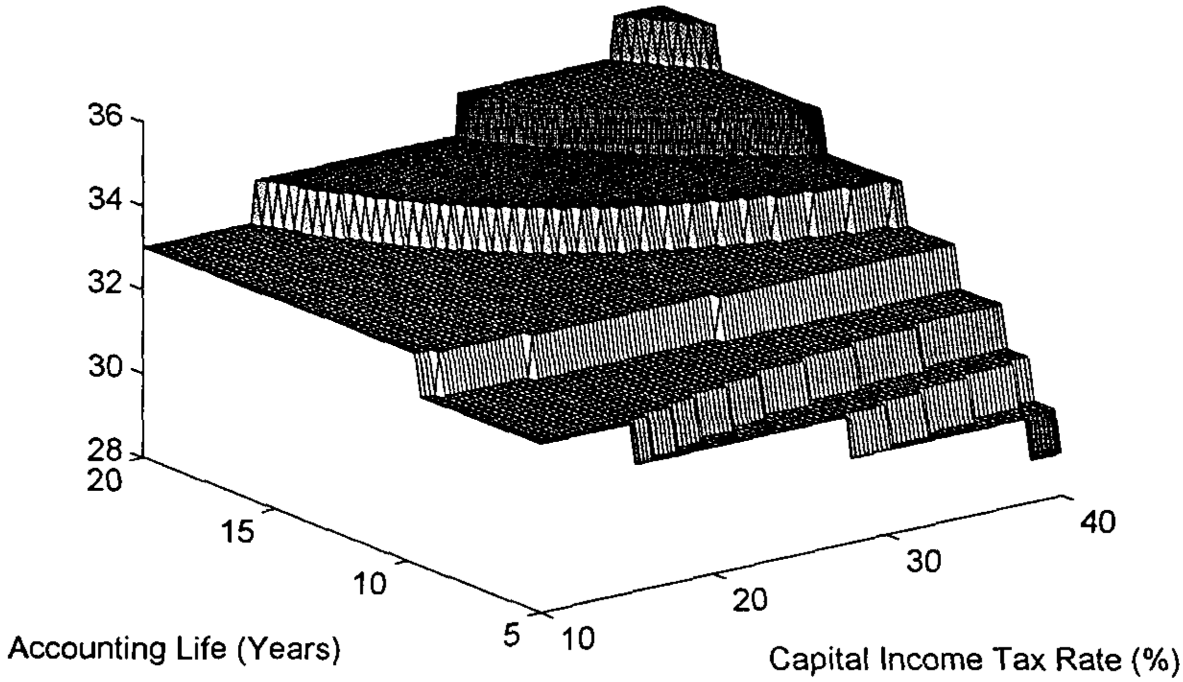


Figure 4- Average Age

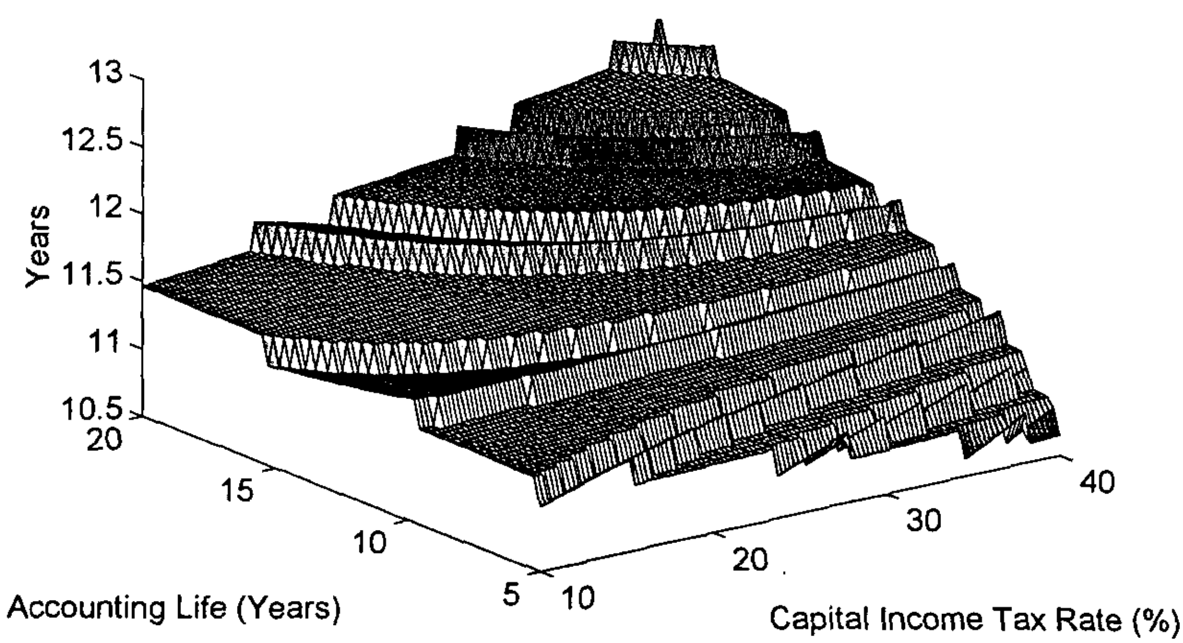


Figure 5- Growth Rate of Output

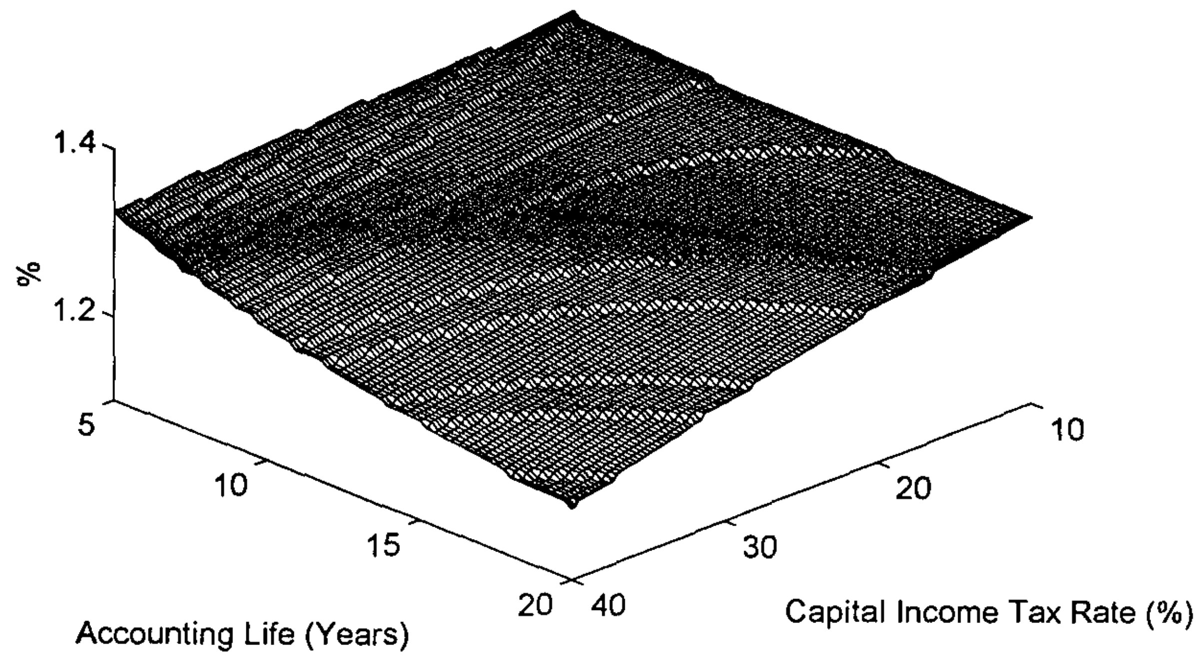
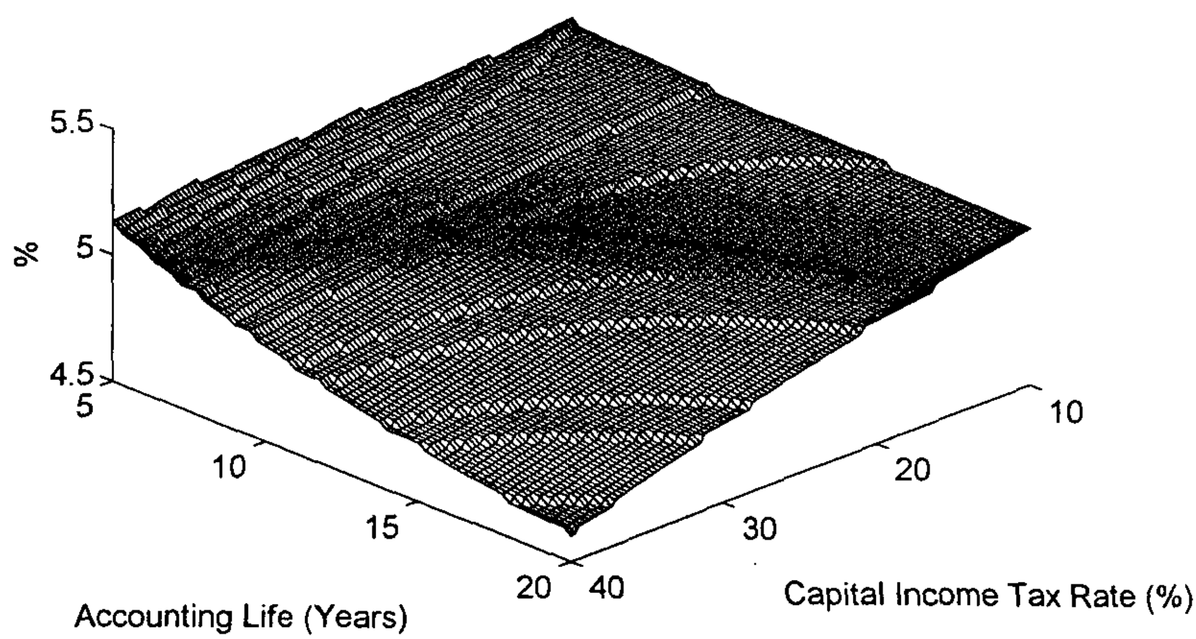


Figure 6- Decline Rate of Price of Capital



is short and is much stronger when accounting lives are longer. Figure 5 shows that accounting life and tax rates have very similar effects on the growth rate. The model predicts a baseline decline in the relative price of capital of 4.81% per year. This is considerably higher than Gordon's estimate of 3%.[18] This rate of decline increase with lower tax rates and shorter accounting lives, but the differences are not dramatic.

While the growth rate effects of these different policies are small, the welfare consequences are quite sizable. The welfare consequences of taxes are measured by calculating the percentage increment to consumption that would be necessary to leave agents as well off with the new policy as they would have been in the baseline policy. The welfare consequences are different for the two types of agent in the model economy. Table 1 and Figure 7 show the weighted aggregate welfare loss and gains (negative numbers) relative to the calibrated growth path measured as a percentage of output. Raising the tax rate from .30 to .40 with a twenty year accounting life lowers welfare by 1.16 percent of output. Decreasing the accounting life of capital produces welfare gains of similar magnitudes. Figure 8 shows the welfare costs of the skilled agents. These are more than five times the costs of the unskilled agents.

The welfare measures just presented don't take account of the fact that government revenue changes with different mixes of taxes. Accordingly, it seems important to consider the fiscal effectiveness of the different taxes. This is measured here by the ratio of dollars of welfare lost to dollars of revenue gained. In the case of the investment tax credit this is the dollar gain in welfare for a dollar loss in revenue. These numbers are referred to as the "excess burden". Table 1 and Figure 9 show that the excess burden changes much more with changes in accounting life than it does with tax rates. Shorter accounting lives have much smaller excess burden and therefore are more efficient in this sense.

The distribution of income has long been thought to have important consequences for economic growth, although the direction of the relationship has been the subject of some debate. In the environment considered in this paper, the distribution of income is endogenous and responds to the mix of taxes on factor incomes. Table 1 and Figure 10 show that the response of the distribution of income to changes in the tax rates is quite pronounced. Raising capital income taxes to 40% produces the most equal distribution of income while increasing the accounting life also yields a more equal distribution of income.

[18]The model is not calibrated to hit a particular number for the price decline, although this is possible.

Figure 7- Welfare Cost

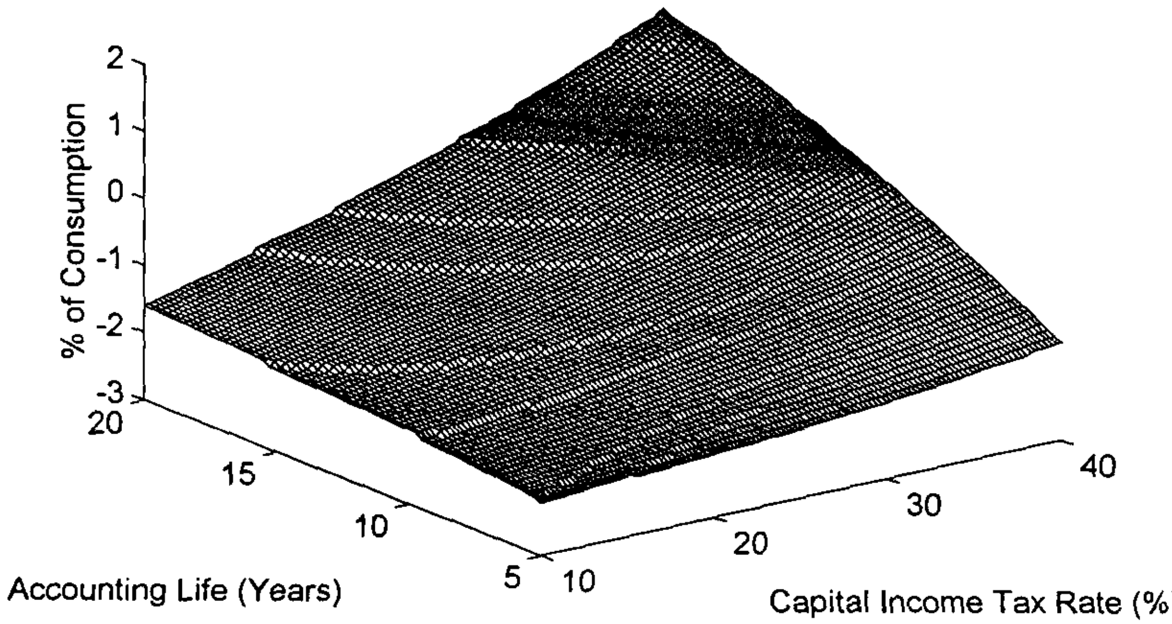


Figure 8- Welfare Cost of Skilled Agents

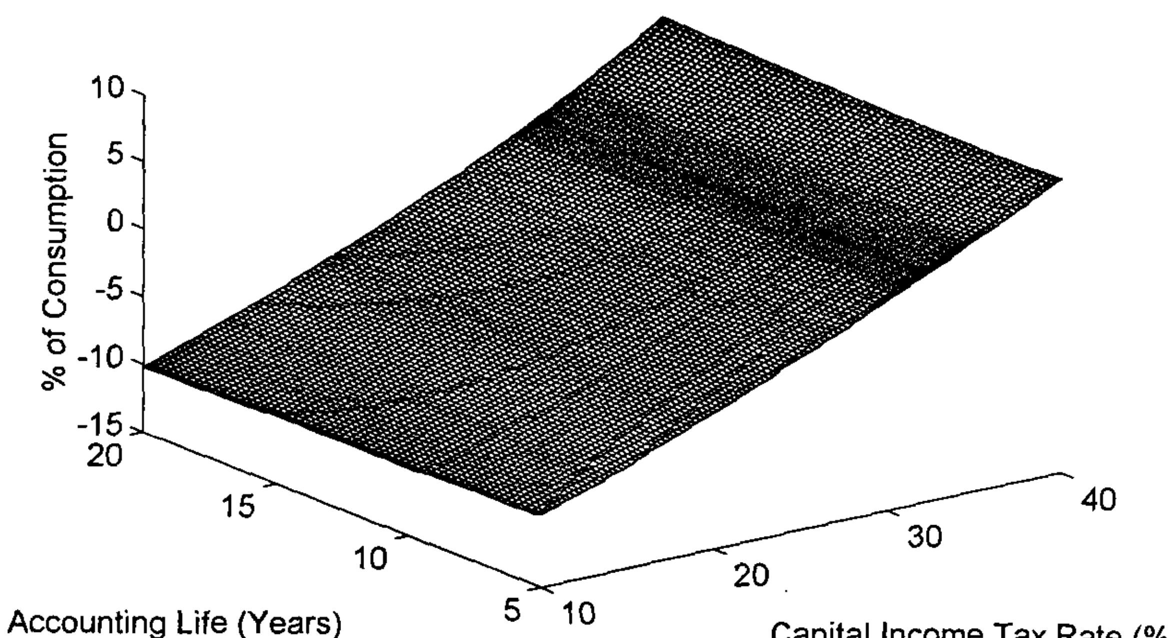


Figure 9- Excess Burden

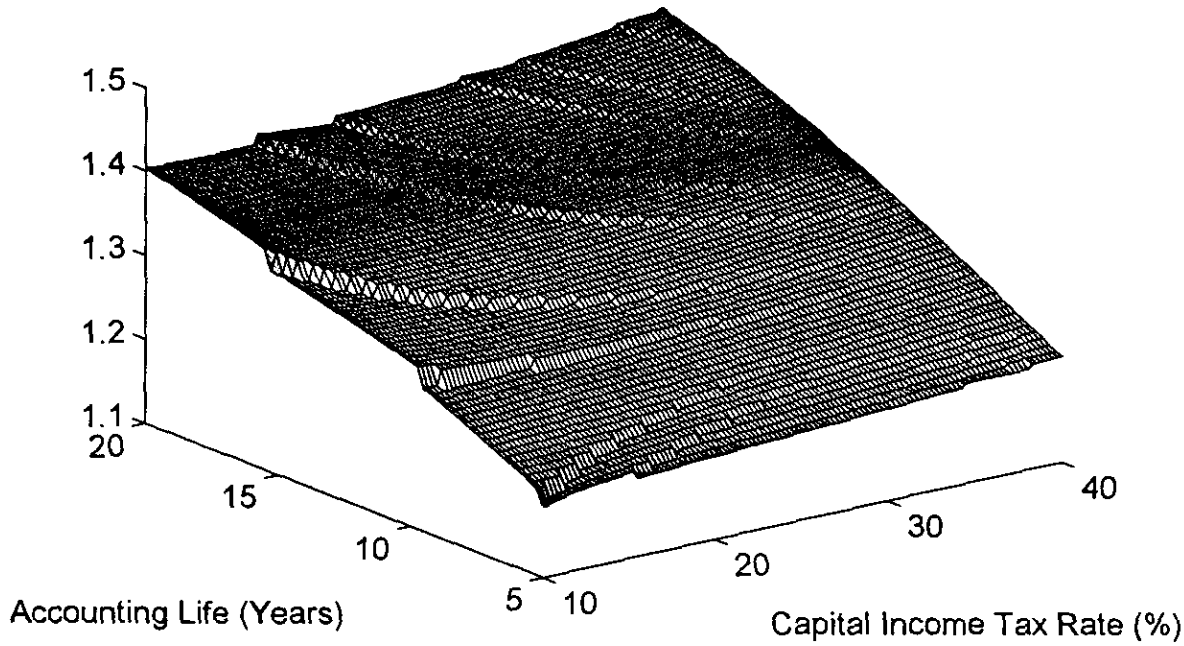
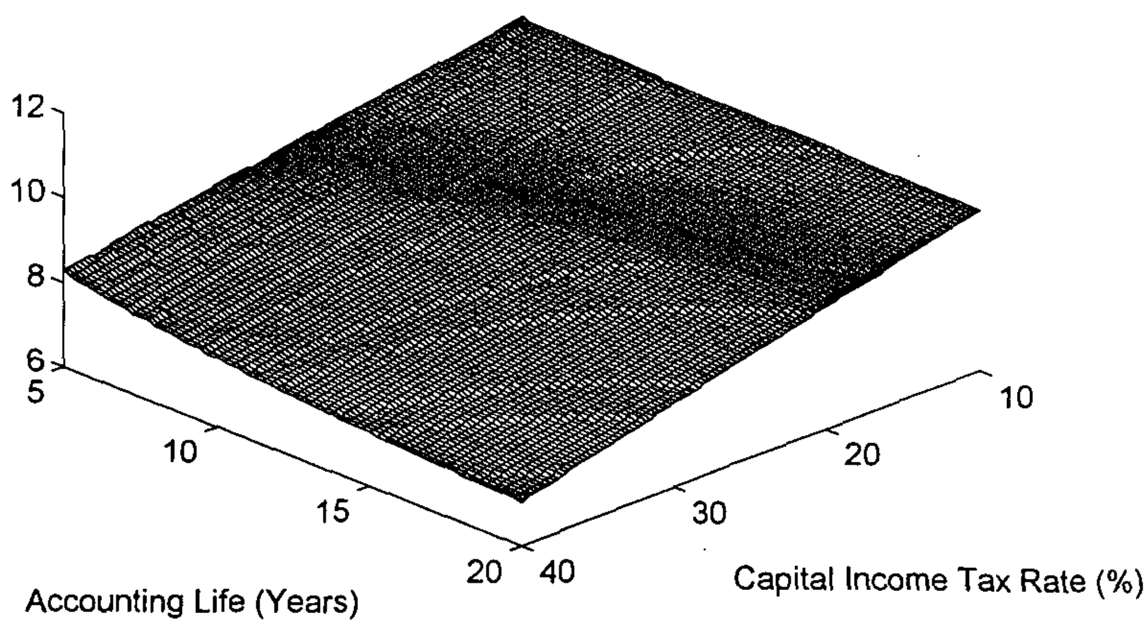


Figure 10- Income Distribution



Investment Tax Credits and Accounting Life

A second important element in the tax treatment of capital has been the use of investment tax credits to stimulate new investment. Such credits have been put in place and removed several different times in the post-war period in the U.S. The argument for adopting the tax credits is always that they stimulate the adoption of new technologies. The arguments for removing them have varied quite a bit. In Table 2 and Figures 11-14 the effects of these credits, their interaction with the accounting life of capital, are illustrated.

Table 2: Response to Investment Tax Credit

Accounting Life	**Average Age**			**Growth Rate**			**Welfare Costs**		
	$\tau_x=$			$\tau_x=$			$\tau_x=$		
	0	.10	.20	0	.10	.20	0	.10	.20
5 Years	11.31	10.83	10.24	1.30	1.34	1.42	-1.07	-2.05	-2.84
10 Years	12.04	11.38	10.89	1.22	1.29	1.37	-0.17	-1.28	-2.22
20 Years	12.63	11.90	11.48	1.17	1.24	1.32	1.05	0	-1.30
	Price Decline			**Excess Burden**			**Income Distribution**		
	$\tau_x=$			$\tau_x=$			$\tau_x=$		
	0	.10	.20	0	.10	.20	0	.10	.20
5 Years	5.03	5.19	5.50	6.34	4.14	2.74	8.19	8.91	9.84
10 Years	4.86	5.00	5.31	3.48	3.03	2.61	7.81	8.50	9.32
20 Years	4.67	4.81	5.10	2.64	2.25	2.06	7.43	8.00	8.75

Investment tax credits are powerful instruments of policy and they interact strongly with the treatment of depreciation. The results in Table 2 and Figure 11 show that a change of 10 percentage points in the investment tax credit can lower the average age of the capital stock by one-half a year or more, depending on the accounting life. Raising the investment tax credit increases the growth rate and improves welfare. The most striking effects of the investment tax credit are on excess burden. Excess burden is high when the tax credit is zero and accounting life is very short. It changes much more sharply with accounting life than it does with the credit.

Figure 11- Average Age

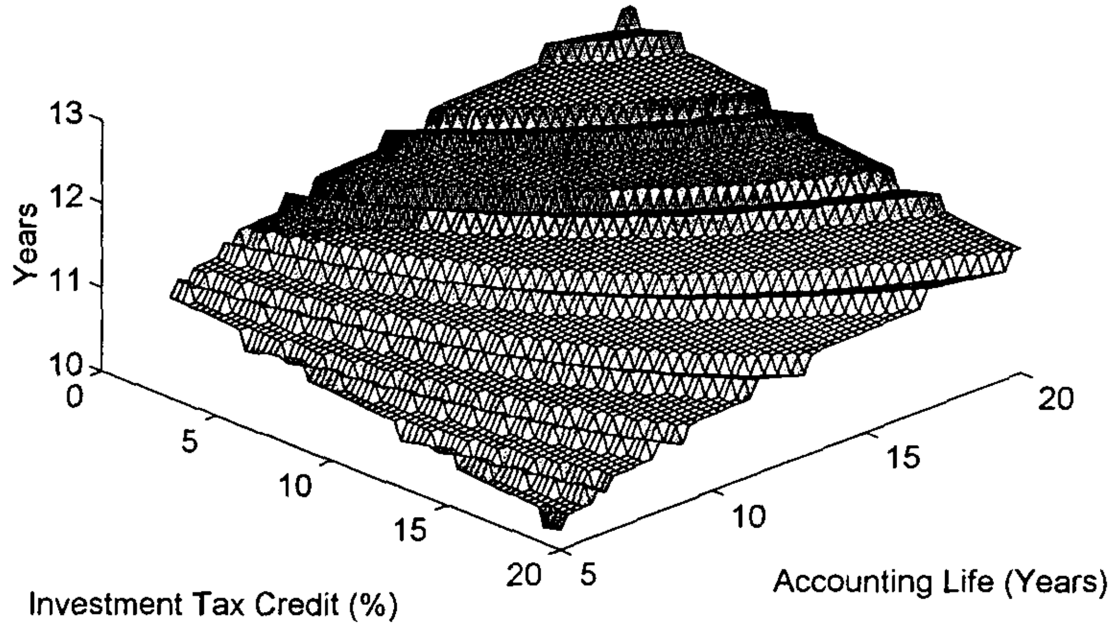


Figure 12- Growth Rate of Output

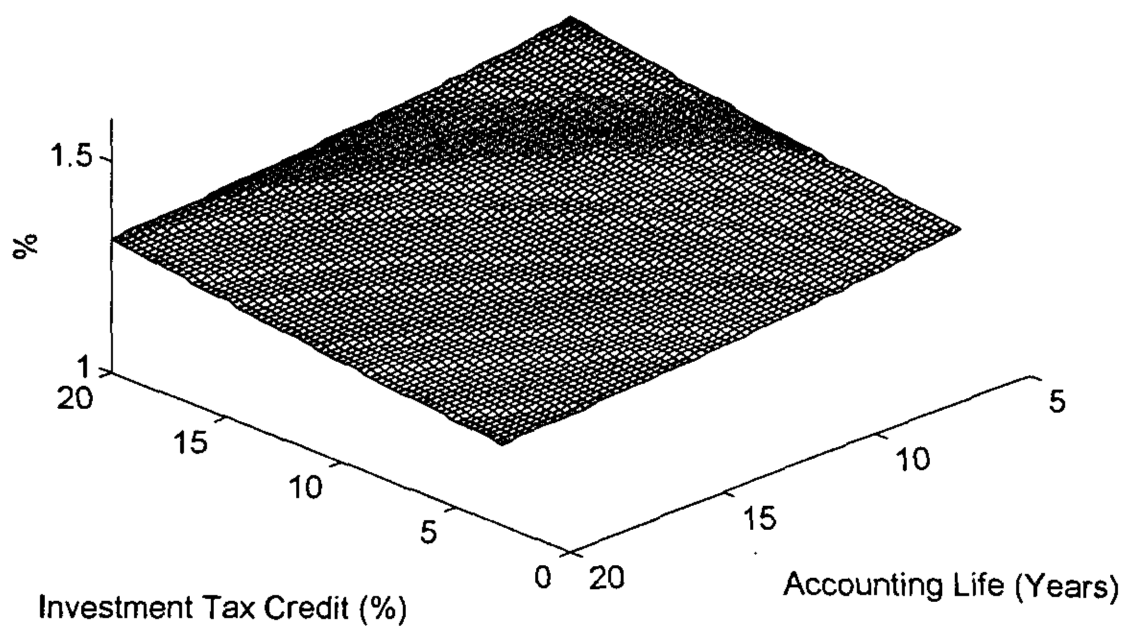


Figure 13- Welfare Cost

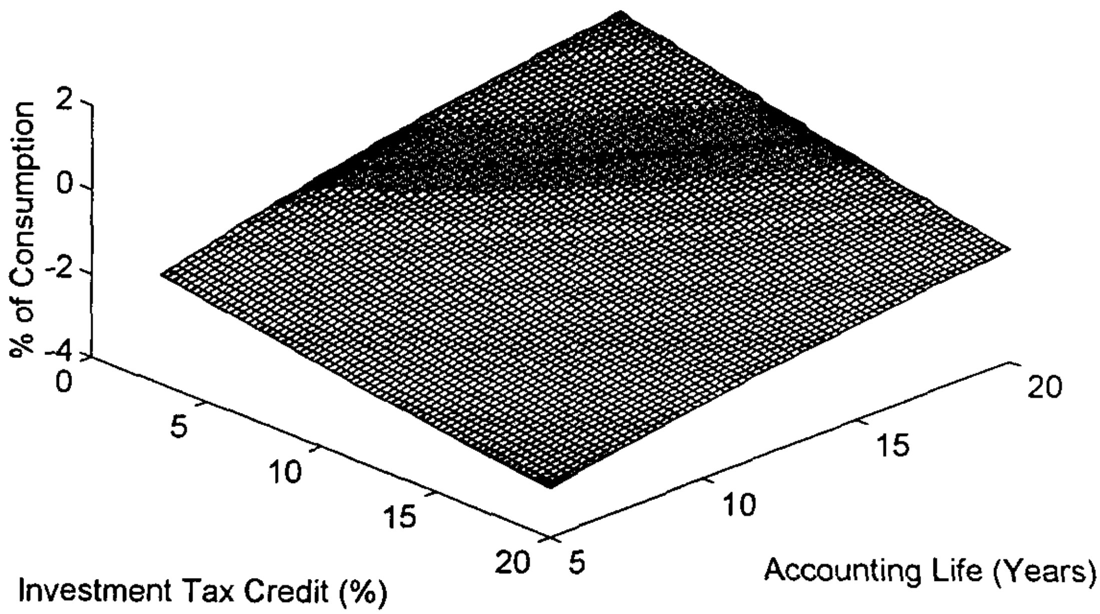
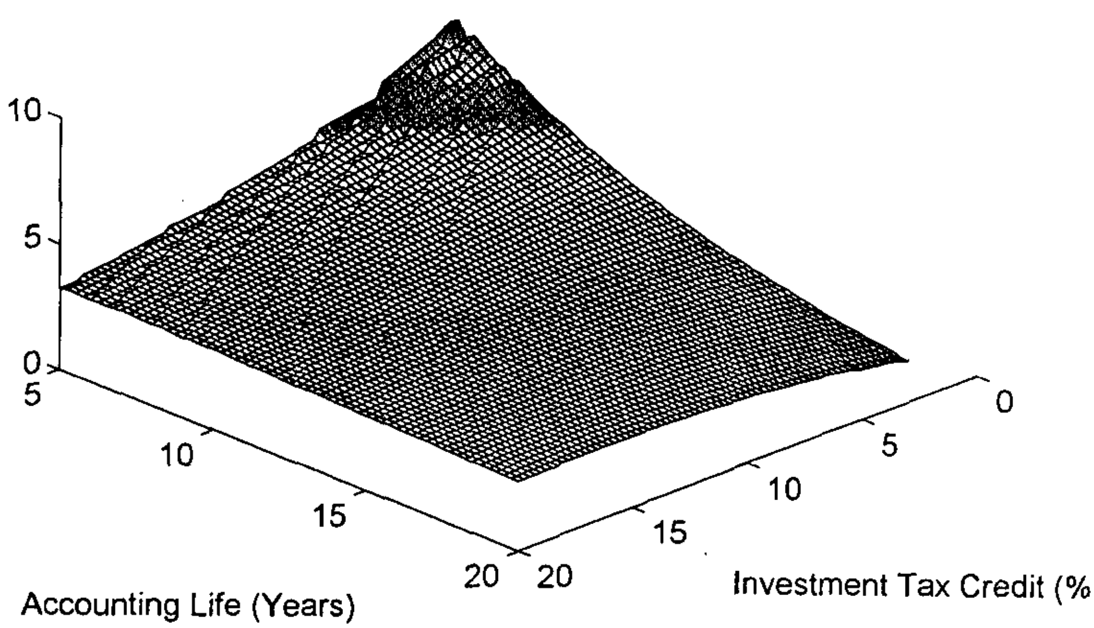


Figure 14- Excess Burden



Tax Holidays

Another element of the tax treatment of capital are tax "holidays", declarations that postpone the taxation of income from new investment for some number of periods. This is a feature that figures more prominently in local public finance and for specialized types of investments like pollution control equipment than at the aggregate federal level. Tax holidays are of some interest because they raise the relative value of new forms of capital. The interaction between tax holidays and the capital income tax are shown in Table 3.

Table 3: Response to Tax Holidays

Tax Holiday Periods	Average Age			Growth Rate			Welfare Costs		
	$\tau_k=$			$\tau_k=$			$\tau_k=$		
	.10	.30	.40	.10	.30	.40	.10	.30	.40
1 Year	11.71	11.57	11.33	1.32	1.27	1.24	-0.50	-0.55	-0.66
2 Years	11.62	10.96	10.61	1.33	1.31	1.29	-0.40	-0.99	-1.31
5 Years	11.40	10.05	9.01	1.35	1.39	1.41	-0.27	-1.68	-2.21
	Price Decline			**Excess Burden**			**Income Distribution**		
	$\tau_k=$			$\tau_k=$			$\tau_k=$		
	.10	.30	.40	.10	.30	.40	.10	.30	.40
1 Year	5.11	4.92	4.81	2.19	1.38	1.32	9.98	8.37	7.60
2 Years	5.15	5.07	5.00	2.23	1.34	1.27	10.09	8.81	8.16
5 Years	5.22	5.37	5.45	2.39	1.29	1.23	10.39	9.87	9.88

The response to tax holidays is quite strong and, somewhat unusual. Tax holidays cause the average age of the capital stock to decrease with increases in the capital tax rate. This is because, with tax holidays, higher tax rates on capital income raise the value of new vintages since they are temporarily exempt from the tax. With a one year tax holiday the growth rate of output still decreases with higher levels of the capital income tax but is everywhere higher than in the baseline case in Table 1. With longer tax holidays the growth rate actually increases with the level of τ_k. Also, somewhat odd is the finding that welfare improves with higher levels of τ_k and that the tax system becomes more efficient as is indicated by it's lower excess burden.

5.2. The Standard Two-Sector Model

To highlight the importance that attaches specifically to the vintage capital structure we consider the behavior of the standard two-sector endogenous growth model. Most of the structure developed above carries over with some alteration to this setting. There is a continuum of firms distributed over the unit interval. Aggregate consumption, $Mc + z$, must satisfy the constraint

$$Mc + z = fk^{\alpha}n^{\beta}, \tag{5.1}$$

where k and n represent inputs of capital and unskilled labor. Likewise, aggregate investment, $k' - (1 - \delta)k$, must satisfy

$$k' - (1 - \delta)k = (1 - f)k^{\alpha}b^{\xi}(\eta m)^{\zeta}, \tag{5.2}$$

where b and m are inputs of unskilled and skilled labor and δ is the fixed exogenous depreciation rate on capital.[19] The depreciation rate on capital is set so that the average age of capital in the standard paradigm is the same as in the vintage model. This involved setting $\delta = .03$. Equilibrium in the skilled and unskilled labor markets imply that $fn + (1 - f)b = Ml$ and $m + e = h$. Table 4 reports the results of the same set of experiments described in Table 1, but now for the standard model. The balanced growth path for the standard model shares most of the features of the vintage model. Growth occurs in this economy because human capital investments are taking place that make the production of capital goods more efficient. A comparison of Table 4 with Table 1 shows that the vintage structure has a significant effect on the the welfare costs of taxation. The welfare effects are roughly twice as large in the vintage economy. These tables also illustrate that vintage structure is important for growth, accounting for between 25 and 40% of the growth effects of policies at the extremes. The conclusion to be drawn from these results is that, although the growth rate effects of policies are small, the vintage structure accounts for an important part of those effects. It also is important for other features of the economy including the income distribution and the excess burden.

[19] It is easy to show that consumption and investment goods producing firms will pick the same level of capital in equilibrium.

Table 4: Response to Tax and Depreciation Policies
Standard Model

Accounting Life	Average Age			Growth Rate			Welfare Costs		
	$\tau_k=$			$\tau_k=$			$\tau_k=$		
	.10	.30	.40	.10	.30	.40	.10	.30	.40
5 Years	11.33	11.41	11.47	1.33	1.33	1.32	-0.87	-0.78	-0.70
10 Years	11.39	11.61	11.77	1.32	1.29	1.26	-0.81	-0.49	-0.23
20 Years	11.47	11.90	12.19	1.31	1.24	1.19	-0.70	0	0.59
	Price Decline			**Excess Burden**			**Income Distribution**		
	$\tau_k=$			$\tau_k=$			$\tau_k=$		
	.10	.30	.40	.10	.30	.40	.10	.30	.40
5 Years	5.19	5.14	5.10	1.18	1.19	1.20	9.60	9.35	9.17
10 Years	5.15	5.00	4.89	1.19	1.20	1.21	9.43	8.75	8.32
20 Years	5.10	4.81	4.62	1.19	1.23	1.25	9.18	8.00	7.32

6. Transitional Dynamics

We now describe the local dynamics of the vintage capital model for an experiment where the economy moves from an initial steady-state with a capital income tax rate of 29% toward the benchmark steady-state where the capital income tax is 30%. To compute the transitional dynamics the transformed model is linearized around the benchmark steady state — the full details are in Appendix A. The difference equation system characterizing the model's dynamics has $2N-1$ eigenvalues with modulus less than one in line with the model's $2N-1$ state variables $p_1, p_2, ...p_{N-1}, \widehat{k}_1, \widehat{k}_2, ..., \widehat{k}_N$.[20] Hence, the transition path is both stable and unique. To serve as a reference point, the transition path for the standard model is also computed. The transitional dynamics displayed by the vintage model are markedly different than the those shown by standard one.

In response to the increase in the capital income tax rate, the economy runs down its capital stock in the transition to the new steady state. This is shown

[20]Note that p_N can be eliminated from the model's state since $p_N = 1 - \sum_{j=1}^{N-1} p_j$.

in Figure 15. The aggregate capital stock in the vintage model is defined by $\widehat{k} = \sum_{j=1}^{N} p_j \widehat{k}_j$. The vertical distance portrays the deviation away from the terminal steady-state as a percentage of the discrepency that needs to be covered. That is, in a figure that plots the time path for some variable x the vertical distance measures $100 \times (x - x^{**})/|x^{**} - x^{*}|$, where $x^{*}$and x^{**}denote the starting and terminal values for x. Observe that the aggregate capital stock in the vintage model behaves non-monotonically. It overshoots its long-run value. This overshooting is due to the dramatic initial decline in aggregate investment that occurs in the vintage model. When the capital income tax rate is raised, aggregate investment in the vintage capital economy drops below the new steady-state value. Now, recall that aggregate investment, $p_1 \widehat{k}_1$, is the product of investment per plant, k_1, and the number of new plants that are being renovated, p_1. If k_1 drops by a factor of $\lambda < 1$ while p_1 falls by a factor $\varepsilon < 1$, then $p_1 k_1$ would decline by the amplified factor of $\lambda\varepsilon < \min(\lambda, \varepsilon)$; i.e. the proportional decline in $p_1 k_1$ is larger than the proportional declines in p_1 and k_1. The impact effect of the increase in the capital income tax rate is to cause both investment per new plant and the number of new plant being renovated to decline and this has an amplified effect on aggregate investment. It is interesting to note that this overshooting behavior in investment is absent when the economy is operating in the intensive, as opposed to the extensive, region. Finally, associated with the overshooting behavior in the capital stock there is overshooting in consumption and output — see Figure 16. The initial decline in investment spending allows consumption to rise in the short run.

It takes the vintage capital model much longer to adjust to the new capital income tax rate than does the standard model. As a measure of the speed of adjustment, define the cumulative λ-life to be the time T at which fraction λ of the total adjustment along the transition path for some variable x of interest has been undertaken. Thus, T solves $\min_T [\sum_{t=0}^{T} |x_t| - \lambda \sum_{t=0}^{\infty} |x_t|]$, where T is some nonnegative integer. As is evident from Table 5, the speed of adjustment for vintage model is much slower than in the standard model — the numbers for the standard model are in parentheses. This is of interest since the standard model has often been criticised for its high speed of adjustment or lack of propagation.

Figure 15

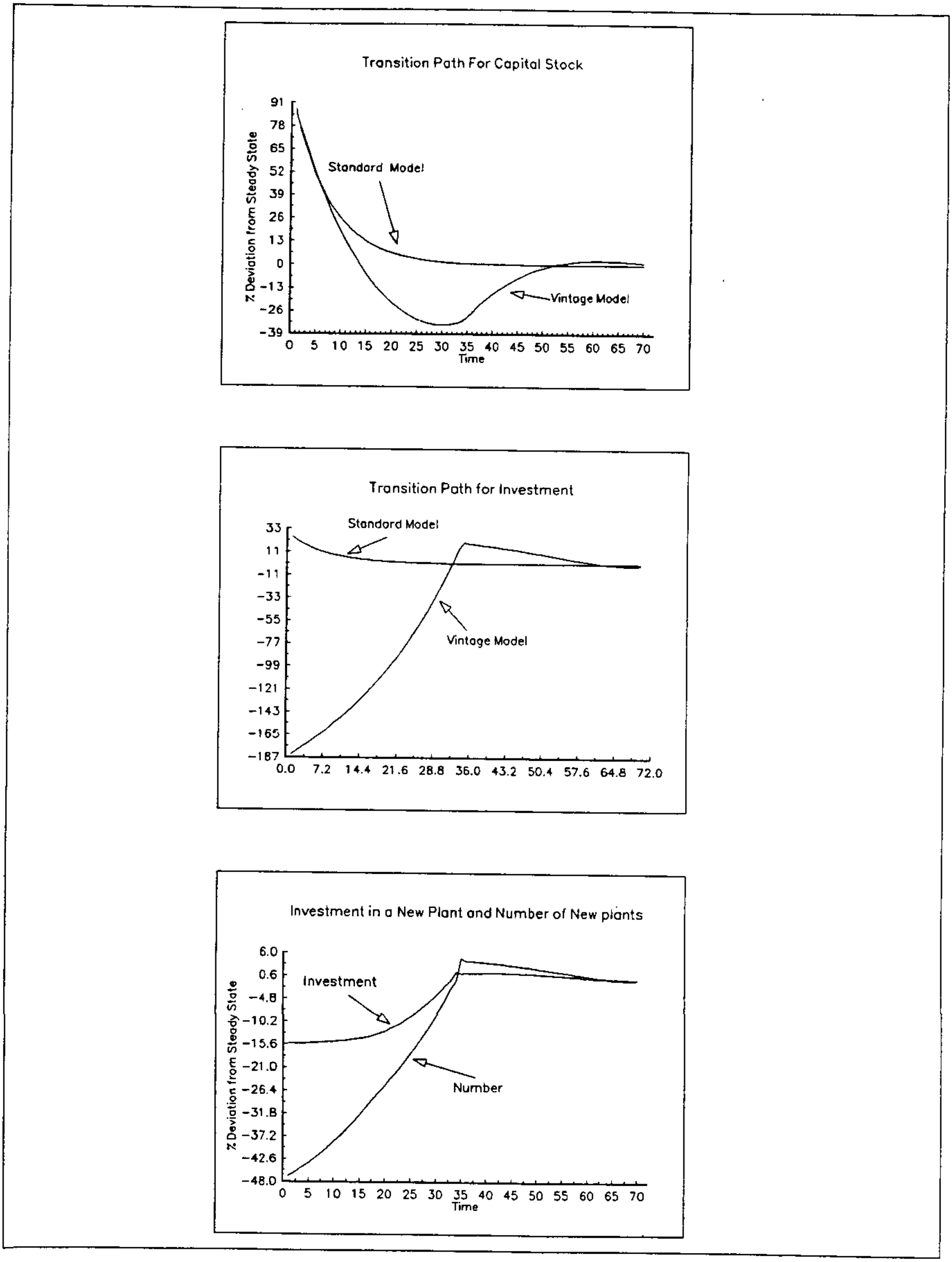

Transition Path For Capital Stock
% Deviation from Steady State
91
78
65
52
39
26
13
0
-13
-26
-39
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70
Time
Standard Model
Vintage Model
Transition Path for Investment
33
11
-11
-33
-55
-77
-99
-121
-143
-165
-187
0.0 7.2 14.4 21.6 28.8 36.0 43.2 50.4 57.6 64.8 72.0
Standard Model
Vintage Model
Investment in a New Plant and Number of New plants
% Deviation from Steady State
6.0
0.6
-4.8
-10.2
-15.6
-21.0
-26.4
-31.8
-37.2
-42.6
-48.0
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70
Time
Investment
Number

Figure 16

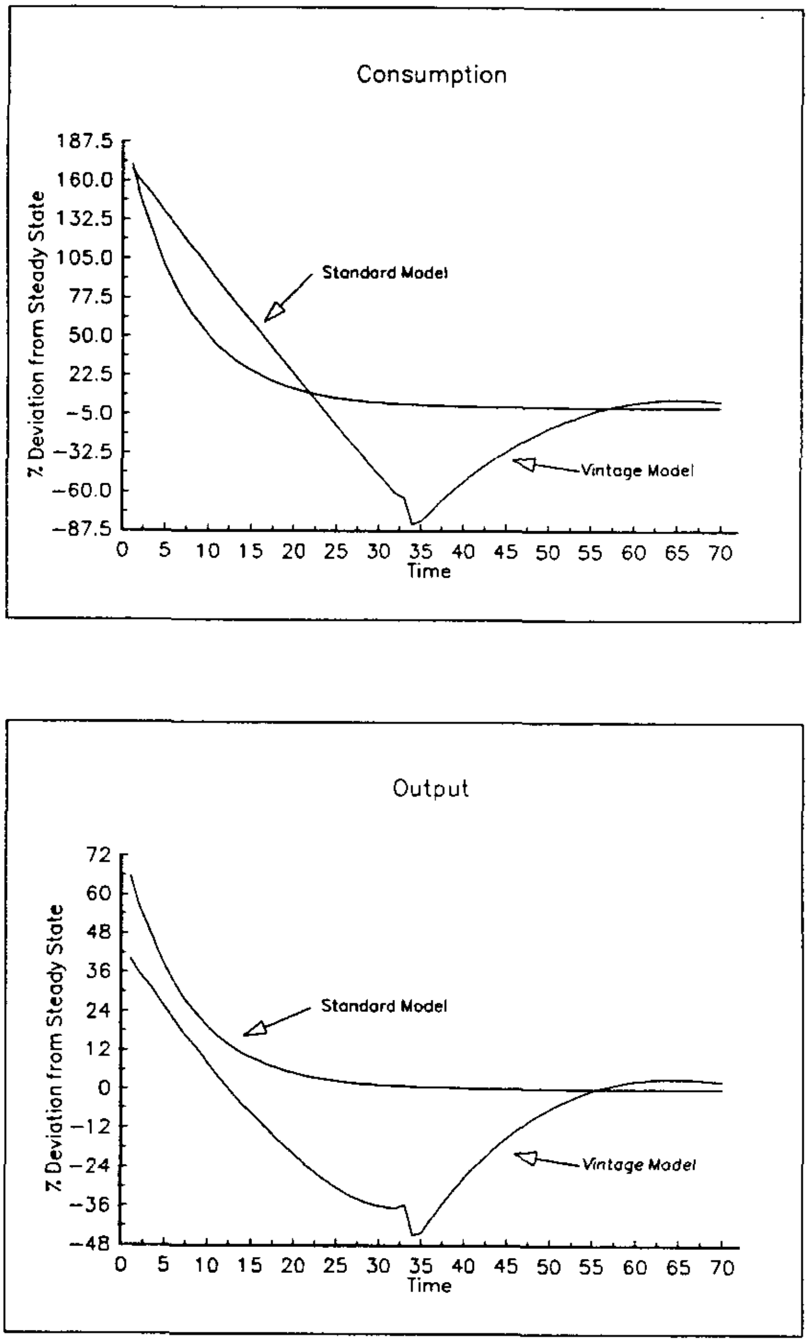

Consumption
187.5
160.0
132.5
105.0
77.5
50.0
22.5
-5.0
-32.5
-60.0
-87.5
% Deviation from Steady State
Standard Model
Vintage Model
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70
Time
Output
72
60
48
36
24
12
0
-12
-24
-36
-48
% Deviation from Steady State
Standard Model
Vintage Model
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70
Time

Table 5: Speed of Adjustment

λ-Life	Capital Stock	Investment	Output
25 percent	05 (2) years	06 (2) years	19 (2) years
50	22 (5)	12 (5)	30 (5)
75	32 (10)	20 (10)	37 (10)

Modeling transitional dynamics tends to lower the welfare effects of tax changes. For instance, in the standard model the welfare cost of raising the capital income tax rate from 29 to 30% is cut by 45% when the transitional dynamics are taken into account. This transpires because consumption initially rises along the adjustment path to the new lower steady-state stock capital stock. Consumption will fall below its initial level at some future date, but this is discounted. In the vintage model this process is extended even further. Taking transitional dynamics into account now reduces the welfare cost of the tax increase by a much larger 60%.

7. Summary

Microevidence suggests that investment at the plant level is lumpy. Additionally, there is macro and micro evidence that technological change is embodied through the introduction of new capital goods. To address these facts, a dynamic general equilibrium model of vintage capital is developed in this paper. Production in the economy is undertaken at a variety of locations or plants. Plants produce either consumption or capital goods using capital and two types of labor, skilled and unskilled. Over time more efficient capital goods can be produced because of investment in human capital by skilled agents. At each point in time, the owners of plants must decide whether to replace their existing old capital with more efficient new capital. In equilibrium, some plants will replace and others won't.

We reported the results of several experiments involving changes in the fiscal environment to illustrate how the model economy works. The quantitative analysis suggests that tax policy can have a significant effect on welfare, the distribution of income and the age distribution of the capital stock. The effects on economic growth are more moderate in this economy, but empirical work on the relationship between taxes and growth has not established the presence of a strong link. The vintage capital structure accounts for between thirty and forty percent of the growth and welfare effects of fiscal policies. Furthermore, the transitional dynamics for a vintage capital economy are very different and much more sluggish.

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A. Appendix

As was mentioned in the text, a key step in solving the model is to deflate all nonstationary period-t variables by functions of η_t to render them stationary. Specifically, let $\widehat{c}_t = c_t/\eta_t^{\alpha\zeta/(1-\alpha)}$, $\widehat{z}_t = z_t/\eta_t^{\alpha\zeta/(1-\alpha)}$, $\widehat{k}_{i,t} = k_{i,t}/\eta_t^{\zeta/(1-\alpha)}$, for $i = 1, ..., N$, $\widehat{q}_t = q_t/\eta_t^{-\zeta}$, where the circumflex over a variable denotes its transformed value. Let $\gamma_{\eta,t+1} = \eta_{t+1}/\eta_t = H(s_t)$, $\gamma_{y,t+1} = \gamma_{\eta,t+1}^{\alpha\zeta/(1-\alpha)}$, and $\gamma_{k,t+1} = \gamma_{\eta,t+1}^{\zeta/(1-\alpha)}$; observe that in general $\gamma_{y,t+1} \neq y_{t+1}/y_t$ and $\gamma_{k,t+1} \neq k_{t+1}/k_t$. Also, note that $w_t/\eta_t^{\alpha\zeta/(1-\alpha)} = \beta\mu\widehat{k}_{1,t}^{\alpha}l_{1,t}^{1-\beta}$, $v_t/\eta_t^{\alpha\zeta/(1-\alpha)-1} = \widehat{q}_t\zeta\widehat{k}_{1,t}^{\alpha}b_{1,t}^{\xi}h_{1,t}^{\zeta-1}$, $P(\cdot_t)/\eta_t^{\alpha\zeta/(1-\alpha)} = \widehat{P}(\cdot_t)$, $V_i(\cdot_t)/\eta_t^{\alpha\zeta/(1-\alpha)} = \widehat{V}_i(\cdot_t)$, $P_1(\cdot_t)/\eta_t^{-\zeta} = \widehat{P}_1(\cdot_t)$, and $V_{N+1}(\cdot_t)/\eta_t^{-\zeta} = \widehat{V}_{N+1}(\cdot_t)$.[21] Finally, it is easy to deduce that $\gamma_{\eta,t+1}^{\zeta/(1-\alpha)}\widehat{k}_{i+1,t+1} = \widehat{k}_{i,t}$. Using these facts, the equations governing the model's dynamics can be represented in the form shown below.

Labor Allocations [cf. (2.3), (2.4), and (2.5)]

$$l_{j,t} = [\widehat{k}_{j,t}/\widehat{k}_{1,t}]^{\alpha/(1-\beta)}l_{1,t}, \text{ for } j = 2, ..., J, \tag{A.1}$$

$$b_{j,t} = [\widehat{k}_{j,t}/\widehat{k}_{1,t}]^{\alpha/(1-\xi-\zeta)}b_{1,t}, \text{ for } j = 2, ..., J, \tag{A.2}$$

$$h_{j,t} = [\widehat{k}_{j,t}/\widehat{k}_{1,t}]^{\alpha/(1-\xi-\zeta)}h_{1,t}, \text{ for } j = 2, ..., J, \tag{A.3}$$

$$\widehat{q}\xi b_{1,t}^{\xi-1}(h_{1,t})^{\zeta} = \beta l_{1,t}^{\beta-1}. \tag{A.4}$$

Euler Equation for Aggregate Consumption [cf. (2.18) and (2.22)]

$$\begin{aligned}\rho[1 + (1 - \tau_k)r_{t+1}]\gamma_{y,t+1}[M\widehat{c}_{t+1} + \widehat{z}_{t+1} - M\tfrac{\Theta}{1+\theta}l_{t+1}^{1+\theta} - \tfrac{\Omega}{1+\omega}h_{t+1}^{1+\omega})] = \\ [M\widehat{c}_t + \widehat{z}_t - M\tfrac{\Theta}{1+\theta}l_t^{1+\theta} - \tfrac{\Omega}{1+\omega}h_t^{1+\omega}].\end{aligned} \tag{A.5}$$

Price of Capital [cf. (2.6)]

$$\widehat{q}_t = \frac{(1-\beta)l_{1,t}^{\beta}}{(1-\xi-\zeta)b_{1,t}^{\xi}h_{1,t}^{\zeta}}. \tag{A.6}$$

[21]The notation $\widehat{F}(\cdot_t)$ indicates that the arguments of $F(\cdot_t)$ are being evaluated at their transformed values in period t.

Plant Renovation [cf. (2.7), (2.10) and (2.11)]

$$(1-\tau_x)\widehat{q}_t(\gamma_{\eta,t+1})^{\zeta/(1-\alpha)}\widehat{k}_{1,t+1}- \\ \frac{1}{[1+(1-\tau_k)r_{t+1}]}\gamma_{y,t+1}[\widehat{V}_{1,t+1}-\widehat{V}_{i,t+1}]\begin{cases} \leq 0, & \text{if } p_{i+1,t+1}=0, \\ =0, & \text{if } 0<p_{i+1,t+1}<p_{1,t+1}, \\ \geq 0, & \text{if } p_{i+1,t+1}=p_{1,t+1}, \end{cases} \quad \text{for } i=2,...,N, \tag{A.7}$$

with

$$\begin{aligned}\widehat{V}_i(\cdot_{t+1}) \;=\;& (1-\tau_k)\widehat{P}(i;\cdot_{t+1})+\max[-(1-\tau_x)\widehat{q}_{t+1}(\gamma_{\eta,t+2})^{\zeta/(1-\alpha)}\widehat{k}_{1,t+2} \\ &+\tfrac{1}{[1+(1-\tau_k)r_{t+2}]}\gamma_{y,t+2}\widehat{V}_1(\cdot_{t+2}),\tfrac{1}{[1+(1-\tau_k)r_{t+2}]}\gamma_{y,t+2}\widehat{V}_{i+1}(\cdot_{t+2})],\end{aligned} \tag{A.8}$$

and

$$\sum_{j=1}^{N} p_{j,t}=1. \tag{A.9}$$

Physical Capital Accumulation [cf. (2.12) and (2.13)]

$$(1-\tau_x)p_{1,t+1}\widehat{q}_t=\frac{1}{[1+(1-\tau_k)r_{t+1}]}(H(e_t))^{-\zeta}(1-\tau_k)\widehat{V}_{N+1}(\cdot_{t+1}), \tag{A.10}$$

with

$$V_{N+1}(\cdot_{t+1})=p_{1,t+1}\alpha\widehat{k}_{1,t+1}^{\alpha-1}l_{1,t+1}^{\beta}+\sum_{j=1}^{N-1}p_{j+1,t+j+1}\frac{\widehat{k}_{1,t+1}^{\alpha-1}l_{j+1,t+1}^{\beta}}{\Pi_{m=1}^{j}[1+(1-\tau_k)r_{t+m+1}]}. \tag{A.11}$$

Labor-Leisure Choices [cf. (2.19) and (2.23)]

$$(1-\tau_l)\overbrace{\beta\widehat{k}_{1,t}^{\alpha}l_{1,t}^{\beta-1}}^{\widehat{w}_t}=\Theta l_t^{\theta}. \tag{A.12}$$

$$(1-\tau_l)\overbrace{\widehat{q}_t\zeta\widehat{k}_{1,t}^{\alpha}b_{1,t}^{\xi}h_{1,t}^{\zeta-1}}^{\widehat{v}_t}=\Omega h_t^{\omega}. \tag{A.13}$$

Human Capital Accumulation [cf. (2.24)]

$$\gamma_y\frac{[(1-\tau_l)\widehat{q}_t\zeta\widehat{k}_{1,t}^{\alpha}b_{1,t}^{\xi}h_{1,t}^{\zeta-1}+\Omega h_t^{\omega}H(e_{t+1})/H_1(e_t)]H_1(e_t)/H(e_t)}{[1+(1-\tau_k)r_{t+1}]}=\Omega h_t^{\omega}. \tag{A.14}$$

Resource and Budget Constraints [cf. (2.27), (2.28), (2.29), and (2.30)]

$$M\widehat{c}_t + \widehat{z}_t = f_t \sum_{j=1}^{N} p_j \widehat{k}_{j,t}^{\alpha} l_{j,t}^{\beta}, \tag{A.15}$$

$$p_{1,t+1}\gamma_{\eta,t+1}^{\zeta/(1-\alpha)}\widehat{k}_{1,t+1} = (1-f_t)\sum_{j=1}^{N} p_j \widehat{k}_{j,t}^{\alpha} b_{j,t}^{\xi} h_{j,t}^{\zeta}, \tag{A.16}$$

$$\sum_{j=1}^{N} p_{j,t}[f_t l_{j,t} + (1-f_t)b_{j,t}] = M l_t, \tag{A.17}$$

$$\sum_{j=1}^{N} p_{j,t}(1-f_t)h_{j,t} = h_t - e_t. \tag{A.18}$$

At time t the state of the transformed system is given by the $2N$ vector $\widehat{s}_t = (p_{1,t}, ..., p_{N,t}, \widehat{k}_{1,t}, ..., \widehat{k}_{N,t})$. Determined in this point in time, as functions of the state of the world, s_t, are: the firm's variables $l_{j,t}$, $b_{j,t}$, $h_{j,t}$, $p_{j,t+1}$, $\widehat{V}_{j,t+1}$, for $j = 1, ..., N$, and $\widehat{k}_{1,t+1}, \widehat{V}_{N+1,t+1}$; the households' variables $M\widehat{c}_t + \widehat{z}_t$, l_t, h_t, e_t; the market variables r_{t+1}, q_t,and f_t. The model's balanced growth path can be solved for using these equations. In balanced growth $\widehat{x}_t = \widehat{x}_{t+1}$, for all time t and variables $\widehat{x}$. Equations (A.1)-(A.18) represent a system of $5N + 9$ equations in $5N + 9$ unknowns. A difficulty associated with computing the balanced growth path is that equation (A.7) does not have to hold with equality; however, Lemma 2 places considerable structure on the range possibilities that can occur. To solve for the model's local dynamics, this system of equations is linearized around the balanced growth path. The resulting set of linearized equations is then represented as system of first-order linear difference equations. The dynamic path will be (locally) stable and unique provided that the system has associated with it exactly $2N - 1$ eigenvalues with modulus less than one — the number of state variables once p_N is solved out for using (A.9). This was the case for all examples studied. While the number of vintages remains fixed along a transition path given the local nature of the analysis, the number of old plants being renovated may vary depending on which of the two zones the economy is operating in. Finally, note that when computing the equilibrium path for the model there is no need to solve for $\widehat{c}_t$ and $\widehat{z}_t$ separately. All that matters is aggregate consumption, $M\widehat{c}_t + \widehat{z}_t$, which appears in (A.5) and (A.15). The aggregate Euler equations obtains from summing over the individual Euler equations, (2.18) and (2.22), and is a consequence of the assumed form for the momentary utility functions, (2.17) and (2.25). The equilibrium path for the model is independent of the distribution of wealth between skilled and unskilled agents.