The CES Production Functions

The constant elasticity of substitution production functions dominates in applied research. The parametric structure is

(1) $$Y = A\,[\,\theta(a_K K)^{\gamma} + (1-\theta)\,(a_N N)^{\gamma}\,]^{1/\gamma}.$$

Here $0 < \theta < 1$ is the share parameter and γ determines the degree of substitutability of the inputs. The parameters A, a_K, and a_N depend upon the units in which the output and inputs are measured and play no important role. The value of γ is less than or equal to 1 and can be $-\infty$. The two extreme cases are when $\gamma = 1$ or $\gamma = -\infty$.

The Case of Perfect Substitution (g= 1): The function is

(2) $$Y = A\,[\,\theta\, a_K K + (1-\theta)\, a_N N\,].$$

The isoquants are straight lines for this production function.

The Case of no Substitution (g= -¥): The function is

(3) $$Y = A \min\{\, a_K K,\, a_N N \,\}.$$

The isoquants are at right angles. Factors are used in fixed proportions

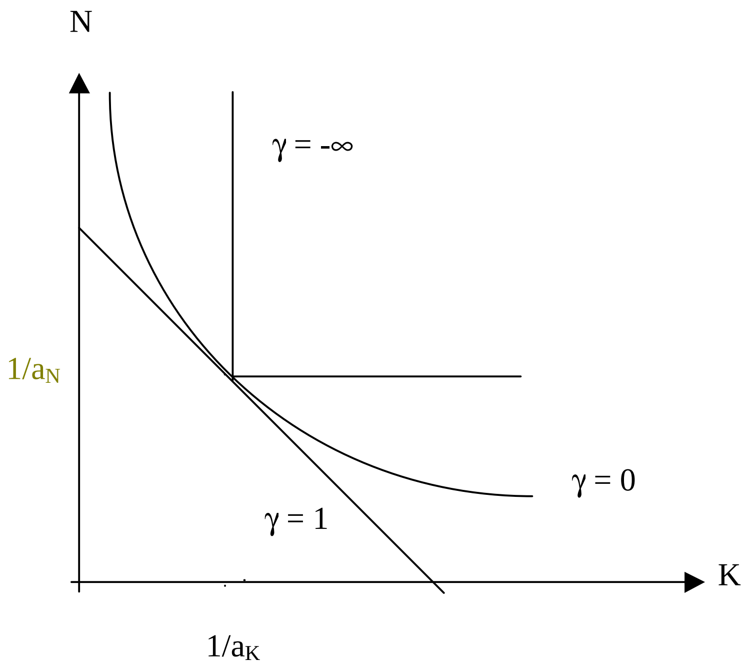
Exercise: Prove that this is the function obtained as $\gamma \to -\infty$ in (1).

The Case of Unit Elasticity of Substitution (g= 0): The function is

(4) $$Y = A\, K^{\theta}\, N^{(1-\theta)}.$$

Figure 1 plots the Y = A isoquant for these three cases.

Figure 1: CES Production Function Isoquants



Exercise: Prove that function (4) is the limit of (1) as $\gamma \rightarrow 0$. [This is an advanced exercise in calculus and requires the use of the L'Hospital's rule.]

Function (4) is the Cobb-Douglas production function and is the one that is most heavily used in aggregate economic analyses. The reason that it is the one used is that, under competition, this is the only production function with the property that factor income shares are independent of relative factor prices. This property of the Cobb-Douglas production function is consistent with the data as, historically, the real wage w in the U.S. has increased by a factor of ten or twenty while the rental price of capital and factor income shares have remained roughly constant.

Exercise: Show that if factors are paid their marginal product that for the Cobb-Douglas production function, factor income shares are θ for capital and $1 - \theta$ for labor.