A Note on Continuity of Expected Utility

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Abstract

This note is about preferences over a set of probability measures on a continuous set. Debreu derived conditions under which preferences are representable by an expected utility function. One might ask when is an expected utility function continuous. Evidently, a necessary condition is that preferences over degenerate measures are continuous. Unfortunately, this condition is not sufficient, as a simple example shows. In this note we derive a slightly stronger condition which will ensure expected utility to be continuous.

Given:

- $(S, \|.\|)$ a normed linear space.
- $\mathcal{B}(S)$ Borel subsets of S.
- $C \subseteq S$
- X: set of probability measures on $\mathcal{B}(S)$ with support $\subseteq C$.
- $\Delta = \{\delta_c : c \in C\} \subseteq X$ where δ_c puts measure 1 on $c \in C$.

  Suppose an individual has regular preferences $\succeq$ over the probability measures that satisfy the following assumptions for all $x, x_1, x_2 \in X$:

A1: $x_1 \prec x_2$ if and only if $\alpha x_1 + (1-\alpha)x \prec \alpha x_2 + (1-\alpha)x$ for all $\alpha \in (0,1)$ and for all $x \in X$.

A2: Given $x_1 \prec x \prec x_2$ there exist $\alpha, \beta \in (0,1)$ such that:

$$x \prec \alpha x_1 + (1-\alpha)x_2 \qquad \wedge \qquad x \succ \beta x_1 + (1-\beta)x_2$$

A3: $\{c \in C : \alpha\delta_c + (1-\alpha)d_1 \preceq \beta d_2 + (1-\beta)d_3\} \in \mathcal{B}(S)$ for all $\alpha, \beta \in (0,1)$ and all $d_1, d_2, d_3 \in \Delta$.

Result by Debreu Given A1, A2, A3, the individual has a utility function $U : X \to \mathbb{R}$ which is representable by:

$$U(x) = \int u(c)x(dc)$$

where $u : C \to \mathbb{R}$.

That is, the utility function U is representable as an expected value of u. We want to find sufficient conditions for the function U to be continuous, that is, we want to answer the question, what guarantees that $x_n \to x$ implies $U(x_n) \to U(x)$. Note, that we are dealing with measures x_n and thus we have to define what it means that a sequence of measures x_n converges to x.

Definition 1 *Weak Convergence: We say that $x_n \to x$ if*

$$\int g(c)x_n(dc) \to \int g(c)x(dc)$$

for all bounded continuous functions g.

If we can find conditions that make u bounded and continuous, then U will be continuous.

We will proceed as follows: First we find a condition on $\succeq$ that makes u continuous. It will be difficult to find conditions on preferences that will guarantee the existence of a bounded u. However, if C is compact, then a continuous function $u : C \to \mathbb{R}$ is necessarily bounded.

Since u is utility of a delta measure, a necessary condition for a continuous u will definitely be:

A4′: $\succeq$ is continuous on Δ, that is, for all $\delta' \in \Delta$, the sets $\{\delta : \delta \succeq \delta'\}$ and $\{\delta : \delta \preceq \delta'\}$ are closed on Δ.

Is A4$'$ enough to guarantee continuity of u? Unfortunately, A4$'$ is only necessary but not sufficient as the following counter example shows: Let $C = [0,1]$. Let $v : C \to \mathbb{R}$ be such that:

$$v(c) = \begin{cases} c & \text{if } 0 \leq c < 1 \\ 2 & \text{if } c = 1 \end{cases}$$

define $\succeq$ on X as:

$$x \succeq x' \Leftrightarrow U(x) = \int v(c)x(dc) \geq \int v(c)x'(dc) = U(x')$$

for all $x, x' \in X$. By construction, U is representable as an expected utility function. Note that $\succeq$ is continuous on C since weak lower and upper contour sets are closed. However, there exists no continuous u such that $\int u(c)x(dc)$ is a utility function for $\succeq$ on X, because only increasing affine transformations of v will work and they will still have a discontinuity at $c = 1$. Take for example $x \in X$ putting mass 0.9 on 1 and mass 0.1 on 0, and $x_n = \delta_{1-\frac{1}{n}}$. Then, evidently, $x \succ x_n$ for all n. However, with a continuous u:

$$\int u(c)x_n(dc) \to u(1) > 0.9u(1) + 0.1u(0)$$

Hence, $\int u(c)x(dc)$ cannot be a utility function for $\succeq$ on X, because it ranks x and x_n incorrectly if n is large enough. The example shows that we need a stronger assumption than A4. Let us try:

A4: Let $r, r', r'', r_n \in \Delta$ for all $n = 1, 2, \ldots$, where $r_n \to r$ as $n \to \infty$. Then, for all $\alpha \in [0,1]$:

$$\alpha r_n + (1-\alpha)r' \preceq r'' \;\; \forall n \Longrightarrow \alpha r + (1-\alpha)r' \preceq r''$$

and

$$\alpha r_n + (1-\alpha)r' \succeq r'' \;\; \forall n \Longrightarrow \alpha r + (1-\alpha)r' \succeq r''$$

Note that A4$'$ is just a special case of A4 if we set $\alpha = 1$. Condition A4 is stronger in the sense that it requires continuity of preferences for a very small class of non-degenerate probability measures, in particular, probability measures that are constructed as a mix of two degenerate probability measures having probability α and $1-\alpha$, respectively.

Proposition 2 *A4 indeed guarantees continuity of* u.

Proof. Fix $r \in \Delta$. Let $S = \{r_n\}_{n=1}^{\infty}$, $r_n \in \Delta$ for all n and $r_n \to r$. Suppose, by way of getting a contradiction that: $\lim\limits_{n\to\infty} u(r_n) \neq u(r)$, that is, there exists a $\zeta > 0$ and a subsequence $S_1 = \{r_{n_k}\}_{k=1}^{\infty}$ of S such that:

$$u(r_{n_k}) \notin [u(r) - \zeta, u(r) + \zeta] \ \ \forall k$$

Without loss of generality, assume that S_1 is such that $u(r_{n_k}) < u(r) - \zeta$ for all k. Pick r', r'' such that:

$$\sup_k u(r_{n_k}) \leq u(r'') < u(r')$$

Define: $u_0 = \sup\limits_k u(r_{n_k}), u_1 = u(r'), u_2 = u(r'')$. Let

$$\alpha = \frac{u_1 - u_2}{u_1 - u_0}$$

Note that $0 < \alpha \leq 1$. Then for all k:

$$\begin{aligned}
\alpha u(r_{n_k}) + (1-\alpha)u(r') &\leq \frac{u_1 - u_2}{u_1 - u_0} u_0 + \frac{u_2 - u_0}{u_1 - u_0} u_1 \\
&= u_2 = u(r'')
\end{aligned}$$

Hence, for all k: $\alpha r_{n_k} + (1-\alpha)r' \preceq r''$. But $r_{n_k} \to r$ as $k \to \infty$ and:

$$\begin{aligned}
\alpha u(r) + (1-\alpha)u(r') &\geq \alpha(u_0 + \zeta) + u_1 \\
&= \alpha u_0 + (1-\alpha)u_1 + \alpha\zeta \\
&= u_2 + \alpha\zeta \\
&> u_2 = u(r'')
\end{aligned}$$

So, $\alpha r + (1-\alpha)r' \succ r''$. This is a violation to the first part of A4. If we assumed $u(r_{n_k}) > u(r) + \zeta$ for all k, then we could have gotten a violation to the second part of A4 after the same kind of reasoning. ■

Proposition 3 *If in addition to A1-A4,* C *is compact, then* U *is continuous.*

Proof. This follows from basic real analysis. ■