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TECHNICAL APPENDIX: Can Sticky Price Models Generate Volatile and Persistent Real Exchange Rates?†

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These notes contain derivations of expressions and results reported in the main text. We also provide intuition for some of the results and details on the computation of equilibria.

1. The Model

The model is a two-country business cycle model. Each country is populated by a large number of identical, infinitely-lived consumers. In both countries, intermediate goods are combined to form final goods which are country specific and cannot be shipped. All trade between the countries is in intermediate goods that are produced by monopolists who can charge different prices in the two countries. We assume that each intermediate goods producer has the exclusive right to sell his own good in the two countries. Thus, there is no possibility for arbitraging away price differences in intermediate goods.

1.1. Notation

Goods produced in the home country are subscripted with an H, while those produced in the foreign country are subscripted with an F. Allocations and prices in the foreign country are denoted with an asterisk. We use a caret over a variable to denote its logged deviation from the mean.

1.2. Uncertainty

In each period t, the economy experiences one of finitely many events s_t. We denote by $s^t = (s_0, \ldots, s_t)$ the history of events up through and including period t. The probability, as of period zero, of any particular history s^t is $\pi(s^t)$. The initial realization s_0 is given.

1.3. The Final Goods Producers

Final goods producers behave competitively and solve a static profit-maximization problem. In the home country in each period producers choose inputs $y_H(i)$ for $i \in [0,1]$ and $y_F(i)$ for $i \in [0,1]$ and output y to maximize profits given by

$$\max \ Py - \int_0^1 P_H(i) y_H(i)\, di - \int_0^1 P_F(i) y_F(i)\, di \tag{1.1}$$

subject to

$$y = \left[\omega_1 \left(\int_0^1 y_H(i)^\theta di\right)^{\rho/\theta} + \omega_2 \left(\int_0^1 y_F(i)^\theta di\right)^{\rho/\theta}\right]^{\frac{1}{\rho}} \tag{1.2}$$

where y is the final good, P is the price of the final good, $y_H(i)$ and $y_F(i)$ are intermediate goods produced in the home and foreign countries, respectively, and $P_H(i)$ and $P_F(i)$ are their prices. These prices (P, P_H, P_F) are in units of the domestic currency.

The first-order conditions of the problem above with respect to $y_H(i)$ and $y_F(i)$ are

$$y_H(i) = \left[\frac{\omega_1 P y^{1-\rho} \left(\int y_H(i)^\theta \, di\right)^{\frac{\rho}{\theta}-1}}{P_H(i)}\right]^{\frac{1}{1-\theta}}$$

$$y_F(i) = \left[\frac{\omega_2 P y^{1-\rho} \left(\int y_F(i)^\theta \, di\right)^{\frac{\rho}{\theta}-1}}{P_F(i)}\right]^{\frac{1}{1-\theta}}$$

If we raise the first expression to the power θ, integrate across i, and solve for $\int y_H(i)^\theta \, di$, we get

$$\begin{aligned}\int y_H(i)^\theta \, di &= \left[\omega_1 P y^{1-\rho}\right]^{\frac{\theta}{1-\theta}} \left(\int y_H(i)^\theta \, di\right)^{\frac{\rho-\theta}{1-\theta}} \int P_H(i)^{\frac{\theta}{\theta-1}} \, di \\ &= \left[\omega_1 P\right]^{\frac{\theta}{1-\rho}} y^\theta \left(\int P_H(i)^{\frac{\theta}{\theta-1}} \, di\right)^{\frac{1-\theta}{1-\rho}}.\end{aligned}$$

Similarly, we can derive an expression for $\int y_F(i)^\theta \, di$. Substituting these expressions into the first-order conditions, we get the input demand functions:

$$y_H(i) = \frac{\left[\omega_1 P\right]^{\frac{1}{1-\rho}} \bar{P}_H^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}}}{P_H(i)^{\frac{1}{1-\theta}}} \ y \tag{1.3}$$

$$y_F(i) = \frac{\left[\omega_2 P\right]^{\frac{1}{1-\rho}} \bar{P}_F^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}}}{P_F(i)^{\frac{1}{1-\theta}}} \ y \tag{1.4}$$

where $\bar{P}_I = \left(\int_0^1 P_I(i)^{\frac{\theta}{\theta-1}}\, di\right)^{\frac{\theta-1}{\theta}}$ for $I = H, F$.

To get the price of the final good, we use the zero-profit condition which implies that

$$P = \left(\omega_1^{\frac{1}{1-\rho}}\, \bar{P}_H^{\frac{\rho}{\rho-1}} + \omega_2^{\frac{1}{1-\rho}}\, \bar{P}_F^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} . \tag{1.5}$$

If we solve the analogous problem for the foreigners, we get:

$$y_F^*(i) = \frac{[\omega_1 P^*]^{\frac{1}{1-\rho}}\, (\bar{P}_F^*)^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}}}{P_F^*(i)^{\frac{1}{1-\theta}}}\; y^* \tag{1.6}$$

$$y_H^*(i) = \frac{[\omega_2 P^*]^{\frac{1}{1-\rho}}\, (\bar{P}_H^*)^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}}}{P_H^*(i)^{\frac{1}{1-\theta}}}\; y^* \tag{1.7}$$

$$P^* = \left(\omega_1^{\frac{1}{1-\rho}} (\bar{P}_F^*)^{\frac{\rho}{\rho-1}} + \omega_2^{\frac{1}{1-\rho}} (\bar{P}_H^*)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} . \tag{1.8}$$

1.4. The Consumer's Problem

The consumer chooses consumption c, labor l, and real balances M/P to maximize:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t)\; U\left(c(s^t), l(s^t), M(s^t)/P(s^t)\right), \tag{1.9}$$

subject to the sequence of budget constraints

$$\begin{aligned} P(s^t)c(s^t) + M(s^t) + \sum_{s_{t+1}} Q(s^{t+1}|s^t)B(s^{t+1}) \\ \leq P(s^t)w(s^t)l(s^t) + M(s^{t-1}) + B(s^t) + \Pi(s^t) + T(s^t) \end{aligned} \tag{1.10}$$

and borrowing constraints $B(s^{t+1}) \geq -P(s^t)\bar{b}$. M and B are their holdings of money and contingent claims, Q is the price of the claims, w is the real wage, Π are profits, and T are government transfers.

The first-order conditions for the consumer are:

$$-\frac{U_l(s^t)}{U_c(s^t)} = w(s^t) \tag{1.11}$$

$$\frac{U_m(s^t)}{P(s^t)} - \frac{U_c(s^t)}{P(s^t)} + \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t) \frac{U_c(s^{t+1})}{P(s^{t+1})} = 0 \tag{1.12}$$

$$Q(s^t|s^{t-1}) = \beta \pi(s^t|s^{t-1}) \frac{U_c(s^t)}{U_c(s^{t-1})} \frac{P(s^{t-1})}{P(s^t)} \tag{1.13}$$

where $U(s^t)$ is shorthand notation for $U(c(s^t), l(s^t), M(s^t)/P(s^t))$.

Let $R(s^t)$ and $r(s^t)$ be the gross and net nominal interest rates, respectively; they are defined as follows:

$$\begin{aligned} \frac{1}{R(s^t)} = \frac{1}{1+r(s^t)} &= \sum_{s_{t+1}} Q(s^{t+1}|s^t) \\ &= \sum_{s_{t+1}} \beta \pi(s^{t+1}|s^t) \frac{U_c(s^{t+1})}{U_c(s^t)} \frac{P(s^t)}{P(s^{t+1})} \end{aligned} \tag{1.14}$$

and $r(s^t) = R(s^t) - 1$. Notice that we can also write the money demand equation in (1.12) more simply,

$$\frac{U_m(s^t)}{U_c(s^t)} = \frac{r(s^t)}{1+r(s^t)} \tag{1.15}$$

using the definition in (1.14).

1.5. The Foreign Consumer's Problem

The foreign consumer solves a problem similar to that of the home consumer, namely to maximize:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t)\ U\left(c^*(s^t), l^*(s^t), M^*(s^t)/P^*(s^t)\right), \tag{1.16}$$

subject to the sequence of budget constraints

$$\begin{aligned} &P^*(s^t)c^*(s^t) + M^*(s^t) + \sum_{s_{t+1}} Q^*(s^{t+1}|s^t)B_F^*(s^{t+1}) + \sum_{s_{t+1}} Q(s^{t+1}|s^t)B_H^*(s^{t+1})/e(s^t) \\ &\quad \leq P^*(s^t)w^*(s^t)l^*(s^t) + M^*(s^{t-1}) + B_F^*(s^t) + B_H^*(s^t)/e(s^t) + \Pi^*(s^t) + T^*(s^t) \end{aligned} \tag{1.17}$$

where $B_H^*(s^t)$ and $B_F^*(s^t)$ denotes the foreign consumer's holdings of home and foreign country bonds in state s^t, respectively. The first-order conditions with respect to bonds are given by:

$$Q(s^t|s^{t-1}) = \beta\pi(s^t|s^{t-1})\frac{U_c^*(s^t)}{U_c^*(s^{t-1})}\frac{P^*(s^{t-1})}{P^*(s^t)}\frac{e(s^{t-1})}{e(s^t)} \tag{1.18}$$

$$Q^*(s^t|s^{t-1}) = \beta\pi(s^t|s^{t-1})\frac{U_c^*(s^t)}{U_c^*(s^{t-1})}\frac{P^*(s^{t-1})}{P^*(s^t)} \tag{1.19}$$

If we equate the bond prices derived here (in (1.18)) and above (in (1.13)), we get

$$\frac{U_c(s^t)}{U_c(s^{t-1})}\frac{P(s^{t-1})}{P(s^t)} = \frac{U_c^*(s^t)}{U_c^*(s^{t-1})}\frac{P^*(s^{t-1})}{P^*(s^t)}\frac{e(s^{t-1})}{e(s^t)}.$$

We can iterate back to 0 and let $q(s^t) = e(s^t)P^*(s^t)/P(s^t)$ denote the real exchange rate. Then we find

$$q(s^t) = \kappa\frac{U_c^*(s^t)}{U_c(s^t)} \tag{1.20}$$

where κ is the real exchange rate at 0 times the ratio of marginal utilities at 0. When computing an equilibrium, we normalize κ to 1.

1.6. The Intermediate Goods Producers

Intermediate goods producers are monopolistically competitive. They set prices for their goods, but they most hold them fixed for N periods. We assume that price-setting is done in a staggered fashion so that $1/N$ of the firms are setting in a particular period. We compute a symmetric equilibrium so we assume that all firms $i \in [0, 1/N]$ behave the same way and all firms $i \in [1/N, 2/N]$ behave the same way, and so on.

The problem solved by the home intermediate goods producers setting prices is to choose sequences of prices P_H, capital stocks k, investments x, and labor inputs l to maximize

$$\begin{aligned}\sum_{\tau=0}^{\infty}\sum_{s^\tau}\tilde{Q}(s^\tau)\big[&P_H(i,s^\tau)y_H(i,s^\tau) + e(s^\tau)P_H^*(i,s^\tau)y_H^*(i,s^\tau)\\ &- P(s^\tau)w(s^\tau)l(i,s^\tau) - P(s^\tau)x(i,s^\tau)\big]\end{aligned} \tag{1.21}$$

subject to the input demands (1.3) and (1.4), the production technology:

$$y_H(i,s^t)+y_H^*(i,s^t)=F(k(i,s^{t-1}),l(i,s^t)), \tag{1.22}$$

the law of motion for capital used in producing good i:

$$k(i,s^t)=(1-\delta)k(i,s^{t-1})+x(i,s^t)-\phi\left(\frac{x(i,s^t)}{k(i,s^{t-1})}\right)k(i,s^{t-1}) \tag{1.23}$$

and the following constraints on prices:

$$\begin{aligned}
P_H(i,s^{t-1}) &= P_H(i,s^t)=\dots P_H(i,s^{t+N-1})\\
P_H(i,s^{t+N}) &= P_H(i,s^{t+N+1})=\dots P_H(i,s^{t+2N-1})\\
&\vdots\\
P_H^*(i,s^{t-1}) &= P_H^*(i,s^t)=\dots P_H^*(i,s^{t+N-1})\\
P_H^*(i,s^{t+N}) &= P_H^*(i,s^{t+N+1})=\dots P_H^*(i,s^{t+2N-1})\\
&\vdots
\end{aligned} \tag{1.24}$$

where $\tilde{Q}(s^\tau)$ is the τth period Arrow-Debreu price (that is, a product of the one-period $Q(s^t|s^{t-1})$'s).

The Lagrangian in this case is

$$\begin{aligned}
\mathcal{L}=\dots+\tilde{Q}(s^t)\Bigg\{\Big[&P_H(i,s^{t-1})^{\frac{\theta}{\theta-1}}\Lambda_H(s^t)+e(s^t)P_H^*(i,s^{t-1})^{\frac{\theta}{\theta-1}}\Lambda_H^*(s^t)\\
&\quad-P(s^t)w(s^t)l(i,s^t)-P(s^t)x(i,s^t)\\
&+\zeta(s^t)\big\{F(k(i,s^{t-1}),l(i,s^t))-\Lambda_H(s^t)P_H(i,s^{t-1})^{\frac{1}{\theta-1}}\\
&\quad-\Lambda_H^*(s^t)P_H^*(i,s^{t-1})^{\frac{1}{\theta-1}}\big\}\\
&+\lambda(s^t)\big\{(1-\delta)k(i,s^{t-1})+x(i,s^t)\\
&\quad-\phi(x(i,s^t)/k(i,s^{t-1}))k(i,s^{t-1})-k(i,s^t)\big\}\Big]\\
+\sum_{s_{t+1}}Q(s^{t+1}|s^t)\Big[&P_H(i,s^{t-1})^{\frac{\theta}{\theta-1}}\Lambda_H(s^{t+1})+e(s^{t+1})P_H^*(i,s^{t-1})^{\frac{\theta}{\theta-1}}\Lambda_H^*(s^{t+1})
\end{aligned}$$

$$- P(s^{t+1})w(s^{t+1})l(i,s^{t+1}) - P(s^{t+1})x(i,s^{t+1})$$

$$+ \zeta(s^{t+1})\big\{F(k(i,s^t),l(i,s^{t+1})) - \Lambda_H(s^{t+1})P_H(i,s^{t-1})^{\frac{1}{\theta-1}}$$

$$- \Lambda_H^*(s^{t+1})P_H^*(i,s^{t-1})^{\frac{1}{\theta-1}}\big\}$$

$$+ \lambda(s^{t+1})\big\{(1-\delta)k(i,s^t) + x(i,s^{t+1})$$

$$- \phi(x(i,s^{t+1})/k(i,s^t))k(i,s^t) - k(i,s^{t+1})\big\}\Big]$$

$$+ \ldots\Bigg\} \tag{1.25}$$

where

$$\Lambda_H(s^t) = \big[\omega_1 P(s^t)\big]^{\frac{1}{1-\rho}}\ \bar{P}_H(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}}\ y(s^t) \tag{1.26}$$

$$\Lambda_H^*(s^t) = \big[\omega_2 P^*(s^t)\big]^{\frac{1}{1-\rho}}\ (\bar{P}_H^*(s^{t-1}))^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}}\ y^*(s^t). \tag{1.27}$$

The variables ζ and λ are multipliers for constraints (1.22) and (1.23), respectively.

Taking the derivative of $\mathcal{L}$ in (1.25) with respect to the monopolist's prices $P_H(i,s^{t-1})$ and $P_H^*(i,s^{t-1})$, we find

$$\sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1})\left\{\theta P_H(i,s^{t-1})^{\frac{1}{\theta-1}}\Lambda_H(s^\tau) - \zeta(s^\tau)P_H(i,s^{t-1})^{\frac{2-\theta}{\theta-1}}\Lambda_H(s^\tau)\right\}=0 \tag{1.28}$$

$$\sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1})\left\{\theta e(s^\tau)P_H^*(i,s^{t-1})^{\frac{1}{\theta-1}}\Lambda_H^*(s^\tau) - \zeta(s^\tau)P_H(i,s^{t-1})^{\frac{2-\theta}{\theta-1}}\Lambda_H^*(s^\tau)\right\}=0. \tag{1.29}$$

Taking the derivative of $\mathcal{L}$ with respect to $l(i,s^t)$, we find:

$$-P(s^t)w(s^t) + \zeta(s^t)F_l(i,s^t) = 0. \tag{1.30}$$

The derivative of $\mathcal{L}$ with respect to $x(i,s^t)$ is:

$$-P(s^t) + \lambda(s^t)\left[1 - \phi'\left(\frac{x(i,s^t)}{k(i,s^{t-1})}\right)\right] = 0. \tag{1.31}$$

Finally, the derivative of $\mathcal{L}$ with respect to $k(i, s^t)$ is:

$$-\lambda(s^t) + \sum_{s_{t+1}} Q(s^{t+1}|s^t)\Big\{\zeta(s^{t+1})F_k(i, s^{t+1}) + \lambda(s^{t+1})\Big[1 - \delta - \phi\left(\frac{x(i, s^{t+1})}{k(i, s^t)}\right) + \phi'\left(\frac{x(i, s^{t+1})}{k(i, s^t)}\right)\frac{x(i, s^{t+1})}{k(i, s^t)}\Big]\Big\} = 0. \tag{1.32}$$

If we substitute expressions for the multipliers using (1.30) and (1.31) into (1.28), (1.29), and (1.32), we get

$$P_H(i, s^{t-1}) = \frac{\sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1})P(s^\tau)mc(i, s^\tau)\Lambda_H(s^\tau)}{\theta \sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1})\Lambda_H(s^\tau)} \tag{1.33}$$

$$P_H^*(i, s^{t-1}) = \frac{\sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1})P(s^\tau)mc(i, s^\tau)\Lambda_H^*(s^\tau)}{\theta \sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1})e(s^\tau)\Lambda_H^*(s^\tau)} \tag{1.34}$$

$$\frac{U_c(s^t)}{1 - \phi'(i, s^t)} = \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t)U_c(s^{t+1})\Big\{mc(i, s^{t+1})F_k(i, s^{t+1}) + \frac{1}{1 - \phi'(i, s^{t+1})}\Big[1 - \delta - \phi(i, s^{t+1}) + \phi'(i, s^{t+1})\frac{x(i, s^{t+1})}{k(i, s^t)}\Big]\Big\}. \tag{1.35}$$

Note that we have used the fact that marginal costs of producer i are given by:

$$mc(i, s^t) = w(s^t)/F_l(i, s^t). \tag{1.36}$$

1.7. The Government

Monetary policy is modeled as an exogenous process for monetary growth rates, that is

$$M(s^t) = \mu(s^t)M(s^{t-1}) \tag{1.37}$$

where μ is a stochastic process. The process that we use is

$$\log \mu(s^t) = \rho_\mu \log \mu(s^{t-1}) + (1 - \rho_\mu) \log \mu + \epsilon_{\mu,t}. \tag{1.38}$$

The home government budget constraint is given by:

$$T(s^t) = M(s^t) - M(s^{t-1}). \tag{1.39}$$

where T are transfers to consumers.

1.8. Additional Equilibrium Conditions

We need some additional conditions before computing an equilibrium. The resource constraint in the home country is given by

$$y(s^t) = c(s^t) + \int_0^1 x(i, s^t)\, di. \tag{1.40}$$

The labor market clearing condition is given by

$$l(s^t) = \int l(i, s^t)\, di. \tag{1.41}$$

There are analogous conditions for the foreign country.

1.9. Aggregates of Interest

Nominal and real net exports are defined as follows:

$$NX(s^t) = e(s^t) \int P_H^*(i, s^{t-1}) y_H^*(i, s^t)\, di - \int P_F(i, s^{t-1}) y_F(i, s^t)\, di \tag{1.42}$$

$$nx(s^t) = NX(s^t)/P(s^t). \tag{1.43}$$

Nominal and real GDP are defined as follows:

$$GDP(s^t) = P(s^t)y(s^t) + NX(s^t) \tag{1.44}$$

$$gdp(s^t) = y(s^t) + \int y_H^*(i, s^t)\, di - \int y_F(i, s^t)\, di. \tag{1.45}$$

1.10. Extensions

The economy just described is our benchmark economy. We now describe four extensions that we also consider.

1.10.1. Taylor Rule as Fed Policy

Above we assumed that there was an exogenous process for the monetary growth rates. We also consider cases where the Fed follows a Taylor-like interest rate-setting rule. In particular, we assume that the nominal interest rate is given by

$$r(s^t) = a' \begin{bmatrix} r(s^{t-1}) \\ r(s^{t-2}) \\ r(s^{t-3}) \\ E_t \log P(s^{t+1}) - \log P(s^t) \\ \log P(s^t) - \log P(s^{t-1}) \\ \log P(s^{t-1}) - \log P(s^{t-2}) \\ \log P(s^{t-2}) - \log P(s^{t-3}) \\ \log gdp(s^t) \\ \log gdp(s^{t-1}) \\ \log gdp(s^{t-2}) \end{bmatrix} + \text{constant} + \epsilon_{r,t} \tag{1.46}$$

with the foreign rate defined analogously. In this case, we back out money from the money demand equation (1.15), $U_m/U_c = r/(1+r)$, once we know consumption, labor, and the interest rate.

1.10.2. Incomplete Asset Markets

When markets are complete, we can use the first-order conditions for state contingent bonds in the two countries (that is, (1.13) and (1.18)) to back out an expression for the real exchange rate, $q(s^t) = U_c^*(s^t)/U_c(s^t)$. In the incomplete-markets extension, we assume that there is a bond market in the home country but no cross-country contingent claims. The budget constraints for the home and foreign country are then given by:

$$\begin{aligned} P(s^t)c(s^t) + M(s^t) + \sum_{s_{t+1}} Q(s^{t+1}|s^t)B(s^{t+1}) + V(s^t)D(s^t) \\ \leq P(s^t)w(s^t)l(s^t) + M(s^{t-1}) + B(s^t) + D(s^{t-1}) + \Pi(s^t) + T(s^t) \end{aligned} \tag{1.47}$$

$$
\begin{aligned}
P^*(s^t)c^*(s^t) + M^*(s^t) + \sum_{s_{t+1}} Q^*(s^{t+1}|s^t)B^*(s^{t+1}) + V(s^t)D^*(s^t)/e(s^t) \\
\leq P^*(s^t)w^*(s^t)l^*(s^t) + M^*(s^{t-1}) + B^*(s^t) + D^*(s^{t-1})/e(s^t) \\
+ \Pi^*(s^t) + T^*(s^t). \qquad (1.48)
\end{aligned}
$$

instead of (1.10) and (1.17). D and D^* are one-period bonds in the home and foreign country respectively. Notice that in the foreign budget constraint, we no longer have the home and foreign contingent claims, B_H^* and B_F^*.

The first-order conditions corresponding to the choices of D and D^* imply the following equations hold in equilibrium:

$$
V(s^t) = \sum_{s_{t+1}} \beta\pi(s^{t+1}|s^t)\frac{U_c(s^{t+1})}{U_c(s^t)}\frac{P(s^t)}{P(s^{t+1})} \qquad (1.49)
$$

$$
V(s^t) = \sum_{s_{t+1}} \beta\pi(s^{t+1}|s^t)\frac{U_c^*(s^{t+1})}{U_c^*(s^t)}\frac{P^*(s^t)}{P^*(s^{t+1})}\frac{e(s^t)}{e(s^{t+1})} \qquad (1.50)
$$

$$
D(s^t) + D^*(s^t) = 0 \qquad (1.51)
$$

Thus, if we eliminate the home budget constraint by Walras' Law and eliminate $D(s^t)$ as a variable using $D + D^* = 0$, we are left with the foreign budget constraint and the two first-order conditions for bonds in the two countries. We keep these two as separate dynamic equations. Then we use the foreign budget constraint to get an expression for the nominal exchange rate:

$$
e(s^t) = \frac{V(s^t)D^*(s^t) - D^*(s^{t-1})}{P^*(s^t)[w^*(s^t)l^*(s^t) - c^*(s^t)] + \Pi^*(s^t)}. \qquad (1.52)
$$

1.10.3. Additional Shocks

Above we assumed that the only shocks in the model were monetary shocks. We also consider extensions of the benchmark economy with changes in government spending and technology.

Because we want to consider alternative assumptions about the revelation of shocks, we will use the following notational convention. Let $s^t = (s_0, \ldots, s_t)$ be the history of events up through and including period t that is observed by consumers and final goods producers. Let $z^t = (z_0, \ldots, z_t)$ be the history of events up through and including period t that is observed by intermediate goods producers when making their pricing decisions. When making investment and hiring decisions, we assume that they have observed all of the information contained in s^t.

Our default assumption is that consumers and final goods producers know current (t) and past realizations ($t-1, t-2, \ldots$) of the three shocks (i.e., money, government spending, and technology) when making their current t decisions. Thus, s_t are realizations of the three shocks. Intermediate goods producers, on the other hand, do not observe the current monetary shock and may or may not observe the current real shocks before making their current pricing decision. If they observe real shocks but don't observe monetary shocks when making pricing decisions, then z_t contains only information on the government spending shock and the technology shock. If they see none of these events before deciding on prices, then z_t contains no information.

With spending included in the model, the resource constraint for the home country is given by

$$y(s^t) = c(s^t) + \int_0^1 x(i, s^t)\, di + g(s^t).$$

with an analogous constraint for the foreigners. With technology shocks, we need to modify the production function as follows:

$$y_H(i, s^t) + y_H^*(i, s^t) = F(k(i, s^{t-1}), A(s^t) l(i, s^t))$$

and therefore the expression for marginal costs,

$$mc(i, s^t) = \frac{w(s^t)}{F_l(i, s^t) A(s^t)}.$$

Throughout, we will assume that $g(s^t)$ and $A(s^t)$ are AR(1) processes with means g and 1, respectively.

1.10.4. Sticky Wages

Finally, we allow for sticky wages in addition to sticky prices. One can think of the economy organized into a continuum of unions indexed by j. Each union j consists of all the consumers in the economy with labor of type j. This union realizes that it faces a downward sloping demand curve for its type of labor. It sets nominal wages for $\mathcal{N}$ periods at t, $t+\mathcal{N}$, $t+2\mathcal{N}$, and so on. Thus, it faces constraints

$$\begin{aligned} W(j, s^{t-1}) &= W(j, s^t) = \ldots = W(j, s^{t+\mathcal{N}-1}) \\ W(j, s^{t+\mathcal{N}}) &= W(j, s^{t+\mathcal{N}+1}) = \ldots = W(j, s^{t+2\mathcal{N}-1}) \end{aligned}$$

and so on in addition to the ones below.

The problem now solved by a consumer of type j is

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t)\, U\left(c(j, s^t), L^s(j, s^t), M^d(j, s^t)/P(s^t)\right),$$

subject to the sequence of budget constraints, the definition of labor supply, and the labor demands of the firms.

$$\begin{aligned} & P(s^t)c(j, s^t) + M^d(j, s^t) + \sum_{s_{t+1}} Q(s^{t+1}|s^t)B(j, s^{t+1}) \\ & \qquad \leq W(j, s^{t-1})L^s(j, s^t) + M^d(j, s^{t-1}) + B(j, s^t) + \Pi(s^t) + T(s^t) \qquad (1.53) \\ & L^s(j, s^t) = \int l(i, j, s^t)\, di \\ & l(i, j, s^t) = \left(\frac{\bar{W}(s^t)}{W(j, s^{t-1})}\right)^{\frac{1}{1-v}} L^d(i, s^t), \quad \text{for all } i. \end{aligned}$$

In this case, the consumer chooses the wage but agrees to supply whatever is demanded at that wage.

The Lagrangian in this case is

$$
\begin{aligned}
\mathcal{L} = \dots \beta^t \sum_{s^t} \pi(s^t) \Big\{ & U\left(c(j,s^t), \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^t), \frac{M^d(j,s^t)}{P(s^t)} \right) \\
& + \zeta(s^t) \big\{ \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{v}{v-1}} L^d(s^t) + M^d(j,s^{t-1}) \\
& \qquad + B(j,s^t) + \Pi(s^t) + T(s^t) \\
& \qquad - P(s^t)c(j,s^t) - M^d(j,s^t) - \sum_{s_{t+1}} Q(s^{t+1}|s^t) B(j,s^{t+1}) \big\} \\
+ \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t) \Big[& U\left(c(j,s^{t+1}), \bar{W}(s^{t+1})^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^{t+1}), \frac{M^d(j,s^{t+1})}{P(s^{t+1})} \right) \\
& + \zeta(s^t) \big\{ \bar{W}(s^{t+1})^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{v}{v-1}} L^d(s^{t+1}) + M^d(j,s^t) \\
& \quad + B(j,s^{t+1}) + \Pi(s^{t+1}) + T(s^{t+1}) \\
& \quad - P(s^{t+1})c(j,s^{t+1}) - M^d(j,s^{t+1}) - \sum_{s_{t+2}} Q(s^{t+2}|s^{t+1}) B(j,s^{t+2}) \big\} \dots
\end{aligned}
$$

where $L^d(s^t) = \int L^d(i,s^t)\,di$.

Taking the derivative of $\mathcal{L}$ with respect to $W(j,s^{t-1})$ we have

$$
\begin{aligned}
0 = \sum_{s^t} \pi(s^t) \Big\{ & \frac{1}{v-1} \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{2-v}{v-1}} L^d(s^t) U_l(j,s^t) \\
& + \frac{v}{v-1} \zeta(s^t) \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^t) \\
+ \frac{1}{v-1} \sum_{s_{t+1}} & \beta\pi(s^{t+1}|s^t) \bar{W}(s^{t+1})^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{2-v}{v-1}} L^d(s^{t+1}) U_l(j,s^{t+1}) \\
& + \frac{v}{v-1} \sum_{s_{t+1}} \beta\pi(s^{t+1}|s^t) \zeta(s^{t+1}) \bar{W}(s^{t+1})^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^{t+1}) + \dots
\end{aligned}
$$

Using the equilibrium bond price and the fact that $\zeta(s^t) = U_c(j,s^t)/P(s^t)$, we can simplify this first order conditions to get:

$$
W(j,s^{t-1}) = -\frac{\sum_{\tau=t}^{t+\mathcal{N}-1} \sum_{s^\tau} Q(s^\tau|s^{t-1}) P(s^\tau) \bar{W}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau) U_l(j,s^\tau)/U_c(j,s^\tau)}{v \sum_{\tau=t}^{t+\mathcal{N}-1} \sum_{s^\tau} Q(s^\tau|s^{t-1}) \bar{W}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau)}. \quad (1.54)
$$

The other first order conditions for the consumer are the same as in the benchmark economy, except for the fact that we need to record the type, e.g.,

$$
\frac{U_m(j,s^t)}{P(s^t)} - \frac{U_c(j,s^t)}{P(s^t)} + \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t) \frac{U_c(j,s^{t+1})}{P(s^{t+1})} = 0 \quad (1.55)
$$

$$Q(s^t|s^{t-1}) = \beta\pi(s^t|s^{t-1})\frac{U_c(j,s^t)}{U_c(j,s^{t-1})}\frac{P(s^{t-1})}{P(s^t)}. \tag{1.56}$$

Notice that these conditions imply that

$$\frac{U_c(j,s^\tau)}{U_c(j,s^t)} = \frac{U_c(k,s^\tau)}{U_c(k,s^t)}$$

for all j and k. So, marginal utilities are equated up to a constant, namely the date 0 Lagrange multiplier on their budget constraint. Here, we assume that initial debts and transfers among the $\mathcal{N}$ types in either country are such that the multipliers are equalized. In that case, we can summarize these first-order conditions as

$$U_c(j,s^t) = U_c(k,s^t) \tag{1.57}$$

$$U_m(j,s^t) = U_m(k,s^t) \tag{1.58}$$

for all j and k. The equating of U_m's from the money demand equations.

The problem solved by the intermediate goods producers setting prices in the home country is to choose sequences of prices $P_H(i)$, capital stocks $k(i)$, investments $x(i)$, and labor inputs $l(i,j)$, $j = 1,\ldots,\mathcal{N}$ to maximize

$$\sum_{\tau=0}^{\infty}\sum_{s^\tau}\tilde{Q}(s^\tau)\big[P_H(i,s^\tau)y_H(i,s^\tau) + e(s^\tau)P_H^*(i,s^\tau)y_H^*(i,s^\tau) \\ - \int W(j,s^{\tau-1})l(i,j,s^\tau)\,dj - P(s^\tau)x(i,s^\tau)\big] \tag{1.59}$$

subject to the input demands (1.3) and (1.4), the production technology:

$$y_H(i,s^t) + y_H^*(i,s^t) = F(k(i,s^{t-1}), L^d(i,s^t)) \tag{1.60}$$

the constraint on labor

$$L^d(i,s^t) \leq \left[\int l(i,j,s^t)^v\,dj\right]^{\frac{1}{v}}, \tag{1.61}$$

the law of motion for capital used in producing good i in (1.23), and the constraints on prices in (1.24). The firms take the wages as given.

The Lagrangian in this case is

$$
\begin{aligned}
\mathcal{L} = \ldots + \tilde{Q}(s^t)\Bigg\{ & \Big[P_H(i,s^{t-1})^{\frac{\theta}{\theta-1}}\Lambda_H(s^t) + e(s^t)P_H^*(i,s^{t-1})^{\frac{\theta}{\theta-1}}\Lambda_H^*(s^t) \\
& \qquad - \int W(j,s^{t-1})l(i,j,s^t)\,dj - P(s^t)x(i,s^t) \\
& + \zeta(s^t)\big\{F(k(i,s^{t-1}),L^d(i,s^t)) - \Lambda_H(s^t)P_H(i,s^{t-1})^{\frac{1}{\theta-1}} \\
& \qquad - \Lambda_H^*(s^t)P_H^*(i,s^{t-1})^{\frac{1}{\theta-1}}\big\} \\
& + \lambda(s^t)\big\{(1-\delta)k(i,s^{t-1}) + x(i,s^t) \\
& \qquad - \phi(x(i,s^t)/k(i,s^{t-1}))k(i,s^{t-1}) - k(i,s^t)\big\}\Big] \\
& + \kappa(s^t)\Big\{\Big[\int l(i,j,s^t)^v\,dj\Big]^{\frac{1}{v}} - L^d(i,s^t)\Big\} \\
+ \sum_{s_{t+1}} Q(s^{t+1}|s^t) & \Big[P_H(i,s^{t-1})^{\frac{\theta}{\theta-1}}\Lambda_H(s^{t+1}) + e(s^{t+1})P_H^*(i,s^{t-1})^{\frac{\theta}{\theta-1}}\Lambda_H^*(s^{t+1}) \\
& \qquad - \int W(j,s^t)l(i,j,s^{t+1})\,dj - P(s^{t+1})x(i,s^{t+1}) \\
& + \zeta(s^{t+1})\big\{F(k(i,s^t),L^d(i,s^{t+1})) - \Lambda_H(s^{t+1})P_H(i,s^{t-1})^{\frac{1}{\theta-1}} \\
& \quad - \Lambda_H^*(s^{t+1})P_H^*(i,s^{t-1})^{\frac{1}{\theta-1}}\big\} \\
& + \lambda(s^{t+1})\big\{(1-\delta)k(i,s^t) + x(i,s^{t+1}) \\
& \qquad - \phi(x(i,s^{t+1})/k(i,s^t))k(i,s^t) - k(i,s^{t+1})\big\}\Big] \\
& + \kappa(s^{t+1})\Big\{\Big[\int l(i,j,s^{t+1})^v\,dj\Big]^{\frac{1}{v}} - L^d(i,s^{t+1})\Big\} \\
& + \ldots\Bigg\} \qquad (1.62)
\end{aligned}
$$

where

$$\Lambda_H(s^t) = \left[\omega_1 P(s^t)\right]^{\frac{1}{1-\rho}} \bar{P}_H(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}} y(s^t) \tag{1.63}$$

$$\Lambda_H^*(s^t) = \left[\omega_2 P^*(s^t)\right]^{\frac{1}{1-\rho}} (\bar{P}_H^*(s^{t-1}))^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}} y^*(s^t). \tag{1.64}$$

The variables ζ, λ, and κ are multipliers for constraints (1.60), (1.23), and (1.61), respectively.

Taking the derivative of $\mathcal{L}$ in (1.25) with respect to the monopolist's prices $P_H(i, s^{t-1})$ and $P_H^*(i, s^{t-1})$, the investment $x(i, s^t)$, and the capital stock $k(i, s^t)$, we find the same results as in the benchmark economy, namely (1.28),(1.29), (1.31), and (1.32).

The difference between the sticky wage economy and the benchmark economy is the labor market. Taking the derivative of $\mathcal{L}$ with respect to $L^d(i, s^t)$, we find:

$$\zeta(s^t) F_l(i, s^t) - \kappa(s^t) = 0.$$

Taking the derivative with respect to $l(i, j, s^t)$, we find:

$$-W(j, s^{t-1}) + \kappa(s^t) l(i, j, s^t)^{v-1} \left[\int l(i, j, s^t)^v \, dj\right]^{\frac{1}{v}-1} = 0$$

or,

$$W(j, s^{t-1}) = \kappa(s^t) l(i, j, s^t)^{v-1} L^d(i, s^t)^{1-v}. \tag{1.65}$$

If we integrate both sides of (1.65) we get

$$\begin{aligned} \kappa(s^t) &= \left[\int W(j, s^{t-1})^{\frac{v}{v-1}} \, dj\right]^{\frac{v-1}{v}} \\ &\equiv \bar{W}(s^t) \end{aligned} \tag{1.66}$$

which implies that the multipler is equal to the aggregate wage. Substituting that back into (1.65), we have

$$l(i, j, s^t) = \left(\frac{\bar{W}(s^t)}{W(j, s^{t-1})}\right)^{\frac{1}{1-v}} L^d(i, s^t).$$

The solution to the decision problem of the intermediate goods producer can also be found in two steps. First, define an intermediate real wage $w(s^t)$ as follows:

$$w(s^t)L^d(i,s^t) \equiv \min_{l(i,j,s^t),\forall j} \int \frac{W(j,s^{t-1})}{P(s^t)} l(i,j,s^t)\,dj$$
$$\text{subject to } L^d(i,s^t) \leq \left[\int l(i,j,s^t)^v\,dj\right]^{\frac{1}{v}}.$$

Solving this problem yields:

$$w(s^t) = \bar{W}(s^t)/P(s^t) \tag{1.67}$$

where $\bar{W}(s^t)$ is defined by (1.66). The second step is to return to (1.21) and everywhere we see $l(i,s^t)$, we replace it with $L^d(i,s^t)$. Thus, the solution for the intermediate goods producer's problem in the benchmark and sticky-wage economies are the same, except that now we have a composite labor input and a composite wage. In other words, the first order conditions are given by (1.33)-(1.36) where F is evaluated at $L^d(i,s^t)$, w is given in (1.67), and $\bar{W}(s^t)$ is a function of the distribution of wages (which are part of the new state vector).

The last equations to be changed in the case of sticky wages are the resource constraints and the equating of money supply and money demand. We replace (1.40) with

$$y(s^t) = \int_0^1 c(j,s^t)\,dj + \int_0^1 x(i,s^t)\,di + g(s^t), \tag{1.68}$$

and we add

$$M^s(s^t) = \int_0^1 M^d(j,s^t)\,dj.$$

Analogous equations hold for the foreign country.

2. Computing an Equilibrium

We now describe how to compute an equilibrium for the four economies: the benchmark economy, the economy with the Fed using a Taylor rule, the incomplete-markets economy, and the sticky-wage economy.[1] In each case, there are four steps taken. First, we normalize variables to make the problem stationary. Second, we derive equations for the steady states of the stationary variables. Third, we linearize the first-order conditions around the steady state. Fourth, we describe in detail the codes used for computing a solution to the linearized system of equations.

2.1. The Benchmark Economy

In this section, we describe the computation for the benchmark economy. Most of the derivations are done in this section since there are many common equations for the three economies that we consider.

To simplify things, we assume from here on (unless noted otherwise) that the ith group of monopolists ($i \in \{1, \dots N\}$) is the one who set prices i periods ago. Thus, in period t, monopolist 1 is assumed to have set prices conditional on seeing s^{t-1}, monopolist 2 set prices conditional on seeing s^{t-2}, and so on.

2.1.1. Normalization in the Benchmark Economy

Since we allow for positive money growth and inflation, we need to normalize prices in

[1] For now we assume that there are just monetary shocks. Below, we extend the analysis to include other shocks.

order to make the system of equations stationary. Let

$$\begin{aligned}
p(s^t) &= P(s^t)/M(s^{t-1}) \\
p_I(i,s^{t-1}) &= P_I(i,s^{t-1})/M(s^{t-i}) \\
\bar{p}_I(s^{t-1}) &= \bar{P}_I(s^{t-1})/M(s^{t-1}) \\
p^*(s^t) &= P^*(s^t)/M^*(s^{t-1}) \\
p_I^*(i,s^{t-1}) &= P_I^*(i,s^{t-1})/M^*(s^{t-i}) \\
\bar{p}_I^*(s^{t-1}) &= \bar{P}_I^*(s^{t-1})/M^*(s^{t-1})
\end{aligned} \tag{2.1}$$

where $I = H$ or F. Notice that we normalize the intermediate goods prices by $M(s^{t-i})$ or $M^*(s^{t-i})$. Since we assume that the ith group set prices i periods ago, we are effectively assuming

$$\frac{P_I(i,s^{t-1})}{M(s^{t-i})} = p_I(i,s^{t-1}) = p_I(s^{t-i}) = \frac{P_I(s^{t-i})}{M(s^{t-i})}.$$

A number of first-order conditions must be changed because they involve nonstationary variables. First, consider the equations derived from the final goods producer's problem. The input demands depend on the prices. Using the normalization above, we can rewrite the input demand equation for the home intermediate goods as follows:

$$\begin{aligned}
y_H(i,s^t) &= \frac{[\omega_1 P(s^t)]^{\frac{1}{1-\rho}}\ \bar{P}_H(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}}}{P_H(i,s^{t-1})^{\frac{1}{1-\theta}}}\ y(s^t) \\
&= \frac{[\omega_1 p(s^t)]^{\frac{1}{1-\rho}}\ \bar{p}_H(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}} M(s^{t-1})^{\frac{1}{1-\theta}}}{p_H(i,s^{t-1})^{\frac{1}{1-\theta}} M(s^{t-i})^{\frac{1}{1-\theta}}}\ y(s^t) \\
&= \frac{[\omega_1 p(s^t)]^{\frac{1}{1-\rho}}\ \bar{p}_H(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}} [\mu(s^{t-i+1})\cdots\mu(s^{t-1})]^{\frac{1}{1-\theta}}}{p_H(i,s^{t-1})^{\frac{1}{1-\theta}}}\ y(s^t).
\end{aligned} \tag{2.2}$$

Note that the first relation is from (1.3) when the state is s^t. We can similarly rewrite y_F, y_H^*, and y_F^*.

The aggregate price in (1.5) is normalized as follows:

$$p(s^t) = \left(\omega_1^{\frac{1}{1-\rho}} \bar{p}_H(s^{t-1})^{\frac{\rho}{\rho-1}} + \omega_2^{\frac{1}{1-\rho}} \bar{p}_F(s^{t-1})^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \tag{2.3}$$

and similarly for the foreign price. These equations replace (1.5) and (1.8).

Normalized first-order conditions for the consumer include

$$U_m(s^t) - U_c(s^t) + \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t) U_c(s^{t+1}) \frac{p(s^t)}{p(s^{t+1})\mu(s^t)} = 0 \tag{2.4}$$

$$Q(s^t|s^{t-1}) = \beta\pi(s^t|s^{t-1}) \frac{U_c(s^t)}{U_c(s^{t-1})} \frac{p(s^{t-1})}{p(s^t)\mu(s^{t-1})} \tag{2.5}$$

and therefore

$$\begin{aligned} 1/R(s^t) &= \sum_{s_{t+1}} Q(s^{t+1}|s^t) \\ &= \sum_{s_{t+1}} \beta\pi(s^{t+1}|s^t) \frac{U_c(s^{t+1})}{U_c(s^t)} \frac{p(s^t)}{p(s^{t+1})\mu(s^t)}. \end{aligned} \tag{2.6}$$

The normalized price of the intermediate goods producer is given by

$$\begin{aligned} p_H(i, s^{t-1}) &= \frac{1}{M(s^{t-1})} \frac{\sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1}) p(s^\tau) mc(i, s^\tau) \lambda_H(s^\tau) M(s^{\tau-1})^{\frac{2-\theta}{1-\theta}}}{\theta \sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1}) \lambda_H(s^\tau) M(s^{\tau-1})^{\frac{1}{1-\theta}}} \\ &= \frac{\sum_\tau \sum_{s^\tau} \beta^{\tau-1} \pi(s^\tau|s^{t-1}) U_c(s^\tau) mc(i, s^\tau) \lambda_H(s^\tau) \left(\mu(s^t) \cdots \mu(s^{\tau-1})\right)^{\frac{1}{1-\theta}}}{\theta \sum_\tau \sum_{s^\tau} \beta^{\tau-1} \pi(s^\tau|s^{t-1}) U_c(s^\tau)/p(s^\tau) \lambda_H(s^\tau) \left(\mu(s^t) \cdots \mu(s^{\tau-1})\right)^{\frac{\theta}{1-\theta}}} \end{aligned} \tag{2.7}$$

where

$$\lambda_H(s^t) = \left[\omega_1 p(s^t)\right]^{\frac{1}{1-\rho}} \bar{p}_H(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}} y(s^t). \tag{2.8}$$

Expressions for p_F, p_H^*, and p_F^* can be derived similarly.

When we aggregate these prices, we get the following intermediate goods price index for the home country:

$$\begin{aligned} \bar{p}_H(s^{t-1}) &= \left[\frac{1}{N} \sum_{j=1}^{N} \left(\frac{P_H(s^{t-j})}{M(s^{t-1})}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\theta-1}{\theta}} \\ &= \left[\frac{1}{N} p_H(s^{t-1})^{\frac{\theta}{\theta-1}} + \frac{1}{N} \left(\frac{p_H(s^{t-2})}{\mu(s^{t-1})}\right)^{\frac{\theta}{\theta-1}} + \cdots \right. \\ &\qquad \left. + \frac{1}{N} \left(\frac{p_H(s^{t-N})}{\mu(s^{t-1}) \cdots \mu(s^{t-N+1})}\right)^{\frac{\theta}{\theta-1}}\right]^{\frac{\theta-1}{\theta}}. \end{aligned} \tag{2.9}$$

2.1.2. Steady State in the Benchmark Economy

When we linearize the first order conditions, we will do so around the steady state values derived below. We will assume that preferences, technologies, and processes for money are the same in the two countries. Therefore, steady state values of home and foreign allocations and prices will be equated.

We take the normalized first-order conditions, drop s^t arguments and solve for a fixed point. Consider doing this iteratively. Start with a guess for $k(i)$, $i = 1, \ldots N$, y, and Pc/M. With the $k(i)'s$, we can back out the investments from the law of motion for capital

$$k(i) = (1-\delta)k(i-1) + x(i) - \phi\left(\frac{x(i)}{k(i-1)}\right)k(i-1), \quad i = 1, \ldots, N.$$

With y, we can get the steady state input demands and total product,

$$y_H(i) = y\left(\frac{\omega_1 p}{\bar{p}_H}\right)^{\frac{1}{1-\rho}}\left(\frac{\bar{p}_H}{p_H}\right)^{\frac{1}{1-\theta}}\mu^{\frac{i-1}{1-\theta}} = y_F^*(i), \quad i = 1, \ldots, N$$

$$y_H^*(i) = y\left(\frac{\omega_2 p}{\bar{p}_H}\right)^{\frac{1}{1-\rho}}\left(\frac{\bar{p}_H}{p_H}\right)^{\frac{1}{1-\theta}}\mu^{\frac{i-1}{1-\theta}} = y_F(i), \quad i = 1, \ldots, N$$

$$F(i) = y\left(\frac{p}{p_H}\right)\left(\frac{\bar{p}_H}{p_H}\right)^{\frac{\theta}{1-\theta}}\mu^{\frac{i-1}{1-\theta}}, \quad i = 1, \ldots, N$$

using (1.22), (2.2), and the fact that the price ratios $p/\bar{p}_H$ and $\bar{p}_H/p_H$ can be written as explicit functions of parameters,

$$p = \left(\omega_1^{\frac{1}{1-\rho}}\bar{p}_H^{\frac{\rho}{\rho-1}} + \omega_2^{\frac{1}{1-\rho}}\bar{p}_F^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} = \bar{p}_H\left(\omega_1^{\frac{1}{1-\rho}} + \omega_2^{\frac{1}{1-\rho}}\right)^{\frac{\rho-1}{\rho}}$$

$$\bar{p}_H = p_H\left[\frac{1}{N}\left(1 + \mu^{\frac{\theta}{1-\theta}} + \cdots + \mu^{\frac{(N-1)\theta}{1-\theta}}\right)\right]^{\frac{\theta-1}{\theta}}.$$

These latter expressions follow directly from (2.3) and (2.9).

With the $F(i)$'s, $k(i)$'s, and $x(i)$'s we back out labor inputs via the production technology $F(i) = F(k(i-1), l(i))$ and marginal costs via the Euler equations for capital:

$$\frac{1}{1-\phi'(i)} = \beta\Big(mc(i+1)F_k(i+1) + \frac{1}{1-\phi'(i+1)}\left[1-\delta-\phi(i+1)+\phi'(i+1)x(i+1)/k(i)\right]\Big), \quad i = 1, \ldots N$$

where $\phi(i) = \phi(x(i)/k(i-1))$.

We can sum up the $l(i)$'s to get aggregate labor,

$$l = \frac{1}{N} \sum_i l(i),$$

and we can sum up the $x(i)$'s to get aggregate investment

$$x = \frac{1}{N} \sum_i x(i).$$

With x and y, we have consumption,

$$c = y - x.$$

With Pc/M, we have P/M and therefore

$$p = \mu P/M = \mu[Pc/M]/c$$

given our definition of p in (2.1).

Now, we can use the following equations to check that we have a fixed point:

$$mc(i) = -U_l/(U_c F_l(i))$$

$$p_H = \frac{p}{\theta} \left(\frac{mc(1) + mc(2)\beta\mu^{\frac{1}{1-\theta}} + mc(3)\beta^2\mu^{\frac{2}{1-\theta}} + \ldots + mc(N)\beta^{N-1}\mu^{\frac{N-1}{1-\theta}}}{1 + \beta\mu^{\frac{\theta}{1-\theta}} + \beta^2\mu^{\frac{2\theta}{1-\theta}} + \ldots + \beta^{N-1}\mu^{\frac{(N-1)\theta}{1-\theta}}} \right)$$

$$U_m = U_c(1 - \beta/\mu)$$

which are the equations for marginal costs ((1.36) with (1.11) substituted in), the intermediate goods price (2.7), and money demand (1.12).

Finally, our assumption about common preferences implies that $q = 1$ in a steady state.

2.1.3. Linearized Equations in the Benchmark Economy

We now use the steady state values in order to linearize the first order conditions. Here again we assume that the cohort 1 are the monopolists setting prices this period (in s^{t-1}), cohort 2 set last period and so on. First-order conditions for the final goods producers can be linearized to yield the following equations:

$$\hat{y}_{H,i,t} = \hat{y}_t + \frac{1}{1-\rho}\hat{p}_t + \frac{\rho-\theta}{(1-\rho)(\theta-1)}\hat{\bar{p}}_{H,t-1} - \frac{1}{1-\theta}\left[\hat{p}_{H,t-i} - \hat{\mu}_{t-i+1} \ldots - \hat{\mu}_{t-1}\right] \tag{2.10}$$

$$\hat{y}_{F,i,t} = \hat{y}_t + \frac{1}{1-\rho}\hat{p}_t + \frac{\rho-\theta}{(1-\rho)(\theta-1)}\hat{\bar{p}}_{F,t-1} - \frac{1}{1-\theta}\left[\hat{p}_{F,t-i} - \hat{\mu}_{t-i+1} \ldots - \hat{\mu}_{t-1}\right] \tag{2.11}$$

$$\hat{y}^*_{F,i,t} = \hat{y}^*_t + \frac{1}{1-\rho}\hat{p}^*_t + \frac{\rho-\theta}{(1-\rho)(\theta-1)}\hat{\bar{p}}^*_{F,t-1} - \frac{1}{1-\theta}\left[\hat{p}^*_{F,t-i} - \hat{\mu}^*_{t-i+1} \ldots - \hat{\mu}^*_{t-1}\right] \tag{2.12}$$

$$\hat{y}^*_{H,i,t} = \hat{y}^*_t + \frac{1}{1-\rho}\hat{p}^*_t + \frac{\rho-\theta}{(1-\rho)(\theta-1)}\hat{\bar{p}}^*_{H,t-1} - \frac{1}{1-\theta}\left[\hat{p}^*_{H,t-i} - \hat{\mu}^*_{t-i+1} \ldots - \hat{\mu}^*_{t-1}\right] \tag{2.13}$$

$$\hat{p}_t = \left[\frac{\bar{p}_I}{p}\right]^{\frac{\rho}{\rho-1}} \left[\omega_1^{\frac{1}{1-\rho}}\hat{\bar{p}}_{H,t-1} + \omega_2^{\frac{1}{1-\rho}}\hat{\bar{p}}_{F,t-1}\right] \tag{2.14}$$

$$\hat{p}^*_t = \left[\frac{\bar{p}_I}{p}\right]^{\frac{\rho}{\rho-1}} \left[\omega_1^{\frac{1}{1-\rho}}\hat{\bar{p}}^*_{F,t-1} + \omega_2^{\frac{1}{1-\rho}}\hat{\bar{p}}^*_{H,t-1}\right] \tag{2.15}$$

$$\begin{aligned}\hat{\bar{p}}_{H,t-1} = \left[1 + \mu^{\frac{\theta}{1-\theta}} + \ldots \mu^{\frac{(N-1)\theta}{1-\theta}}\right]^{-1} &\left[\hat{p}_{H,t-1} + \mu^{\frac{\theta}{1-\theta}}(\hat{p}_{H,t-2} - \hat{\mu}_{t-1}) + \ldots\right. \\ &\left. + \mu^{\frac{(N-1)\theta}{1-\theta}}(\hat{p}_{H,t-N} - \hat{\mu}_{t-1} - \ldots - \hat{\mu}_{t-N+1})\right]\end{aligned} \tag{2.16}$$

$$\begin{aligned}\hat{\bar{p}}_{F,t-1} = \left[1 + \mu^{\frac{\theta}{1-\theta}} + \ldots \mu^{\frac{(N-1)\theta}{1-\theta}}\right]^{-1} &\left[\hat{p}_{F,t-1} + \mu^{\frac{\theta}{1-\theta}}(\hat{p}_{F,t-2} - \hat{\mu}_{t-1}) + \ldots\right. \\ &\left. + \mu^{\frac{(N-1)\theta}{1-\theta}}(\hat{p}_{F,t-N} - \hat{\mu}_{t-1} - \ldots - \hat{\mu}_{t-N+1})\right]\end{aligned} \tag{2.17}$$

$$\begin{aligned}\hat{\bar{p}}^*_{F,t-1} = \left[1 + \mu^{*\frac{\theta}{1-\theta}} + \ldots \mu^{*\frac{(N-1)\theta}{1-\theta}}\right]^{-1} &\left[\hat{p}^*_{F,t-1} + \mu^{*\frac{\theta}{1-\theta}}(\hat{p}^*_{F,t-2} - \hat{\mu}^*{}_{t-1}) + \ldots\right. \\ &\left. + \mu^{*\frac{(N-1)\theta}{1-\theta}}(\hat{p}^*_{F,t-N} - \hat{\mu}^*_{t-1} - \ldots - \hat{\mu}^*_{t-N+1})\right]\end{aligned} \tag{2.18}$$

$$\begin{aligned}\hat{\bar{p}}^*_{H,t-1} = \left[1 + \mu^{*\frac{\theta}{1-\theta}} + \ldots \mu^{*\frac{(N-1)\theta}{1-\theta}}\right]^{-1} &\left[\hat{p}_{H,t-1} + \mu^{*\frac{\theta}{1-\theta}}(\hat{p}_{H,t-2} - \hat{\mu}^*_{t-1}) + \ldots\right. \\ &\left. + \mu^{*\frac{(N-1)\theta}{1-\theta}}(\hat{p}_{H,t-N} - \hat{\mu}^*_{t-1} - \ldots - \hat{\mu}^*_{t-N+1})\right]\end{aligned} \tag{2.19}$$

where p_I is the steady state value for the intermediate goods (i.e., for $p_{H,t}$, $p_{F,t}$, $p^*_{H,t}$, and $p^*_{F,t}$) and p is the steady state value of the final goods (i.e., for p_t and p^*_t).

First-order conditions for the consumer can be linearized to yield the following equations:

$$\hat{w}_t = \left(\frac{U_{cl}}{U_l} - \frac{U_{cc}}{U_c}\right) c\,\hat{c}_t + \left(\frac{U_{ll}}{U_l} - \frac{U_{cl}}{U_c}\right) l\,\hat{l}_t + \left(\frac{U_{lm}}{U_l} - \frac{U_{cm}}{U_c}\right) M/P(\hat{\mu}_t - \hat{p}_t) \tag{2.20}$$

$$\begin{aligned}\frac{U_m}{U_c}\left\{\left(\frac{U_{cc}}{U_c} - \frac{U_{cm}}{U_m}\right) c\,\hat{c}_t + \left(\frac{U_{cl}}{U_c} - \frac{U_{lm}}{U_m}\right) l\,\hat{l}_t + \left(\frac{U_{cm}}{U_c} - \frac{U_{mm}}{U_m}\right) M/P(\hat{\mu}_t - \hat{p}_t)\right\}\\ = \beta E_t\Bigg(\frac{U_{cc}c}{U_c}(\hat{c}_{t+1} - \hat{c}_t) + \frac{U_{cl}l}{U_c}(\hat{l}_{t+1} - \hat{l}_t)\\ + \frac{U_{cm}M/P}{U_c}(\hat{\mu}_{t+1} - \hat{p}_{t+1} - \hat{\mu}_t + \hat{p}_t) + \hat{p}_t - \hat{p}_{t+1} - \hat{\mu}_t\Bigg).\end{aligned} \tag{2.21}$$

Equation (2.20) is the linearization of wages from (1.11) and equation (2.21) is the money demand equation from (1.12).

We can also linearize the interest rate r using the definition in (1.14) to get

$$\begin{aligned}-\frac{\beta}{\mu}(r_t - r) = E_t\Bigg(\frac{U_{cc}c}{U_c}(\hat{c}_{t+1} - \hat{c}_t) + \frac{U_{cl}l}{U_c}(\hat{l}_{t+1} - \hat{l}_t)\\ + \frac{U_{cm}M/P}{U_c}(\hat{\mu}_{t+1} - \hat{p}_{t+1} - \hat{\mu}_t + \hat{p}_t) + \hat{p}_t - \hat{p}_{t+1} - \hat{\mu}_t\Bigg),\end{aligned} \tag{2.22}$$

where r_t is the level of the interest rate and all other variables are logged. Note also that the linearized money demand equation can be written

$$\begin{aligned}\frac{\beta(r_t - r)}{\mu r} = \left(\frac{U_{cm}}{U_m} - \frac{U_{cc}}{U_c}\right) c\,\hat{c}_t + \left(\frac{U_{lm}}{U_m} - \frac{U_{cl}}{U_c}\right) l\,\hat{l}_t\\ + \left(\frac{U_{mm}}{U_m} - \frac{U_{cm}}{U_c}\right) M/P(\hat{\mu}_t - \hat{p}_t)\end{aligned} \tag{2.23}$$

if we use the dynamic money demand equation (2.21) and the definition of r in (2.22).

From the foreign consumer's problem we can linearize the expression for the real exchange rates:

$$\hat{q}_t = -\frac{U_{cc}c}{U_c}(\hat{c}_t - \hat{c}^*_t) - \frac{U_{cl}l}{U_c}(\hat{l}_t - \hat{l}^*_t) - \frac{U_{cm}M/P}{U_c}(\hat{\mu}_t - \hat{p}_t - \hat{\mu}^*_t + \hat{p}^*_t) \tag{2.24}$$

We next want to log-linearize the pricing equations such as (2.7). If there is positive inflation, the formulas are a bit messy so we do this in several steps. Consider log-linearizing the deterministic analogue of (2.7):

$$p_H(i, s^{t-1}) = \frac{\sum_\tau \beta^{\tau-1} U_c(s^\tau) mc(i, s^\tau) \lambda_H(s^\tau) \left(\mu(s^t) \cdots \mu(s^{\tau-1})\right)^{\frac{1}{1-\theta}}}{\theta \sum_\tau \beta^{\tau-1} U_c(s^\tau)/p(s^\tau) \lambda_H(s^\tau) \left(\mu(s^t) \cdots \mu(s^{\tau-1})\right)^{\frac{\theta}{1-\theta}}} \tag{2.25}$$

where

$$\lambda_H(s^t) = \left[\omega_1 p(s^t)\right]^{\frac{1}{1-\rho}} \bar{p}_H(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}} y(s^t).$$

We will add an expectations operator when we are done since we assume that $E_{t-1} f(x(s^t)) \approx E_{t-1} f'(x) x \hat{x}_t$. But, for now, we ignore the expectations operator.

First, rewrite (2.25) as:

$$\begin{aligned} \theta p_H(i, s^{t-1}) \Big[\ldots + \beta^j U_c(s^{t+j})/p(s^{t+j}) \lambda_H(s^{t+j}) \left(\mu(s^t) \cdots \mu(s^{t+j-1})\right)^{\frac{\theta}{1-\theta}} + \ldots \Big] \\ = \ldots + \beta^j U_c(s^{t+j}) mc(i, s^{t+j}) \lambda_H(s^{t+j}) \left(\mu(s^t) \cdots \mu(s^{t+j-1})\right)^{\frac{1}{1-\theta}} \end{aligned} \tag{2.26}$$

and then do the linearization of (2.26) in pieces:

$$\begin{aligned} & U_c(s^{t+j})/p(s^{t+j}) \lambda_H(s^{t+j}) (\mu(s^t) \cdots \mu(s^{t+j-1}))^{\frac{\theta}{1-\theta}} \\ & \approx U_c/p \lambda_H \mu^{\frac{\theta j}{1-\theta}} \Bigg[\frac{U_{cc} c}{U_c} \hat{c}_{t+j} + \frac{U_{cl} l}{U_c} \hat{l}_{t+j} + \frac{U_{cm} M/P}{U_c} (\hat{\mu}_{t+j} - \hat{p}_{t+j}) \\ & \quad - \hat{p}_{t+j} + \hat{\lambda}_{H,t+j} + \frac{\theta}{1-\theta} (\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}) \Bigg] \end{aligned}$$

$$\begin{aligned} & U_c(s^{t+j}) mc(i, s^{t+j}) \lambda_H(s^{t+j}) (\mu(s^t) \cdots \mu(s^{t+j-1}))^{\frac{1}{1-\theta}} \\ & \approx U_c mc(i, j) \lambda_H \mu^{\frac{j}{1-\theta}} \Bigg[\frac{U_{cc} c}{U_c} \hat{c}_{t+j} + \frac{U_{cl} l}{U_c} \hat{l}_{t+j} + \frac{U_{cm} M/P}{U_c} (\hat{\mu}_{t+j} - \hat{p}_{t+j}) \\ & \quad + \hat{mc}_{i,t+j} + \hat{\lambda}_{H,t+j} + \frac{1}{1-\theta} (\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}) \Bigg] \end{aligned}$$

$$\hat{\lambda}_{H,t} = \frac{1}{1-\rho} \hat{p}_t + \frac{\rho-\theta}{(1-\rho)(\theta-1)} \hat{p}_{H,t-1} + \hat{y}_t. \tag{2.27}$$

Therefore, the full equation is:

$$
\begin{aligned}
&\theta p_H U_c \lambda_H / p(1+\beta\mu^{\frac{\theta}{1-\theta}}+\ldots)\hat{p}_{H,i,t-1} \\
&\quad + \theta p_H U_c \lambda_H / p \Bigg\{ \ldots + (\beta\mu^{\frac{\theta}{1-\theta}})^j \Bigg[\frac{U_{cc}c}{U_c}\hat{c}_{t+j} + \frac{U_{cl}l}{U_c}\hat{l}_{t+j} + \frac{U_{cm}M/P}{U_c}(\hat{\mu}_{t+j}-\hat{p}_{t+j}) \\
&\qquad\qquad - \hat{p}_{t+j} + \hat{\lambda}_{H,t+j} + \frac{\theta}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}) \Bigg] + \ldots \Bigg\} \\
&\quad = U_c \lambda_H \Bigg\{ \ldots + (\beta\mu^{\frac{1}{1-\theta}})^{j-1} mc(i,j) \Bigg[\frac{U_{cc}c}{U_c}\hat{c}_{t+j} + \frac{U_{cl}l}{U_c}\hat{l}_{t+j} + \frac{U_{cm}M/P}{U_c}(\hat{\mu}_{t+j}-\hat{p}_{t+j}) \\
&\qquad\qquad + \hat{mc}_{i,t+j} + \hat{\lambda}_{H,t+j} + \frac{1}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}) \Bigg] + \ldots \Bigg\}.
\end{aligned} \tag{2.28}
$$

Crossing out common coefficients in (2.28) and dividing by the coefficient on p_H, we get

$$
\begin{aligned}
&\hat{p}_{H,i,t-1} = \\
&\quad \frac{p}{\theta p_H(1+\beta\mu^{\frac{\theta}{1-\theta}}+\ldots)} \Bigg\{ \ldots + (\beta\mu^{\frac{1}{1-\theta}})^j mc(i,j) \Bigg[\frac{U_{cc}c}{U_c}\hat{c}_{t+j} + \frac{U_{cl}l}{U_c}\hat{l}_{t+j} \\
&\qquad + \frac{U_{cm}M/P}{U_c}(\hat{\mu}_{t+j}-\hat{p}_{t+j}) + \hat{mc}_{i,t+j} + \hat{\lambda}_{H,t+j} + \frac{1}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}) \Bigg] + \ldots \Bigg\} \\
&\quad - \frac{1}{(1+\beta\mu^{\frac{\theta}{1-\theta}}+\ldots)} \Bigg\{ \ldots + (\beta\mu^{\frac{\theta}{1-\theta}})^j \Bigg[\frac{U_{cc}c}{U_c}\hat{c}_{t+j} + \frac{U_{cl}l}{U_c}\hat{l}_{t+j} \\
&\qquad + \frac{U_{cm}M/P}{U_c}(\hat{\mu}_{t+j}-\hat{p}_{t+j}) - \hat{p}_{t+j} + \hat{\lambda}_{H,t+j} + \frac{\theta}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}) \Bigg] + \ldots \Bigg\}
\end{aligned} \tag{2.29}
$$

Finally, we use the steady state equation for p in (2.29) to get

$$
\begin{aligned}
&\hat{p}_{H,i,t-1} = \\
&\quad \Bigg\{ \ldots + \frac{(\beta\mu^{\frac{1}{1-\theta}})^j mc(i,j)}{\sum(\beta\mu^{\frac{1}{1-\theta}})^j mc(i,j)} \Bigg[\frac{U_{cc}c}{U_c}\hat{c}_{t+j} + \frac{U_{cl}l}{U_c}\hat{l}_{t+j} + \frac{U_{cm}M/P}{U_c}(\hat{\mu}_{t+j}-\hat{p}_{t+j}) \\
&\qquad + \hat{mc}_{i,t+j} + \hat{\lambda}_{H,t+j} + \frac{1}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}) \Bigg] + \ldots \Bigg\} \\
&\quad - \Bigg\{ \ldots + \frac{(\beta\mu^{\frac{\theta}{1-\theta}})^j}{\sum(\beta\mu^{\frac{\theta}{1-\theta}})^j} \Bigg[\frac{U_{cc}c}{U_c}\hat{c}_{t+j} + \frac{U_{cl}l}{U_c}\hat{l}_{t+j} + \frac{U_{cm}M/P}{U_c}(\hat{\mu}_{t+j}-\hat{p}_{t+j}) \\
&\qquad - \hat{p}_{t+j} + \hat{\lambda}_{H,t+j} + \frac{\theta}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}) \Bigg] + \ldots \Bigg\}.
\end{aligned}
$$

The derivations are similar for the other three prices.

Putting expectations back in and doing the same exercise for the remaining prices yields the following linearized price equations:

$$\hat{p}_{H,t-1} = E_{t-1}\sum_{j=0}^{N-1}\omega_{1,j}\left(\frac{\hat{U}_{c,t+j}}{U_c} + \hat{\lambda}_{H,t+j} + \hat{mc}_{j+1,t+j} + \frac{1}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1})\right)$$

$$- E_{t-1}\sum_{j=0}^{N-1}\omega_{2,j}\left(\frac{\hat{U}_{c,t+j}}{U_c} + \hat{\lambda}_{H,t+j} - \hat{p}_{t+j} + \frac{\theta}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1})\right) \tag{2.30}$$

$$\hat{p}_{F,t-1} = E_{t-1}\sum_{j=0}^{N-1}\omega_{1,j}\left(\frac{\hat{U}^*_{c,t+j}}{U_c} + \hat{\lambda}_{F,t+j} + \hat{mc}^*_{j+1,t+j} + \frac{1}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1})\right)$$

$$- E_{t-1}\sum_{j=0}^{N-1}\omega_{2,j}\left(\frac{\hat{U}^*_{c,t+j}}{U_c} + \hat{\lambda}_{F,t+j} - \hat{p}_{t+j} - \hat{q}_{t+j} + \frac{\theta}{1-\theta}(\hat{\mu}_t + \ldots \hat{\mu}_{t+j-1})\right) \tag{2.31}$$

$$\hat{p}^*_{F,t-1} = E_{t-1}\sum_{j=0}^{N-1}\omega_{1,j}\left(\frac{\hat{U}^*_{c,t+j}}{U_c} + \hat{\lambda}^*_{F,t+j} + \hat{mc}^*_{j+1,t+j} + \frac{1}{1-\theta}(\hat{\mu}^*_t + \ldots \hat{\mu}^*_{t+j-1})\right)$$

$$- E_{t-1}\sum_{j=0}^{N-1}\omega_{2,j}\left(\frac{\hat{U}^*_{c,t+j}}{U_c} + \hat{\lambda}^*_{F,t+j} - \hat{p}^*_{t+j} + \frac{\theta}{1-\theta}(\hat{\mu}^*_t + \ldots \hat{\mu}^*_{t+j-1})\right) \tag{2.32}$$

$$\hat{p}^*_{H,t-1} = E_{t-1}\sum_{j=0}^{N-1}\omega_{1,j}\left(\frac{\hat{U}_{c,t+j}}{U_c} + \hat{\lambda}^*_{H,t+j} + \hat{mc}_{j+1,t+j} + \frac{1}{1-\theta}(\hat{\mu}^*_t + \ldots \hat{\mu}^*_{t+j-1})\right)$$

$$- E_{t-1}\sum_{j=0}^{N-1}\omega_{2,j}\left(\frac{\hat{U}_{c,t+j}}{U_c} + \hat{\lambda}^*_{H,t+j} - \hat{p}^*_{t+j} + \hat{q}_{t+j} + \frac{\theta}{1-\theta}(\hat{\mu}^*_t + \ldots \hat{\mu}^*_{t+j-1})\right) \tag{2.33}$$

where $\omega_{1,j} = (\beta\mu^{\frac{1}{1-\theta}})^j mc(j+1)/\sum(\beta\mu^{\frac{1}{1-\theta}})^j mc(j+1)$ and $\omega_{2,j} = (\beta\mu^{\frac{\theta}{1-\theta}})^j/\sum(\beta\mu^{\frac{1}{1-\theta}})^j$ and $\hat{U}_{c,t}$ is shorthand for the log-linearized marginal utility. Note that in the case with zero-inflation, the linearized pricing equations simplify to:

$$\hat{p}_{H,t-1} = E_{t-1}\sum_{j=0}^{N-1}\beta^j\left(\hat{p}_{t+j} + \hat{mc}_{j+1,t+j} + \hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}\right)/\sum_{j=0}^{N-1}\beta^j$$

$$\hat{p}_{F,t-1} = E_{t-1}\sum_{j=0}^{N-1}\beta^j\left(\hat{p}_{t+j} + \hat{mc}^*_{j+1,t+j} + \hat{q}_{t+j} + \hat{\mu}_t + \ldots \hat{\mu}_{t+j-1}\right)/\sum_{j=0}^{N-1}\beta^j$$

$$\hat{p}^*_{F,t-1} = E_{t-1} \sum_{j=0}^{N-1} \beta^j \left(\hat{p}^*_{t+j} + \hat{mc}^*_{j+1,t+j} + \hat{\mu}^*_t + \ldots \hat{\mu}^*_{t+j-1} \right) / \sum_{j=0}^{N-1} \beta^j$$

$$\hat{p}^*_{H,t-1} = E_{t-1} \sum_{j=0}^{N-1} \beta^j \left(\hat{p}^*_{t+j} + \hat{mc}_{j+1,t+j} - \hat{q}_{t+j} + \hat{\mu}^*_t + \ldots \hat{\mu}^*_{t+j-1} \right) / \sum_{j=0}^{N-1} \beta^j .$$

The remaining equations for the monopolists are as follows:

$$y_H(i)\, \hat{y}_{H,i,t} + y^*_H(i)\, \hat{y}^*_{H,i,t} = F_k(i) k(i-1)\, \hat{k}_{i-1,t-1} + F_l(i) l(i)\, \hat{l}_{i,t} \tag{2.34}$$

$$\hat{k}_{i,t} = \hat{k}_{i-1,t-1} + (\phi'(i) - 1) x(i)/k(i) \left[\hat{k}_{i-1,t-1} - \hat{x}_{i,t} \right] . \tag{2.35}$$

$$\begin{aligned}
0 = E_t \Bigg\{ & \frac{U_{cc} c}{U_c} (\hat{c}_{t+1} - \hat{c}_t) + \frac{U_{cl} l}{U_c} (\hat{l}_{t+1} - \hat{l}_t) \\
& + \frac{U_{cm} M/P}{U_c} (\hat{\mu}_{t+1} - \hat{p}_{t+1} - \hat{\mu}_t + \hat{p}_t) \\
& + \beta (1 - \phi'(i)) mc(i+1) F_k(i+1) \Bigg[\hat{w}_{t+1} + \left(\frac{F_{kk}(i+1)}{F_k(i+1)} - \frac{F_{kl}(i+1)}{F_l(i+1)} \right) k(i)\, \hat{k}_{i,t} \\
& \qquad + \left(\frac{F_{kl}(i+1)}{F_k(i+1)} - \frac{F_{ll}(i+1)}{F_l(i+1)} \right) l(i+1)\, \hat{l}_{i+1,t+1} \\
& \qquad - \frac{\phi''(i)}{1 - \phi'(i)} \frac{x(i)}{k(i-1)} (\hat{x}_{i,t} - \hat{k}_{i-1,t-1}) \Bigg] \\
& + \beta \left(\frac{1 - \phi'(i)}{1 - \phi'(i+1)} \right) \left(1 - \delta - \phi(i+1) + \phi'(i+1) \frac{x(i+1)}{k(i)} \right) \\
& \qquad \Bigg[\frac{-\phi''(i)}{1 - \phi'(i)} \frac{x(i)}{k(i-1)} (\hat{x}_{i,t} - \hat{k}_{i-1,t-1}) \\
& \qquad + \frac{\phi''(i+1)}{1 - \phi'(i+1)} \frac{x(i+1)}{k(i)} (\hat{x}_{i+1,t+1} - \hat{k}_{i,t}) \Bigg] \\
& + \beta \frac{1 - \phi'(i)}{1 - \phi'(i+1)} \phi''(i+1) \left(\frac{x(i+1)}{k(i)} \right)^2 (\hat{x}_{i+1,t+1} - \hat{k}_{i,t}) \Bigg\}
\end{aligned}$$

$$\begin{aligned}
= E_t \Bigg\{ & \frac{U_{cc} c}{U_c} (\hat{c}_{t+1} - \hat{c}_t) + \frac{U_{cl} l}{U_c} (\hat{l}_{t+1} - \hat{l}_t) \\
& + \frac{U_{cm} M/P}{U_c} (\hat{\mu}_{t+1} - \hat{p}_{t+1} - \hat{\mu}_t + \hat{p}_t)
\end{aligned}$$

$$
\begin{aligned}
&+ \beta(1-\phi'(i))mc(i+1)F_k(i+1)\Big[\hat{w}_{t+1} + \left(\frac{F_{kk}(i+1)}{F_k(i+1)} - \frac{F_{kl}(i+1)}{F_l(i+1)}\right) k(i)\,\hat{k}_{i,t} \\
&\qquad + \left(\frac{F_{kl}(i+1)}{F_k(i+1)} - \frac{F_{ll}(i+1)}{F_l(i+1)}\right) l(i+1)\,\hat{l}_{i+1,t+1}\Big] \\
&- \frac{\phi''(i)}{1-\phi'(i)}\frac{x(i)}{k(i-1)}(\hat{x}_{i,t} - \hat{k}_{i-1,t-1}) \\
&+ (1-\beta(1-\phi'(i))mc(i+1)F_k(i+1)) \\
&\qquad \left[\frac{\phi''(i+1)}{1-\phi'(i+1)}\frac{x(i+1)}{k(i)}(\hat{x}_{i+1,t+1} - \hat{k}_{i,t})\right] \\
&+ \beta\frac{1-\phi'(i)}{1-\phi'(i+1)}\phi''(i+1)\left(\frac{x(i+1)}{k(i)}\right)^2(\hat{x}_{i+1,t+1} - \hat{k}_{i,t})\Big\}
\end{aligned} \tag{2.36}
$$

$$
\hat{mc}_{i,t} = \hat{w}_t - F_{kl}(i)k(i-1)/F_l(i)\hat{k}_{i-1,t-1} - F_{ll}(i)l(i)/F_l(i)\hat{l}_{i,t} \tag{2.37}
$$

These are linearizations of (1.22), (1.23), (1.35), and (1.36). Note that in deriving (2.36) we use the steady state Euler equation for capital to simplify terms.

Finally we need the labor market clearing condition and the resource constraint:

$$
\hat{l}_t = (l(1)\,\hat{l}_{1,t} + l(2)\,\hat{l}_{2,t} + \ldots l(N)\,\hat{l}_{N,t})/\sum_i l(i) \tag{2.38}
$$

$$
\hat{c}_t = (y\,\hat{y}_t - [x(1)\,\hat{x}_{1,t} + \ldots x(N)\,\hat{x}_{N,t}]/N)/c \tag{2.39}
$$

which are linearizations of (1.41) and (1.40).

2.1.4. Solving the Linearized System in the Benchmark Economy

The system of equations that we solve has $2N+6$ **dynamic** equations:

- 4 pricing equations, (2.30)-(2.33);
- $2N$ Euler equations for capital ((2.36) for home and similar for foreign);
- 2 money demand equations ((2.21) for home and similar for foreign);
- and **static** equations and definitions that determine:
    - $\hat{y}_{H,i}$, $y_{F,i}$, $y^*_{F,i}$, $y^*_{H,i}$ from (2.10) and analogues;

- $\hat{p}$, $\hat{p}^*$ from (2.14)-(2.15);
- $\hat{\bar{p}}_H$, $\hat{\bar{p}}_F$, $\hat{\bar{p}}^*_F$, $\hat{\bar{p}}^*_H$ from (2.16)-(2.19)
- $\hat{w}$, $\hat{w}^*$ from (2.20) and foreign analogue;
- $\hat{\mu} - \hat{p}$, $\hat{\mu}^* - \hat{p}^*$ for money demands;
- $\hat{q}$ from (2.24);
- $\hat{\lambda}_H$, $\hat{\lambda}_F$, $\hat{\lambda}^*_F$, $\hat{\lambda}^*_H$ from (2.27) and analogues.
- $\hat{l}_i$, $\hat{l}^*_i$ from (2.34) and foreign analogue;
- $\hat{x}_i$, $\hat{x}^*_i$ from (2.35) and foreign analogue;
- $\hat{mc}_i$, $\hat{mc}^*_i$ from (2.37) and foreign analogue;
- $\hat{l}$, $\hat{l}^*$ from (2.38) and foreign analogue;
- $\hat{c}$, $\hat{c}^*$ from (2.39) and foreign analogue;

We can write the system of equations in terms of a subset of our variables and back out all variables via the static conditions listed above. We turn to this next.

We will use the following vectors in our computation:

$$z_t = [\hat{p}_{H,t-1}, \hat{p}_{F,t-1}, \hat{p}^*_{F,t-1}, \hat{p}^*_{H,t-1}, \hat{k}_{1,t}, \dots \hat{k}_{N,t}, \hat{k}^*_{1,t}, \dots \hat{k}^*_{N,t}, \hat{y}_t, \hat{y}^*_t]' \qquad (n_z \times 1)$$

$$\begin{aligned} X_t = [&\hat{p}_{H,t-2}, \dots, \hat{p}_{H,t-N}, \hat{p}_{F,t-2}, \dots, \hat{p}_{F,t-N}, \hat{p}^*_{F,t-2}, \dots, \hat{p}^*_{F,t-N}, \hat{p}^*_{H,t-2}, \dots, \hat{p}^*_{H,t-N}, \\ &\hat{k}_{1,t-1}, \dots, \hat{k}_{N,t-1}, \hat{k}^*_{1,t-1}, \dots, \hat{k}^*_{Nt-1}] \qquad (n_X \times 1) \end{aligned}$$

$$\mathcal{Z}_t = [z_{t+N-1}, z_{t+N-2}, ...z_t, X_t, \hat{\mu}_{t+N-1}, \dots, \hat{\mu}_{t-N+1}, \hat{\mu}^*_{t+N-1}, \dots, \hat{\mu}^*_{t-N+1}]$$

$$Z_t = [z_t, z_{t-1}, \dots, z_{t-N+2}]' \qquad (n_Z \times 1)$$

$$S_t = [\hat{\mu}_t, \dots, \hat{\mu}_{t-N+1}, \hat{\mu}^*_t, \dots, \hat{\mu}^*_{t-N+1}]' \qquad (n_S \times 1)$$

The vector z_t contains the choice variables at time t. It has $n_z = 2N + 6$ elements. The vector X_t are the state variables at time t. There are $n_X = 6N - 4$ state variables. The vector $\mathcal{Z}_t$ contains all variables that appear in the residual equations. The vectors Z_t and S_t are used when we characterize the solution since it will take the form

$$Z_t = AZ_{t-1} + BS_t \tag{2.40}$$

where Z has $n_Z = (N-1)n_z$ elements and S has $n_S = 2N$ elements.

The residual equations can be written succinctly as follows:

$$\mathcal{E}\left[A_1\begin{bmatrix}X_{t+1}\\ Z_{t+N-1}\end{bmatrix} + A_2\begin{bmatrix}X_t\\ Z_{t+N-2}\end{bmatrix} + \text{shock terms}|\Omega_t\right] = 0$$

where $\mathcal{E}$ implies that expectations are taken – but we will assume that different information sets for the different residual equations. For our example, the residuals are denoted $R(\mathcal{Z})$ and the matrix A_1 is given by

$$A_1 = \begin{bmatrix} I_{n_X,n_X} & 0_{n_X,n_z} & 0_{n_X,n_z} & \dots & 0_{n_X,n_z} \\ 0_{n_z,n_X} & \frac{dR}{d\mathcal{Z}}(:,1:n_z) & \frac{dR}{d\mathcal{Z}}(:,n_z{+}1:2n_z) & \dots & \frac{dR}{d\mathcal{Z}}(:,(N{-}2)n_z{+}1:(N{-}1)n_z) \\ 0_{n_z,n_X} & 0_{n_z,n_z} & I_{n_z,n_z} & \dots & 0_{n_z,n_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_z,n_X} & 0_{n_z,n_z} & 0_{n_z,n_z} & \dots & I_{n_z,n_z} \end{bmatrix} \tag{2.41}$$

and matrix A_2 is given by:

$$A_2 = \begin{bmatrix} -\mathcal{I}_1 & 0_{n_X,n_z} & \dots & 0_{n_X,n_z} & -\mathcal{I}_2 \\ \frac{dR}{d\mathcal{Z}}(:,Nn_z{+}1:Nn_z{+}n_X) & 0_{nz,nz} & \dots & 0_{n_z,n_z} & \frac{dR}{d\mathcal{Z}}(:,(N{-}1)n_z{+}1:Nn_z) \\ 0_{n_z,n_X} & I_{n_z,n_z} & 0_{n_z,n_z} & \dots & 0_{n_z,n_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_z,n_X} & 0_{n_z,n_z} & \dots & I_{n_z,n_z} & 0_{n_z,n_z} \end{bmatrix}. \tag{2.42}$$

The matrices $\mathcal{I}_1$ and $\mathcal{I}_2$ in A_2 are given by

$$\mathcal{I}_1 = \begin{bmatrix} I_{4,4}\otimes\begin{bmatrix}0 & \dots & 0 & 0\\ 1 & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 1 & 0\end{bmatrix}, & 0_{4N-4,2N} \\ 0_{2N,4N-4} & 0_{2N,2N} \end{bmatrix}$$

$$\mathcal{I}_2 = \begin{bmatrix} I_{4,4}\otimes\begin{bmatrix}1\\ 0_{N-2,1}\end{bmatrix}, & 0_{4N-4,2N+2} \\ 0_{2N,4} & [I_{2N,2N}, 0_{2N,2}] \end{bmatrix}$$

Using the method laid out in Blanchard and Kahn (1980), we construct eigenvalues of $-A_1^{-1}A_2$ if A_1 is invertible and generalized eigenvalues otherwise. Then, ignoring shock terms, we have

$$\begin{bmatrix} X_{t+1} \\ Z_{t+N-1} \end{bmatrix} = V\Lambda V^{-1} \begin{bmatrix} X_t \\ Z_{t+N-2} \end{bmatrix}.$$

We can sort eigenvalues inside and outside the unit circle. If there are n_X stable eigenvalues (which is the number of state variables in X), then we have a locally determinate system. Suppose that the eigenvectors in V and eigenvalues in Λ are sorted so that the upper left partition of Λ contains the stable eigenvalues. Then,

$$X_{t+1} = V_{11}\Lambda_1 V_{11}^{-1} X_t$$

$$Z_{t+N-2} = V_{21} V_{11}^{-1} X_t.$$

The last n_z elements of Z_{t+N-2} are those of z_t. Therefore, we have a relationship between our decision variables z and the state variables X. If we want to write the system as (2.40), then we can use this relationship between z and X to fill in the elements of A. In particular, we set

$$\begin{aligned}
A(1:n_z, 1:n_z-2) \quad &= A_{zX}(:, [1, N, 2N-1, 3N-2, 4N-3:6N-4]) \\
A(1:n_z, n_z+1:n_z:n_Z) \ &= A_{zX}(:, 2:N-1) \\
A(1:n_z, n_z+2:n_z:n_Z) \ &= A_{zX}(:, N+1:2N-2) \\
A(1:n_z, n_z+3:n_z:n_Z) \ &= A_{zX}(:, 2N:3N-3) \\
A(1:n_z, n_z+4:n_z:n_Z) \ &= A_{zX}(:, 3N-1:4N-4) \\
A(n_z+1:n_Z, 1:n_Z-n_z) &= I_{n_Z-n_z, n_Z-n_z}
\end{aligned}$$

where A_{zX} comes from $z_t = A_{zX} X_t$.

The next step is to compute B. In this case, we use undetermined coefficients. This method boils down to solving a system of equations in the elements of B_1 where B_1 is the first partition of B which has dimension $n_z \times n_S$, i.e.,

$$B = \begin{bmatrix} B_1 \\ 0_{n_z,n_S} \\ \vdots \\ 0_{n_z,n_S} \end{bmatrix} = \begin{bmatrix} I_{n_z,n_z} \\ 0_{n_z,n_z} \\ \vdots \\ 0_{n_z,n_z} \end{bmatrix} B_1 \equiv \mathcal{S} B_1.$$

We will use $\mathcal{S}$ below in order to reduce the problem of computing B to one of computing B_1.

The system of equations for B_1 turns out to be linear so it is a simple computational problem. The only tricky part of the problem is getting the specification of the equations correct since expectations of intermediate goods producers and the other agents in the economy depend on different information sets.

To derive expressions for the elements of B, we first note that the residuals can be written as follows:

$$\mathcal{E}\Big[a_0 Z_{t+N-1} + a_1 Z_{t+N-2} + \ldots + a_{N-1} Z_t + a_N Z_{t-1} \\ + b_0 S_{t+N-1} + b_1 S_{t+N-2} .... + b_{N-1} S_t | \Omega_t\Big] = 0$$

We didn't originally write them this way because we wanted to avoid lots of redundancies when computing the eigenvalues described above. However, here it is convenient to write it this way to show how we derive B. Using the definitions of Z and $\mathcal{Z}$, we can write:

$$a_0 = dR/d\mathcal{Z}(:, 1 : (N-1)n_z)$$

$$a_{N-1}(:, 1 : n_z) = dR/d\mathcal{Z}(:, (N-1)n_z + 1 : Nn_z)$$

$$b_0(:, 1 : N) = dR/d\mathcal{Z}(:, Nn_z + n_X + 1 : Nn_z + n_X + N)$$

$$b_0(:, N+1 : 2N) = dR/d\mathcal{Z}(:, Nn_z + n_X + 2N : Nn_z + n_X + 3N - 1)$$

$$b_{N-1}(:, 2 : N) = dR/d\mathcal{Z}(:, Nn_z + n_X + N + 1 : Nn_z + n_X + 2N - 1)$$

$$b_{N-1}(:, N+2 : 2N) = dR/d\mathcal{Z}(:, Nn_z + n_X + 3N : Nn_z + n_X + 4N - 2) \quad (2.43)$$

with all other coefficients but a_N set equal to 0. The matrix a_N is nonzero but it is not used in computing B.

Using the solution in (2.40) we get:

$$\mathcal{E}\Big[a_0\left(A^N Z_{t-1} + BS_{t+N-1} + ABS_{t+N-2} + \ldots + A^{N-1}BS_t\right)$$
$$+ a_1\left(A^{N-1}Z_{t-1} + BS_{t+N-2} + ABS_{t+N-3} + \ldots + A^{N-2}BS_t\right) + \ldots$$
$$+ a_{N-1}\left(AZ_{t-1} + BS_t\right) + a_N Z_{t-1}$$
$$+ b_0 S_{t+N-1} + b_1 S_{t+N-2} \ldots + b_{N-1}S_t|\Omega_t\Big] = 0 \tag{2.44}$$

We next derive expressions for $\mathcal{E}[\mathcal{M}S_{t+j}|\Omega_t]$ as a function of S_t, where $\mathcal{M}$ is assumed to be one of the coefficients in (2.44). First, using the fact that $S_{t+1} = \mathcal{P}S_t + \epsilon_{t+1}$ we have

$$\mathcal{E}[\mathcal{M}S_{t+j}|\Omega_t] = \mathcal{M}\mathcal{P}^j\mathcal{E}[S_t|\Omega_t].$$

For example, in our model,

$$\mathcal{P} = \begin{bmatrix} \begin{bmatrix} \rho_\mu & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} & 0_{N,N} \\ 0_{N,N} & \begin{bmatrix} \rho_\mu & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \end{bmatrix}. \tag{2.45}$$

Next, define $\hat{\mathcal{M}}$ as follows:

$$\mathcal{E}[\mathcal{M}S_t|\Omega_t] = \hat{\mathcal{M}}S_t.$$

If $\Omega_t = \{\hat{\mu}_s, \hat{\mu}_s^*\}_{s=0}^{t-1}$ (as is the case for the pricing equations in our model) and $\mathcal{P}$ is given by (2.45) then

$$\mathcal{E}[\mathcal{M}_1\hat{\mu}_t + \ldots + \mathcal{M}_N\hat{\mu}_{t-N+1} + \mathcal{M}_{N+1}\hat{\mu}_t^* + \ldots + \mathcal{M}_{2N}\hat{\mu}_{t-N+1}^*|\Omega_t]$$
$$= [0,\, \mathcal{M}_1\rho_\mu + \mathcal{M}_2,\, \mathcal{M}_3,\, \ldots\, \mathcal{M}_N,\, 0,\, \mathcal{M}_{N+1}\rho_\mu + \mathcal{M}_{N+2},\, \mathcal{M}_{N+3},\, \ldots\, \mathcal{M}_{2N}]\, S_t.$$

If $\Omega_t = \{\hat{\mu}_s, \hat{\mu}_s^*\}_{s=0}^{t}$ (as is the case for the capital Euler equations and the money demand equations), then $\hat{\mathcal{M}} = \mathcal{M}$. For our example, we have

$$\mathcal{E}\Big[\Big((a_0B + b_0)\mathcal{P}^{N-1} + (a_0AB + a_1B + b_1)\mathcal{P}^{N-2} + \ldots +$$
$$(a_0A^{N-1}B + a_1A^{N-2}B + \ldots a_{N-1}B + b_{N-1})\mathcal{P}^0\Big)S_t|\Omega_t\Big] = \mathcal{E}[\mathcal{M}S_t|\Omega_t] = \hat{\mathcal{M}}S_t$$

where $\mathcal{M}$ and $\hat{\mathcal{M}}$ both have dimension $n_z \times n_S$. Applying the method of undetermined coefficients, we want to find B_1 such that every element of $\hat{\mathcal{M}}$ is equal to 0. Because of the timing of the pricing decisions, this will imply $n_z \times n_S - 8$ equations in $n_z \times n_S - 8$ unknowns. In other words, the coefficients on $\hat{\mu}_t$ and $\hat{\mu}_t^*$ in the first four rows of B_1 will be set equal to 0 because prices cannot respond immediately to the monetary shocks.

The following steps are taken to set up the system of equations. First, we stack the nonzero elements of $\hat{\mathcal{M}}$ in a vector. Second, we construct a matrix $\mathcal{D}$ that relates this vector to vec$(\mathcal{M}')$. In our case, this relation is:

$$\begin{bmatrix} \mathcal{M}_{1,1}\rho_\mu + \mathcal{M}_{1,2} \\ \mathcal{M}_{1,3} \\ \vdots \\ \mathcal{M}_{1,N} \\ \mathcal{M}_{1,N+1}\rho_\mu + \mathcal{M}_{1,N+2} \\ \mathcal{M}_{1,N+3} \\ \vdots \\ \mathcal{M}_{1,n_S} \\ \vdots \\ \mathcal{M}_{4,1}\rho_\mu + \mathcal{M}_{4,2} \\ \mathcal{M}_{4,3} \\ \vdots \\ \mathcal{M}_{4,N} \\ \mathcal{M}_{4,N+1}\rho_\mu + \mathcal{M}_{4,N+2} \\ \mathcal{M}_{4,N+3} \\ \vdots \\ \mathcal{M}_{4,n_S} \\ \mathcal{M}_{5,1} \\ \mathcal{M}_{5,2} \\ \vdots \\ \mathcal{M}_{n_z,n_S} \end{bmatrix} = \underbrace{\begin{bmatrix} I_{8,8} \otimes \Psi & 0_{8N-8,n_z n_S-8N} \\ 0_{n_z n_S-8N,8N} & I_{n_z n_S-8N} \end{bmatrix}}_{\mathcal{D}} \underbrace{\begin{bmatrix} \mathcal{M}_{1,1} \\ \mathcal{M}_{1,2} \\ \mathcal{M}_{1,3} \\ \vdots \\ \mathcal{M}_{1,n_S} \\ \mathcal{M}_{2,1} \\ \mathcal{M}_{2,2} \\ \vdots \\ \mathcal{M}_{n_Z,n_S} \end{bmatrix}}_{\text{vec}(\mathcal{M}')} \tag{2.46}$$

where

$$\Psi = \begin{bmatrix} \rho_\mu & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \qquad (N-1 \times N).$$

Third, we set $\mathcal{D}\mathrm{vec}(\mathcal{M}')$ equal to zero (which ensures that $\hat{\mathcal{M}} = 0$),

$$\begin{aligned}\mathcal{D}\mathrm{vec}(\mathcal{M}') = \mathcal{D}\mathrm{vec}\big([a_0\mathcal{S}B_1\mathcal{P}^{N-1}]' + [a_0A\mathcal{S}B_1\mathcal{P}^{N-2} + a_1\mathcal{S}B_1\mathcal{P}^{N-2}]' + \ldots + \\ [a_0A^{N-1}\mathcal{S}B_1\mathcal{P}^0 + a_1A^{N-2}\mathcal{S}B_1\mathcal{P}^0 + \ldots + a_{N-1}\mathcal{S}B_1\mathcal{P}^0]'\big) \\ + \mathcal{D}\mathrm{vec}\left([b_0\mathcal{P}^{N-1} + b_1\mathcal{P}^{N-2} + \ldots + b_{N-1}\mathcal{P}^0]'\right) \\ \\ \equiv \mathcal{Q}\mathrm{vec}(B_1') + \mathcal{R}.\end{aligned}$$

To construct $\mathcal{Q}$ we need to use the fact that $\mathrm{vec}(ABC)$ is equal to $[C' \otimes A]\mathrm{vec}(B)$. At this point, we can write the equation explicitly in terms of B_1 – or more precisely, the nonzero elements of B_1:

$$\mathrm{vec}(B1')(\text{nonzero elements}) = -\left[\mathcal{Q}(:, \text{nonzero elements})\right]^{-1}\mathcal{R}.$$

For our example, the zero elements of B_1 are: (i,1) and (i,N+1) for i =1, 2, 3, and 4. These are the coefficients on contemporaneous shocks in the pricing decision rules. All other elements are assumed to be nonzero.

2.2. The Taylor-Rule Extension

We now consider our first extension to the benchmark economy: the Fed follows (1.46) and money is determined residually from the money demand equation (1.15). The main differences in the computation from the benchmark economy are these:

- we add outputs and interest rates to the state vector;
- we add the Taylor Rules as residuals;
- we add r and r^* to our choice variables;
- we define M/P via the money demand equation rather than as μ_t/p_t;
- μ is constant.

2.2.1. Normalization for the Taylor-Rule Extension

In the Taylor-Rule case, we assume that prices grow at the rate μ. To keep everything consistent with the equations derived above, we use the following normalization for prices:

$$\begin{aligned}
p(s^t) &= P(s^t)/\mu^{t-1} \\
p_I(i, s^{t-1}) &= P_I(i, s^{t-1})/\mu^{t-i} \\
\bar{p}_I(s^{t-1}) &= \bar{P}_I(s^{t-1})/\mu^{t-1} \\
p^*(s^t) &= P^*(s^t)/\mu^{t-1} \\
p_I^*(i, s^{t-1}) &= P_I^*(i, s^{t-1})/\mu^{t-i} \\
\bar{p}_I^*(s^{t-1}) &= \bar{P}_I^*(s^{t-1})/\mu^{t-1}
\end{aligned}$$

where $I = H$ or F. In the equations of the benchmark economy, we replace all state dependent μ's with a constant μ.

2.2.2. Steady State for the Taylor-Rule Extension

The steady state is the same as in the benchmark economy.

2.2.3. Linearized Equations for the Taylor-Rule Extension

We add the following equation to the set of linearized equations derived above:

$$r_t = a' \begin{bmatrix} r_{t-1} \\ r_{t-2} \\ r_{t-3} \\ E_t\hat{p}_{t+1} - \hat{p}_t \\ \hat{p}_t - \hat{p}_{t-1} \\ \hat{p}_{t-1} - \hat{p}_{t-2} \\ \hat{p}_{t-2} - \hat{p}_{t-3} \\ \hat{gdp}_t \\ \hat{gdp}_{t-1} \\ \hat{gdp}_{t-2} \end{bmatrix} + \epsilon_{r,t} \tag{2.47}$$

and its foreign analogue, where

$$\hat{gdp}_t = \hat{y}_t + \sum_i y_H^*(i)\,(\hat{y}^*_{H,i,t} - \hat{y}_{F,i,t})/(Ny).$$

Again, note that the interest rate is in levels while all other variables are logged.

We back out money (also normalized by μ^t) from the money demand equation (1.15):

$$\hat{m}_t - \hat{p}_t = \left[\left(\frac{U_{cm}}{U_m} - \frac{U_{cc}}{U_c} \right) c\, \hat{c}_t + \left(\frac{U_{lm}}{U_m} - \frac{U_{cl}}{U_c} \right) l\, \hat{l}_t \right.$$

$$\left. - \frac{\beta}{r}(r_t - r) \right] / \left[\left(\frac{U_{mm}}{U_m} - \frac{U_{cm}}{U_c} \right) M/P \right] \tag{2.48}$$

where $\hat{m}_t = \log(M(s^t)/\mu^t)$.

2.2.4. Solving the Linearized System for the Taylor-Rule Extension

The system of equations that we solve has $2N + 8$ **dynamic** equations:

- 2 Taylor rules ((2.47) for home and similar for foreign)
- 4 pricing equations, (2.30)-(2.33)
- $2N$ Euler equations for capital ((2.36) for home and similar for foreign)
- 2 equations for interest rates ((2.22) for home and similar for foreign)
- and **static** equations that are the same as the benchmark economy, except for real balances which are now
  - $\hat{m} - \hat{p}$, $\hat{m}^* - \hat{p}^*$ from (2.48) and foreign analogue;

As before, we can write the system of equations in terms of a subset of our variables and back out all variables via the static first-order conditions.

When computing the Taylor-rule extension, we use the following vectors:

$$z_t = [\hat{p}_{H,t-1}, \hat{p}_{F,t-1}, \hat{p}^*_{F,t-1}, \hat{p}^*_{H,t-1}, \hat{k}_{1,t}, \ldots \hat{k}_{N,t}, \hat{k}^*_{1,t}, \ldots \hat{k}^*_{N,t}, \hat{y}_t, \hat{y}^*_t, r_t, r^*_t]' \quad (n_z \times 1)$$

$$\begin{aligned} X_t = [&\hat{p}_{H,t-2}, \ldots, \hat{p}_{H,t-(N+3)}, \hat{p}_{F,t-2}, \ldots, \hat{p}_{F,t-(N+3)}, \hat{p}^*_{F,t-2}, \ldots, \hat{p}^*_{F,t-(N+3)}, \\ &\hat{p}^*_{H,t-2}, \ldots, \hat{p}^*_{H,t-(N+3)}, \hat{k}_{1,t-1}, \ldots, \hat{k}_{N,t-1}, \hat{k}^*_{1,t-1}, \ldots, \hat{k}^*_{Nt-1}, \\ &\hat{y}_{t-1}, \hat{y}_{t-2}, \hat{y}^*_{t-1}, \hat{y}^*_{t-2}, r_{t-1}, r_{t-2}, r_{t-3}, r^*_{t-1}, r^*_{t-2}, r^*_{t-3}]' \quad (n_X \times 1) \end{aligned}$$

$$\mathcal{Z}_t = [z_{t+N-1}, z_{t+N-2}, \ldots z_t, X_t, \epsilon_{r,t}, \epsilon^*_{r,t}]'$$

$$Z_t = [z_t, z_{t-1}, \ldots, z_{t-N-1}]' \quad (n_Z \times 1)$$

$$S_t = [\epsilon_{r,t}, \epsilon_{r,t-1}, \epsilon^*_{r,t}, \epsilon^*_{r,t-1}]' \quad (n_S \times 1)$$

The vector z_t contains the choice variables at time t. It has $n_z = 2N + 8$ elements. The vector X_t are the state variables at time t. There are $n_X = 6N + 18$ state variables. The vector $\mathcal{Z}_t$ contains all variables that appear in the residual equations. The vectors Z_t and S_t are used when we characterize the solution, $Z_t = AZ_{t-1} + BS_t$ where Z has $n_Z = (N + 2)n_z$ elements and S has $n_S = 4$ elements.

The residual equations can be written as follows:

$$\mathcal{E}\left[A_1 \begin{bmatrix} X_{t+1} \\ z_{t+N-1} \\ \vdots \\ z_{t+1} \end{bmatrix} + A_2 \begin{bmatrix} X_t \\ z_{t+N-2} \\ \vdots \\ z_t \end{bmatrix} + \text{shock terms} | \Omega_t \right] = 0.$$

For our example, the residuals are denoted $R(\mathcal{Z})$ and the matrix A_1 is given by

$$A_1 = \begin{bmatrix} I_{n_X,n_X} & 0_{n_X,n_z} & 0_{n_X,n_z} & \ldots & 0_{n_X,n_z} \\ 0_{n_z,n_X} & \frac{dR}{d\mathcal{Z}}(:,1:n_z) & \frac{dR}{d\mathcal{Z}}(:,n_z{+}1:2n_z) & \ldots & \frac{dR}{d\mathcal{Z}}(:,(N{-}2)n_z{+}1:(N{-}1)n_z) \\ 0_{n_z,n_X} & 0_{n_z,n_z} & I_{n_z,n_z} & \ldots & 0_{n_z,n_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_z,n_X} & 0_{n_z,n_z} & 0_{n_z,n_z} & \ldots & I_{n_z,n_z} \end{bmatrix}$$

and matrix A_2 is given by:

$$A_2 = \begin{bmatrix} -\mathcal{I}_1 & 0_{n_X,n_z} & \ldots & 0_{n_X,n_z} & -\mathcal{I}_2 \\ \frac{dR}{d\mathcal{Z}}(:,Nn_z{+}1:Nn_z{+}n_X) & 0_{nz,nz} & \ldots & 0_{n_z,n_z} & \frac{dR}{d\mathcal{Z}}(:,(N{-}1)n_z{+}1:Nn_z) \\ 0_{n_z,n_X} & I_{n_z,n_z} & 0_{n_z,n_z} & \ldots & 0_{n_z,n_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_z,n_X} & 0_{n_z,n_z} & \ldots & I_{n_z,n_z} & 0_{n_z,n_z} \end{bmatrix}.$$

The matrices $\mathcal{I}_1$ and $\mathcal{I}_2$ in A_2 are given by

$$\mathcal{I}_1 = \begin{bmatrix} I_{4,4} \otimes \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}, & 0_{4N+8,2N} & 0_{4N+8,4} & 0_{4N+8,6} \\ 0_{2N,4N+8} & 0_{2N,2N} & 0_{2N,4} & 0_{2N,6} \\ 0_{4,4N+8} & 0_{4,2N} & I_{2,2} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & 0_{4,6} \\ 0_{6,4N+8} & 0_{6,2N} & 0_{6,4} & I_{2,2} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix}$$

$$\mathcal{I}_2 = \begin{bmatrix} I_{4,4} \otimes \begin{bmatrix} 1 \\ 0_{N+1,1} \end{bmatrix}, & 0_{4N+8,2N} & 0_{4N+8,2} & 0_{4N+8,2} \\ 0_{2N,4} & I_{2N,2N} & 0_{2N,2} & 0_{2N,2} \\ 0_{4,4} & 0_{4,2N} & I_{2,2} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0_{4,2} \\ 0_{6,4} & 0_{6,2N} & 0_{6,2} & I_{2,2} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

Again, using the method laid out in Blanchard and Kahn (1980), we construct eigenvalues of $-A_1^{-1}A_2$ if A_1 is invertible and generalized eigenvalues otherwise. Then, ignoring shock terms, we have

$$\begin{bmatrix} X_{t+1} \\ z_{t+N-1} \\ \vdots \\ z_{t+1} \end{bmatrix} = V \Lambda V^{-1} \begin{bmatrix} X_t \\ z_{t+N-2} \\ \vdots \\ z_t \end{bmatrix}.$$

We can sort eigenvalues inside and outside the unit circle. If there are n_X stable eigenvalues (which is the number of state variables in X), then we have a locally determinate system. Suppose that the eigenvectors in V and eigenvalues in Λ are sorted so that the upper left partition of Λ contains the stable eigenvalues. Then,

$$\begin{aligned} X_{t+1} &= V_{11} \Lambda_1 V_{11}^{-1} X_t \\ \begin{bmatrix} z_{t+N-2} \\ \vdots \\ z_t \end{bmatrix} &= V_{21} V_{11}^{-1} X_t. \end{aligned}$$

Now we have a relationship between our decision variables z and the state variables X. If we want to write the system as (2.40), then we can use this relationship between z and X to fill in the elements of A. In particular, we set

$$
\begin{aligned}
A(1:n_z, 1:n_z) \quad &= A_{zX}(:, [1, N+3, 2N+5, 3N+7, 4N+9:6N+9, ... \\
&\qquad 6N+11, 6N+13, 6N+16]) \\
A(1:n_z, n_z+1:n_z:n_Z) &= A_{zX}(:, 2:N+2) \\
A(1:n_z, n_z+2:n_z:n_Z) &= A_{zX}(:, N+4:2N+4) \\
A(1:n_z, n_z+3:n_z:n_Z) &= A_{zX}(:, 2N+6:3N+6) \\
A(1:n_z, n_z+4:n_z:n_Z) &= A_{zX}(:, 3N+8:4N+8) \\
A(1:n_z, 2n_z-3) &= A_{zX}(:, 6N+10) \\
A(1:n_z, 2n_z-2) &= A_{zX}(:, 6N+12) \\
A(1:n_z, 2n_z-1) &= A_{zX}(:, 6N+14) \\
A(1:n_z, 2n_z) &= A_{zX}(:, 6N+17) \\
A(1:n_z, 3n_z-1) &= A_{zX}(:, 6N+15) \\
A(1:n_z, 3n_z) &= A_{zX}(:, 6N+18) \\
A(n_z+1:n_Z, 1:n_Z-n_z) &= I_{n_Z-n_z, n_Z-n_z}
\end{aligned}
$$

where A_{zX} comes from $z_t = A_{zX}X_t$.

The next step is to compute B. In this case, we use undetermined coefficients. This method boils down to solving a system of equations in the elements of B_1 where B_1 is the first partition of B which has dimension $n_z \times n_S$, i.e.,

$$
B = \begin{bmatrix} B_1 \\ 0_{n_z,n_S} \\ \vdots \\ 0_{n_z,n_S} \end{bmatrix} = \begin{bmatrix} I_{n_z,n_z} \\ 0_{n_z,n_z} \\ \vdots \\ 0_{n_z,n_z} \end{bmatrix} B_1 \equiv \mathcal{S} B_1.
$$

We will use $\mathcal{S}$ below in order to reduce the problem of computing B to one of computing B_1.

To derive expressions for the elements of B, we first note that the residuals can be written as follows:

$$\mathcal{E}\Big[a_0 Z_{t+N-1} + a_1 Z_{t+N-2} + \ldots + a_{N-1} Z_t + a_N Z_{t-1}$$
$$+ b_0 S_{t+N-1} + b_1 S_{t+N-2} \ldots . + b_{N-1} S_t | \Omega_t\Big] = 0.$$

Using the definitions of Z and $\mathcal{Z}$, we can write:

$$a_0(:, 1 : Nn_z) \qquad = dR/d\mathcal{Z}(:, 1 : Nn_z)$$

$$b_{N-1}(:, 1) \qquad = dR/d\mathcal{Z}(:, Nn_z + n_X + 1)$$

$$b_{N-1}(:, 3) \qquad = dR/d\mathcal{Z}(:, Nn_z + n_X + 2)$$

with all other coefficients but a_N set equal to 0. The matrix a_N is nonzero but it is not used in computing B.

Again, using the solution in (2.40) we get:

$$\mathcal{E}\Big[a_0 \left(A^N Z_{t-1} + BS_{t+N-1} + ABS_{t+N-2} + \ldots + A^{N-1} BS_t\right)$$
$$+ a_1 \left(A^{N-1} Z_{t-1} + BS_{t+N-2} + ABS_{t+N-3} + \ldots + A^{N-2} BS_t\right) + \ldots$$
$$+ a_{N-1} \left(AZ_{t-1} + BS_t\right) + a_N Z_{t-1}$$
$$+ b_0 S_{t+N-1} + b_1 S_{t+N-2} \ldots . + b_{N-1} S_t | \Omega_t\Big] = 0$$

We next derive expressions for $\mathcal{E}[\mathcal{M} S_{t+j} | \Omega_t]$ as a function of S_t, where $\mathcal{M}$ is assumed to be one of the coefficients in (2.44). First, using the fact that $S_{t+1} = \mathcal{P} S_t + \epsilon_{t+1}$ we have

$$\mathcal{E}[\mathcal{M} S_{t+j} | \Omega_t] = \mathcal{M} \mathcal{P}^j \mathcal{E}[S_t | \Omega_t].$$

For example, if the policy shocks $\epsilon_{r,t}$ and $\epsilon^*_{r,t}$ are both serially correlated, then

$$\mathcal{P} = \begin{bmatrix} \rho_\epsilon & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \rho_\epsilon & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.49}$$

Next, define $\hat{\mathcal{M}}$ as follows:

$$\mathcal{E}[\mathcal{M}S_t|\Omega_t] = \hat{\mathcal{M}}S_t.$$

If $\Omega_t = \{\epsilon_{r,s}, \epsilon^*_{r,s}\}_{s=0}^{t-1}$ (as is the case for the pricing equations in our model) and $\mathcal{P}$ is given by (2.49) then

$$\begin{aligned}&\mathcal{E}[\mathcal{M}_1\epsilon_{r,t} + \mathcal{M}_2\epsilon_{r,t-1} + \mathcal{M}_3\epsilon^*_{r,t} + \mathcal{M}_4\epsilon^*_{r,t-1}|\Omega_t]\\ &= [0,\ \mathcal{M}_1\rho_\epsilon + \mathcal{M}_2,\ 0,\ \mathcal{M}_3\rho_\epsilon + \mathcal{M}_4]\ S_t.\end{aligned}$$

If $\Omega_t = \{\epsilon_{r,s}, \epsilon^*_{r,s}\}_{s=0}^{t}$ (as is the case for the capital Euler equations and the money demand equations), then $\hat{\mathcal{M}} = \mathcal{M}$. For our example, we have

$$\begin{aligned}&\mathcal{E}\Bigg[\Bigg((a_0B + b_0)\mathcal{P}^{N-1} + (a_0AB + a_1B + b_1)\mathcal{P}^{N-2} + ...+\\ &\qquad (a_0A^{N-1}B + a_1A^{N-2}B + \ldots a_{N-1}B + b_{N-1})\mathcal{P}^0\Bigg)S_t|\Omega_t\Bigg] = \mathcal{E}[\mathcal{M}S_t|\Omega_t] = \hat{\mathcal{M}}S_t\end{aligned}$$

where $\mathcal{M}$ and $\hat{\mathcal{M}}$ both have dimension $n_z \times n_S$. Applying the method of undetermined coefficients, we want to find B_1 such that every element of $\hat{\mathcal{M}}$ is equal to 0. Because of the timing of the pricing decisions, this will imply $n_z \times n_S - 8$ equations in $n_z \times n_S - 8$ unknowns. In other words, the coefficients on μ_t and μ^*_t in the first four rows of B_1 will be set equal to 0 because prices cannot respond immediately to the monetary shocks.

The following steps are taken to set up the system of equations. First, we stack the nonzero elements of $\hat{\mathcal{M}}$ in a vector. Second, we construct a matrix $\mathcal{D}$ that relates this vector to vec$(\mathcal{M}')$. In our case, this relation is:

$$\begin{bmatrix} \mathcal{M}_{1,1}\rho_\epsilon + \mathcal{M}_{1,2} \\ \mathcal{M}_{1,3}\rho_\epsilon + \mathcal{M}_{1,4} \\ \mathcal{M}_{2,1}\rho_\epsilon + \mathcal{M}_{2,2} \\ \mathcal{M}_{2,3}\rho_\epsilon + \mathcal{M}_{2,4} \\ \vdots \\ \mathcal{M}_{4,1}\rho_\epsilon + \mathcal{M}_{4,2} \\ \mathcal{M}_{4,3}\rho_\epsilon + \mathcal{M}_{4,4} \\ \mathcal{M}_{5,1} \\ \mathcal{M}_{5,2} \\ \mathcal{M}_{5,3} \\ \mathcal{M}_{5,4} \\ \vdots \\ \mathcal{M}_{n_z,4} \end{bmatrix} = \underbrace{\begin{bmatrix} I_{8,8} \otimes [\rho_\epsilon,\ 1] & 0_{8,8N+16} \\ 0_{8N+16,16} & I_{8N+16,8N+16} \end{bmatrix}}_{\mathcal{D}} \underbrace{\begin{bmatrix} \mathcal{M}_{1,1} \\ \mathcal{M}_{1,2} \\ \mathcal{M}_{1,3} \\ \mathcal{M}_{1,4} \\ \vdots \\ \mathcal{M}_{n_z,4} \end{bmatrix}}_{\text{vec}(\mathcal{M}')}.$$

Third, we set $\mathcal{D}\text{vec}(\mathcal{M}')$ equal to zero (which ensures that $\hat{\mathcal{M}} = 0$),

$$\begin{aligned}\mathcal{D}\text{vec}(\mathcal{M}') = \mathcal{D}\text{vec}\big([a_0\mathcal{S}B_1\mathcal{P}^{N-1}]' + [a_0A\mathcal{S}B_1\mathcal{P}^{N-2} + a_1\mathcal{S}B_1\mathcal{P}^{N-2}]' + \ldots + \\ [a_0A^{N-1}\mathcal{S}B_1\mathcal{P}^0 + a_1A^{N-2}\mathcal{S}B_1\mathcal{P}^0 + \ldots + a_{N-1}\mathcal{S}B_1\mathcal{P}^0]'\big) \\ + \mathcal{D}\text{vec}\left([b_0\mathcal{P}^{N-1} + b_1\mathcal{P}^{N-2} + \ldots + b_{N-1}\mathcal{P}^0]'\right)\end{aligned}$$

$$\equiv \mathcal{Q}\text{vec}(B_1') + \mathcal{R}.$$

To construct $\mathcal{Q}$ we need to use the fact that $\text{vec}(ABC)$ is equal to $[C' \otimes A]\text{vec}(B)$. At this point, we can write the equation explicitly in terms of B_1 – or more precisely, the nonzero elements of B_1:

$$\text{vec}(B1')(\text{nonzero elements}) = -\left[\mathcal{Q}(:,\text{nonzero elements})\right]^{-1}\mathcal{R}.$$

For our example, the zero elements of B_1 are: (i,1) and (i,3) for i =1, 2, 3, and 4. These are the coefficients on contemporaneous shocks in the pricing decision rules. All other elements are assumed to be nonzero.

2.3. The Incomplete-Markets Extension

We turn next to the second extension: asset markets are incomplete. The main differences in the computation from the benchmark economy are these:

- we add foreign bond holdings (D^*_{-1}) as a state variable;
- we add equation relating right hand sides of (1.49) and (1.50) to our residuals;
- we add D^* to our choice variables;
- we use $\hat{q}$ derived below rather than (2.24).

2.3.1. Normalization for the I-M Extension

In the foreign budget constraint (1.48) impose $T^*(s^t) = M^*(s^t) - M^*(s^{t-1})$ and $B(s^t) = 0$. If we normalize variables as follows:

$$p^*(s^t) = \frac{P^*(s^t)}{M^*(s^{t-1})},\ \pi^*(s^t) = \frac{\Pi^*(s^t)}{M^*(s^{t-1})},\ d^*(s^t) = \frac{D^*(s^t)}{M(s^t)},\ \varepsilon(s^t) = \frac{e(s^t)M^*(s^{t-1})}{M(s^{t-1})}$$

then the foreign budget constraint is given by:

$$p^*(s^t)c^*(s^t) + V(s^t)d^*(s^t)\frac{\mu(s^t)}{\varepsilon(s^t)} = p^*(s^t)w^*(s^t)l^*(s^t) + \frac{d^*(s^{t-1})}{\varepsilon(s^t)} + \pi^*(s^t).$$

Solving for normalized nominal exchange rate $\varepsilon(s^t)$, we get

$$\varepsilon(s^t) = \frac{V(s^t)d^*(s^t)\mu(s^t) - d^*(s^{t-1})}{p^*(s^t)[w^*(s^t)l^*(s^t) - c^*(s^t)] + \pi^*(s^t)}. \tag{2.50}$$

Now substitute out for profits in (2.50). Profits for the ith producer are

$$\Pi^*(i,s^t) = \frac{P_F(i,s^{t-1})}{e(s^t)}y_F(i,s^t) + P_F^*(i,s^{t-1})y_F^*(i,s^t) - P^*(s^t)[w^*(s^t)l^*(i,s^t) + x^*(i,s^t)].$$

If we normalize these profits by the money supply we get

$$\begin{aligned}\frac{\Pi^*(i,s^t)}{M^*(s^{t-1})} = {} & \frac{P_F(i,s^{t-1})}{M^*(s^{t-1})}\frac{M^*(s^{t-1})}{\varepsilon(s^t)M(s^{t-1})}y_F(i,s^t) + \frac{P_F^*(i,s^{t-1})}{M^*(s^{t-1})}y_F^*(i,s^t) \\ & - \frac{P^*(s^t)}{M^*(s^{t-1})}[w^*(s^t)l^*(i,s^t) + x^*(i,s^t)]\end{aligned}$$

or, using the definitions above,

$$\begin{aligned}\pi^*(i,s^t) = {} & \frac{P_F(i,s^{t-1})}{M(s^{t-1})}\frac{M(s^{t-i})}{\varepsilon(s^t)M(s^{t-1})}y_F(i,s^t) + \frac{P_F^*(i,s^{t-i})}{M^*(s^{t-1})}\frac{M^*(s^{t-i})}{M^*(s^{t-i})}y_F^*(i,s^t) \\ & - p^*(s^t)[w^*(s^t)l^*(i,s^t) + x^*(i,s^t)] \\ = {} & p_F(i,s^{t-1})y_F(i,s^t)\frac{1}{\varepsilon(s^t)}\frac{M(s^{t-i})}{M(s^{t-1})} + p_F^*(i,s^{t-1})y_F^*(i,s^t)\frac{M^*(s^{t-i})}{M^*(s^{t-1})} \\ & - p^*(s^t)[w^*(s^t)l^*(i,s^t) + x^*(i,s^t)] \\ = {} & \frac{p_F(i,s^{t-1})y_F(i,s^t)}{\varepsilon(s^t)\mu(s^{t-1})\cdots\mu(s^{t-i+1})} + \frac{p_F^*(i,s^{t-1})y_F^*(i,s^t)}{\mu^*(s^{t-1})\cdots\mu^*(s^{t-i+1})} \\ & - p^*(s^t)[w^*(s^t)l^*(i,s^t) + x^*(i,s^t)]\end{aligned}$$

The last equation follows from the fact that $M(s^{t-1}) = \mu(s^{t-1}) \cdots \mu(s^{t-i+1}) M(s^{t-i})$. If we integrate across firms, we get

$$\pi^*(s^t) = \frac{1}{\varepsilon(s^t)} \frac{1}{N} \sum_{i=1}^{N} \frac{p_F(i, s^{t-1}) y_F(i, s^t)}{\mu(s^{t-1}) \cdots \mu(s^{t-i+1})} + \frac{1}{N} \sum_{i=1}^{N} \frac{p_F^*(i, s^{t-1}) y_F^*(i, s^t)}{\mu^*(s^{t-1}) \cdots \mu^*(s^{t-i+1})}$$
$$- p^*(s^t)[w^*(s^t) l^*(s^t) + x^*(s^t)] \tag{2.51}$$

since $\pi^*(s^t) = \sum_{i=1}^{N} \pi^*(i, s^t)/N$, $l^* = \sum_{i=1}^{N} l^*(i)/N$, and $x^* = \sum_{i=1}^{N} x^*(i)/N$.

Substitute for (2.51) in (2.50) to get our final equation for the normalized nominal exchange rate:

$$\varepsilon(s^t) = \frac{V(s^t) d^*(s^t) \mu(s^t) - d^*(s^{t-1}) - \frac{1}{N} \sum \frac{p_F(i, s^{t-1}) y_F(i, s^t)}{\mu(s^{t-1}) \cdots \mu(s^{t-i+1})}}{\frac{1}{N} \sum \frac{p_F^*(i, s^{t-1}) y_F^*(i, s^t)}{\mu^*(s^{t-1}) \cdots \mu^*(s^{t-i+1})} - p^*(s^t) y^*(s^t)}. \tag{2.52}$$

The real exchange rate is then given by

$$q(s^t) = \frac{e(s^t) P^*(s^t)}{P(s^t)} = \frac{\varepsilon(s^t) p^*(s^t)}{p(s^t)}. \tag{2.53}$$

Finally, we normalize the equation for bond prices (1.50):

$$\begin{aligned} V(s^t) &= \sum_{s_{t+1}} \beta \pi(s^{t+1}|s^t) \frac{U_c^*(s^{t+1})}{U_c^*(s^t)} \frac{P^*(s^t)}{P^*(s^{t+1})} \frac{e(s^t)}{e(s^{t+1})} \\ &= \sum_{s_{t+1}} \beta \pi(s^{t+1}|s^t) \frac{U_c^*(s^{t+1})}{U_c^*(s^t)} \frac{P(s^t)}{P(s^{t+1})} \frac{q(s^t)}{q(s^{t+1})} \\ &= \sum_{s_{t+1}} \beta \pi(s^{t+1}|s^t) \frac{U_c^*(s^{t+1})}{U_c^*(s^t)} \frac{p(s^t)}{p(s^{t+1}) \mu(s^t)} \frac{q(s^t)}{q(s^{t+1})}. \end{aligned} \tag{2.54}$$

2.3.2. Steady State for the I-M Extension

The steady state is the same as in the benchmark economy. The additional variables are equal to

$$\varepsilon = 1$$
$$d^* = 0$$
$$V = \beta/\mu$$

in the steady state.

2.3.3. Linearized Equations for the I-M Extension

In addition to the equations for the benchmark economy, we have to linearize the nominal exchange rate (2.52) and the bond price (2.54).

Let's start with the exchange rate. Rewrite the equation as

$$\begin{aligned}\varepsilon(s^t)\,\frac{1}{N}\sum \frac{p_F^*(i,s^{t-1})y_F^*(i,s^t)}{\mu^*(s^{t-1})\cdots\mu^*(s^{t-i+1})} - \varepsilon(s^t)p^*(s^t)y^*(s^t)\\ = V(s^t)d^*(s^t)\mu(s^t) - d^*(s^{t-1}) - \frac{1}{N}\sum\frac{p_F(i,s^{t-1})y_F(i,s^t)}{\mu(s^{t-1})\cdots\mu(s^{t-i+1})}\end{aligned}$$

Linearizing this yields

$$\begin{aligned}&\left(\frac{p_I}{N}\sum_i \frac{y_F^*(i)}{\mu^{i-1}} - py\right)\hat{\varepsilon}_t - py(\hat{p}_t^* + \hat{y}_t^*)\\ &\quad + \frac{p_I}{N}\left(y_F^*(1)[\hat{p}^*_{F,t-1} + \hat{y}^*_{F,1,t}] + \frac{y_F^*(2)}{\mu}[\hat{p}^*_{F,t-2} + \hat{y}^*_{F,2,t} - \hat{\mu}^*_{t-1}] + \dots\right)\\ &\quad = \beta d_t^* - d_{t-1}^* - \frac{p_I}{N}\left(y_F(1)[\hat{p}_{F,t-1} + \hat{y}_{F,1,t}] + \frac{y_F(2)}{\mu}[\hat{p}_{F,t-2} + \hat{y}_{F,2,t} - \hat{\mu}_{t-1}] + \dots\right)\end{aligned}$$

or

$$\begin{aligned}\hat{\varepsilon}_t = \frac{1}{\Delta}\Bigg[&\beta d_t^* - d_{t-1}^* + py(\hat{p}_t^* + \hat{y}_t^*)\\ &- \frac{p_I}{N}\left(\sum_{i=1}^N y_F(i)[\hat{p}_{F,t-i} + \hat{y}_{F,i,t} - \hat{\mu}_{t-1} - \dots\hat{\mu}_{t-i+1}]\right)\\ &- \frac{p_I}{N}\left(\sum_{i=1}^N y_F^*(i)[\hat{p}^*_{F,t-i} + \hat{y}^*_{F,i,t} - \hat{\mu}^*_{t-1} - \dots\hat{\mu}^*_{t-i+1}]\right)\Bigg]\end{aligned}$$

where $\Delta = p_I \left(\frac{1}{N} \sum_i \frac{y_F^*(i)}{\mu^{i-1}} \right) - py$, and p_I is the steady value for all of the intermediate goods prices. Note that we have used the fact that the countries have the same steady state values (e.g., $p = p^*$) when writing these expressions. Given $\hat{\varepsilon}_t$, we can compute the real exchange rate,

$$\hat{q}_t = \hat{\varepsilon}_t + \hat{p}_t^* - \hat{p}_t. \tag{2.55}$$

Next, we linearize (2.54) to get

$$\begin{aligned} -\frac{\beta}{\mu}(r_t - r) = E_t \Bigg(& \frac{U_{cc} c}{U_c} (\hat{c}_{t+1}^* - \hat{c}_t^*) + \frac{U_{cl} l}{U_c} (\hat{l}_{t+1}^* - \hat{l}_t^*) \\ & + \frac{U_{cm} M/P}{U_c} (\hat{\mu}_{t+1}^* - \hat{p}_{t+1}^* - \hat{\mu}_t^* + \hat{p}_t^*) \\ & + \hat{p}_t + \hat{q}_t - \hat{p}_{t+1} - \hat{q}_{t+1} - \hat{\mu}_t \Bigg). \end{aligned} \tag{2.56}$$

where we have used the fact that $V(s^t) = 1/(1 + r(s^t))$.

Given that V is the inverse of the gross real interest rate, we can also define it as $V(s^t) = 1 - U_m(s^t)/U_c(s^t)$ and therefore,

$$V(V_t - V) = \frac{U_m}{U_c} \left\{ \left(\frac{U_{cc}}{U_c} - \frac{U_{cm}}{U_m} \right) c\, \hat{c}_t + \left(\frac{U_{cl}}{U_c} - \frac{U_{lm}}{U_m} \right) l\, \hat{l}_t + \left(\frac{U_{cm}}{U_c} - \frac{U_{mm}}{U_m} \right) M/P (\hat{\mu}_t - \hat{p}_t) \right\}. \tag{2.57}$$

2.3.4. Solving the Linearized System for the I-M Extension

The system of equations that we solve has $2N + 7$ **dynamic** equations:

- 4 pricing equations, (2.30)-(2.33)
- $2N$ Euler equations for capital ((2.36) for home and similar for foreign)
- 2 money demand equations ((2.21) for home and similar for foreign)
- 1 equation relating bond prices (set (2.22) equal to (2.56), eliminating r)
- and **static** equations and definitions as in the benchmark economy except
  - $\hat{q}$ from (2.55);

- V from (2.57).

We turn next to the computation. We will use the following vectors:

$$z_t = [\hat{p}_{H,t-1}, \hat{p}_{F,t-1}, \hat{p}^*_{F,t-1}, \hat{p}^*_{H,t-1}, \hat{k}_{1,t}, \ldots \hat{k}_{N,t}, \hat{k}^*_{1,t}, \ldots \hat{k}^*_{N,t}, d^*_t, \hat{y}_t, \hat{y}^*_t]' \qquad (n_z \times 1)$$

$$X_t = [\hat{p}_{H,t-2}, \ldots, \hat{p}_{H,t-N}, \hat{p}_{F,t-2}, \ldots, \hat{p}_{F,t-N}, \hat{p}^*_{F,t-2}, \ldots, \hat{p}^*_{F,t-N}, \hat{p}^*_{H,t-2}, \ldots, \hat{p}^*_{H,t-N},$$
$$\hat{k}_{1,t-1}, \ldots, \hat{k}_{N,t-1}, \hat{k}^*_{1,t-1}, \ldots, \hat{k}^*_{Nt-1}, d^*_{t-1}] \qquad (n_X \times 1)$$

$$\mathcal{Z}_t = [z_{t+N-1}, z_{t+N-2}, ...z_t, X_t, \hat{\mu}_{t+N-1}, \ldots, \hat{\mu}_{t-N+1}, \hat{\mu}^*_{t+N-1}, \ldots, \hat{\mu}^*_{t-N+1}]$$

$$Z_t = [z_t, z_{t-1}, \ldots, z_{t-N+2}]' \qquad (n_Z \times 1)$$

$$S_t = [\hat{\mu}_t, \ldots, \hat{\mu}_{t-N+1}, \hat{\mu}^*_t, \ldots, \hat{\mu}^*_{t-N+1}]' \qquad (n_S \times 1)$$

The vector z_t contains the choice variables at time t. It has $n_z = 2N + 7$ elements. The vector X_t are the state variables at time t. There are $n_X = 6N - 3$ state variables. The vector $\mathcal{Z}_t$ contains all variables that appear in the residual equations. As before, the vectors Z_t and S_t are used when we characterize the solution. (See (2.40).)

As in the benchmark economy, the residual equations can be written succinctly as follows:

$$\mathcal{E}\left[A_1 \begin{bmatrix} X_{t+1} \\ Z_{t+N-1} \end{bmatrix} + A_2 \begin{bmatrix} X_t \\ Z_{t+N-2} \end{bmatrix} + \text{shock terms}|\Omega_t\right] = 0.$$

The A_1 and A_2 matrices can again be written as in (2.41) and (2.42), respectively. However, $\mathcal{I}_1$ and $\mathcal{I}_2$ in the incomplete-markets example are given by

$$\mathcal{I}_1 = \begin{bmatrix} I_{4,4} \otimes \begin{bmatrix} 0 & \ldots & 0 & 0 \\ 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}, & 0_{4N-4,2N+1} \\ 0_{2N+1,4N-4} & 0_{2N+1,2N+1} \end{bmatrix}$$

$$\mathcal{I}_2 = \begin{bmatrix} I_{4,4} \otimes \begin{bmatrix} 1 \\ 0_{N-2,1} \end{bmatrix}, & 0_{4N-4,2N+3} \\ 0_{2N+1,4} & [I_{2N+1,2N+1}, 0_{2N+1,2}] \end{bmatrix}$$

We now construct eigenvalues of $-A_1^{-1}A_2$ if A_1 is invertible and generalized eigenvalues otherwise. Then, ignoring shock terms, we have

$$\begin{bmatrix} X_{t+1} \\ Z_{t+N-1} \end{bmatrix} = V \Lambda V^{-1} \begin{bmatrix} X_t \\ Z_{t+N-2} \end{bmatrix}.$$

We can sort eigenvalues inside and outside the unit circle. If there are n_X stable eigenvalues (which is the number of state variables in X), then we have a locally determinate system. Suppose that the eigenvectors in V and eigenvalues in Λ are sorted so that the upper left partition of Λ contains the stable eigenvalues. Then,

$$X_{t+1} = V_{11}\Lambda_1 V_{11}^{-1} X_t$$

$$Z_{t+N-2} = V_{21}V_{11}^{-1} X_t.$$

The last n_z elements of Z_{t+N-2} are those of z_t. Therefore, we have a relationship between our decision variables z and the state variables X. If we want to write the system as (2.40), then we can use this relationship between z and X to fill in the elements of A. In particular, we set

$$\begin{aligned}
A(1:n_z, 1:n_z-2) \quad & = A_{zX}(:,[1, N, 2N-1, 3N-2, 4N-3:6N-3]) \\
A(1:n_z, n_z+1:n_z:n_Z) & = A_{zX}(:, 2:N-1) \\
A(1:n_z, n_z+2:n_z:n_Z) & = A_{zX}(:, N+1:2N-2) \\
A(1:n_z, n_z+3:n_z:n_Z) & = A_{zX}(:, 2N:3N-3) \\
A(1:n_z, n_z+4:n_z:n_Z) & = A_{zX}(:, 3N-1:4N-4) \\
A(n_z+1:n_Z, 1:n_Z-n_z) & = I_{n_Z-n_z, n_Z-n_z}
\end{aligned}$$

where A_{zX} comes from $z_t = A_{zX}X_t$.

The next step is to compute B. The computation is exactly the same as in the benchmark case except that the values for the a_0's and b_0's in (2.43) are different. In the incomplete-markets case, they are:

$$a_0 = dR/d\mathcal{Z}(:, 1:(N-1)n_z)$$

$$a_{N-1}(:, 1:n_z) = dR/d\mathcal{Z}(:, (N-1)n_z+1:Nn_z)$$

$$b_0(:, 1:N) = dR/d\mathcal{Z}(:, Nn_z+n_X+1:Nn_z+n_X+N)$$

$$b_0(:, N+1:2N) = dR/d\mathcal{Z}(:, Nn_z + n_X + 2N : Nn_z + n_X + 3N - 1)$$

$$b_{N-1}(:, 2:N) = dR/d\mathcal{Z}(:, Nn_z + n_X + N + 1 : Nn_z + n_X + 2N - 1)$$

$$b_{N-1}(:, N+2:2N) = dR/d\mathcal{Z}(:, Nn_z + n_X + 3N : Nn_z + n_X + 4N - 2)$$

with all other coefficients but a_N set equal to 0. The matrix a_N is nonzero but it is not used in computing B. From here on, the steps of the computation of B are the same as in the benchmark case and therefore, the codes look the same.

2.4. The Extension with Additional Shocks

It is easy to include additional shocks to the three models described above. We need to make the following changes:

- Replace $c = y - x$ by $c = y - x - g$ in the steady state calculation.
- Replace (2.34) by

$$y_H(i)\,\hat{y}_{H,i,t} + y_H^*(i)\,\hat{y}_{H,i,t}^* = F_k(i)k(i-1)\,\hat{k}_{i-1,t-1} + F_l(i)l(i)\,(\hat{l}_{i,t} + \hat{A}_t).$$

- Replace (2.37) by

$$\hat{mc}_{i,t} = \hat{w}_t - F_{kl}(i)k(i-1)/F_l(i)\hat{k}_{i-1,t-1} - F_{ll}(i)l(i)/F_l(i)(\hat{l}_{i,t} + \hat{A}_t) - \hat{A}_t.$$

- Replace (2.39) by

$$\hat{c}_t = (y\,\hat{y}_t - [x(1)\,\hat{x}_{1,t} + \ldots x(N)\,\hat{x}_{N,t}]/N - g\,\hat{g}_t)/c.$$

Notice that we did not adjust the Euler equations for capital since the A term in mc cancels with the A term in F_k leaving the linearization unchanged.

2.4.1. New Code for the Benchmark Economy

When we rewrite the code with multiple shocks, we need to adjust the $\mathcal{Z}$ and S vectors as follows:

$$\begin{aligned}\mathcal{Z}_t = [&z_{t+N-1}, z_{t+N-2}, ...z_t, X_t, \hat{\mu}_{t+N-1}, \ldots, \hat{\mu}_{t-N+1}, \hat{\mu}^*_{t+N-1}, \ldots, \hat{\mu}^*_{t-N+1},\\ &\hat{g}_{t+N-1}, \ldots, \hat{g}_t, \hat{g}^*_{t+N-1}, \ldots, \hat{g}^*_t, \hat{a}_{t+N-1}, \ldots, \hat{a}_t, \hat{a}^*_{t+N-1}, \ldots, \hat{a}^*_t]'\end{aligned}$$

$$\begin{aligned}S_t = [&\hat{\mu}_t, \ldots, \hat{\mu}_{t-N+1}, \hat{\mu}^*_t, \ldots, \hat{\mu}^*_{t-N+1},\\ &\hat{g}_t, \hat{g}_{t-1}, \hat{g}^*_t, \hat{g}^*_{t-1}, \hat{a}_t, \hat{a}_{t-1}, \hat{a}^*_t, \hat{a}^*_{t-1}]'.\end{aligned}$$

The dimension of S is now $2N + 8 \times 1$.

Although the elements of A in (2.40) will be different when we turn on the other shocks, we will not have to change the code used to calculate it. The calculation of B on the other hand will change. We need to use new code for the b_i's as follows:

$$b_0(:, 1 : N) = dR/d\mathcal{Z}(:, Nn_z + n_X + 1 : Nn_z + n_X + N)$$

$$b_0(:, N + 1 : 2N) = dR/d\mathcal{Z}(:, Nn_z + n_X + 2N : Nn_z + n_X + 3N - 1)$$

$$b_0(:, 2N + 1) = dR/d\mathcal{Z}(:, Nn_z + n_X + 4N - 1)$$

$$b_0(:, 2N + 3) = dR/d\mathcal{Z}(:, Nn_z + n_X + 5N - 1)$$

$$b_0(:, 2N + 5) = dR/d\mathcal{Z}(:, Nn_z + n_X + 6N - 1)$$

$$b_0(:, 2N + 7) = dR/d\mathcal{Z}(:, Nn_z + n_X + 7N - 1)$$

$$b_1(:, 2N + 1) = dR/d\mathcal{Z}(:, Nn_z + n_X + 4N)$$

$$b_1(:, 2N + 3) = dR/d\mathcal{Z}(:, Nn_z + n_X + 5N)$$

$$b_1(:, 2N + 5) = dR/d\mathcal{Z}(:, Nn_z + n_X + 6N)$$

$$b_1(:, 2N + 7) = dR/d\mathcal{Z}(:, Nn_z + n_X + 7N)$$

$$\vdots$$

$$b_{N-2}(:, 2N+1) = dR/d\mathcal{Z}(:, Nn_z + n_X + 5N - 3)$$

$$b_{N-2}(:, 2N+3) = dR/d\mathcal{Z}(:, Nn_z + n_X + 6N - 3)$$

$$b_{N-2}(:, 2N+5) = dR/d\mathcal{Z}(:, Nn_z + n_X + 7N - 3)$$

$$b_{N-2}(:, 2N+7) = dR/d\mathcal{Z}(:, Nn_z + n_X + 8N - 3)$$

$$b_{N-1}(:, 2:N) = dR/d\mathcal{Z}(:, Nn_z + n_X + N + 1 : Nn_z + n_X + 2N - 1)$$

$$b_{N-1}(:, N+2:2N) = dR/d\mathcal{Z}(:, Nn_z + n_X + 3N : Nn_z + n_X + 4N - 2)$$

$$b_{N-1}(:, 2N+1) = dR/d\mathcal{Z}(:, Nn_z + n_X + 5N - 2)$$

$$b_{N-1}(:, 2N+3) = dR/d\mathcal{Z}(:, Nn_z + n_X + 6N - 2)$$

$$b_{N-1}(:, 2N+5) = dR/d\mathcal{Z}(:, Nn_z + n_X + 7N - 2)$$

$$b_{N-1}(:, 2N+7) = dR/d\mathcal{Z}(:, Nn_z + n_X + 8N - 2)$$

The matrix $\mathcal{P}$ in $S_{t+1} = \mathcal{P}S_t + \epsilon_{t+1}$ will also change. With multiple shocks we have

$$\mathcal{P} = \begin{bmatrix} I_{2,2} \otimes \begin{bmatrix} \rho_\mu & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} & 0_{2N,8} \\ 0_{8,2N} & \begin{bmatrix} \rho_g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{bmatrix} \tag{2.58}$$

To set up the matrix $\mathcal{D}$ needed to compute B in (2.40), we need to make assumptions about the timing of decisions.[2] Suppose that Ω_t is the information set of the monopolists when setting prices. Let's start with the case: $\Omega_t = \{\omega_s\}_{s=0}^{t-1}$, where $\omega_t =$

[2] See (2.46) for the benchmark case.

$[\hat{\mu}_t, \hat{\mu}_t^*, \hat{g}_t, \hat{g}_t^*, \hat{a}_t, \hat{a}_t^*]'$. The interpretation in this case is that monopolists see none of the period t shocks before choosing their period t prices. In this case,

$$\begin{aligned}
\mathcal{E}[&\mathcal{M}_1\hat{\mu}_t + \ldots + \mathcal{M}_N\hat{\mu}_{t-N+1} + \mathcal{M}_{N+1}\hat{\mu}_t^* + \ldots + \mathcal{M}_{2N}\hat{\mu}_{t-N+1}^* \\
&+ \mathcal{M}_{2N+1}\hat{g}_t + \mathcal{M}_{2N+2}\hat{g}_{t-1} + \mathcal{M}_{2N+3}\hat{g}_t^* + \mathcal{M}_{2N+4}\hat{g}_{t-1}^* \\
&+ \mathcal{M}_{2N+5}\hat{a}_t + \mathcal{M}_{2N+6}\hat{a}_{t-1} + \mathcal{M}_{2N+7}\hat{a}_t^* + \mathcal{M}_{2N+8}\hat{a}_{t-1}^*|\Omega_t]
\end{aligned}$$

$$\begin{aligned}
= \big[&0,\ \mathcal{M}_1\rho_\mu + \mathcal{M}_2,\ \mathcal{M}_3,\ \ldots\ \mathcal{M}_N,\ 0,\ \mathcal{M}_{N+1}\rho_\mu + \mathcal{M}_{N+2},\ \mathcal{M}_{N+3},\ \ldots\ \mathcal{M}_{2N}, \\
&0,\ \mathcal{M}_{2N+1}\rho_g + \mathcal{M}_{2N+2},\ 0,\ \mathcal{M}_{2N+3}\rho_g + \mathcal{M}_{2N+4}, \\
&0,\ \mathcal{M}_{2N+5}\rho_a + \mathcal{M}_{2N+6},\ 0,\ \mathcal{M}_{2N+7}\rho_a + \mathcal{M}_{2N+8}\big]\, S_t.
\end{aligned}$$

In this case, $\mathcal{D}$ would be given by

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-6),n_z n_S - 4n_S} \\ 0_{n_z n_S - 4n_S, 4n_S} & I_{n_z n_S - 4n_S} \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} \begin{bmatrix} \Psi & 0_{N-1,N} \\ 0_{N-1,N} & \Psi \end{bmatrix} & 0_{2N-2,8} \\ 0_{4,2N} & \begin{bmatrix} \rho_g & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_a & 1 \end{bmatrix} \end{bmatrix}$$

$$\Psi = \begin{bmatrix} \rho_\mu & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

The matrix Φ has dimensions $2N + 2 \times 2N + 8$ while Ψ has $N - 1 \times N$.

Now consider the case with $\Omega_t = \{\{\omega_s\}_{s=0}^{t-1}, a_t, a_t^*\}$ so that monopolists observe current technology shocks when choosing prices. In this case,

$$\begin{aligned}
\mathcal{E}[&\mathcal{M}_1\hat{\mu}_t + \ldots + \mathcal{M}_N\hat{\mu}_{t-N+1} + \mathcal{M}_{N+1}\hat{\mu}_t^* + \ldots + \mathcal{M}_{2N}\hat{\mu}_{t-N+1}^* \\
&+ \mathcal{M}_{2N+1}\hat{g}_t + \mathcal{M}_{2N+2}\hat{g}_{t-1} + \mathcal{M}_{2N+3}\hat{g}_t^* + \mathcal{M}_{2N+4}\hat{g}_{t-1}^*
\end{aligned}$$

$$+ \mathcal{M}_{2N+5}\hat{a}_t + \mathcal{M}_{2N+6}\hat{a}_{t-1} + \mathcal{M}_{2N+7}\hat{a}^*_t + \mathcal{M}_{2N+8}\hat{a}^*_{t-1}|\Omega_t]$$

$$\begin{aligned}= \big[&0,\, \mathcal{M}_1\rho_\mu + \mathcal{M}_2,\, \mathcal{M}_3,\, \ldots\, \mathcal{M}_N,\, 0,\, \mathcal{M}_{N+1}\rho_\mu + \mathcal{M}_{N+2},\, \mathcal{M}_{N+3},\, \ldots\, \mathcal{M}_{2N},\\ &0,\, \mathcal{M}_{2N+1}\rho_g + \mathcal{M}_{2N+2},\, 0,\, \mathcal{M}_{2N+3}\rho_g + \mathcal{M}_{2N+4},\\ &\mathcal{M}_{2N+5}, \mathcal{M}_{2N+6},\, \mathcal{M}_{2N+7}, \mathcal{M}_{2N+8}\big]\, S_t.\end{aligned}$$

In this case, $\mathcal{D}$ would be given by

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-4),n_z n_S - 4n_S} \\ 0_{n_z n_S - 4n_S, 4n_S} & I_{n_z n_S - 4n_S} \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} \begin{bmatrix} \Psi & 0_{N-1,N} \\ 0_{N-1,N} & \Psi \end{bmatrix} & 0_{2N-2,8} \\ 0_{6,2N} & \begin{bmatrix} \rho_g & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

and Ψ is the same as above. The matrix Φ has dimensions $2N + 4 \times 2N + 8$.

Finally, suppose that $\Omega_t = \{\{\omega_s\}_{s=0}^{t-1}, \hat{g}_t, \hat{g}^*_t, \hat{a}_t, \hat{a}^*_t\}$ so that monopolists observe current real shocks when choosing prices. In this case,

$$\begin{aligned}\mathcal{E}[&\mathcal{M}_1\hat{\mu}_t + \ldots + \mathcal{M}_N\hat{\mu}_{t-N+1} + \mathcal{M}_{N+1}\hat{\mu}^*_t + \ldots + \mathcal{M}_{2N}\hat{\mu}^*_{t-N+1}\\ &+ \mathcal{M}_{2N+1}\hat{g}_t + \mathcal{M}_{2N+2}\hat{g}_{t-1} + \mathcal{M}_{2N+3}\hat{g}^*_t + \mathcal{M}_{2N+4}\hat{g}^*_{t-1}\\ &+ \mathcal{M}_{2N+5}\hat{a}_t + \mathcal{M}_{2N+6}\hat{a}_{t-1} + \mathcal{M}_{2N+7}\hat{a}^*_t + \mathcal{M}_{2N+8}\hat{a}^*_{t-1}|\Omega_t]\end{aligned}$$

$$\begin{aligned}= \big[&0,\, \mathcal{M}_1\rho_\mu + \mathcal{M}_2,\, \mathcal{M}_3,\, \ldots\, \mathcal{M}_N,\, 0,\, \mathcal{M}_{N+1}\rho_\mu + \mathcal{M}_{N+2},\, \mathcal{M}_{N+3},\, \ldots\, \mathcal{M}_{2N},\\ &\mathcal{M}_{2N+1}, \mathcal{M}_{2N+2},\, \mathcal{M}_{2N+3}, \mathcal{M}_{2N+4},\\ &\mathcal{M}_{2N+5}, \mathcal{M}_{2N+6},\, \mathcal{M}_{2N+7}, \mathcal{M}_{2N+8}\big]\, S_t.\end{aligned}$$

In this case, $\mathcal{D}$ would be given by

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-2),n_z n_S - 4n_S} \\ 0_{n_z n_S - 4n_S, 4n_S} & I_{n_z n_S - 4n_S} \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} \begin{bmatrix} \Psi & 0_{N-1,N} \\ 0_{N-1,N} & \Psi \end{bmatrix} & 0_{2N-2,8} \\ 0_{8,2N} & I_{8,8} \end{bmatrix}$$

The matrix Φ has dimensions $2N+6 \times 2N+8$.

2.4.2. New Code for the Taylor-Rule Extension

The new $\mathcal{Z}$ and S vectors for the Taylor-Rule case are as follows:

$$\mathcal{Z}_t = [z_{t+N-1}, z_{t+N-2}, ...z_t, X_t, \epsilon_{r,t}, \epsilon^*_{r,t}, \hat{g}_{t+N-1}, \ldots, \hat{g}_t, \hat{g}^*_{t+N-1}, \ldots, \hat{g}^*_t,$$

$$\hat{a}_{t+N-1}, \ldots, \hat{a}_t, \hat{a}^*_{t+N-1}, \ldots, \hat{a}^*_t]'$$

$$S_t = [\epsilon_{r,t}, \epsilon_{r,t-1}, \epsilon^*_{r,t}, \epsilon^*_{r,t-1}, \hat{g}_t, \hat{g}_{t-1}, \hat{g}^*_t, \hat{g}^*_{t-1}, \hat{a}_t, \hat{a}_{t-1}, \hat{a}^*_t, \hat{a}^*_{t-1}]'$$

The dimension of S is now 12×1.

To calculate the B matrix in (2.40) we need to update the b_i's to take into account the real shocks. These are now:

$$b_0(:,5) = dR/d\mathcal{Z}(:, n_Z + n_X + 3)$$

$$b_0(:,7) = dR/d\mathcal{Z}(:, n_Z + n_X + N + 3)$$

$$b_0(:,9) = dR/d\mathcal{Z}(:, n_Z + n_X + 2N + 3)$$

$$b_0(:,11) = dR/d\mathcal{Z}(:, n_Z + n_X + 3N + 3)$$

$$b_1(:,5) = dR/d\mathcal{Z}(:, n_Z + n_X + 4)$$

$$b_1(:,7) = dR/d\mathcal{Z}(:, n_Z + n_X + N + 4)$$

$$b_1(:,9) = dR/d\mathcal{Z}(:, n_Z + n_X + 2N + 4)$$

$$b_1(:,11) = dR/d\mathcal{Z}(:, n_Z + n_X + 3N + 4)$$

$$\vdots$$

$$b_{N-1}(:,1) = dR/d\mathcal{Z}(:, n_Z + n_X + 1)$$

$$b_{N-1}(:,3) = dR/d\mathcal{Z}(:,n_Z+n_X+2)$$

$$b_{N-1}(:,5) = dR/d\mathcal{Z}(:,n_Z+n_X+N+2)$$

$$b_{N-1}(:,7) = dR/d\mathcal{Z}(:,n_Z+n_X+2N+2)$$

$$b_{N-1}(:,9) = dR/d\mathcal{Z}(:,n_Z+n_X+3N+2)$$

$$b_{N-1}(:,11) = dR/d\mathcal{Z}(:,n_Z+n_X+4N+2)$$

with all of the other b coefficients equal to 0.

Next, we adjust the $\mathcal{P}$ matrix. For this case it is given by

$$\mathcal{P} = \begin{bmatrix} \begin{bmatrix} \rho_\epsilon & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \rho_\epsilon & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & 0_{4,8} \\ 0_{8,4} & \begin{bmatrix} \rho_g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{bmatrix} \tag{2.59}$$

We next set up the matrix $\mathcal{D}$. If $\Omega_t = \{\epsilon_{r,s}, \epsilon^*_{r,s}, g_s, g^*_s, a_s, a^*_s\}_{s=0}^{t-1}$, then

$$\begin{aligned} &\mathcal{E}[\mathcal{M}_1\epsilon_{r,t} + \mathcal{M}_2\epsilon_{r,t-1} + \mathcal{M}_3\epsilon^*_{r,t} + \mathcal{M}_4\epsilon^*_{r,t-1} \\ &\qquad + \mathcal{M}_5\hat{g}_t + \mathcal{M}_6\hat{g}_{t-1} + \mathcal{M}_7\hat{g}^*_t + \mathcal{M}_8\hat{g}^*_{t-1} \\ &\qquad + \mathcal{M}_9\hat{a}_t + \mathcal{M}_{10}\hat{a}_{t-1} + \mathcal{M}_{11}\hat{g}^*_t + \mathcal{M}_{12}\hat{a}^*_{t-1}|\Omega_t] \\ &= \big[0,\, \mathcal{M}_1\rho_\epsilon + \mathcal{M}_2,\, 0,\, \mathcal{M}_3\rho_\epsilon + \mathcal{M}_4,\, 0,\, \mathcal{M}_5\rho_g + \mathcal{M}_6,\, 0,\, \mathcal{M}_7\rho_g + \mathcal{M}_8, \\ &\qquad 0,\, \mathcal{M}_9\rho_a + \mathcal{M}_{10},\, 0,\, \mathcal{M}_{11}\rho_a + \mathcal{M}_{12}\big]\, S_t. \end{aligned}$$

For this case,

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-6),n_zn_S-4n_S} \\ 0_{n_zn_S-4n_S,4n_S} & I_{n_zn_S-4n_S} \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} \rho_\epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_\epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_g & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_g & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_a & 1 \end{bmatrix}$$

Now consider the case with $\Omega_t = \{\{\omega_s\}_{s=0}^{t-1}, a_t, a_t^*\}$ with $\omega_s = [\epsilon_{r,s}, \epsilon_{r,s}^*, g_s, g_s^*, a_s, a_s^*]'$. Here, the monopolists observe the current technology shocks before changing prices. For this case,

$$\begin{aligned} &\mathcal{E}[\mathcal{M}_1\epsilon_{r,t} + \mathcal{M}_2\epsilon_{r,t-1} + \mathcal{M}_3\epsilon_{r,t}^* + \mathcal{M}_4\epsilon_{r,t-1}^* \\ &\qquad + \mathcal{M}_5\hat{g}_t + \mathcal{M}_6\hat{g}_{t-1} + \mathcal{M}_7\hat{g}_t^* + \mathcal{M}_8\hat{g}_{t-1}^* \\ &\qquad + \mathcal{M}_9\hat{a}_t + \mathcal{M}_{10}\hat{a}_{t-1} + \mathcal{M}_{11}\hat{g}_t^* + \mathcal{M}_{12}\hat{a}_{t-1}^*|\Omega_t] \\ &= \big[0,\, \mathcal{M}_1\rho_\epsilon + \mathcal{M}_2,\, 0,\, \mathcal{M}_3\rho_\epsilon + \mathcal{M}_4,\, 0,\, \mathcal{M}_5\rho_g + \mathcal{M}_6,\, 0,\, \mathcal{M}_7\rho_g + \mathcal{M}_8, \\ &\qquad \mathcal{M}_9, \mathcal{M}_{10}, \mathcal{M}_{11}, \mathcal{M}_{12}\big]\, S_t \end{aligned}$$

and

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-4),n_z n_S - 4n_S} \\ 0_{n_z n_S - 4n_S, 4n_S} & I_{n_z n_S - 4n_S} \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} \rho_\epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_\epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_g & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_g & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, consider the case with $\Omega_t = \{\{\omega_s\}_{s=0}^{t-1}, \hat{a}_t, \hat{a}_t^*, \hat{g}_t, \hat{g}_t^*\}$. Here, the monopolists observe the current real shocks before changing prices. For this case,

$$\begin{aligned} &\mathcal{E}[\mathcal{M}_1\epsilon_{r,t} + \mathcal{M}_2\epsilon_{r,t-1} + \mathcal{M}_3\epsilon_{r,t}^* + \mathcal{M}_4\epsilon_{r,t-1}^* \\ &\qquad + \mathcal{M}_5\hat{g}_t + \mathcal{M}_6\hat{g}_{t-1} + \mathcal{M}_7\hat{g}_t^* + \mathcal{M}_8\hat{g}_{t-1}^* \\ &\qquad + \mathcal{M}_9\hat{a}_t + \mathcal{M}_{10}\hat{a}_{t-1} + \mathcal{M}_{11}\hat{g}_t^* + \mathcal{M}_{12}\hat{a}_{t-1}^*|\Omega_t] \\ &= \big[0,\, \mathcal{M}_1\rho_\epsilon + \mathcal{M}_2,\, 0,\, \mathcal{M}_3\rho_\epsilon + \mathcal{M}_4,\, \mathcal{M}_5, \mathcal{M}_6,\, \mathcal{M}_7, \mathcal{M}_8, \\ &\qquad \mathcal{M}_9, \mathcal{M}_{10}, \mathcal{M}_{11}, \mathcal{M}_{12}\big]\, S_t \end{aligned}$$

and

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-2),n_z n_S - 4n_S} \\ 0_{n_z n_S - 4n_S, 4n_S} & I_{n_z n_S - 4n_S} \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} \begin{bmatrix} \rho_\epsilon & 1 & 0 & 0 \\ 0 & 0 & \rho_\epsilon & 1 \end{bmatrix} & 0_{2,8} \\ 0_{8,4} & I_{8,8} \end{bmatrix}$$

2.4.3. New Code for the I-M Extension

The $\mathcal{Z}$ and S vectors and the b, and D matrices are the same in the benchmark economy and the incomplete-markets economy. Therefore, the extension of the incomplete-markets economy allowing for real shocks follows exactly as in the benchmark economy. (See above.)

2.4.4. Allowing for Accommodative Monetary Policy

Consider the following money growth rate processes:

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + \gamma \hat{A}_t + \sigma_\mu \epsilon_{\mu,t} + \sigma_{\mu\mu^*} \epsilon_{\mu^*,t} \tag{2.60}$$

$$\hat{\mu}_t^* = \rho_\mu \hat{\mu}_{t-1}^* + \gamma \hat{A}_t^* + \sigma_{\mu^*} \epsilon_{\mu^*,t}. \tag{2.61}$$

With the technology shocks given by

$$\begin{bmatrix} \hat{A}_t \\ \hat{A}_t^* \end{bmatrix} = \begin{bmatrix} \rho_a & 0 \\ 0 & \rho_a \end{bmatrix} \begin{bmatrix} \hat{A}_{t-1} \\ \hat{A}_{t-1}^* \end{bmatrix} + \begin{bmatrix} \sigma_a & \sigma_{a,a^*} \\ 0 & \sigma_{a^*} \end{bmatrix} \begin{bmatrix} \epsilon_{a,t} \\ \epsilon_{a^*,t} \end{bmatrix}$$

we can rewrite (2.60) as follows:

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + \gamma \rho_a \hat{A}_{t-1} + \sigma_\mu \epsilon_{\mu,t} + \sigma_{\mu\mu^*} \epsilon_{\mu^*,t} + \gamma \sigma_a \epsilon_{a,t} + \gamma \sigma_{aa^*} \epsilon_{a^*,t}$$

and the foreign growth rate as

$$\hat{\mu}_t^* = \rho_\mu \hat{\mu}_{t-1}^* + \gamma \rho_a \hat{A}_{t-1}^* + \sigma_{\mu^*} \epsilon_{\mu^*,t} + \gamma \sigma_{a^*} \epsilon_{a^*,t}$$

The complete system (ignoring g) looks like

$$\begin{bmatrix} \hat{\mu}_t \\ \hat{\mu}_t^* \\ \hat{A}_t \\ \hat{A}_t^* \end{bmatrix} = \begin{bmatrix} \rho_\mu & 0 & \gamma\rho_a & 0 \\ 0 & \rho_\mu & 0 & \gamma\rho_a \\ 0 & 0 & \rho_a & 0 \\ 0 & 0 & 0 & \rho_a \end{bmatrix} \begin{bmatrix} \hat{\mu}_{t-1} \\ \hat{\mu}_{t-1}^* \\ \hat{A}_{t-1} \\ \hat{A}_{t-1}^* \end{bmatrix} + \begin{bmatrix} \sigma_\mu & \sigma_{\mu\mu^*} & \gamma\sigma_a & \gamma\sigma_{aa^*} \\ 0 & \sigma_{\mu^*} & 0 & \gamma\sigma_{a^*} \\ 0 & 0 & \sigma_a & \sigma_{aa^*} \\ 0 & 0 & 0 & \sigma_{a^*} \end{bmatrix} \begin{bmatrix} \hat{\epsilon}_{\mu,t} \\ \hat{\epsilon}_{\mu^*,t} \\ \hat{\epsilon}_{a,t} \\ \hat{\epsilon}_{a^*,t} \end{bmatrix} \tag{2.62}$$

or $x_t = Ax_{t-1} + C\epsilon_t$ where

$$A = \begin{bmatrix} \rho_\mu I_2 & \gamma\rho_a I_2 \\ 0_2 & \rho_a I_2 \end{bmatrix}, \quad C = \begin{bmatrix} \Sigma_\mu & \gamma\Sigma_a \\ 0_2 & \Sigma_a \end{bmatrix}.$$

Note that the matrices Σ_μ and Σ_a are the coefficient matrices on the disturbances in (2.62).

Let V be the variance-covariance matrix for this system. This 4×4 matrix solves

$$V = AVA' + CC'.$$

Doing the algebra, we have

$$\begin{aligned} V_{11} &= \rho_\mu^2 V_{11} + 2\gamma\rho_\mu\rho_a V_{12} + \gamma^2\rho_a^2 V_{22} + \Sigma_\mu\Sigma_\mu' + \gamma^2\Sigma_a\Sigma_a' \\ V_{12} &= \rho_\mu\rho_a V_{12} + \gamma\rho_a^2 V_{22} + \gamma\Sigma_a\Sigma_a' \\ V_{22} &= \rho_a^2 V_{22} + \Sigma_a\Sigma_a' \end{aligned}$$

We want to set parameters so that we can govern the correlation between the growth rates of money and the technology shocks across countries. To do this, we first introduce parameters ϱ_a and ϱ_μ which satisfy:

$$\begin{aligned} \sigma_{aa^*} &= \varrho_a\sigma_a/\sqrt{1-\varrho_a^2} \\ \sigma_{a^*} &= \sigma_a/\sqrt{1-\varrho_a^2} \\ \sigma_{\mu\mu^*} &= \varrho_\mu\sigma_\mu/\sqrt{1-\varrho_\mu^2} \\ \sigma_{\mu^*} &= \sigma_\mu/\sqrt{1-\varrho_\mu^2} \end{aligned}$$

and are used to define the elements of Σ_μ and Σ_a.

It turns out that ϱ_a is simply the correlation between $\hat{A}_t$ and $\hat{A}_t^*$. To see this, solve for V_{22} above,

$$V_{22} = \frac{\sigma_a^2}{(1-\rho_a^2)(1-\varrho_a^2)} \begin{bmatrix} 1 & \varrho_a \\ \varrho_a & 1 \end{bmatrix}$$

and note that the correlation between $\hat{A}_t$ and $\hat{A}_t^*$ is equal to the (1,2) element of V_{22} divided by the square roots of the (1,1) and (2,2) elements.

Unfortunately, constructing the correlation of $\hat{\mu}$ and $\hat{\mu}^*$ is not so simple when there is feedback from technology shocks. So, we will calculate this correlation in steps. First, note that

$$V_{11} = \frac{1}{1-\rho_\mu^2}\left[\frac{\gamma^2(1+\rho_\mu\rho_a)}{(1-\rho_\mu\rho_a)(1-\rho_a^2)}\frac{\sigma_a^2}{(1-\varrho_a^2)}\begin{bmatrix} 1 & \varrho_a \\ \varrho_a & 1 \end{bmatrix} + \frac{\sigma_\mu^2}{(1-\varrho_\mu^2)}\begin{bmatrix} 1 & \varrho_\mu \\ \varrho_\mu & 1 \end{bmatrix}\right]$$

$$= \kappa_1 \begin{bmatrix} 1 & \varrho_a \\ \varrho_a & 1 \end{bmatrix} + \frac{\kappa_2}{(1-\varrho_\mu^2)}\begin{bmatrix} 1 & \varrho_\mu \\ \varrho_\mu & 1 \end{bmatrix}$$

after substituting in for V_{22} and then for V_{12}. We want to set ϱ_μ so that the correlation between $\hat{\mu}$ and $\hat{\mu}^*$ can be set (i.e., the (1,2) element of V_{11} divided by either the (1,1) element or the (2,2) element),

$$\text{cor}(\hat{\mu}, \hat{\mu}^*) = \frac{\kappa_1\varrho_a + \kappa_2\varrho_\mu/(1-\varrho_\mu^2)}{\kappa_1 + \kappa_2/(1-\varrho_\mu^2)}.$$

Rewriting this, we have a quadratic equation in ϱ_μ:

$$\varrho_\mu^2 + \left[\frac{\kappa_2}{\kappa_1\left(\text{cor}(\hat{\mu},\hat{\mu}^*) - \varrho_a\right)}\right]\varrho_\mu - \left[1 + \text{cor}(\hat{\mu},\hat{\mu}^*)\frac{\kappa_2}{\kappa_1\left(\text{cor}(\hat{\mu},\hat{\mu}^*) - \varrho_a\right)}\right] = 0$$

If we choose $\text{cor}(\hat{\mu}, \hat{\mu}^*)$, we can back out the value for ϱ_μ that ensures this (assuming there is only one root inside the unit circle).

Next, we modify our definitions for $\mathcal{P}$ and $\mathcal{D}$. Recall that S_t is given by

$$S_t = [\hat{\mu}_t, \ldots, \hat{\mu}_{t-N+1}, \hat{\mu}_t^*, \ldots, \hat{\mu}_{t-N+1}^*, \\ \hat{g}_t, \hat{g}_{t-1}, \hat{g}_t^*, \hat{g}_{t-1}^*, \hat{a}_t, \hat{a}_{t-1}, \hat{a}_t^*, \hat{a}_{t-1}^*]'$$

and $\mathcal{P}$ is the coefficient matrix in $S_{t+1} = \mathcal{P}S_t + \epsilon_{t+1}$. With accommodative money, we have

$$\mathcal{P} = \begin{bmatrix} I_{2,2} \otimes \begin{bmatrix} \rho_\mu & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & \gamma\rho_a & 0 & 0 & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma\rho_a & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ 0_{8,2N} & \begin{bmatrix} \rho_g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{bmatrix}$$

Now consider setting up the matrix $\mathcal{D}$. Let's start with the assumption that: $\Omega_t = \{\omega_s\}_{s=0}^{t-1}$, where $\omega_t = [\hat{\mu}_t, \hat{\mu}_t^*, \hat{g}_t, \hat{g}_t^*, \hat{a}_t, \hat{a}_t^*]'$. That is, we assume that monopolists see none of the period t shocks before choosing their period t prices. In this case,

$$\begin{aligned}
&\mathcal{E}[\mathcal{M}_1\hat{\mu}_t + \ldots + \mathcal{M}_N\hat{\mu}_{t-N+1} + \mathcal{M}_{N+1}\hat{\mu}_t^* + \ldots + \mathcal{M}_{2N}\hat{\mu}_{t-N+1}^* \\
&\qquad + \mathcal{M}_{2N+1}\hat{g}_t + \mathcal{M}_{2N+2}\hat{g}_{t-1} + \mathcal{M}_{2N+3}\hat{g}_t^* + \mathcal{M}_{2N+4}\hat{g}_{t-1}^* \\
&\qquad + \mathcal{M}_{2N+5}\hat{a}_t + \mathcal{M}_{2N+6}\hat{a}_{t-1} + \mathcal{M}_{2N+7}\hat{a}_t^* + \mathcal{M}_{2N+8}\hat{a}_{t-1}^*|\Omega_t]
\end{aligned}$$

$$\begin{aligned}
&= \big[0,\, \mathcal{M}_1\rho_\mu + \mathcal{M}_2,\, \mathcal{M}_3,\, \ldots\, \mathcal{M}_N,\, 0,\, \mathcal{M}_{N+1}\rho_\mu + \mathcal{M}_{N+2},\, \mathcal{M}_{N+3},\, \ldots\, \mathcal{M}_{2N}, \\
&\qquad 0,\, \mathcal{M}_{2N+1}\rho_g + \mathcal{M}_{2N+2},\, 0,\, \mathcal{M}_{2N+3}\rho_g + \mathcal{M}_{2N+4}, \\
&\qquad 0,\, \mathcal{M}_1\gamma\rho_a + \mathcal{M}_{2N+5}\rho_a + \mathcal{M}_{2N+6},\, 0,\, \mathcal{M}_{N+1}\gamma\rho_a + \mathcal{M}_{2N+7}\rho_a + \mathcal{M}_{2N+8}\big]\, S_t.
\end{aligned}$$

In this case, $\mathcal{D}$ would be given by

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-6),n_z n_S - 4n_S} \\ 0_{n_z n_S - 4n_S, 4n_S} & I_{n_z n_S - 4n_S} \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} \begin{bmatrix} \Psi & 0_{N-1,N} \\ 0_{N-1,N} & \Psi \end{bmatrix} & 0_{2N-2,8} \\ \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \gamma\rho_a \\ 0 \end{bmatrix} & 0_{4,N-1} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ \gamma\rho_a \end{bmatrix} & 0_{4,N-1} \end{bmatrix} & \begin{bmatrix} \rho_g & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_a & 1 \end{bmatrix} \end{bmatrix}$$

$$\Psi = \begin{bmatrix} \rho_\mu & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Now consider the case: $\Omega_t = \{\{\omega_s\}_{s=0}^{t-1}, a_t, a_t^*\}$ so that monopolists observe current technology shocks when choosing prices. In this case,

$$\begin{aligned}
&\mathcal{E}[\mathcal{M}_1\hat{\mu}_t + \ldots + \mathcal{M}_N\hat{\mu}_{t-N+1} + \mathcal{M}_{N+1}\hat{\mu}_t^* + \ldots + \mathcal{M}_{2N}\hat{\mu}_{t-N+1}^* \\
&\qquad + \mathcal{M}_{2N+1}\hat{g}_t + \mathcal{M}_{2N+2}\hat{g}_{t-1} + \mathcal{M}_{2N+3}\hat{g}_t^* + \mathcal{M}_{2N+4}\hat{g}_{t-1}^*
\end{aligned}$$

$$+\mathcal{M}_{2N+5}\hat{a}_t+\mathcal{M}_{2N+6}\hat{a}_{t-1}+\mathcal{M}_{2N+7}\hat{a}^*_t+\mathcal{M}_{2N+8}\hat{a}^*_{t-1}|\Omega_t]$$

$$\begin{aligned}=\big[&0,\,\mathcal{M}_1\rho_\mu+\mathcal{M}_2,\,\mathcal{M}_3,\,\ldots\,\mathcal{M}_N,\,0,\,\mathcal{M}_{N+1}\rho_\mu+\mathcal{M}_{N+2},\,\mathcal{M}_{N+3},\,\ldots\,\mathcal{M}_{2N},\\ &0,\,\mathcal{M}_{2N+1}\rho_g+\mathcal{M}_{2N+2},\,0,\,\mathcal{M}_{2N+3}\rho_g+\mathcal{M}_{2N+4},\\ &\mathcal{M}_1\gamma+\mathcal{M}_{2N+5},\mathcal{M}_{2N+6},\,\mathcal{M}_{N+1}\gamma+\mathcal{M}_{2N+7},\mathcal{M}_{2N+8}\big]\,S_t.\end{aligned}$$

In this case, $\mathcal{D}$ would be given by

$$\mathcal{D}=\begin{bmatrix} I_{4,4}\otimes\Phi & 0_{4(n_S-4),n_zn_S-4n_S}\\ 0_{n_zn_S-4n_S,4n_S} & I_{n_zn_S-4n_S}\end{bmatrix}$$

where

$$\Phi=\begin{bmatrix} \begin{bmatrix}\Psi & 0_{N-1,N}\\ 0_{N-1,N} & \Psi\end{bmatrix} & 0_{2N-2,8}\\ \begin{bmatrix}\begin{bmatrix}0\\0\\\gamma\\0\\0\\0\end{bmatrix} & 0_{6,N-1} & \begin{bmatrix}0\\0\\0\\0\\\gamma\\0\end{bmatrix} & 0_{6,N-1}\end{bmatrix} & \begin{bmatrix}\rho_g&1&0&0&0&0&0&0\\0&0&\rho_g&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\end{bmatrix}$$

and Ψ is the same as above.

Finally, suppose that $\Omega_t=\{\{\omega_s\}_{s=0}^{t-1},\hat{a}_t,\hat{a}^*_t,\hat{g}_t,\hat{g}^*_t\}$ so that monopolists observe current real shocks when choosing prices. In this case,

$$\begin{aligned}\mathcal{E}[&\mathcal{M}_1\hat{\mu}_t+\ldots+\mathcal{M}_N\hat{\mu}_{t-N+1}+\mathcal{M}_{N+1}\hat{\mu}^*_t+\ldots+\mathcal{M}_{2N}\hat{\mu}^*_{t-N+1}\\ &+\mathcal{M}_{2N+1}\hat{g}_t+\mathcal{M}_{2N+2}\hat{g}_{t-1}+\mathcal{M}_{2N+3}\hat{g}^*_t+\mathcal{M}_{2N+4}\hat{g}^*_{t-1}\\ &+\mathcal{M}_{2N+5}\hat{a}_t+\mathcal{M}_{2N+6}\hat{a}_{t-1}+\mathcal{M}_{2N+7}\hat{a}^*_t+\mathcal{M}_{2N+8}\hat{a}^*_{t-1}|\Omega_t]\end{aligned}$$

$$\begin{aligned}=\big[&0,\,\mathcal{M}_1\rho_\mu+\mathcal{M}_2,\,\mathcal{M}_3,\,\ldots\,\mathcal{M}_N,\,0,\,\mathcal{M}_{N+1}\rho_\mu+\mathcal{M}_{N+2},\,\mathcal{M}_{N+3},\,\ldots\,\mathcal{M}_{2N},\\ &\mathcal{M}_{2N+1},\mathcal{M}_{2N+2},\,\mathcal{M}_{2N+3},\mathcal{M}_{2N+4},\\ &\mathcal{M}_1\gamma+\mathcal{M}_{2N+5},\mathcal{M}_{2N+6},\,\mathcal{M}_{N+1}\gamma+\mathcal{M}_{2N+7},\mathcal{M}_{2N+8}\big]\,S_t.\end{aligned}$$

In this case, $\mathcal{D}$ would be given by

$$\mathcal{D}=\begin{bmatrix} I_{4,4}\otimes\Phi & 0_{4(n_S-2),n_zn_S-4n_S}\\ 0_{n_zn_S-4n_S,4n_S} & I_{n_zn_S-4n_S}\end{bmatrix}$$

where

$$
\Phi = \begin{bmatrix} \begin{bmatrix} \Psi & 0_{N-1,N} \\ 0_{N-1,N} & \Psi \end{bmatrix} & 0_{2N-2,8} \\ \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma \\ 0 \\ 0 \\ 0 \end{bmatrix} & 0_{8,N-1} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \gamma \\ 0 \end{bmatrix} & 0_{8,N-1} \end{bmatrix} & I_{8,8} \end{bmatrix}
$$

2.5. The Extension with Sticky Wages

We now consider extending the benchmark economy to include sticky wages. For completeness, we will allow for both monetary and real shocks. The main differences in the computation from the benchmark economy with additional shocks are these:

- we add $W(s^{t-2}), \ldots, W(s^{t-\mathcal{N}})$ and foreign analogues as state variables;
- we add equation (1.54) and one for W^* to our residuals;
- we add $W(s^{t-1})$ and $W^*(s^{t-1})$ as choice variables;
- we drop $w = -U_l/U_c$;
- we add $\mathcal{N} - 1$ static equations $U_c(j) = U_c(k)$ to determine $c(j, s^t)$'s and do the same for c^*;
- we add $\mathcal{N} - 1$ static equations $U_m(j) = U_m(k)$ to determine $M^d(j, s^t)$'s and do the same for M^{d^*};
- we add $\mathcal{N}$ static equations $L^s(j) = (\bar{W}/W(j))^{\frac{1}{1-v}} \int L^d(i)\, di$ to determine the $L^s(j, s^t)$'s, and we do the same for the foreign country.

2.5.1. Normalization in the Sticky-Wage Economy

In addition to normalizing prices as we had in (2.1), we need to normalize the wages of the j consumer-types. Let

$$\omega(j,s^{t-1}) = W(j,s^{t-1})/M^s(s^{t-j})$$

$$\omega(s^{t-1}) = W(s^{t-1})/M^s(s^{t-1})$$

$$\bar{\omega}(s^t) = \bar{W}(s^t)/M^s(s^{t-1})$$

$$\omega^*(j,s^{t-1}) = W^*(j,s^{t-1})/M^{s*}(s^{t-j})$$

$$\omega^*(s^{t-1}) = W^*(s^{t-1})/M^{s*}(s^{t-1})$$

$$\bar{\omega}^*(s^t) = \bar{W}^*(s^t)/M^{s*}(s^{t-1})$$

$$m^d(j,s^t) = M^d(j,s^t)/P(s^t) = \left(\frac{M^d(j,s^t)}{M^s(s^t)}\right)\left(\frac{M^s(j,s^{t-1})\mu(s^t)}{P(s^t)}\right)$$

$$m^{d^*}(j,s^t) = M^{d^*}(j,s^t)/P^*(s^t) = \left(\frac{M^{d^*}(j,s^t)}{M^{s*}(s^t)}\right)\left(\frac{M^{s*}(j,s^{t-1})\mu^*(s^t)}{P^*(s^t)}\right)$$

As in the case of prices, we assume that cohort 1 is the group changing wages. Therefore, when we normalize the wage equation, we have

$$\omega(s^{t-1})M^s(s^{t-1})$$
$$= \frac{-\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})\bar{\omega}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau)U_l(\tau-t+1,s^\tau)M^s(s^{\tau-1})^{\frac{1}{1-v}}}{v\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})\bar{\omega}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau)U_c(\tau-t+1,s^\tau)/p(s^\tau)M^s(s^{\tau-1})^{\frac{v}{1-v}}}$$

or

$$\omega(s^{t-1}) \tag{2.63}$$
$$= \frac{-\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})\bar{\omega}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau)U_l(\tau-t+1,s^\tau)\left(\mu(s^t)\ldots\mu(s^{\tau-1})\right)^{\frac{1}{1-v}}}{v\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})\bar{\omega}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau)U_c(\tau-t+1,s^\tau)/p(s^\tau)\left(\mu(s^t)\ldots\mu(s^{\tau-1})\right)^{\frac{v}{1-v}}}$$

Notice that the indices for the marginal utilities are $\tau-t+1$ which is $1,2,\ldots,\mathcal{N}$ when we write out the sums.

The relationship between the aggregate and individual wages is normalized as follows:

$$\bar{\omega}(s^t) = \left[\frac{1}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \left(\frac{W(s^{t-j})}{M^s(s^{t-1})} \right)^{\frac{v}{v-1}} \right]^{\frac{v-1}{v}}$$

$$= \left[\frac{1}{\mathcal{N}} \omega(s^{t-1})^{\frac{v}{v-1}} + \frac{1}{\mathcal{N}} \left(\frac{\omega(s^{t-2})}{\mu(s^{t-1})} \right)^{\frac{v}{v-1}} + \cdots \right.$$

$$\left. + \frac{1}{\mathcal{N}} \left(\frac{\omega(s^{t-\mathcal{N}})}{\mu(s^{t-1}) \ldots \mu(s^{t-\mathcal{N}+1})} \right)^{\frac{v}{v-1}} \right]^{\frac{v-1}{v}}$$

2.5.2. Steady State in the Sticky-Wage Economy

Let's follow the same strategy as in the benchmark and change what we need to. First, guess the capital stocks, $k(i)$, $i = 1, \ldots N$, output y, the consumption levels $c(j)$, $j = 2, \ldots \mathcal{N}$, and the money demands $m^d(j)$, $j = 1, \ldots \mathcal{N}$. Next get the $x(i)$'s, the $y_H(i)$'s, the $y_H^*(i)$'s, and the $F(i)$'s as in the benchmark case. Using $F(i)$'s and $k(i)$'s, we can back out the labor demands for each firm i, i.e., $L^d(i)$, using the production technology.

Having the $L^d(i)$'s we can determine the $F_k(i)$'s and then back out marginal costs via the usual capital Euler equations. The marginal cost of i is the real wage $(\bar{w}/p)$ divided by the marginal product $F_l(i)$. We have everything needed to construct the marginal product but the real wage is a function of all of the $U_l(j)$'s and $U_c(j)$'s – which we don't yet have. But to get these marginal utilities, we need the $c(1)$, the $L^s(j)$'s and the $M^d(j)$'s.

The first consumption level is derived with the resource constraint

$$c(1) = \mathcal{N}(y - x - g) - \sum_{j=2}^{\mathcal{N}} c(j).$$

We can back out $L^s(j)$'s from the labor demand functions:

$$L^s(j) = \left(\frac{\mu^{j-1}\bar{\omega}}{\omega} \right)^{\frac{1}{1-v}} \frac{1}{N} \sum_{i=1}^{N} L^d(i) = \left(\frac{\mu^{j-1}\bar{\omega}}{\omega} \right)^{\frac{1}{1-v}} L^d$$

Note that $L^s(j)$ is a function of the μ's and the total labor demand which we know. We can back out the aggregate price from the definition of $m^d(j)$ as follows:

$$p = \mu / [\frac{1}{\mathcal{N}} \sum_j m^d(j)].$$

With all of the consumptions, labor supplies, and money demands we can compute marginal utilities and, in turn, the steady state wage:

$$\omega = -\frac{p}{v}\left(\frac{U_l(1) + U_l(2)\beta\mu^{\frac{1}{1-v}} + U_l(3)\beta^2\mu^{\frac{2}{1-v}} + \ldots + U_l(\mathcal{N})\beta^{\mathcal{N}-1}\mu^{\frac{\mathcal{N}-1}{1-v}}}{U_c(1) + U_c(2)\beta\mu^{\frac{v}{1-v}} + U_c(3)\beta^2\mu^{\frac{2v}{1-v}} + \ldots + U_c(\mathcal{N})\beta^{\mathcal{N}-1}\mu^{\frac{(\mathcal{N}-1)v}{1-v}}}\right)$$

With ω, we can derive the steady state aggregate wage as follows:

$$\bar{\omega} = \omega\left[\frac{1}{\mathcal{N}}\left(1 + \mu^{\frac{v}{1-v}} + \ldots\mu^{\frac{(\mathcal{N}-1)v}{1-v}}\right)\right]^{\frac{v-1}{v}}.$$

We can use the following equations to check that we have a fixed point:

$$mc(i) = \bar{\omega}/(pF_l(i)), \quad i = 1, \ldots, N$$

$$p_H = \frac{p}{\theta}\left(\frac{mc(1) + mc(2)\beta\mu^{\frac{1}{1-\theta}} + mc(3)\beta^2\mu^{\frac{2}{1-\theta}} + \ldots + mc(N)\beta^{N-1}\mu^{\frac{N-1}{1-\theta}}}{1 + \beta\mu^{\frac{\theta}{1-\theta}} + \beta^2\mu^{\frac{2\theta}{1-\theta}} + \ldots + \beta^{N-1}\mu^{\frac{(N-1)\theta}{1-\theta}}}\right)$$

$$U_m(1) = U_c(1)(1-\beta/\mu)$$

$$U_c(j) = U_c(1), \quad j = 2, \ldots, \mathcal{N}$$

$$U_m(j) = U_m(1), \quad j = 2, \ldots, \mathcal{N}.$$

Finally, our assumption about common preferences implies that $q = 1$ in a steady state.

2.5.3. Linearized Equations in the Sticky-Wage Economy

Most of the first-order conditions look the same in the benchmark and sticky-wage economies. The utility functions must be indexed but the form of the linearized equations remain the same. In this section, we derive formulas for the additional equations needed in the sticky-wage economy.

Let's start with the deterministic wage equation:

$$\omega(s^{t-1}) = \frac{-\sum_\tau \beta^{\tau-1}\bar{\omega}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau) U_l(\tau-t+1, s^\tau)\left(\mu(s^t)\ldots\mu(s^{\tau-1})\right)^{\frac{1}{1-v}}}{v\sum_\tau \beta^{\tau-1}\bar{\omega}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau) U_c(\tau-t+1, s^\tau)/p(s^\tau)\left(\mu(s^t)\ldots\mu(s^{\tau-1})\right)^{\frac{v}{1-v}}}$$

First, rewrite this as:

$$v\omega(s^{t-1})\left[\ldots+\beta^{i}U_c(i+1,s^{t+i})/p(s^{t+i})\bar{\omega}(s^{t+i})^{\frac{1}{1-v}}L^d(s^{t+i})\left(\mu(s^t)\cdots\mu(s^{t+i-1})\right)^{\frac{v}{1-v}}+\ldots\right]$$
$$=\ldots-\beta^{i}U_l(i+1,s^{t+i})\bar{\omega}(s^{t+i})^{\frac{1}{1-v}}L^d(s^{t+i})\left(\mu(s^t)\cdots\mu(s^{t+i-1})\right)^{\frac{1}{1-v}}-\ldots \quad (2.64)$$

and then do the linearization of (2.64) in pieces:

$$U_c(i+1,s^{t+i})/p(s^{t+i})\bar{\omega}(s^{t+i})^{\frac{1}{1-v}}L^d(s^{t+i})\left(\mu(s^t)\cdots\mu(s^{t+i-1})\right)^{\frac{v}{1-v}}$$
$$\approx U_c/p\bar{\omega}^{\frac{1}{1-v}}L^d\mu^{\frac{vi}{1-v}}\left[\frac{U_{cc}(i+1)c(i+1)}{U_c}\hat{c}_{i+1,t+i}+\frac{U_{cl}(i+1)L^s(i+1)}{U_c}\hat{L}^s_{i+1,t+i}\right.$$
$$+\frac{U_{cm}(i+1)M^d(i+1)/P}{U_c(i+1)}\hat{m}^d_{i+1,t+i}-\hat{p}_{t+i}+\frac{1}{1-v}\hat{\bar{\omega}}_{t+i}+\hat{L}^d_{t+i}$$
$$\left.+\frac{v}{1-v}(\hat{\mu}_t+\ldots\hat{\mu}_{t+i-1})\right]$$

$$U_l(i+1,s^{t+i})\bar{\omega}(s^{t+i})^{\frac{1}{1-v}}L^d(s^{t+i})\left(\mu(s^t)\cdots\mu(s^{t+i-1})\right)^{\frac{1}{1-v}}$$
$$\approx U_l(i+1)\bar{\omega}^{\frac{1}{1-v}}L^d\mu^{\frac{i}{1-v}}\left[\frac{U_{cl}(i+1)c(i+1)}{U_l(i+1)}\hat{c}_{i+1,t+i}+\frac{U_{ll}(i+1)L^s(i+1)}{U_l(i+1)}\hat{L}^s_{i+1,t+i}\right.$$
$$+\frac{U_{lm}M^d(i+1)/P}{U_l(i+1)}\hat{m}^d_{i+1,t+i}+\frac{1}{1-v}\hat{\bar{\omega}}_{t+i}+\hat{L}^d_{t+i}$$
$$\left.+\frac{1}{1-v}(\hat{\mu}_t+\ldots\hat{\mu}_{t+i-1})\right]$$

Therefore, the full equation is:

$$v\omega/p\bar{\omega}^{\frac{1}{1-v}}L^dU_c(1+\beta\mu^{\frac{v}{1-v}}+\ldots)\hat{\omega}_{t-1}$$
$$+v\omega/p\bar{\omega}^{\frac{1}{1-v}}L^d\left\{\ldots+(\beta\mu^{\frac{v}{1-v}})^iU_c\left[\frac{U_{cc}(i+1)c(i+1)}{U_c}\hat{c}_{i+1,t+i}\right.\right.$$
$$+\frac{U_{cl}(i+1)L^s(i+1)}{U_c}\hat{L}^s_{i+1,t+i}+\frac{U_{cm}(i+1)M^d(i+1)/P}{U_c}\hat{m}^d_{i+1,t+i}$$
$$\left.\left.-\hat{p}_{t+i}+\frac{1}{1-v}\hat{\bar{\omega}}_{t+i}+\hat{L}^d_{t+i}+\frac{v}{1-v}(\hat{\mu}_t+\ldots\hat{\mu}_{t+i-1})\right]+\ldots\right\}$$
$$=\bar{\omega}^{\frac{1}{1-v}}L^d\left\{\ldots-(\beta\mu^{\frac{1}{1-v}})^iU_l(i+1)\left[\frac{U_{cl}(i+1)c(i+1)}{U_l(i+1)}\hat{c}_{i+1,t+i}\right.\right.$$

$$+\frac{U_{ll}(i+1)L^s(i+1)}{U_l(i+1)}\hat{L}^s_{t+i}+\frac{U_{lm}(i+1)M^d(i+1)/P}{U_l(i+1)}\hat{m}^d_{i+1,t+i}$$
$$\left.\left.+\frac{1}{1-v}\hat{\hat{\omega}}_{t+i}+\hat{L}^d_{t+i}+\frac{1}{1-v}(\hat{\mu}_t+\ldots\hat{\mu}_{t+i-1})\right]-\ldots\right\} \tag{2.65}$$

Crossing out common coefficients in (2.65) and dividing by the coefficient on ω, we get

$$\begin{aligned}
&\hat{\omega}_{t-1}= \\
&\quad\frac{p}{v\omega U_c(1+\beta\mu^{\frac{v}{1-v}}+\ldots)}\left\{\ldots-(\beta\mu^{\frac{1}{1-v}})^iU_l(i+1)\left[\frac{U_{cl}(i+1)c(i+1)}{U_l(i+1)}\hat{c}_{i+1,t+i}\right.\right. \\
&\qquad+\frac{U_{ll}(i+1)L^s(i+1)}{U_l(i+1)}\hat{L}^s_{t+i}+\frac{U_{lm}(i+1)M^d(i+1)/P}{U_l(i+1)}\hat{m}^d_{i+1,t+i} \\
&\qquad\left.\left.+\frac{1}{1-v}\hat{\hat{\omega}}_{t+i}+\hat{L}^d_{t+i}+\frac{1}{1-v}(\hat{\mu}_t+\ldots\hat{\mu}_{t+i-1})\right]-\ldots\right\} \\
&\quad-\frac{1}{1+\beta\mu^{\frac{v}{1-v}}+\ldots}\left\{\ldots+(\beta\mu^{\frac{v}{1-v}})^i\left[\frac{U_{cc}(i+1)c(i+1)}{U_c}\hat{c}_{i+1,t+i}\right.\right. \\
&\qquad+\frac{U_{cl}(i+1)L^s(i+1)}{U_c}\hat{L}^s_{i+1,t+i}+\frac{U_{cm}(i+1)M^d(i+1)/P}{U_c}\hat{m}^d_{i+1,t+i} \\
&\qquad\left.\left.-\hat{p}_{t+i}+\frac{1}{1-v}\hat{\hat{\omega}}_{t+i}+\hat{L}^d_{t+i}+\frac{v}{1-v}(\hat{\mu}_t+\ldots\hat{\mu}_{t+i-1})\right]+\ldots\right\}.
\end{aligned} \tag{2.66}$$

Finally, we use the steady state equation for p in (2.66) to get

$$\begin{aligned}
&\hat{\omega}_{t-1}= \\
&\quad\left\{\ldots+\frac{(\beta\mu^{\frac{1}{1-v}})^iU_l(i+1)}{\sum(\beta\mu^{\frac{1}{1-v}})^iU_l(i+1)}\left[\frac{U_{cl}(i+1)c(i+1)}{U_l(i+1)}\hat{c}_{i+1,t+i}\right.\right. \\
&\qquad+\frac{U_{ll}(i+1)L^s(i+1)}{U_l(i+1)}\hat{L}^s_{t+i}+\frac{U_{lm}(i+1)M^d(i+1)/P}{U_l(i+1)}\hat{m}^d_{i+1,t+i} \\
&\qquad\left.\left.+\frac{1}{1-v}\hat{\hat{\omega}}_{t+i}+\hat{L}^d_{t+i}+\frac{1}{1-v}(\hat{\mu}_t+\ldots\hat{\mu}_{t+i-1})\right]-\ldots\right\} \\
&\quad-\left\{\ldots+\frac{(\beta\mu^{\frac{v}{1-v}})^i}{\sum(\beta\mu^{\frac{v}{1-v}})^i}\left[\frac{U_{cc}(i+1)c(i+1)}{U_c}\hat{c}_{i+1,t+i}\right.\right. \\
&\qquad+\frac{U_{cl}(i+1)L^s(i+1)}{U_c}\hat{L}^s_{i+1,t+i}+\frac{U_{cm}(i+1)M^d(i+1)/P}{U_c}\hat{m}^d_{i+1,t+i} \\
&\qquad\left.\left.-\hat{p}_{t+i}+\frac{1}{1-v}\hat{\hat{\omega}}_{t+i}+\hat{L}^d_{t+i}+\frac{v}{1-v}(\hat{\mu}_t+\ldots\hat{\mu}_{t+i-1})\right]+\ldots\right\}.
\end{aligned}$$

Putting expectations back in, we get the following linearized wage equation:

$$\hat{\omega}_{t-1} = E_{t-1} \sum_{i=0}^{\mathcal{N}-1} \omega_{1,i} \Big(\frac{\hat{U}_{l,i+1,t+i}}{U_l(i+1)} + \frac{1}{1-v} \hat{\bar{\omega}}_{t+i} + \hat{L}^d_{t+i} + \frac{1}{1-v} (\hat{\mu}_t + \dots \hat{\mu}_{t+i-1}) \Big)$$

$$- E_{t-1} \sum_{i=0}^{\mathcal{N}-1} \omega_{2,i} \Big(\frac{\hat{U}_{c,i+1,t+i}}{U_c} - \hat{p}_{t+i} + \frac{1}{1-v} \hat{\bar{\omega}}_{t+i} + \hat{L}^d_{t+i}$$

$$+ \frac{v}{1-v} (\hat{\mu}_t + \dots \hat{\mu}_{t+i-1}) \Big) \tag{2.67}$$

where

$$\omega_{1,i} = \frac{(\beta \mu^{\frac{1}{1-v}})^i U_l(i+1)}{\sum (\beta \mu^{\frac{1}{1-v}})^i U_l(i+1)}, \quad \omega_{2,i} = \frac{(\beta \mu^{\frac{v}{1-v}})^i}{\sum (\beta \mu^{\frac{v}{1-v}})^i}$$

and $\hat{U}_{l,i,t}$, $U_{c,i,t}$ is shorthand for the log-linearized marginal utilities. Note that in the case with zero-inflation, the linearized pricing equation simplifies to:

$$\hat{\omega}_{t-1} = E_{t-1} \sum_{i=0}^{\mathcal{N}-1} \beta^i (\hat{p}_{t+i} + \frac{\hat{U}_{l,i+1,t+i}}{U_l} - \frac{\hat{U}_{c,i+1,t+i}}{U_c} + \hat{\mu}_t + \dots \hat{\mu}_{t+i-1}) / \sum_{i=0}^{\mathcal{N}-1} \beta^i$$

From the first-order conditions of consumer j, we also get a relationship for labor supply which when linearized is

$$\hat{L}^s_{j,t} = L^d_t + \frac{1}{1-v} (\hat{\bar{\omega}}_t - \hat{\omega}_{t-j} + \hat{\mu}_{t-j+1} + \dots + \hat{\mu}_{t-1}). \tag{2.68}$$

This equation depends on the individual wages, which will be in the state vector, and the aggregate given by

$$\hat{\bar{\omega}}_t = \Big[1 + \mu^{\frac{v}{1-v}} + \dots \mu^{\frac{(\mathcal{N}-1)v}{1-v}} \Big]^{-1} \big[\hat{\omega}_{t-1} + \mu^{\frac{v}{1-v}} (\hat{\omega}_{t-2} - \hat{\mu}_{t-1}) + \dots$$

$$+ \mu^{\frac{(\mathcal{N}-1)v}{1-v}} (\hat{\omega}_{t-\mathcal{N}} - \hat{\mu}_{t-1} - \dots - \hat{\mu}_{t-\mathcal{N}+1}) \big]. \tag{2.69}$$

Note that the labor demands are going to be derived by the same equation as in the benchmark, except that we use the notation L^d here:

$$y_H(i)\, \hat{y}_{H,i,t} + y^*_H(i)\, \hat{y}^*_{H,i,t} = F_k(i) k(i-1)\, \hat{k}_{i-1,t-1} + F_l(i) L^d(i)\, (\hat{L}^d_{i,t} + \hat{A}_t). \tag{2.70}$$

Similarly, with marginal costs, we use L^d:

$$\hat{mc}_{i,t} = \hat{w}_t - F_{kl}(i) k(i-1) / F_l(i) \hat{k}_{i-1,t-1} - F_{ll}(i) L^d(i) / F_l(i) (\hat{L}^d_{i,t} + \hat{A}_t) - \hat{A}_t \tag{2.71}$$

where $\hat{w}_t$ is now:

$$\hat{w}_t = \hat{\bar{\omega}}_t - \hat{p}_t. \tag{2.72}$$

From the first-order conditions (1.57) and (1.58), we get

$$\begin{aligned} &U_{cc}(j)c(j)\hat{c}_{j,t} + U_{cl}(j)L^s(j)\hat{L}^s_{j,t} + U_{cm}(j)M^d(j)/P\hat{m}^d_{j,t} \\ &\quad = U_{cc}(k)c(k)\hat{c}_{k,t} + U_{cl}(k)L^s(k)\hat{L}^s_{k,t} + U_{cm}(k)M^d(k)/P\hat{m}^d_{k,t} \end{aligned} \tag{2.73}$$

$$\begin{aligned} &U_{cm}(j)c(j)\hat{c}_{j,t} + U_{lm}(j)L^s(j)\hat{L}^s_{j,t} + U_{mm}(j)M^d(j)/P\hat{m}^d_{j,t} \\ &\quad = U_{cm}(k)c(k)\hat{c}_{k,t} + U_{lm}(k)L^s(k)\hat{L}^s_{k,t} + U_{mm}(k)M^d(k)/P\hat{m}^d_{k,t} \end{aligned} \tag{2.74}$$

for all $j, k \in \{1, \ldots, \mathcal{N}\}$.

Finally, we need the labor market clearing condition, the money market clearing condition, and the resource constraint:

$$\hat{L}^d_t = (L^d(1)\,\hat{L}^d_{1,t} + L^2(2)\,\hat{L}^d_{2,t} + \ldots L^d(N)\,\hat{L}^d_{N,t})/\sum_i L^d(i) \tag{2.75}$$

$$\hat{\mu}_t - \hat{p}_t = \frac{1}{\mathcal{N}}\left(\hat{m}^d_{1,t} + \hat{m}^d_{2,t} + \ldots + \hat{m}^d_{\mathcal{N},t}\right) \tag{2.76}$$

$$\hat{y}_t = ([c(1)\,\hat{c}_{1,t} + \ldots c(\mathcal{N})\,\hat{c}_{\mathcal{N},t}]/\mathcal{N} + [x(1)\,\hat{x}_{1,t} + \ldots x(N)\,\hat{x}_{N,t}]/N + g\,\hat{g}_t)/y \tag{2.77}$$

2.5.4. Solving the Linearized System in the Sticky-Wage Economy

The system of equations that we solve has $2N + 7$ **dynamic** equations:

- 4 pricing equations, (2.30)-(2.33);
- $2N$ Euler equations for capital ((2.36) for home and similar for foreign);
- 2 money demand equations ((1.55) for home and similar for foreign);
- 2 wage-setting equations, ((2.67) for home and similar for foreign);
- and **static** equations and definitions that determine:
  - $\hat{y}_{H,i}$, $y_{F,i}$, $y^*_{F,i}$, $y^*_{H,i}$ from (2.10) and analogues;

- $\hat{p}$, $\hat{p}^*$ from (2.14)-(2.15);
- $\hat{\bar{p}}_H$, $\hat{\bar{p}}_F$, $\hat{\bar{p}}^*_F$, $\hat{\bar{p}}^*_H$ from (2.16)-(2.19)
- $\hat{q}$ from (2.24);
- $\hat{\lambda}_H$, $\hat{\lambda}_F$, $\hat{\lambda}^*_F$, $\hat{\lambda}^*_H$ from (2.27) and analogues.
- $\hat{L}^d_i$, $\hat{L^d}^*_i$ from (2.70) and foreign analogue;
- $\hat{x}_i$, $\hat{x}^*_i$ from (2.35) and foreign analogue;
- $\hat{mc}_i$, $\hat{mc}^*_i$ from (2.71) and foreign analogue;
- $\hat{L}^s_j$, $\hat{L^s}^*_j$ from (2.68) and foreign analogue;
- $\hat{w}$, $\hat{w}^*$ from (2.72) and foreign analogue;
- $\hat{c}_j$, $\hat{m}^d_j$, $\hat{c}^*_j$, $\hat{m^d}^*_j$ from (2.73), (2.74), (2.76), (2.77) and the foreign analogues.

We can write the system of equations in terms of a subset of our variables and back out all variables via the static conditions listed above. We turn to this next.

We introduce a new index $\aleph = \max(N, \mathcal{N})$ because we will need to record sufficient lags and leads of the variables. We will use the following vectors in our computation:

$$z_t = [\hat{p}_{H,t-1}, \hat{p}_{F,t-1}, \hat{p}^*_{F,t-1}, \hat{p}^*_{H,t-1}, \hat{k}_{1,t}, \dots \hat{k}_{N,t}, \hat{k}^*_{1,t}, \dots \hat{k}^*_{N,t}, \hat{y}_t, \hat{y}^*_t, \hat{\omega}_{t-1}, \hat{\omega}^*_{t-1}]' \qquad (n_z \times 1)$$

$$\begin{aligned} X_t = [&\hat{p}_{H,t-2}, \dots, \hat{p}_{H,t-N}, \hat{p}_{F,t-2}, \dots, \hat{p}_{F,t-N}, \hat{p}^*_{F,t-2}, \dots, \hat{p}^*_{F,t-N}, \hat{p}^*_{H,t-2}, \dots, \hat{p}^*_{H,t-N}, \\ &\hat{k}_{1,t-1}, \dots, \hat{k}_{N,t-1}, \hat{k}^*_{1,t-1}, \dots, \hat{k}^*_{Nt-1}, \\ &\hat{\omega}_{t-2}, \dots, \hat{\omega}_{t-\mathcal{N}}, \hat{\omega}^*_{t-2}, \dots, \hat{\omega}^*_{t-\mathcal{N}}] \qquad (n_X \times 1) \end{aligned}$$

$$\begin{aligned} \mathcal{Z}_t = [&z_{t+\aleph-1}, z_{t+\aleph-2}, \dots, z_t, X_t, \hat{\mu}_{t+\aleph-1}, \dots, \hat{\mu}_{t-\aleph+1}, \hat{\mu}^*_{t+\aleph-1}, \dots, \hat{\mu}^*_{t-\aleph+1}, \\ &\hat{g}_{t+\aleph-1}, \dots, \hat{g}_t, \hat{g}^*_{t+\aleph-1}, \dots, \hat{g}^*_t, \hat{a}_{t+\aleph-1}, \dots, \hat{a}_t, \hat{a}^*_{t+\aleph-1}, \dots, \hat{a}^*_t]' \end{aligned}$$

$$Z_t = [z_t, z_{t-1}, \dots, z_{t-\aleph+2}]' \qquad (n_Z \times 1)$$

$$\begin{aligned} S_t = [&\hat{\mu}_t, \dots, \hat{\mu}_{t-\aleph+1}, \hat{\mu}^*_t, \dots, \hat{\mu}^*_{t-\aleph+1}, \\ &\hat{g}_t, \hat{g}_{t-1}, \hat{g}^*_t, \hat{g}^*_{t-1}, \hat{a}_t, \hat{a}_{t-1}, \hat{a}^*_t, \hat{a}^*_{t-1}]' \qquad (n_S \times 1) \end{aligned}$$

The vector z_t contains the choice variables at time t. It has $n_z = 2N + 8$ elements. The vector X_t are the state variables at time t. There are $n_X = 6N + 2\mathcal{N} - 6$ state variables.

The vector $\mathcal{Z}_t$ contains all variables that appear in the residual equations. The vectors Z_t and S_t are used when we characterize the solution, $Z_t = AZ_{t-1} + BS_t$, where Z has $n_Z = (\aleph - 1)n_z$ elements and S has $n_S = 2\aleph + 8$ elements.

The residual equations can be written succinctly as follows:

$$\mathcal{E}\left[A_1\begin{bmatrix} X_{t+1} \\ Z_{t+\aleph-1}\end{bmatrix} + A_2\begin{bmatrix} X_t \\ Z_{t+\aleph-2}\end{bmatrix} + \text{shock terms}|\Omega_t\right] = 0$$

where $\mathcal{E}$ implies that expectations are taken – but we will assume that different information sets for the different residual equations. For our example, the residuals are denoted $R(\mathcal{Z})$ and the matrix A_1 is given by

$$A_1 = \begin{bmatrix} I_{n_X,n_X} & 0_{n_X,n_z} & 0_{n_X,n_z} & \cdots & 0_{n_X,n_z} \\ 0_{n_z,n_X} & \frac{dR}{d\mathcal{Z}}(:,1:n_z) & \frac{dR}{d\mathcal{Z}}(:,n_z{+}1:2n_z) & \cdots & \frac{dR}{d\mathcal{Z}}(:,(\aleph{-}2)n_z{+}1:(\aleph{-}1)n_z) \\ 0_{n_z,n_X} & 0_{n_z,n_z} & I_{n_z,n_z} & \cdots & 0_{n_z,n_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_z,n_X} & 0_{n_z,n_z} & 0_{n_z,n_z} & \cdots & I_{n_z,n_z} \end{bmatrix} \tag{2.78}$$

and matrix A_2 is given by:

$$A_2 = \begin{bmatrix} -\mathcal{I}_1 & 0_{n_X,n_z} & \cdots & 0_{n_X,n_z} & -\mathcal{I}_2 \\ \frac{dR}{d\mathcal{Z}}(:,\aleph n_z{+}1:\aleph n_z{+}n_X) & 0_{nz,nz} & \cdots & 0_{n_z,n_z} & \frac{dR}{d\mathcal{Z}}(:,(\aleph{-}1)n_z{+}1:\aleph n_z) \\ 0_{n_z,n_X} & I_{n_z,n_z} & 0_{n_z,n_z} & \cdots & 0_{n_z,n_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_z,n_X} & 0_{n_z,n_z} & \cdots & I_{n_z,n_z} & 0_{n_z,n_z} \end{bmatrix}. \tag{2.79}$$

The matrices $\mathcal{I}_1$ and $\mathcal{I}_2$ in A_2 are given by

$$\mathcal{I}_1 = \begin{bmatrix} I_{4,4} \otimes \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{N-1}, & 0_{4N-4,2N} & 0_{4N-4,2\mathcal{N}-2} \\ 0_{2N,4N-4} & 0_{2N,2N} & 0_{2N,2\mathcal{N}-2} \\ 0_{2\mathcal{N}-2,4N-4} & 0_{2\mathcal{N}-2,2N} & I_{2,2} \otimes \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{\mathcal{N}-1} \end{bmatrix}$$

$$\mathcal{I}_2 = \begin{bmatrix} I_{4,4} \otimes \begin{bmatrix} 1 \\ 0_{N-2,1} \end{bmatrix}, & 0_{4N-4,2N} & 0_{4N-4,2} & 0_{4N-4,2} \\ 0_{2N,4} & I_{2N,2N} & 0_{2N,2} & 0_{2N,2} \\ 0_{2\mathcal{N}-2,4} & 0_{2\mathcal{N}-2,2N} & 0_{2\mathcal{N}-2,2} & I_{2,2} \otimes \begin{bmatrix} 1 \\ 0_{\mathcal{N}-2,1} \end{bmatrix} \end{bmatrix}$$

Using the method laid out in Blanchard and Kahn (1980), we construct eigenvalues of $-A_1^{-1}A_2$ if A_1 is invertible and generalized eigenvalues otherwise. Then, ignoring shock terms, we have

$$\begin{bmatrix} X_{t+1} \\ Z_{t+\aleph-1} \end{bmatrix} = V\Lambda V^{-1} \begin{bmatrix} X_t \\ Z_{t+\aleph-2} \end{bmatrix}.$$

We can sort eigenvalues inside and outside the unit circle. If there are n_X stable eigenvalues (which is the number of state variables in X), then we have a locally determinate system. Suppose that the eigenvectors in V and eigenvalues in Λ are sorted so that the upper left partition of Λ contains the stable eigenvalues. Then,

$$X_{t+1} = V_{11}\Lambda_1 V_{11}^{-1} X_t$$

$$Z_{t+\aleph-2} = V_{21}V_{11}^{-1} X_t.$$

The last n_z elements of $Z_{t+\aleph-2}$ are those of z_t. Therefore, we have a relationship between our decision variables z and the state variables X. If we want to write the system as (2.40), then we can use this relationship between z and X to fill in the elements of A. In particular, we set

$$\begin{aligned}
A(1:n_z, 1:n_z-4) &= A_{zX}(:, [1, N, 2N-1, 3N-2, 4N-3:6N-4]) \\
A(1:n_z, n_z+1:n_z:(N-1)n_z) &= A_{zX}(:, 2:N-1) \\
A(1:n_z, n_z+2:n_z:(N-1)n_z) &= A_{zX}(:, N+1:2N-2) \\
A(1:n_z, n_z+3:n_z:(N-1)n_z) &= A_{zX}(:, 2N:3N-3) \\
A(1:n_z, n_z+4:n_z:(N-1)n_z) &= A_{zX}(:, 3N-1:4N-4) \\
A(1:n_z, n_z-1:n_z:(\mathcal{N}-1)n_z) &= A_{zX}(:, 6N-3:6N+\mathcal{N}-5) \\
A(1:n_z, n_z:n_z:(\mathcal{N}-1)n_z) &= A_{zX}(:, 6N+\mathcal{N}-4:6N+2\mathcal{N}-6) \\
A(n_z+1:n_Z, 1:n_Z-n_z) &= I_{n_Z-n_z, n_Z-n_z}
\end{aligned}$$

where A_{zX} comes from $z_t = A_{zX} X_t$.

The next step is to compute B:

$$B = \begin{bmatrix} B_1 \\ 0_{n_z,n_S} \\ \vdots \\ 0_{n_z,n_S} \end{bmatrix} = \begin{bmatrix} I_{n_z,n_z} \\ 0_{n_z,n_z} \\ \vdots \\ 0_{n_z,n_z} \end{bmatrix} B_1 \equiv \mathcal{S} B_1.$$

We will use $\mathcal{S}$ below in order to reduce the problem of computing B to one of computing B_1.

To derive expressions for the elements of B, we first note that the residuals can be written as follows:

$$\mathcal{E}\Big[a_0 Z_{t+\aleph-1} + a_1 Z_{t+\aleph-2} + \ldots + a_{\aleph-1} Z_t + a_\aleph Z_{t-1} \\ + b_0 S_{t+\aleph-1} + b_1 S_{t+\aleph-2} + \ldots + b_{\aleph-1} S_t | \Omega_t\Big] = 0$$

Using the definitions of Z and $\mathcal{Z}$, we can write:

$$a_0 = dR/d\mathcal{Z}(:, 1 : (\aleph - 1)n_z)$$

$$a_{\aleph-1}(:, 1 : n_z) = dR/d\mathcal{Z}(:, (\aleph - 1)n_z + 1 : \aleph n_z)$$

$$b_0(:, 1 : \aleph) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 1 : \aleph n_z + n_X + \aleph)$$

$$b_0(:, \aleph + 1 : 2\aleph) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 2\aleph : \aleph n_z + n_X + 3\aleph - 1)$$

$$b_0(:, 2\aleph + 1) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 4\aleph - 1)$$

$$b_0(:, 2\aleph + 3) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 5\aleph - 1)$$

$$b_0(:, 2\aleph + 5) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 6\aleph - 1)$$

$$b_0(:, 2\aleph + 7) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 7\aleph - 1)$$

$$b_1(:, 2\aleph + 1) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 4\aleph)$$

$$b_1(:, 2\aleph + 3) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 5\aleph)$$

$$b_1(:, 2\aleph + 5) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 6\aleph)$$

$$b_1(:, 2\aleph + 7) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 7\aleph)$$

$$\vdots$$

$$b_{\aleph-2}(:, 2\aleph + 1) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 5\aleph - 3)$$

$$b_{\aleph-2}(:, 2\aleph + 3) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 6\aleph - 3)$$

$$b_{\aleph-2}(:, 2\aleph + 5) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 7\aleph - 3)$$

$$b_{\aleph-2}(:, 2\aleph + 7) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 8\aleph - 3)$$

$$b_{\aleph-1}(:, 2 : \aleph) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + \aleph + 1 : \aleph n_z + n_X + 2\aleph - 1)$$

$$b_{\aleph-1}(:, \aleph + 2 : 2\aleph) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 3\aleph : \aleph n_z + n_X + 4\aleph - 2)$$

$$b_{\aleph-1}(:, 2\aleph + 1) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 5\aleph - 2)$$

$$b_{\aleph-1}(:, 2\aleph + 3) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 6\aleph - 2)$$

$$b_{\aleph-1}(:, 2\aleph + 5) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 7\aleph - 2)$$

$$b_{\aleph-1}(:, 2\aleph + 7) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 8\aleph - 2)$$

with all other coefficients but $a_\aleph$ set equal to 0. The matrix $a_\aleph$ is nonzero but it is not used in computing B.

Using the solution in (2.40) we get:

$$\begin{aligned}\mathcal{E}\Big[&a_0\left(A^{\aleph}Z_{t-1} + BS_{t+\aleph-1} + ABS_{t+\aleph-2} + \ldots + A^{\aleph-1}BS_t\right)\\ &+ a_1\left(A^{\aleph-1}Z_{t-1} + BS_{t+\aleph-2} + ABS_{t+\aleph-3} + \ldots + A^{\aleph-2}BS_t\right) + \ldots\\ &+ a_{\aleph-1}\left(AZ_{t-1} + BS_t\right) + a_\aleph Z_{t-1}\\ &+ b_0 S_{t+\aleph-1} + b_1 S_{t+\aleph-2} + \ldots + b_{\aleph-1}S_t|\Omega_t\Big] = 0\end{aligned} \tag{2.80}$$

The matrix $\mathcal{P}$ in $S_{t+1} = \mathcal{P}S_t + \epsilon_{t+1}$ is the same as in the benchmark case except that we use $\aleph$ in place of N. (See equation (2.58).)

If we assume that the timing of the household/union wages are the same as that for the monopolist, then we have to do a little more work to get $\mathcal{D}$. Suppose first that households and firms see $\Omega_t = \{\hat{\mu}_s, \hat{\mu}_s^*, \hat{g}_s, \hat{g}_s^*, \hat{a}_s, \hat{a}_s^*\}_{s=0}^{t-1}$ when making their pricing decisions. Taking expectations is the same as in the benchmark economy with multiple shocks except here we have 2 additional equations to restrict – those related to the wage rates. The matrix $\mathcal{D}$ in this case is

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-6),n_z n_S - 6n_S} & 0_{4(n_S-6),2n_S} \\ 0_{n_z n_S - 6n_S,4n_S} & I_{n_z n_S - 6n_S} & 0_{n_z n_S - 6n_S,2n_S} \\ 0_{2(n_S-6),4n_S} & 0_{2(n_S-6),n_z n_S - 6n_S} & I_{2,2} \otimes \Phi \end{bmatrix}$$

If agents can see technology shocks, then

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-4),n_z n_S - 6n_S} & 0_{4(n_S-4),2n_S} \\ 0_{n_z n_S - 6n_S,4n_S} & I_{n_z n_S - 6n_S} & 0_{n_z n_S - 6n_S,2n_S} \\ 0_{2(n_S-4),4n_S} & 0_{2(n_S-4),n_z n_S - 6n_S} & I_{2,2} \otimes \Phi \end{bmatrix}$$

Finally, if agents can see both technology and government spending shocks, then

$$\mathcal{D} = \begin{bmatrix} I_{4,4} \otimes \Phi & 0_{4(n_S-2),n_z n_S - 6n_S} & 0_{4(n_S-2),2n_S} \\ 0_{n_z n_S - 6n_S,4n_S} & I_{n_z n_S - 6n_S} & 0_{n_z n_S - 6n_S,2n_S} \\ 0_{2(n_S-2),4n_S} & 0_{2(n_S-2),n_z n_S - 6n_S} & I_{2,2} \otimes \Phi \end{bmatrix}$$

3. Formulas for Preferences

We will consider three alternative functional forms for the utility function. In these notes, I refer to them as ‘separable preferences,’ ‘nonseparable preferences I’ and ‘nonseparable preferences II’.

3.1. Separable Preferences

The separable preferences are given by

$$
\begin{aligned}
U(c,l,m) &= \frac{1}{1-\sigma}\left[\left(\omega\, c^{\frac{\eta-1}{\eta}} + (1-\omega)\, m^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}\right]^{1-\sigma} + \psi(1-l)^{1-\xi}/(1-\xi) \quad (3.1)\\
&\equiv \frac{1}{1-\sigma}\Phi(c,m)^{1-\sigma} + \psi(1-l)^{1-\xi}/(1-\xi)
\end{aligned}
$$

where $\Phi(c,m) = \Psi^{\frac{\eta}{\eta-1}}$. The first and second partials of this function are as follows

$$
\begin{aligned}
U_c &= \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}\omega c^{-\frac{1}{\eta}}\\
U_l &= -\psi(1-l)^{-\xi}\\
U_m &= \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}(1-\omega)m^{-\frac{1}{\eta}}\\
U_{cc} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}\omega c^{-\frac{1}{\eta}})^2\\
&\quad + \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega^2/\eta c^{-\frac{2}{\eta}}\\
&\quad - \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}\omega/\eta c^{-\frac{1}{\eta}-1}\\
U_{cl} &= 0\\
U_{lm} &= 0\\
U_{ll} &= -\psi\xi(1-l)^{-\xi-1}\\
U_{cm} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}\omega c^{-\frac{1}{\eta}})(\Psi^{\frac{1}{\eta-1}}(1-\omega)m^{-\frac{1}{\eta}})\\
&\quad + \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega(1-\omega)/\eta c^{-\frac{1}{\eta}}m^{-\frac{1}{\eta}}\\
U_{mm} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}(1-\omega)m^{-\frac{1}{\eta}})^2\\
&\quad + \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega^2/\eta m^{-\frac{2}{\eta}}\\
&\quad - \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}(1-\omega)/\eta m^{-\frac{1}{\eta}-1}.
\end{aligned}
$$

Again, ignoring m, consider calculating the labor supply elasticity. In this case, the elasticity holding U_c fixed will be the same as that holding c fixed since U_c is not a function of l. With $w = \psi(1-l)^{-\xi}c^{\sigma}$, we get

$$
\frac{d\ln\, l}{d\ln\, w} = \frac{U_l}{U_{ll}l} = \frac{1-l}{\xi l}.
$$

If $l = 1/4$ and $\xi = 1.5$ then $d\ln\, l/d\ln\, w = 2$.

3.2. Nonseparable Preferences I

For the nonseparable preferences, type I, we assume

$$\begin{aligned} U(c,l,m) &= \frac{1}{1-\sigma}\left[\left(\omega\, c^{\frac{\eta-1}{\eta}} + (1-\omega)\, m^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} (1-l)^{\psi}\right]^{1-\sigma} \\ &\equiv \frac{1}{1-\sigma}\Phi(c,l,m)^{1-\sigma} \end{aligned} \tag{3.2}$$

where

$$\Phi(c,l,m) = \Psi(c,m)^{\frac{\eta}{\eta-1}}(1-l)^{\psi}$$

where m denotes real balances (M/P). The first and second partial derivatives of this function are as follows

$$U_c = \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}\omega c^{-\frac{1}{\eta}}(1-l)^{\psi} = (1-\sigma)\omega U c^{-\frac{1}{\eta}}/\Psi$$

$$U_l = -\Phi^{-\sigma}\Psi^{\frac{\eta}{\eta-1}}\psi(1-l)^{\psi-1} = -(1-\sigma)\psi\frac{U}{1-l}$$

$$U_m = \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}(1-\omega)m^{-\frac{1}{\eta}}(1-l)^{\psi}$$

$$\begin{aligned} U_{cc} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}\omega c^{-\frac{1}{\eta}}(1-l)^{\psi})^2 \\ &+ \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega^2/\eta c^{-\frac{2}{\eta}}(1-l)^{\psi} \\ &- \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}\omega/\eta c^{-\frac{1}{\eta}-1}(1-l)^{\psi} \\ &= \frac{U_cU_c}{U}\frac{1-\eta\sigma}{\eta(1-\sigma)} - \frac{1}{\eta}\frac{U_c}{c} \end{aligned}$$

$$\begin{aligned} U_{cl} &= \sigma\Phi^{-\sigma-1}\Psi^{\frac{\eta}{\eta-1}}\psi(1-l)^{\psi-1}\Psi^{\frac{1}{\eta-1}}\omega c^{-\frac{1}{\eta}}(1-l)^{\psi} \\ &- \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}\omega c^{-\frac{1}{\eta}}\psi(1-l)^{\psi-1} \\ &= \frac{U_cU_l}{U} \end{aligned}$$

$$\begin{aligned} U_{lm} &= \sigma\Phi^{-\sigma-1}\Psi^{\frac{\eta}{\eta-1}}\psi(1-l)^{\psi-1}\Psi^{\frac{1}{\eta-1}}(1-\omega)m^{-\frac{1}{\eta}}(1-l)^{\psi} \\ &- \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}(1-\omega)m^{-\frac{1}{\eta}}\psi(1-l)^{\psi-1} \end{aligned}$$

$$\begin{aligned} U_{ll} &= -\sigma\Phi^{-\sigma-1}(\psi(1-l)^{\psi-1}\Psi^{\frac{\eta}{\eta-1}})^2 + \Phi^{-\sigma}\Psi^{\frac{\eta}{\eta-1}}\psi(\psi-1)(1-l)^{\psi-2} \\ &= \frac{U_lU_l}{U} + \frac{U_l}{1-l} \end{aligned}$$

$$
\begin{aligned}
U_{cm} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}\omega c^{-\frac{1}{\eta}}(1-l)^{\psi})(\Psi^{\frac{1}{\eta-1}}(1-\omega)m^{-\frac{1}{\eta}}(1-l)^{\psi}) \\
&+ \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega(1-\omega)/\eta c^{-\frac{1}{\eta}}m^{-\frac{1}{\eta}}(1-l)^{\psi} \\
&= \frac{U_cU_m}{U}\frac{1-\eta\sigma}{\eta(1-\sigma)} \\
U_{mm} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}(1-\omega)m^{-\frac{1}{\eta}}(1-l)^{\psi})^2 \\
&+ \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega^2/\eta m^{-\frac{2}{\eta}}(1-l)^{\psi} \\
&- \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}(1-\omega)/\eta m^{-\frac{1}{\eta}-1}(1-l)^{\psi}
\end{aligned}
$$

where the arguments of the utility function and its derivatives have been dropped for convenience.

Ignoring m, consider two calculations of the labor supply elasticity. First, hold U_c fixed. If $\lambda = U_c$, then

$$\lambda = c^{-\sigma}(1-l)^{\psi(1-\sigma)}.$$

Inverting this we have

$$c = \left[\frac{(1-l)^{\psi(1-\sigma)}}{\lambda}\right]^{\frac{1}{\sigma}}.$$

The equilibrium equation is $w = -U_l/U_c = \psi c/(1-l)$. With c substituted in, we get

$$w = \frac{\psi}{\lambda^{\frac{1}{\sigma}}}(1-l)^{\frac{\psi(1-\sigma)}{\sigma}-1}$$

Totally differentiating, we get

$$
\begin{aligned}
dw &= -\frac{\psi}{\lambda^{\frac{1}{\sigma}}}\left[\frac{\psi(1-\sigma)}{\sigma}-1\right](1-l)^{\frac{\psi(1-\sigma)}{\sigma}-2}\,dl \\
&= \left[1-\frac{\psi(1-\sigma)}{\sigma}\right]\frac{w}{1-l}\,dl.
\end{aligned}
$$

Rearranging this equation we get the following elasticity:

$$\frac{d\ln l}{d\ln w} = \left[\frac{\sigma}{\sigma-\psi(1-\sigma)}\right]\frac{1-l}{l}$$

If $l = 1/4$ and $\sigma = 1$, the elasticity is 3. Suppose that we set $\psi = 2$. As σ increases from 1 to 10, the elasticity falls from 3 to roughly 1.

The labor elasticity holding c fixed is simpler. Again, ignoring m, we have $w = \psi c/(1-l)$ and, therefore,

$$\left.\frac{d\ln l}{d\ln w}\right|_{\text{c fixed}} = \frac{d(1-\psi\bar{c}/w)}{dw}\frac{w}{l} = \frac{1-l}{l}$$

If $l = 1/4$, the elasticity is 3.

3.3. Nonseparable Preferences II

The nonseparable preferences, type II, are given by

$$\begin{aligned} U(c,l,m) &= \frac{1}{1-\sigma}\left[\frac{1}{1-\kappa}c^{1-\kappa} + \frac{\omega}{1-1/\eta}m^{1-\frac{1}{\eta}} - \frac{\psi}{1+\xi}l^{1+\xi}\right]^{1-\sigma} \\ &\equiv \frac{1}{1-\sigma}\Phi(c,m,l)^{1-\sigma}. \end{aligned} \tag{3.3}$$

The first and second partials of this function are as follows

$$\begin{aligned}
U_c &= \Phi^{-\sigma}c^{-\kappa} \\
U_l &= \Phi^{-\sigma}l^{\xi}(-\psi) \\
U_m &= \Phi^{-\sigma}m^{-1/\eta}\omega \\
U_{cc} &= -U_c(\sigma/\Phi c^{-\kappa} + \kappa/c) \\
U_{cl} &= U_c\sigma/\Phi l^{\xi}\psi \\
U_{lm} &= U_m\sigma/\Phi l^{\xi}\psi \\
U_{ll} &= U_l(\sigma/\Phi l^{\xi}\psi + \xi/l) \\
U_{cm} &= -U_c\sigma/\Phi m^{-1/\eta}\omega \\
U_{mm} &= -U_m(\sigma/\Phi m^{-1/\eta}\omega + 1/(\eta m))
\end{aligned}$$

Again, ignoring m, consider calculating the labor supply elasticity. The elasticity holding U_c fixed is complicated when σ is greater than 0. For now, let's do the simpler case of holding c fixed. Since $w = \psi l^{\xi}c^{\kappa}$,

$$\frac{d\ln l}{d\ln w} = \frac{1}{\xi}$$

If $\xi = 1$, then the elasticity is 1. If $\xi = .5$, the elasticity is 2. Note that this is the elasticity for the $\sigma = 0$ (i.e., completely separable preferences) case.

4. Are Real Exchange Rates Volatile and Persistent?

When we log-linearize the real exchange rate in the benchmark economy, we get

$$\hat{q}_t = -\frac{U_{cc}c}{U_c}(\hat{c}_t - \hat{c}_t^*) - \frac{U_{cl}l}{U_c}(\hat{l}_t - \hat{l}_t^*) - \frac{U_{cm}M/P}{U_c}(\hat{m}_t - \hat{m}_t^*) \tag{4.1}$$

where $\hat{m}_t = \hat{\mu}_t - \hat{p}_t$. Suppose that we chose preferences in such a way as to guarantee that increases in money would not have a large effect on utility or marginal utility; that is, we assume that quantitatively the third term in (4.1) is small. In that case we could think of preferences defined over consumption c and labor l. In this case, the variance of $\hat{q}_t$ is given by

$$\operatorname{var}\hat{q} = \left(-U_{cc}c/U_c\right)^2 \sigma_{\Delta c}^2 + \left(-U_{cl}l/U_c\right)^2 \sigma_{\Delta l}^2 + 2\left(U_{cc}U_{cl}cl/U_c^2\right)\sigma_{\Delta c,\Delta l}^2 \tag{4.2}$$

where $\sigma_{\Delta c}^2 = \operatorname{var}(\hat{c} - \hat{c}^*)$, $\sigma_{\Delta l}^2 = \operatorname{var}(\hat{l} - \hat{l}^*)$, and $\sigma_{\Delta c,\Delta l}^2 = \operatorname{cov}(\hat{c} - \hat{c}^*, \hat{l} - \hat{l}^*)$.

If we construct variances of the real exchange rate, the difference in logged and detrended consumption, and the difference in logged and detrended employment for U.S. and European data, we find that the variance of $\hat{q}$ is relatively large. Therefore, to get the model to generate sufficiently volatile real exchange rates, we require that the coefficient on $\hat{c} - \hat{c}^*$ or $\hat{l} - \hat{l}^*$ in (4.1) is large in absolute value with U_{cl} either positive or not too negative.

Consider the signs of partial derivatives with respect to c and l. We know that $U_c > 0$ and $U_{cc} < 0$ so that the first coefficient in (4.1) is positive. The cross-product U_{cl} can be negative or positive as long as $U_{cc}U_{ll} > U_{cl}^2$ (for concavity). That implies that the second coefficient in (4.1) is either positive or negative. If it is too negative, then we cannot generate much volatility in exchange rates since the covariance term in (4.2) would be negative (with $U_{cc} < 0$ and $U_{cl} > 0$). So, for now, we will assume that $-U_{cc}c/U_c$ is a large positive number and $-U_{cl}l/U_c$ is either positive or not very negative. This ensures volatility.

What about persistence? As we show in our earlier work, it is crucial that prices don't jump in response to changes in marginal costs. Otherwise, the effects of monetary shocks are short-lived. If the production function is Cobb-Douglas with a capital share equal to α, then the linearized expression for firm i's marginal cost is:

$$\hat{mc}_{i,t} = \hat{w}_t - \alpha \hat{k}_{i-1,t-1} + \alpha \hat{l}_{i,t}. \tag{4.3}$$

This expression depends on the wage $\hat{w}_t$ which, when linearized, is given by

$$\hat{w}_t = \left(\frac{U_{cl}}{U_l} - \frac{U_{cc}}{U_c} \right) c\, \hat{c}_t + \left(\frac{U_{ll}}{U_l} - \frac{U_{cl}}{U_c} \right) l\, \hat{l}_t + \left(\frac{U_{lm}}{U_l} - \frac{U_{cm}}{U_c} \right) M/P\, \hat{m}_t. \tag{4.4}$$

Again, let's assume that the coefficients on money are quantitatively small so that we can ignore the last term in (4.4). What can we say about the other two given the restrictions above. Since preferences are concave in consumption and leisure ($-l$), we know that $-U_{cc}c/U_c$ and $U_{ll}l/U_l$ are both positive. And, above, we assume that $-U_{cc}c/U_c$ was large. If $U_{cl} < 0$, then the first coefficient in (4.4) is large and positive and the second coefficient is positive. This follows from the fact that $U_c > 0$ and $U_l < 0$. With at least one large coefficient on c or l, the theory won't generate persistent responses to positive monetary shocks because marginal costs will jump up immediately and firms will respond by immediately increasing their prices. What if $U_{cl} > 0$? If this cross-product is large enough, then we can get small coefficients in (4.4). However, having $U_{cl} > 0$ leads to less volatility in $\hat{q}_t$. (Recall that above we wanted $-U_{cl}l/U_c$ positive.)

The arguments make clear the tension: there is no obvious way to ensure both volatility and persistence of real exchange rates – unless we choose parameters to get volatility and then assume that prices are stuck for long periods.

Let's consider the specific preferences of the last section to see this tension more clearly. For the separable utility function in (3.1), we have

$$\begin{aligned}\hat{q}_t = &\left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi} \left(\sigma - \frac{1}{\eta} \right) + \frac{1}{\eta} \right] (\hat{c}_t - \hat{c}_t^*) \\ &+ \left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi} \left(\sigma - \frac{1}{\eta} \right) \right] (\hat{m}_t - \hat{m}_t^*).\end{aligned}$$

In this case $U_{cl} = 0$. If we choose a value of ω near 1, then all that we have to do to guarantee a lot of volatility is to choose a large value for σ.

Now consider the wage rate for the separable-utility case:

$$\hat{w}_t = \left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right) + \frac{1}{\eta}\right]\hat{c}_t + \frac{\xi l}{1-l}\hat{l}_t + \left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right)\right]\hat{m}_t \qquad (4.5)$$

with U defined in (3.1). With ω near 1 and σ large, we will find that wages, and in turn marginal costs, will jump when there is an increase in the growth rate of money. This is because the coefficient (what we called γ in our earlier) on output in the price equation will be large if the coefficients on $\hat{c}$ and $\hat{l}$ in (4.5) are large.

What about nonseparable preferences of type I in (3.2)? In this case, the real exchange rate is

$$\hat{q}_t = \left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right) + \frac{1}{\eta}\right](\hat{c}_t - \hat{c}_t^*) + \frac{(1-\sigma)\psi l}{1-l}(\hat{l}_t - \hat{l}_t^*)$$
$$+ \left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right)\right](\hat{m}_t - \hat{m}_t^*)$$

where $\hat{l}$ appears because U_{cl} is not equal to 0. Notice though that a large σ implies a very negative value for U_{cl}. Thus, we can't generate volatile real exchange rates by simply increasing σ. And, if money is not playing an important role, there is really no way to get volatile real exchange rates. (To see this, note that $\omega c^{(\eta-1)/\eta} \approx \Psi$ so that the coefficient on $\hat{c}_t - \hat{c}_t^*$ is approximately equal to σ.)

In summary, we see that standard choices for preferences lead to a negative result: it is not easy to generate both volatile and persistent real exchange rates in the benchmark economy.

What happens when we assume that markets are incomplete? When markets are incomplete, we replace $q_t = U_{ct}^*/U_{ct}$ with the following condition

$$E_t \frac{U_c(s^{t+1})P(s^t)}{U_c(s^t)P(s^{t+1})} = E_t \frac{U_c^*(s^{t+1})P(s^t)q(s^t)}{U_c(s^t)P(s^{t+1})q(s^{t+1})}.$$

When we linearize this, we get

$$E_t\hat{q}_{t+1} - \hat{q}_t = E_t\left[-\frac{U_{cc}c}{U_c}(\hat{c}_{t+1} - \hat{c}_{t+1}^*) - \frac{U_{cl}l}{U_c}(\hat{l}_{t+1} - \hat{l}_{t+1}^*) - \frac{U_{cm}M/P}{U_c}(\hat{m}_{t+1} - \hat{m}_{t+1}^*)\right]$$

$$+\frac{U_{cc}c}{U_c}(\hat{c}_t - \hat{c}_t^*) + \frac{U_{cl}l}{U_c}(\hat{l}_t - \hat{l}_t^*) + \frac{U_{cm}M/P}{U_c}(\hat{m}_t - \hat{m}_t^*)$$

This is only a slight modification of what we had before – it is simply an expectational difference of (4.1). As before, we see that the sign of U_{cl} will matter. If it is positive, then the model will generate persistence. If it is negative, then the model will generate volatility. But, again, it will be hard to generate both.

What happens when we allow for sticky wages? Consider the simplest case with $\mu = 1$, $\beta \approx 1$, and $\mathcal{N} = 2$. In that case, the normalized nominal wage set by household type 1 is equal to

$$\hat{\omega}_{t-1} = \frac{1}{2}E_{t-1}\Bigg[\hat{p}_t + \hat{p}_{t+1} + \hat{\mu}_t + \left(\frac{U_{cl}}{U_l} - \frac{U_{cc}}{U_c}\right)c(\hat{c}_{1,t} + \hat{c}_{2,t+1})$$
$$+\left(\frac{U_{ll}}{U_l} - \frac{U_{cl}}{U_c}\right)l(\hat{l}_{1,t} + \hat{l}_{2,t+1}) + \left(\frac{U_{lm}}{U_l} - \frac{U_{cm}}{U_c}\right)\frac{M}{P}(\hat{m}_{1,t} + \hat{m}_{2,t+1})\Bigg].$$

Note that the μ_t term appears because of our choice of normalizing by M. The linearized real wage in the sticky-wage case is given by

$$\hat{w}_t = \hat{\bar{\omega}}_t - \hat{p}_t = \frac{1}{2}(\hat{\omega}_{t-1} + \hat{\omega}_{t-2} - \hat{\mu}_{t-1}) - \hat{p}_t.$$

Thus, we have lags and leads of the terms in (4.4). But, to get persistence of real exchange rates, we still require that firms do not raise prices too quickly. The sticky wages will slow the response down somewhat – but not entirely. Unless U_{cl} is sufficiently negative (to offset $-U_{cc}/U_c$ or U_{ll}/U_l), we will have the same problem as before: in response to an increase in μ, wages will rise quickly, marginal costs will rise quickly, and firms will increase prices. This in turn will imply no persistence in consumption and, therefore, no persistence in real exchange rates.

5. Some Analytics for Special Cases

5.1. Labor-Only Case with Nonseparable U, Exogenous Money, and $N = 2$

Let's first consider the case that we studied in our earlier work. Assume that preferences are given by (3.2) and that $F(k, l) = l$. In the labor-only case, we have $c(s^t) = y(s^t)$ and therefore $\hat{c}_t = \hat{y}_t$. Assume that prices and monies are not growing over time so that P and M are stationary variables. In this case $\mu = 1$. Finally, assume that there are two cohorts of firms ($N = 2$). With $N = 2$, aggregate labor is

$$
\begin{aligned}
l(s^t) &= (l(1, s^t) + l(2, s^t))/2 \\
&= (y_H(1, s^t) + y_H^*(1, s^t) + y_H(2, s^t) + y_H^*(2, s^t))/2 \\
&= \frac{1}{2}\left[\omega_1 P(s^t)\right]^{\frac{1}{1-\rho}} \bar{P}_H(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}} y(s^t) \left(P_H(s^{t-1})^{\frac{1}{\theta-1}} + P_H(s^{t-2})^{\frac{1}{\theta-1}}\right) + \\
&\qquad \frac{1}{2}\left[\omega_1 P^*(s^t)\right]^{\frac{1}{1-\rho}} \bar{P}_H^*(s^{t-1})^{\frac{\rho-\theta}{(1-\rho)(\theta-1)}} y^*(s^t) \left(P_H^*(s^{t-1})^{\frac{1}{\theta-1}} + P_H^*(s^{t-2})^{\frac{1}{\theta-1}}\right).
\end{aligned}
$$

If we linearize this, we get

$$
\begin{aligned}
l\hat{l}_t &= y_H \left(\frac{1}{1-\rho}\hat{P}_t + \frac{\rho-\theta}{(1-\rho)(\theta-1)}\hat{\bar{P}}_{H,t-1} + \hat{y}_t + \frac{1}{\theta-1}(\hat{P}_{H,t-1} + \hat{P}_{H,t-2})/2\right) \\
&\quad + y_H^* \left(\frac{1}{1-\rho}\hat{P}_t^* + \frac{\rho-\theta}{(1-\rho)(\theta-1)}\hat{\bar{P}}^*_{H,t-1} + \hat{y}_t^* + \frac{1}{\theta-1}(\hat{P}^*_{H,t-1} + \hat{P}^*_{H,t-2})/2\right) \\
&= \alpha l \left(\frac{1}{1-\rho}(\hat{P}_t - \hat{\bar{P}}_{H,t-1}) + \hat{y}_t\right) + \alpha^* l \left(\frac{1}{1-\rho}(\hat{P}_t^* - \hat{\bar{P}}^*_{H,t-1}) + \hat{y}_t^*\right) \\
&= \alpha l \left(\frac{\alpha^*}{1-\rho}\left[\hat{\bar{P}}_{F,t-1} - \hat{\bar{P}}_{H,t-1}\right] + \hat{y}_t\right) + \alpha^* l \left(\frac{\alpha}{1-\rho}\left[\hat{\bar{P}}^*_{F,t-1} - \hat{\bar{P}}^*_{H,t-1}\right] + \hat{y}_t^*\right) \\
&= \frac{\alpha\alpha^* l}{1-\rho}\left[\hat{\bar{P}}_{F,t-1} + \hat{\bar{P}}^*_{F,t-1} - \hat{\bar{P}}_{H,t-1} - \hat{\bar{P}}^*_{H,t-1}\right] + \alpha l\,\hat{y}_t + \alpha^* l\,\hat{y}_t^*
\end{aligned}
$$

where the parameter α corresponds to the ratio $y_H/(y_H + y_H^*)$. (In the paper we chose this to be 0.984 because imports from Europe are roughly 1-.984 or 1.6% of GDP.) The parameter α^* is $1 - \alpha$. We will need the above expression for the labor input when we write out the pricing equations.

We will also need the wage rate which is given by $-U_l(s^t)/U_c(s^t)$ and therefore depends on consumption, the labor input, and real balances. The linearized wage equation is given by

$$\begin{aligned}
\hat{w}_t &= \left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi}\left(\frac{\eta-1}{\eta}\right)+\frac{1}{\eta}\right]\hat{c}_t+\frac{l}{1-l}\hat{l}_t+\left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi}\left(\frac{\eta-1}{\eta}\right)\right](\hat{M}_t-\hat{P}_t)\\
&= \left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi}\left(\frac{\eta-1}{\eta}\right)+\frac{1}{\eta}\right]\hat{y}_t\\
&\quad+\left(\frac{l}{1-l}\right)\left\{\frac{\alpha\alpha^*}{1-\rho}\left[\hat{\bar{P}}_{F,t-1}+\hat{\bar{P}}^*_{F,t-1}-\hat{\bar{P}}_{H,t-1}-\hat{\bar{P}}^*_{H,t-1}\right]+\alpha\,\hat{y}_t+\alpha^*\,\hat{y}^*_t\right\}\\
&\quad+\left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi}\left(\frac{\eta-1}{\eta}\right)\right](\hat{M}_t-\hat{P}_t).
\end{aligned}$$

where we have used the formulas above for partial derivatives of the nonseparable (type I) utility function.

The nominal exchange rate appears in two of the pricing equations. We can write this in terms of the real exchange rate since $e = qP/P^*$. The real exchange rate is a ratio of the marginal utilities. Using the nonseparable (type I) preferences, we get the following expression for the real exchange rate:

$$\begin{aligned}
\hat{q}_t &= \left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma-\frac{1}{\eta}\right)+\frac{1}{\eta}\right](\hat{c}_t-\hat{c}^*_t)+\frac{(1-\sigma)\psi l}{1-l}(\hat{l}_t-\hat{l}^*_t)\\
&\quad+\left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma-\frac{1}{\eta}\right)\right](\hat{M}_t-\hat{P}_t-\hat{M}^*_t+\hat{P}^*_t).
\end{aligned}$$

We can substitute in $c = y$ and the formula for the labor input derived above to get q in terms of outputs, prices, and money.

The money demand equation is given by

$$\hat{M}_t-\hat{P}_t=\hat{y}_t-\eta\beta(r_t/r-1).$$

If $\eta \approx 0$ then money demand is interest-inelastic and output and real balances are equal in equilibrium. Suppose this is true. Then wages are equal to:

$$\hat{w}_t=\left(1+\frac{\alpha l}{1-l}\right)(\hat{M}_t-\hat{P}_t)+\left(\frac{\alpha^* l}{1-l}\right)(\hat{M}^*_t-\hat{P}^*_t)$$

$$+\left(\frac{\alpha\alpha^* l}{(1-l)(1-\rho)}\right)\left[\hat{\bar{P}}_{F,t-1}+\hat{\bar{P}}^*_{F,t-1}-\hat{\bar{P}}_{H,t-1}-\hat{\bar{P}}^*_{H,t-1}\right]$$

and the real exchange rate is equal to

$$\hat{q}_t=\left(\sigma+(1-\sigma)\frac{\psi(\alpha-\alpha^*)l}{1-l}\right)(\hat{M}_t-\hat{P}_t-\hat{M}^*_t+\hat{P}^*_t)$$

$$+\left(\frac{2\psi(1-\sigma)\alpha\alpha^* l}{(1-l)(1-\rho)}\right)\left[\hat{\bar{P}}_{F,t-1}+\hat{\bar{P}}^*_{F,t-1}-\hat{\bar{P}}_{H,t-1}-\hat{\bar{P}}^*_{H,t-1}\right]$$

We will now derive the pricing equation for P_H assuming that money demand is interest-inelastic. The intermediate goods price is a function of nominal marginal costs. In this example, the linearized marginal nominal cost is given by

$$\hat{P}_t+\hat{w}_t=\left(1+\frac{\alpha l}{1-l}\right)\hat{M}_t+\left(\frac{\alpha^* l}{1-l}\right)\hat{M}^*_t$$

$$-\left(\frac{\alpha l}{1-l}\right)\left[\alpha\hat{\bar{P}}_{H,t-1}+\alpha^*\hat{\bar{P}}_{F,t-1}\right]-\left(\frac{\alpha^* l}{1-l}\right)\left[\alpha\hat{\bar{P}}^*_{F,t-1}+\alpha^*\hat{\bar{P}}^*_{H,t-1}\right]$$

$$+\left(\frac{\alpha\alpha^* l}{(1-l)(1-\rho)}\right)\left[\hat{\bar{P}}_{F,t-1}+\hat{\bar{P}}^*_{F,t-1}-\hat{\bar{P}}_{H,t-1}-\hat{\bar{P}}^*_{H,t-1}\right]$$

If we substitute this expression into the pricing equation we get

$$\hat{P}_{H,t-1}=\frac{1}{1+\beta}E_{t-1}\Bigg\{\left(1+\frac{\alpha l}{1-l}\right)\hat{M}_t+\left(\frac{\alpha^* l}{1-l}\right)\hat{M}^*_t$$

$$-\left(\frac{\alpha\alpha^* l}{1-l}\right)\Bigg(\left(\frac{1}{1-\rho}+\frac{\alpha}{\alpha^*}\right)\left(\hat{P}_{H,t-1}+\hat{P}_{H,t-2}\right)/2$$

$$+\left(\frac{1}{1-\rho}+\frac{\alpha^*}{\alpha}\right)\hat{\bar{P}}^*_{H,t-1}-\frac{\rho}{1-\rho}\left(\hat{\bar{P}}_{F,t-1}+\hat{\bar{P}}^*_{F,t-1}\right)\Bigg)$$

$$+\beta\Bigg[\left(1+\frac{\alpha l}{1-l}\right)\hat{M}_{t+1}+\left(\frac{\alpha^* l}{1-l}\right)\hat{M}^*_{t+1}$$

$$-\left(\frac{\alpha\alpha^* l}{1-l}\right)\Bigg(\left(\frac{1}{1-\rho}+\frac{\alpha}{\alpha^*}\right)\left(\hat{P}_{H,t}+\hat{P}_{H,t-1}\right)/2$$

$$+\left(\frac{1}{1-\rho}+\frac{\alpha^*}{\alpha}\right)\hat{\bar{P}}^*_{H,t}-\frac{\rho}{1-\rho}\left(\hat{\bar{P}}_{F,t}+\hat{\bar{P}}^*_{F,t}\right)\Bigg)\Bigg]\Bigg\}$$

Thus, we get finally,

$$\hat{P}_{H,t-1} = \frac{1}{1+\beta}\left[1 + \frac{1}{2}\left(\frac{\alpha\alpha^* l}{1-l}\right)\left(\frac{1}{1-\rho} + \frac{\alpha}{\alpha^*}\right)\right]^{-1}$$

$$\times E_{t-1}\Bigg\{\left(1 + \frac{\alpha l}{1-l}\right)(\hat{M}_t + \beta\hat{M}_{t+1}) + \left(\frac{\alpha^* l}{1-l}\right)(\hat{M}^*_t + \beta\hat{M}^*_{t+1})$$

$$- \left(\frac{\alpha\alpha^* l}{1-l}\right)\Bigg(\left(\frac{1}{1-\rho} + \frac{\alpha}{\alpha^*}\right)(\hat{P}_{H,t-2} + \beta\hat{P}_{H,t})/2$$

$$+ \left(\frac{1}{1-\rho} + \frac{\alpha^*}{\alpha}\right)(\hat{\bar{P}}^*_{H,t-1} + \beta\hat{\bar{P}}^*_{H,t})$$

$$- \frac{\rho}{1-\rho}\left(\hat{\bar{P}}_{F,t-1} + \hat{\bar{P}}^*_{F,t-1} + \beta\hat{\bar{P}}_{F,t} + \beta\hat{\bar{P}}^*_{F,t}\right)\Bigg)\Bigg\}$$

If we drive α and β both to 1 and let $\gamma = 1/(1-l)$, then we get the following difference equation:

$$E_{t-1}\hat{P}_{H,t} - 2\frac{1+\gamma}{1-\gamma}\hat{P}_{H,t-1} + \hat{P}_{H,t-2} = -\frac{2\gamma}{1-\gamma}E_{t-1}(\hat{M}_t + \hat{M}_{t+1})$$

which is exactly what we found for our closed economy example. Recall that the solution to this is:

$$\hat{P}_{H,t-1} = a\hat{P}_{H,t-2} + (1-a)\hat{M}_{t-1}$$

where $a = (1-\sqrt{\gamma})/(1+\sqrt{\gamma})$.

To derive the equations for P^*_H and P_F we need to linearize Pw/e and P^*w^*e, respectively. When money demand is interest-insensitive, the linearization of Pw/e is given by

$$\hat{P}_t + \hat{w}_t - \hat{e}_t = \hat{P}^*_t + \hat{w}_t - \hat{q}_t$$

$$= \left(1 - \sigma + (\alpha - \psi(1-\sigma)(\alpha-\alpha^*))\frac{l}{1-l}\right)\hat{M}_t$$

$$+ \left(\sigma + (\alpha^* + \psi(1-\sigma)(\alpha-\alpha^*))\frac{l}{1-l}\right)\hat{M}^*_t$$

$$
\begin{aligned}
&-\left(1-\sigma+(\alpha-\psi(1-\sigma)(\alpha-\alpha^*))\frac{l}{1-l}\right)\left[\alpha\hat{\bar{P}}_{H,t-1}+\alpha^*\hat{\bar{P}}_{F,t-1}\right]\\
&+\left(1-\sigma-(\alpha^*+\psi(1-\sigma)(\alpha-\alpha^*))\frac{l}{1-l}\right)\left[\alpha\hat{\bar{P}}^*_{F,t-1}+\alpha^*\hat{\bar{P}}^*_{H,t-1}\right]\\
&+\left(\frac{\alpha\alpha^* l[1-2\psi(1-\sigma)]}{(1-l)(1-\rho)}\right)\left[\hat{\bar{P}}_{F,t-1}+\hat{\bar{P}}^*_{F,t-1}-\hat{\bar{P}}_{H,t-1}-\hat{\bar{P}}^*_{H,t-1}\right].
\end{aligned}
$$

The linearization of $P^* w^* e$ is given by

$$
\begin{aligned}
\hat{P}^*_t+\hat{w}^*_t+\hat{e}_t &= \hat{P}_t+\hat{w}^*_t+\hat{q}_t\\
&=\left(1-\sigma+(\alpha-\psi(1-\sigma)(\alpha-\alpha^*))\frac{l}{1-l}\right)\hat{M}^*_t\\
&\quad+\left(\sigma+(\alpha^*+\psi(1-\sigma)(\alpha-\alpha^*))\frac{l}{1-l}\right)\hat{M}_t\\
&\quad-\left(1-\sigma+(\alpha-\psi(1-\sigma)(\alpha-\alpha^*))\frac{l}{1-l}\right)\left[\alpha\hat{\bar{P}}^*_{F,t-1}+\alpha^*\hat{\bar{P}}^*_{H,t-1}\right]\\
&\quad+\left(1-\sigma-(\alpha^*+\psi(1-\sigma)(\alpha-\alpha^*))\frac{l}{1-l}\right)\left[\alpha\hat{\bar{P}}_{H,t-1}+\alpha^*\hat{\bar{P}}_{F,t-1}\right]\\
&\quad+\left(\frac{\alpha\alpha^* l[1-2\psi(1-\sigma)]}{(1-l)(1-\rho)}\right)\left[\hat{\bar{P}}^*_{H,t-1}+\hat{\bar{P}}_{H,t-1}-\hat{\bar{P}}^*_{F,t-1}-\hat{\bar{P}}_{F,t-1}\right]
\end{aligned}
$$

which looks the same as the first except that we swap domestic and foreign variables. Notice that these formulas simplify considerably in the case of $\sigma = 1$, for example:

$$
\begin{aligned}
\hat{P}_t+\hat{w}_t-\hat{e}_t = \frac{\alpha l}{1-l}\hat{M}_t &+ \left(1+\frac{\alpha^* l}{1-l}\right)\hat{M}^*_t\\
&-\frac{\alpha l}{1-l}\left[\alpha\hat{\bar{P}}_{H,t-1}+\alpha^*\hat{\bar{P}}_{F,t-1}\right]\\
&-\frac{\alpha^* l}{1-l}\left[\alpha\hat{\bar{P}}^*_{F,t-1}+\alpha^*\hat{\bar{P}}^*_{H,t-1}\right]\\
&+\left(\frac{\alpha\alpha^* l}{(1-l)(1-\rho)}\right)\left[\hat{\bar{P}}_{F,t-1}+\hat{\bar{P}}^*_{F,t-1}-\hat{\bar{P}}_{H,t-1}-\hat{\bar{P}}^*_{H,t-1}\right]
\end{aligned}
$$

because in this case $\hat{e} = \hat{M} - \hat{M}^*$. Therefore, when $\sigma = 1$ the expressions for P_H and P^*_H differ only in the money terms.

What if we set $\eta = 0$, $\beta = 1$, $\alpha = \alpha^* = 1/2$ and $\rho = 1/3$? Then the equation for P_H is given by

$$\hat{P}_{H,t-1} = \frac{1}{2+5(\gamma-1)/8} E_{t-1}\Bigg\{ \frac{1}{2}(1+\gamma)(\hat{M}_t + \hat{M}_{t+1}) + \frac{1}{2}(\gamma-1)(\hat{M}_t^* + \hat{M}_{t+1}^*)$$

$$- \frac{1}{4}(\gamma-1)\Bigg[\frac{5}{2}\left(\hat{P}_{H,t-2} + \hat{P}_{H,t}\right)/2 + \frac{5}{2}\left(\hat{\hat{P}}^*_{H,t-1} + \hat{\hat{P}}^*_{H,t}\right)$$

$$- \frac{1}{2}\left(\hat{\hat{P}}_{F,t-1} + \hat{\hat{P}}^*_{F,t-1} + \hat{\hat{P}}_{F,t} + \hat{\hat{P}}^*_{F,t}\right)\Bigg]\Bigg\}.$$

Notice that this can be rewritten as

$$E_{t-1}P_{H,t} + \phi P_{H,t-1} + P_{H,t-2} = \text{r.h.s. terms}$$

where $\phi = [2 + 5(\gamma-1)/8]/[5/16(\gamma-1)]$ which is greater than 2. (This follows because $\gamma = 1/(1-l) > 1$.) The root satisfying $\lambda^2 + \phi\lambda + 1$ that is inside the unit circle lies somewhere between -1 and 0. Therefore, to generate persistence, we need that the other prices have a quantitatively important affect on P_H. The equation with the right hand side terms written out is

$$E_{t-1}P_{H,t} + 2\left(1 + \frac{16}{5(\gamma-1)}\right) P_{H,t-1} + P_{H,t-2}$$

$$= E_{t-1}\Bigg\{ \frac{8}{5}(1+\gamma)(\hat{M}_t + \hat{M}_{t+1}) + \frac{8}{5}(\gamma-1)(\hat{M}_t^* + \hat{M}_{t+1}^*)$$

$$+ 2(\hat{\hat{P}}^*_{H,t-1} + \hat{\hat{P}}^*_{H,t}) - \frac{2}{5}\left(\hat{\hat{P}}_{F,t-1} + \hat{\hat{P}}^*_{F,t-1} + \hat{\hat{P}}_{F,t} + \hat{\hat{P}}^*_{F,t}\right)\Bigg\}$$

5.2. Labor-Only Case with Separable U, Exogenous Money, and $N = 2$

We need to rederive the equation for the wage rate and the real exchange rate. The linearized wage equation with separable preferences is given by

$$\hat{w}_t = \left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right) + \frac{1}{\eta}\right]\hat{c}_t + \frac{\xi l}{1-l}\hat{l}_t + \left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right)\right](\hat{M}_t - \hat{P}_t)$$

$$= \left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right) + \frac{1}{\eta}\right]\hat{y}_t$$

$$+ \left(\frac{\xi l}{1-l}\right)\left\{\frac{\alpha\alpha^*}{1-\rho}\left[\hat{\bar{P}}_{F,t-1} + \hat{\bar{P}}^*_{F,t-1} - \hat{\bar{P}}_{H,t-1} - \hat{\bar{P}}^*_{H,t-1}\right] + \alpha\,\hat{y}_t + \alpha^*\,\hat{y}^*_t\right\}$$

$$+ \left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right)\right](\hat{M}_t - \hat{P}_t).$$

The real exchange rate is now

$$\hat{q}_t = \left[\frac{\omega c^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right) + \frac{1}{\eta}\right](\hat{c}_t - \hat{c}^*_t)$$

$$+ \left[\frac{(1-\omega)(M/P)^{\frac{\eta-1}{\eta}}}{\Psi}\left(\sigma - \frac{1}{\eta}\right)\right](\hat{M}_t - \hat{P}_t - \hat{M}^*_t + \hat{P}^*_t)$$

which is the same as before except that now there is no term with the labor inputs.

If we assume that money demand is interest-inelastic, then the wage and real exchange rate simplify to:

$$\hat{w}_t = \left(\sigma + \frac{\xi\alpha l}{1-l}\right)(\hat{M}_t - \hat{P}_t) + \left(\frac{\xi\alpha^* l}{1-l}\right)(\hat{M}^*_t - \hat{P}^*_t)$$

$$+ \left(\frac{\xi\alpha\alpha^* l}{(1-l)(1-\rho)}\right)\left[\hat{\bar{P}}_{F,t-1} + \hat{\bar{P}}^*_{F,t-1} - \hat{\bar{P}}_{H,t-1} - \hat{\bar{P}}^*_{H,t-1}\right]$$

$$\hat{q}_t = \sigma(\hat{M}_t - \hat{P}_t - \hat{M}^*_t + \hat{P}^*_t).$$

If we drive α and β both to 1 and let $\gamma = 1/(1-l)$, then we get the following difference equation for P_H:

$$E_{t-1}\hat{P}_{H,t} - 2\frac{1+\hat{\gamma}}{1-\hat{\gamma}}\hat{P}_{H,t-1} + \hat{P}_{H,t-2} = -\frac{2\hat{\gamma}}{1-\hat{\gamma}}E_{t-1}(\hat{M}_t + \hat{M}_{t+1})$$

where

$$\hat{\gamma} = \sigma + \xi(\gamma - 1).$$

Therefore, if $\sigma = 1$ and $\xi = 1$, we have the same equation as in the nonseparabe utility case. If σ and ξ are both small then we can get persistence – changes in outputs and other firms prices do not imply that a producer adjusts his price immediately. However, we lose volatility of exchange rates since σ must be large in order to amplify q.