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DUALITY AND ARBITRAGE WITH TRANSACTIONS COSTS:  
THEORY AND APPLICATIONS

Michael J. Stutzer

Federal Reserve Bank of Minneapolis  
and Carlson School of Management,  
University of Minnesota

ABSTRACT

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Recent advances in duality theory have made it easier to discover relationships between asset prices and the portfolio choices based on them. But this approach to arbitrage-free securities markets has yet to be extended and applied to economies with transactions costs. This paper does so, within the context of a general state-preference model of securities markets. Several applications are developed to illustrate the nature of the theory and its potential to resolve a host of issues surrounding the effects of transactions costs on securities markets.

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What effects do transactions costs have on the structure of asset prices, the nature of securities trading, and the types of securities issued? Can the corpus of financial theory be modified to readily address these questions? To find an answer to these questions, this paper reexamines one of the major developments in financial theory during the past 25 years: the discovery and application of linear valuation operators, which govern relative asset prices in arbitrage-free economies.

These operators, also dubbed state price measures or equivalent Martingale measures, are typically used in two ways. First, they are used to establish relationships among asset prices, which follow from the seemingly innocuous assumption that markets eliminate arbitrage opportunities. For example, with the additional assumptions of complete markets, no short sales constraints, and no transactions costs, their use yields simple proofs of such results as the Modigliani-Miller Theorem, the spot-futures parity relation, and the put-call parity relation of options pricing. Second, under these assumptions, the valuation operator is unique, in which case it is represented by the familiar normalized, Arrow-Debreu state prices. As such, via the so-called duality approach, the operator can be used to simplify a trader's consumption-portfolio choice problem by decomposing it into two, often simpler, problems:

1. the choice of a utility maximizing, state-dependent consumption plan among those feasible at Arrow-Debreu prices.
2. the calculation of a trading strategy to guarantee the attainment of this consumption.<sup>1</sup>

In recent years, He and Pearson (1988a,b), Pagés (1987), and related papers have extended this duality framework to incorporate the effects of incomplete markets and short sales constraints. This is achieved by forming a dual minimization problem whose solution, called a minimax measure, provides the analog of the normalized Arrow-Debreu prices in problem 1 above.

In this paper, this duality framework is modified to include the important effects of transactions costs. Adopting the representation of transactions costs effectively used by Garman and Ohlson (1981), a dual problem is developed whose solution is a minimax measure for the portfolio choice problem with transactions costs, incomplete markets, and short sales constraints in a finite horizon, discrete time setting. The nature and utility of the duality approach are made evident by a number of applications in static finance theory. Finally, utilizing a format adopted by Breeden (1987), the last section generalizes the single-period results to the familiar finite horizon, discrete time, event-tree model. Future work will develop applications of the multi-period results to questions involving the effects of transactions costs on securities trading.

### I. A One-Period Model

A standard, one-period, state-preference approach is adopted.<sup>2</sup> Traders are endowed with  $e$  units of a single consumption good  $c$  at either the beginning of the period, the end, or both. Uncertain states drawn from a set  $\Omega = \{\omega_1, \dots, \omega_K\}$  determine both end-of-period endowments (if any) and marketed asset payoffs. There are  $N$  primary assets available for trade at the beginning of the period; asset  $i$  pays  $X_i(\omega_j)$  of the consumption good to the buyer from the seller when state  $j$

occurs at the period's end. Following Garman and Ohlson (1981), non-negative transactions costs incurred in trading assets are paid to third parties, are proportional to the number of units traded, and are allowed to differ across assets and across states. In addition, transactions costs incurred may differ when buying than when selling, and they may also differ between the beginning of the period (when trading takes place) and the end of the period (when payouts are made).

**Table I**  
**Transactions Costs Per Unit Traded**

Transaction Costs Per Unit of Asset i:	Beginning of Period	End of Period
Bought at beginning	$T_i^+$	$t_i^+(\omega_j)$
Sold at beginning	$T_i^-$	$t_i^-(\omega_j)$

Table I summarizes the notation used for the various possibilities. To clarify matters, let  $\theta_i^+$  denote the nonnegative number of units of asset  $i$  bought at the beginning of the period. This incurs a beginning-of-period transactions cost of  $\theta_i^+ T_i^+$ , which must be paid to the agents brokering the transaction. The total purchasing cost is then  $\theta_i^+ T_i^+ + \theta_i^+ P_i$ , where  $P_i$  is the asset's price, which is paid to the seller. At the period's end in state  $\omega_j$ , this purchaser must pay  $\theta_i^+ t_i^+(\omega_j)$  to close out the position, and receives asset proceeds of  $\theta_i^+ X_i(\omega_j)$ . A seller of  $\theta_i^-$  units of the asset incurs a beginning-of-period transactions cost of  $\theta_i^- T_i^-$  and an end-of-period transactions cost of  $\theta_i^- t_i^-(\omega_j)$  when the short position is closed. The inability to sell asset  $i$  short can then be modeled by requiring  $\theta_i^- = 0$ . Without loss of

generality, we assume that the last L assets can't be shorted.

A trader's feasible consumption by asset trading must then satisfy the following condition.

**DEFINITION 1:** The bundle  $(c_0, c)$  is in the feasible consumption set  $C(e_0, e)$  if and only if there exists  $\theta$  satisfying

$$\begin{pmatrix} P(T) \\ A(t) \end{pmatrix} \theta \geq \begin{pmatrix} c_0 - e_0 \\ c - e \end{pmatrix} \text{ and } \begin{pmatrix} I_N & 0 \\ 0 & I_{N-L} \end{pmatrix} \theta \geq 0 \quad (1)$$

where a subscripted I is used to denote the identity matrix of subscripted order, and

$$P(T) = [-(P_1 + T_1^+), \dots, -(P_N + T_N^+), P_1 - T_1^-, \dots, P_{N-L} - T_{N-L}^-]$$

$$A(t) = \begin{pmatrix} X_1(\omega_1) - t_1^+(\omega_1), \dots, X_N(\omega_1) - t_N^+(\omega_1), -X_1(\omega_1) - t_1^-(\omega_1), \dots, -X_{N-L}(\omega_1) - t_{N-L}^-(\omega_1) \\ \vdots \\ X_1(\omega_k) - t_1^+(\omega_k), \dots, X_N(\omega_k) - t_N^+(\omega_k), -X_1(\omega_k) - t_1^-(\omega_k), \dots, -X_{N-L}(\omega_k) - t_{N-L}^-(\omega_k) \end{pmatrix}$$

$$\theta = (\theta_1^+, \dots, \theta_N^+, \theta_1^-, \dots, \theta_{N-L}^-)'$$

$$c - e = (c(\omega_1) - e(\omega_1), \dots, c(\omega_k) - e(\omega_k))'$$

The first inequality in (1) requires that the beginning-of-period consumption  $c_0$  not exceed beginning-of-period endowment  $e_0$ , plus any beginning-of-period revenue generated by short sales, minus any costs of taking long positions and transactions costs incurred in all positions. The other inequalities similarly constrain end-of-period consumption in each state after all positions are closed.

A trader's portfolio-choice problem would then be to solve

$$\max_{\theta} U(c_0, c) \quad \text{subject to (1)} \quad (2)$$

where  $U$  is always strictly increasing in all of its arguments.

Any condition necessary for (2) to have a solution must also be necessary for any equilibrium notion in which traders solve (2). The following familiar, no-arbitrage property of  $P(T)$  and  $A$  is such a condition.

**DEFINITION 2:** There are no arbitrage opportunities when there are no  $\theta$  satisfying

$$\begin{pmatrix} P(T) \\ A(t) \end{pmatrix} \theta \neq 0 \quad \text{and} \quad \begin{pmatrix} I_N & 0 \\ 0 & I_{N-L} \end{pmatrix} \theta \geq 0. \quad (3)$$

A glance at (1) shows that the existence of  $\theta$  satisfying (3) would permit consumption in excess of endowment. By choosing an arbitrarily large number  $\lambda > 0$ , the portfolio  $\lambda\theta$  would then provide arbitrarily large feasible consumption, and a solution to (2) thus would not exist.

For the case without short sale constraints, Garman and Ohlson (1981, p. 274) point out that Tucker's Theorem of the Alternative (Mangasarian (1969), p. 29) could be used to provide a dual representation of the no-arbitrage condition (3). This dual representation demonstrates the existence of a strictly positive, state price vector useful for bounding asset prices. To see this, Tucker's Theorem states that there is no  $\theta$  satisfying (3) if and only if  $V^1 \equiv (V_0^1, V_1^1, \dots, V_k^1) > 0$  and  $V^2 \equiv (V_1^2, \dots, V_{N-L}^2, \dots, V_{2N-L}^2) \geq 0$  satisfying

$$(P(T)'A')V^1 + \begin{pmatrix} I_N & 0 \\ 0 & I_{N-L} \end{pmatrix} V^2 = 0. \quad (4)$$

Divide (4) by  $V_0^1 > 0$ , and define scalars  $q_j \equiv V_j^1/V_0^1 > 0$ ,  $j = 1, \dots, k$ ;  $w_i \equiv V_i^2/V_0^1 \geq 0$ ,  $i = 1, \dots, N$ ; and  $u_i \equiv V_{N+i}^2/V_0^1 \geq 0$ ,  $i = 1, \dots, N - L$ . Substituting these scalars into (4) and defining state price probabilities  $Q_j = q_j / \sum_{j=1}^k q_j$  yields the following analog of Garman and Ohlson's Proposition 2.

**THEOREM 1.** There are no arbitrage opportunities if and only if there exist  $w_i \geq 0$ ,  $i = 1, \dots, N$ ;  $u_i \geq 0$ ,  $i = 1, \dots, N - L$ ; and strictly positive probabilities  $Q_j$ ,  $j = 1, \dots, k$  satisfying

$$P_i + T_i^+ = \frac{\sum_{j=1}^k [X_i(\omega_j) - t_i^+(\omega_j)] Q_j}{1+r} + w_i \geq \frac{E_Q[X_i - t_i^+]}{1+r}$$

$$P_i - T_i^- = \frac{\sum_{j=1}^k [X_i(\omega_j) + t_i^-(\omega_j)] Q_j}{1+r} - u_i \leq \frac{E_Q[X_i + t_i^-]}{1+r}$$

for  $i = 1, \dots, N - L$  (5)

$$P_i + T_i^+ = \frac{\sum_{j=1}^k [X_i(\omega_j) - t_i^+(\omega_j)] Q_j}{1+r} + w_i \geq \frac{E_Q[X - t_i^+]}{1+r}$$

for  $i = N - L + 1, \dots, N$

where the riskless discount factor  $1/(1+r) = \sum_{j=1}^k q_j$  denotes the price of a hypothetical, riskless asset paying 1 unit in all states without transactions cost.  $E_Q[\cdot]$  denotes the mathematical expectation with respect to  $Q$ .

The probability measure  $Q$  is analogous to an equivalent Martingale measure because it relates (net of transactions cost) asset

payoffs to asset prices via an expected discounted value operator. But here, with transactions costs paid to third parties, there are error terms  $w_i$  and  $u_i$  in the cum-transactions cost price to payoff relationships.<sup>3</sup> When positive, these error terms cause the expected discounted value operator to underestimate the beginning-of-period cost of buying an asset and to overestimate the net revenues earned from selling it short.

To interpret these error terms, consider a trading strategy long 1 unit of asset  $i$  and short 1 unit of asset  $i$ . Using (5), subtract  $P_i - T_i^-$  from  $P_i + T_i^+$  to find the expression

$$(T_i^+ + T_i^-) + \sum_{j=1}^k \frac{(t_i^+(\omega_j) + t_i^-(\omega_j))}{1+r} Q_j = w_i + u_i. \quad (6)$$

The first term in (6) is the beginning-of-period portfolio transactions cost, while the second term is the expected discounted value of end-of-period transactions costs (with respect to the probability measure  $Q$ ). Thus,  $w_i + u_i$  is the round-trip transactions cost of the portfolio valued at state prices. Because both  $w_i$  and  $u_i$  are always nonnegative, any asset that can be traded without transactions costs will have both  $w_i$  and  $u_i$  equal to zero. In other words, there will be no error terms in its pricing equation.

Rearranging the inequalities in (5) yields bounds for the deviation of asset prices from their  $Q$ -expected discounted payoffs:

$$-(T_i^+ + E_Q[t_i^+]/(1+r)) \leq P_i - E_Q[X_i]/(1+r) \leq T_i^- + E_Q[t_i^-]/(1+r)$$

for  $i = 1, \dots, N - L$

(7)

$$-(T_i^+ + E_Q[t_i^+]/(1+r)) \leq P_i - E_Q[X_i]/(1+r)$$

for  $i = N - L + 1, \dots, N$ .

From (7), it is easy to see that if there were no transactions costs incurred in trading the first  $N - L$  assets,  $Q$  would be an equivalent Martingale measure for pricing assets spanned by them. If, in addition, there were no short sales constraints, i.e.,  $L = 0$ , then  $Q$  would be an equivalent Martingale measure for all marketed assets. This special case of Theorem 1 has been developed by Green and Srivastava (1985) using more complex separation arguments. Finally, if in addition there are  $N = k$  linearly independent assets (i.e., complete markets), the equations  $P_i = E_Q[X_i]/(1+r)$  implicitly define a unique measure  $Q^*$  of normalized, Arrow-Debreu state prices.

In fact, it is possible to strengthen Theorem 1 using the following proposition, which is critical to applications developed later.

**PROPOSITION 1:** No trader would take simultaneous long and short positions in any asset  $i$  subject to transactions costs.

Proof: Suppose not, i.e., suppose there exists  $1 \leq i \leq N - L$  such that  $\theta_i^+ > 0$  and  $\theta_i^- > 0$ . Consider an alternative trading strategy produced by replacing the simultaneous long and short position with the net position  $\theta_i^+ - \theta_i^-$ . By economizing on transactions costs, the alternative strategy's beginning-of-period cost is no more than the hypothesized strat-

egy's cost, and it will actually cost less when either  $T_1^+$  or  $T_1^-$  is positive. Yet, the alternative strategy's state-dependent payoff is never less than the hypothesized strategy's payoff, and it will actually be more if either  $t_1^+(\omega_j)$  or  $t_1^-(\omega_j)$  is positive for some  $j = 1, \dots, k$ . Any trader with preferences strictly increasing in consumption will prefer the alternative strategy, contrary to the hypothesis that simultaneous positions are held.

Partition the portfolio  $\theta = (\theta^+, \theta^-)$ . Proposition 1 asserts that there is no loss in generality in restricting feasible trading strategies to those satisfying  $\theta^+ \cdot \theta^- = 0$ , where " $\cdot$ " denotes the inner product. These are called orthogonal strategies.

The proof of Proposition 1 shows that the existence of a nonorthogonal arbitrage opportunity implies the existence of an orthogonal arbitrage opportunity (i.e., substitute net positions for simultaneous ones). The contrapositive of this is that there are no orthogonal arbitrage opportunities when there are no nonorthogonal ones. But the converse may not be true: There may be orthogonal arbitrage opportunities when there are no nonorthogonal ones. This prospect permits us to sharpen the criteria by searching for the existence of separate solutions to subsystems of (5) defined by the orthogonality restriction.

More specifically, examine subsystems of  $N - L$  equations chosen from the first  $2(N-L)$  equation in (5), in which exactly one of each pair  $(w_i, u_i)$  occurs. For example, when  $N = 2$  and  $L = 0$ , there are four, or  $2^{N-L}$ , such subsystems whose left-hand sides are

$$\begin{pmatrix} P_1 + T_1^+ \\ P_2 + T_2^+ \end{pmatrix} \begin{pmatrix} P_1 - T_1^- \\ P_2 - T_2^- \end{pmatrix} \begin{pmatrix} P_1 + T_1^+ \\ P_2 - T_2^- \end{pmatrix} \begin{pmatrix} P_1 - T_1^- \\ P_2 + T_2^+ \end{pmatrix}.$$

These are the subsystems dual to orthogonal trading strategies. Let  $\pi_m$  for  $m = 1, \dots, 2^{N-L}$  denote the set of all triples,  $(Q, w, u) \in$  unit  $k - 1$  simplex  $X R_+^N \times R_+^N$ , which solves the  $m^{\text{th}}$  subsystem. Let  $\pi_L$  denote the set of triples solving the last  $L$  equations. The following result sharpens Theorem 1.

**THEOREM 2.** There are no orthogonal arbitrage opportunities if and only if

$$\pi_m \cap \pi_L \neq \emptyset \quad \text{for } m = 1, \dots, 2^{N-L}.$$

Proof: Common to each hypothetical, orthogonal arbitrage opportunity is the nonexistence of solutions to a subsystem of  $N$  equations consisting of  $N - L$  equations chosen as above and the last  $L$  equations of (5).

Thus, the nonemptiness of each  $\pi_m \cap \pi_L$  is both necessary and sufficient to rule out orthogonal arbitrage opportunities and, therefore, is also sufficient to rule out nonorthogonal arbitrage opportunities.

Theorem 2 only requires that each  $\pi_m \cap \pi_L \neq \emptyset$ , while Theorem 1 requires that  $\bigcap_{m=1}^{2^{N-L}} \pi_m \cap \pi_L \neq \emptyset$ . The no-arbitrage criterion of Theorem 2 is weaker than that of Theorem 1. And while Theorem 2 requires one to examine multiple subsystems, the dimension of each subsystem is smaller than in the single system of Theorem 1.

Furthermore, some solutions for some subsystems can be inferred by inspection of their duals. In the example of  $N = 2$  and  $L = 0$  above, the first two subsystems usually won't admit arbitrage opportunities. Strategies dual to the first listed subsystem have both assets long, and hence, they must incur a positive beginning-of-period

cost. By definition, these can't be arbitrage opportunities. Suppose that both assets have strictly positive payoffs in excess of transactions costs. Then the second subsystem, which corresponds to short positions in both assets, must also have a solution because its dual has negative payoffs in all states, and hence, it can't be an arbitrage possibility. We depict this case in Figure 1.

In the special case shown in Figure 1, the vectors  $(P_1+T_1^+, P_2-T_2^-)$  and  $(P_1-T_1^-, P_2+T_2^+)$  both lie between their respective cum-transactions cost, asset-payoff columns. As such, not only is there no arbitrage, but there exist equivalent Martingale measures, that is, neither  $w$ 's nor  $u$ 's need be present in the pricing equations of (5). However, had  $(P_1+T_1^+, P_2-T_2^-)$  lay to the right of  $Z_1$  while  $(P_1-T_1^-, P_2+T_2^+)$  lay to the left of  $\hat{Z}_1$ , then  $w_1 > 0$  would enter the pricing equation for asset 1. Had  $(P_1+T_1^+, P_2-T_2^-)$  lay to the left of  $\hat{Z}_2$  while  $(P_1+T_1^+, P_2-T_2^-)$  lay to the right of  $Z_2$ , then  $w_2 > 0$  would enter the pricing equation for asset 2. Price vectors lying further outside the columns admit orthogonal arbitrage opportunities.

Although the main purpose of this paper is to develop the theory of transactions costs, several applications of the measures defined by (5) will be presented later in the paper, in the spirit of Ross (1978). But to do so, we first need to develop the duality approach to problem (2).

## II. A Duality Approach to Problem (2)

As shown below, an alternative characterization of the feasible consumption set (1) is possible under the maintained assumption (3) of no arbitrage. It is this characterization that is required for the duality approach to solving problem (2). The following modification of a nonhomogenous version of Farkas' Theorem is needed to derive this.

**LEMMA 1:** A Strictly Positive, Nonhomogenous Farkas' Theorem.

For a given  $p \times n$  matrix  $M$ , vectors  $b \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^p$ , and a scalar  $\beta$ , either

$$b'x > \beta, Mx \leq d \quad \text{has a strictly positive solution } x > 0, \text{ or} \quad (I)$$

i.  $M'y \geq b, d'y \leq \beta$ , or

(II)

ii.  $\begin{pmatrix} M' \\ -d' \end{pmatrix} y \geq 0$  has a nonnegative solution  $y \geq 0$

but (I) and (II) never both hold.

Proof: Following the general method of Mangasarian (1969, p. 32), introduce a strictly positive scalar  $\epsilon$  and rewrite (I) as

$$\begin{aligned} \epsilon &> 0 \\ b'x - \beta\epsilon &> 0 \\ x &> 0 \\ -Mx + \epsilon d &> 0 \end{aligned} \quad (I')$$

which in matrix form is

$$\begin{pmatrix} b' & -\beta \\ \hat{0}' & 1 \\ I_n & \hat{0} \end{pmatrix} \begin{pmatrix} x \\ \epsilon \end{pmatrix} > 0$$

where  $\hat{0}$  is an  $n \times 1$  vector of zeroes and

$$(-M \ d) \begin{pmatrix} x \\ \epsilon \end{pmatrix} \geq 0.$$

By Motzkin's Theorem of the Alternative (Mangasarian (1969), p. 28) either (I') has a solution  $\begin{pmatrix} x \\ \epsilon \end{pmatrix}$  or there exist vectors  $y_1 \geq 0$  and  $y_3 \geq 0$  satisfying

$$\begin{pmatrix} b & \hat{0} & I_n \\ -\beta & 1 & \hat{0}' \end{pmatrix} \begin{pmatrix} y_1^0 \\ y_1^1 \\ y_1^2 \\ \vdots \\ y_1^{1+n} \end{pmatrix} + \begin{pmatrix} -M' \\ d' \end{pmatrix} \begin{pmatrix} y_3^1 \\ \vdots \\ y_3^p \end{pmatrix} = 0 \quad (\text{II}')$$

but never both.

There are two possible forms of solution to (II'). If  $y_1^0 = 0$ , then (II') requires that  $\sum_{i=1}^p M_{ij} y_3^i = y_1^{1+j} \geq 0$  for  $j = 1, \dots, n$  and  $\sum_{i=1}^p d_i y_3^i = -y_1^1 \leq 0$ , with a least one of the  $n + 1$  inequalities strict, i.e.,  $\begin{matrix} M' \\ -d' \end{matrix} y \geq 0$ .

Alternatively, if  $y_1^0 > 0$  then divide each equation in (II') by  $y_1^0$  and redefine  $y = y/y_1^0$  to obtain  $M'y \geq b$  and  $d'y \leq \beta$ . The lemma is thus proven.

The desired alternative representation of the feasible set (1) is the budget feasible set (He and Pearson (1988a), p. 14) defined below:

**DEFINITION 3:** The budget feasible set for Q is

$$B_Q(e_0, e) = \{(c_0, c) | E_Q[c-e]/(1+r) \leq e_0 - c_0\}.$$

That is,  $(c_0, c)$  is budget feasible for Q if its Q-expected discounted value doesn't exceed the expected discounted value of the endowment.

**DEFINITION 4:** The budget feasible set

$$B(e_0, e) = \bigcap_{Q:(5)} B_Q(e_0, e)$$

where  $Q:(5)$  means that the intersection is taken over all probability measures  $Q$  satisfying (5). In other words,  $(c_0, c)$  is budget feasible if and only if it is budget feasible for all probability measures  $Q$  satisfying (5).

Using Lemma 1, it is easy to show the relationship between  $B(e_0, e)$  and the feasible consumption set  $C(e_0, e)$  defined by (1).

**THEOREM 3.** No-arbitrage implies  $B(e_0, e) = C(e_0, e)$ .

Proof: Replace the objects in Lemma 1 with the following objects defined on page 4:

$$\begin{array}{lll} M = A(t)' & x = Q/1+r \equiv q & d = -P(T)' \\ b = c - e & \beta = e_0 - c_0 & y = \theta. \end{array}$$

Then, system (I) requires  $(c_0, c)$  to be budget infeasible at all probability measures  $Q$  satisfying (5). System (II(i)) requires that  $(c_0, c) \in C(e_0, e)$ , while system (II(ii)) requires that  $\theta$  is an arbitrage opportunity satisfying (3).

To show that  $C(e_0, e) \subset B(e_0, e)$ , let  $(c_0, c) \in C(e_0, e)$  so that (II(i)) has a solution. Then, by Lemma 1, system (I) must not have a solution, that is, a probability measure  $Q$  satisfying (5) for which  $(c_0, c)$  is budget infeasible must not exist. Thus,  $(c_0, c) \in B(e_0, e)$ .

To show that  $B(e_0, e) \subset C(e_0, e)$ , let  $(c_0, c) \in B(e_0, e)$  so that system (I) has no solution. Lemma 1 then requires that either (II(i)) or (II(ii)) has a solution. But the assumption of no arbitrage means that (II(ii)) has no solution. Therefore, (II(i)) must have a solution, or  $(c_0, c) \in C(e_0, e)$ .  $\square$

By Theorem 3, the portfolio-consumption problem (2) is equivalent to problem (8):

$$\max_{c_0, c} U(c_0, c) \quad \text{s.t.} \quad (c_0, c) \in B(e_0, e) = \bigcap_{Q:(5)} B_Q(e_0, e). \quad (8)$$

The duality approach to solving (8) starts with the following problem (9) with a single constraint, and its associated value function  $V$ :

**DEFINITION 5:** The problem  $(P'_Q)$  is

$$V(Q) \equiv \max_{c_0, c} U(c_0, c) \quad \text{s.t.} \quad (c_0, c) \in B_Q(e_0, e). \quad (9)$$

Following He and Pearson (1988b, p. 21), define

**DEFINITION 6:** A minimax probability measure  $Q^*$  satisfies (5) and  $\text{argmax}(P'_{Q^*}) = \text{argmax}(8)$ .

If a minimax measure can be found, then solutions to problem (8) with an infinite number of constraints (which, by Theorem 3, coincide with solutions to problem (2) with a finite number of constraints (1)), coincide with solutions to the singly constrained problem  $(P'_{Q^*})$ . The latter, which may be much simpler to study, is one of the main benefits of the duality approach. The term minimax is motivated by the following theorem.

**THEOREM 4.** Define the dual problem to be

$$\min_{Q:(5)} V(Q). \quad (10)$$

Then  $Q^*$  is a minimax probability measure only if it solves (10). If, in addition,  $U$  is strictly concave and if  $Q^*$  solves (10), then  $Q^*$  is a minimax probability measure.

Proof: The proof is along the lines suggested by He and Pearson (1988b, p. 22), where there are no transactions costs. To prove necessity, suppose  $Q^*$  is a minimax probability measure. Then by definition  $U(\text{argmax}(8)) = U(\text{argmax}(P'_{Q^*})) \equiv V(Q^*)$ . For purposes of contradiction, hypothesize that  $Q^*$  doesn't solve (10), that is, a  $Q$  exists which satisfies (5) with  $V(Q^*) > V(Q)$ , so that  $U(\text{argmax}(8)) > V(Q)$ . However, because  $(P'_Q)$  has only one of the constraints present in (8),  $U(\text{argmax}(8)) \leq V(Q)$ , which is a contradiction.

To prove sufficiency, suppose  $Q^*$  solves (10). Let  $(c_0^*, c^*) = \text{argmax}(P'_{Q^*})$ . Add a constraint and examine the more heavily constrained problem  $(P'_{Q^* \cap Q})$  defined by:

$$\max U(c_0, c) \quad \text{s.t.} \quad (c_0, c) \in B_{Q^*} \cap B_Q.$$

If  $\text{argmax}(P'_{Q^* \cap Q}) = \text{argmax}(P'_{Q^*})$ , can be proven, then  $\text{argmax}(P'_{Q^*}) \in B_{Q^* \cap Q}$  for the arbitrarily-chosen  $Q$  satisfying (5). Hence,

$$\text{argmax}(P'_{Q^*}) \in \bigcap_{Q:(5)} B_Q(e_0, e) = B(e_0, e)$$

and it must solve the more heavily constrained (8) as well. By Definition 6,  $Q^*$  would then be a minimax probability measure, and sufficiency would be proven. So it only remains to establish:

**Lemma 2:**  $\text{argmax}(P'_{Q^* \cap Q}) = \text{argmax}(P'_{Q^*})$ .

Proof: Hypothesize, for purposes of contradiction, that this is not so. Strict concavity of  $U$  ensures that there is at most one maximizer to the concave program  $(P'_{Q^*})$ , so that  $U(\text{argmax}(P'_{Q^*})) < V(Q^*)$ . If there exists  $Q^{**}$  satisfying (5) for which  $\text{argmax}(P'_{Q^*}) = \text{argmax}(P'_{Q^{**}})$ , then the above inequality yields  $U(\text{argmax}(P'_{Q^{**}})) < V(Q^*)$ , and it contradicts the theorem's supposition that  $Q^* = \text{argmin}(10)$ .

The desired  $Q^{**}$  can be explicitly constructed. Denote the two nonnegative, Kuhn-Tucker multipliers for the two linear inequality constraints of  $(P'_{Q^*})$  as  $\mu_{Q^*}$  and  $\mu_Q$ , respectively. Because  $U$  is strictly increasing, one of the constraints must bind, so that generically  $\mu_{Q^*} + \mu_Q > 0$ . Let  $\lambda_1 = \mu_{Q^*}/(\mu_{Q^*} + \mu_Q)$ , and define the probability measure  $Q^{**} = \lambda_1 Q^* + (1 - \lambda_1)Q$ . Because (5) defines a convex set of probability measures,  $Q^{**}$  satisfies (5). Now multiply and divide the Lagrangian saddle point condition for  $(P'_{Q^*})$  by  $\mu_{Q^*} + \mu_Q$  to show that  $(\text{argmax}(P'_{Q^*}), \mu_{Q^*} + \mu_Q)$  is the Lagrangian saddle point for the concave program  $(P'_{Q^{**}})$ . Therefore,  $Q^{**}$  is indeed the required measure.  $\square$

Theorem 4 can be used to complete the duality approach to problem (2). Using the notation on p. 4, I now prove Proposition 2.

**PROPOSITION 2:** Assuming the utility function is continuously differentiable,  $\theta^* = \lambda^*/\mu_{Q^*}$ , where  $\lambda^*$  is a vector of Kuhn-Tucker multipliers of (10), and  $\mu_{Q^*}$  is the multiplier in problem  $(P'_{Q^*})$ , or the marginal utility of beginning-of-period consumption.

Proof: The first-order condition for (10) is

$$\text{grad}_{Q^*} V + A(t)\lambda^*/(1+r) = 0 \quad (11)$$

where the first  $N$  components of  $\lambda^*$  are the multipliers on the constraints for long positions, and the last  $N - L$  multipliers are associated with short positions. Applying the envelope theorem to the problem  $(P'_{Q^*})$ , compute

$$\text{grad}_{Q^*} V = -\mu_{Q^*}(c^* - e)/(1+r) \quad (12)$$

where the first-order condition of  $(P'_{Q^*})$  yields

$$\mu_{Q^*} = \partial U / \partial c_0(c_0^*, c^*) > 0. \quad (13)$$

Substituting (12) into (11) yields

$$c^* - e = A(t)\lambda^* / \mu_{Q^*}. \quad (14)$$

Because  $U$  is strictly increasing, the feasibility constraint (1) must be binding. Coupled with (14), (1) forces the optimal vector of asset demands  $\theta^*$  to satisfy

$$\theta^* = \lambda^* / \mu_{Q^*} \quad (15)$$

and

$$c_0^* = P(T)\theta^* + e_0. \quad (16)$$

Analogous to the duality approach to the microeconomics of consumer demand, the researcher is free to specify any economically rational, indirect utility  $V$ . For example, those  $V$  leading to econometrically useful functional forms for  $\lambda^*$  might be useful in empirical studies.

### III. Asset Pricing

It is not surprising that few specific, asset pricing formulas can be derived in a framework as general as this. The following propositions relate minimax measures to asset prices, and will be used in the following applications section.

**PROPOSITION 3:** Assume that dual problem (10) is nondegenerate, that is, it satisfies the strict complementarity condition<sup>4</sup> that positive multipliers are associated with binding constraints and that zero multipliers are associated with nonbinding constraints. Then, there exists a partition of the set  $\{1, \dots, N-L\}$  into the cells B, S, and I satisfying

$$\begin{aligned}
 P_b + T_b^+ &= E_{Q^*}[X_b - t_b^+]/(1+r) && b \in B \\
 P_b - T_b^- &< E_{Q^*}[X_b + t_b^-]/(1+r) \\
 P_s - T_s^- &= E_{Q^*}[X_s + t_s^-]/(1+r) && s \in S \\
 P_s + T_s^+ &> E_{Q^*}[X_s - t_s^+]/(1+r) \\
 P_i + T_i^+ &> E_{Q^*}[X_i - t_i^+]/(1+r) && i \in I \\
 P_i - T_i^- &< E_{Q^*}[X_i + t_i^-]/(1+r).
 \end{aligned} \tag{17}$$

Proof: Combining Propositions 1 and 2 yields the condition

$$\lambda_i^* \lambda_{N+i}^* = 0, \quad i = 1, \dots, N - L. \tag{18}$$

With strict complementarity, B corresponds to assets held long with  $\lambda_i^* > 0$ , S corresponds to assets held short with  $\lambda_{N+i}^* > 0$ , and I corresponds to assets not held with  $\lambda_i^* = \lambda_{N+i}^* = 0$ .

The following proposition is an extension of an argument in Varian (1985).

**PROPOSITION 4:** Assume that U is additively separable in  $c_0$  and is a Von Neumann-Morganstern expected utility over c. Then

$$\begin{aligned}
 P_i + T_i^+ &\geq \frac{\text{cov}_\pi [X_i - t_i^+, Q^*/\pi] + E_\pi [X_i - t_i^+]}{1 + r} & i = 1, \dots, N - L \\
 P_i - T_i^- &\leq \frac{\text{cov}_\pi [X_i + t_i^-, Q^*/\pi] + E_\pi [X_i + t_i^-]}{1 + r} & i = 1, \dots, N - L
 \end{aligned}
 \tag{19}$$

where  $\pi$  is the trader's subjective probability measure over states, and  $\text{cov}_\pi [\cdot]$  and  $E_\pi [\cdot]$  are the covariance and expectations operators taken with respect to  $\pi$ .

Proof: Solve the first order condition for the Von Neumann-Morganstern case of  $(P'_{Q^*})$  to find an expression for  $Q^*$ . Then, follow the derivation of Varian (1985).

Unfortunately, much more restrictive assumptions are needed to go beyond this. The only strengthening of (19), attributed to Rubinstein (1976) and Breeden and Litzenberger (1978) by Varian (1985), is for the case of either no transactions costs, short sales constraints, or incomplete markets. Thus, it is possible to show that if all traders have the same subjective probability distribution  $\pi$  over states,  $Q^*/\pi$  in (19) is unique and can be replaced by some function of aggregate consumption.

With otherwise heterogeneous agents, even this modest step can't be achieved in the presence of any of the three previously mentioned market imperfections. The reason for this is the variation of  $Q^*$

across traders. Unless traders have identical preferences and endowments, it is not generally possible to guarantee that (10) will have the same solution for each trader. Of course, with complete and perfect markets, there is a unique minimax measure  $Q^*$  satisfying (5), which is the only feasible point for each trader's problem (10). But outside of this special case, restrictions on preferences and endowments must be made to ensure that  $Q^*$  is invariant across traders.

#### IV. Applications

In the spirit of Ross (1978), several applications of the theory are developed below.

##### A. Default Premia

Suppose two of the primary assets are a default-free, zero-coupon bond and a risky bond with the same face value,  $F$ . Let  $D \subset \{\omega_1, \dots, \omega_k\}$  denote a nonempty subset of default states for the risky bond. The costs associated with redeeming the risky bond in default states are assumed to be no less than those incurred in redeeming the default-free bond. Denoting the default-free bond as asset 1 and the risky bond as asset 2, the payoffs are

$$\begin{aligned} X_1(\omega_j) &= F \quad j = 1, \dots, k \\ X_2(\omega_j) &= F, \quad t_2^+(\omega_j) = t_1^+(\omega_j) \quad j \notin D \\ X_2(\omega_j) &< F, \quad t_2^+(\omega_j) \geq t_1^+(\omega_j) \quad j \in D. \end{aligned} \tag{A1}$$

Using (17), a minimax measure  $Q^*$  for a trader holding a riskless bond must satisfy

$$P_1 + T_1^+ = E_{Q^*}[X_1 - t_1^+]/(1+r) = F/(1+r) - E_{Q^*}[t_1^+]/(1+r) \tag{A2}$$

while a minimax measure  $\hat{Q}$  for a trader holding a risky bond must satisfy

$$P_2 + T_2^+ = E_{\hat{Q}}[X_2 - t_2^+]/(1+r) = E_{\hat{Q}}[x_2]/(1+r) - E_{\hat{Q}}[t_2^+]/(1+r). \quad (A3)$$

By assumption (A1), the right-hand side of (A2) is strictly greater than the right-hand side of (A3). So

$$P_1 + T_1^+ > P_2 + T_2^+ \quad (A4)$$

as long as both assets are held somewhere in the economy.

That is, the cum-transactions cost of buying a default-free bond must exceed the cum-transactions costs of buying a risky bond with the same face value. The latter's cum-transactions cost yield to maturity is thus higher, reflecting a positive default premium. Note that this must be true regardless of the relative sizes of the transactions costs  $T_1^+$  and  $T_2^+$  incurred in purchasing the bonds.

#### B. Modigliani-Miller Theorem

Suppose a firm must raise  $C$  dollars to finance its only venture, which will payoff  $f(\omega_j)$ . To finance the cost, the firm sells  $\theta_1$  bonds with face value  $F$  per bond, while the rest is financed by issuing  $\theta_2$  shares of equity. Denoting the default set  $D = \{j: f(\omega_j) < \theta_1 F - \theta_1 t_1^-(\omega_j) - \theta_2 t_2^-(\omega_j)\}$ , the payoffs are

$$X_1(\omega_j) = F,$$

$$X_2(\omega_j) = [f(\omega_j) - \theta_1 F - \theta_1 t_1^-(\omega_j) - \theta_2 t_2^-(\omega_j)]/\theta_2 \quad j \notin D \quad (B1)$$

$$X_1(\omega_j) = [f(\omega_j) - \theta_1 t_1^-(\omega_j) - \theta_2 t_2^-(\omega_j)]/\theta_1, \quad X_2(\omega_j) = 0 \quad j \in D$$

and immediately note that

$$\theta_1 X_1 + \theta_2 X_2 = f - \theta_1 t_1^- - \theta_2 t_2^- \quad (B2)$$

The financing constraint is

$$C = \theta_1 (P_1 - T_1^-) + \theta_2 (P_2 - T_2^-). \quad (B3)$$

Assume that the firm issuing debt and equity is one of the models' traders. Use (17) to compute

$$\theta_i (P_i - T_i^-) = \theta_i E_Q [X_i + t_i^-] / (1+r) \quad \text{for } i = 1, 2 \quad (B4)$$

where  $Q$  is the firm/trader's minimax measure. Add across  $i$  and use (B2) and (B3) to find

$$C = E_Q [f] / (1+r). \quad (B5)$$

The Modigliani-Miller Theorem holds because changes in  $\theta_1$  and  $\theta_2$  don't affect  $E_Q[f]$ , and therefore affect the cost of debt and equity used to finance it. Note that this derivation isn't valid in the absence of the assumption that the firm is a trader.

### C. Project Selection Criterion

Should the firm in (B) undertake the venture with payoff  $f$  costing  $C$ ? Assume that there exists a portfolio  $\hat{\theta}$  with payoff  $-f = A(t)\hat{\theta}$  after transaction costs are paid, meaning that  $-f$  is in the cone generated by the columns of  $A(t)$ . The revenue earned from portfolio short sales net of all costs is  $P(T)\hat{\theta}$ . If  $C < P(T)\hat{\theta}$ , then it is beneficial to undertake the venture because financing it by  $P(T)\hat{\theta}$  leaves  $P(T)\hat{\theta} - C > 0$  available for additional beginning-of-period consumption, at no loss ( $V - V = 0$ ) of end-of-period consumption. From (5), compute

$$\begin{aligned}
 P(T)\hat{\theta} &= \sum_i \theta_i^-(P_i - T_i^-) - \sum_i \theta_i^+(P_i + T_i^+) \\
 &\leq \sum_i \theta_i^- E_Q[X_i + t_i^-]/(1+r) - \sum_i \theta_i^+ E_Q[X_i - t_i^+]/(1+r) \quad (C1) \\
 &= E_Q\left[\sum_i \theta_i^-(X_i + t_i^-) - \sum_i \theta_i^+(X_i - t_i^+)\right]/(1+r) \\
 &= E_Q[f]/(1+r).
 \end{aligned}$$

Thus,  $C < E_Q[f]/(1+r)$  is a necessary condition for  $C < P(T)\hat{\theta}$ . But unlike the case of no transactions costs or short sales constraints, it is no longer a sufficient condition for project selection. However, if some trader holds  $\hat{\theta}$ , then (17) yields  $P(T)\hat{\theta} = E_Q[f]/(1+r)$ , so the condition is also sufficient.

#### D. A Simple Test For Welfare Improvement

In the absence of complete markets without transactions costs, it is well known that the issuance of additional assets may leave some traders worse off in the new equilibrium. Also, changes in transactions costs and/or asset prices might occur, with unexpected welfare effects.

To investigate this, examine the effect on a single trader of changes in  $P(T)$  and/or  $A(t)$ , including the addition or subtraction of assets changing their dimensions. Any such changes will cause the minimax measure solving (10) to change from  $Q^*$  to  $\hat{Q}$ . Assuming a convex, differentiable  $V$ , the change in trader welfare is

$$V(\hat{Q}) - V(Q^*) \geq (\hat{Q} - Q^*)' \text{grad}_{Q^*} V. \quad (D1)$$

Substituting (12) into (D1) and simplifying, a sufficient condition for the right-hand side of (D1) to be positive is

$$E_{Q^*}[c^* - e] > E_{\hat{Q}}[c^* - e] \quad (D2)$$

or equivalently, using (11) and (15),

$$E_{Q^*}[A(t)\theta^*] > E_{\hat{Q}}[A(t)\theta^*]. \quad (D3)$$

That is, the trader's welfare will definitely increase if the expected payoff of the pre-change portfolio exceeds its expected value using the post-change minimax measure.

This in accord with intuition. In states  $j$ , where traders' need more consumption,  $c^*(w_j) - e(w_j) > 0$ , the trader benefits when the state price,  $Q_j$ , falls. In states where the trader desires less consumption,  $c^*(w_j) - e(w_j) < 0$ , the trader would like  $Q_j$  to rise. This explains the plausibility of (D2).

By (5),  $-P(T)' \geq A(t)'Q^*/(1+r)$ . Directly compute to find the more observable, sufficient condition

$$-P(T)\theta^* > E_{\hat{Q}}[A(t)\theta^*]/(1+r) \quad (D4)$$

that is, welfare improves if the old portfolio's initial market value, after transactions costs are paid, exceeds the old portfolio's expected discounted value using the post-change minimax measure.

#### V. A Finite Horizon, Discrete Time Model

A standard,  $T$ -period, state preference approach will now be modified (see Dothan (forthcoming), Huang and Litzenberger (1988), or Duffie (1988)). There exists an event-tree which graphically depicts the resolution of uncertainty over time, starting from the knowledge of  $\Omega = \{\omega_1, \dots, \omega_k\}$  at  $t = 0$  and ending with the knowledge of the particular state occurring at time  $T$ . Such information is common to all traders, and is captured by the notion of an information structure.

**DEFINITION 7:** An information structure  $\{f_t\}$  is a sequence of partitions  $f_0, \dots, f_T$  satisfying

(i)  $f_0 = \Omega = \{\omega_1, \dots, \omega_k\}$

(ii)  $f_T = \{\{\omega_1\}, \{\omega_2\}, \dots, \{\omega_k\}\}$

(iii)  $f_{t+1}$  is finer than  $f_t$  for  $0 \leq t \leq T - 1$ .

(See Dothan (1990).) An event-tree corresponding to a particular information structure is illustrated in Figure 2.

A trader's consumption and endowment vectors at each time  $t$  must be constant in cells of  $f_t$ ; one says they are adapted to the information structure. In addition, there are  $N$  assets, whose prices, dividends, and/or other payoffs are also adapted to the information structure.

Finally, all transaction costs incurred in trading are adapted to the information structure. As such, consumption, endowment, asset payoffs, and transactions costs can be represented by vectors, with each component corresponding to a different cell in the information structure.

Denote the number of cells in  $f_t$  by  $k_t$ , and let  $\sum_{t=0}^T k_t = 1 + \sum_{t=1}^T k_t \equiv K + 1$  be the total number of cells in the information structure. Then a trader's consumption and endowment vectors  $(c_0, c)$  and  $(e_0, e)$  can each be represented by a  $K + 1$  vector, whose first component corresponds to cell 0 (at  $t=0$ ). Label other components of these vectors by numbering cells in the event-tree from top to bottom and from left to right, as illustrated in Figure 2.

Investment opportunities may also be represented as  $K + 1$  vectors. We adopt the following notation for trader cash flows arising from the six conceivable occurrences listed in Table II.

**Table II**  
Cash Flows Per Unit Asset  $i$  Traded In Cell  $j > 0$

Transaction	Cash Flow
Buy	$-[X_i(j) + T_i^+(j)]$
Short Sell	$X_i(j) - T_i^-(j)$
Sell From Holdings	$X_i(j) - t_i^+(j)$
Cover Previous Short	$-[X_i(j) + t_i^-(j)]$
Long Position Dividend	$D_i(j)$
Short Position Foregone Dividend	$-D_i(j)$

To illustrate this notation, denote two investment opportunities possible in the event-tree depicted in Figure 2:

**Investment Opportunity A:** Buy asset 1 at  $t = 1$  in cell 2, collect dividends and/or other intermediate cash flows (for example, coupons) for one period, and sell it in period 3.

**Investment Opportunity B:** Short sell asset 1 at  $t = 1$  in cell 2, forego dividends and/or other intermediate cash flows for one period, and cover the short in period 3.

Table III lists the cash flow vectors associated with the two investment opportunities.

**Table III**  
**Vector Representation of Investment Opportunities**

<u>Time</u>	<u>Cell #</u>	<u>Cash Flow "A"</u>	<u>Cash Flow "B"</u>
0	0	0	0
1	1	0	0
1	2	$-[X_1(2) + T_1^+(2)]$	$X_1(2) - T_1^-(2)$
2	3	0	0
2	4	$D_1(4)$	$-D_1(4)$
2	5	$D_1(5)$	$-D_1(5)$
3	6	$X_1(6) - t_1^+(6)$	$-[X_1(6) + t_1^-(6)]$
		⋮	⋮
3	7	$X_1(7) - t_1^+(7)$	⋮
		⋮	⋮
3	8	⋮	⋮
		⋮	⋮
3	9	⋮	⋮
		⋮	⋮
3	10	⋮	⋮
		⋮	⋮
3	11	$X_1(11) - t_1^+(11)$	$-[X_1(11) + t_1^-(11)]$

Clearly, there is a relatively large number of investment opportunities. At  $t = 0$ , there are  $T$  possible long positions for each of the  $N$  assets, or one position for each possible holding period through time  $T$ . In the absence of constraints, there would also be  $T$  analogous short positions for each asset. For each asset at any time

$0 < t \leq T$ , there are  $T - t$  long positions for each of the  $k_t$  cells: Each position corresponds to buying an asset in some possible cell with some possible holding period. In total, there are  $N \sum_{t=0}^T (T-t)k_t$  long investment opportunities that are possible, and  $(N-L) \sum_{t=0}^T (T-t)k_t$  short investment opportunities that are possible, where  $L$  is the number of assets the trader may not short.

In the example of Figure 2, suppose that there are  $N = 2$  assets and  $L = 0$  short sales constraints. Thus, there are 20 long investment opportunities: The first six are buy-and-hold strategies at  $t = 0$  (three for each asset), the next eight are four buy-and-hold strategies (two for each asset at each of the two cells at  $t=1$ ), and the last six are buy-and-hold strategies (one for each asset at each of the three cells at  $t=2$ ). Because  $L = 0$ , there are also 20 short investment opportunities, paired with the long opportunities, as in the example of Table III. The total number of investment opportunities is  $S$ .

As before, denote  $t = 0$  prices and transactions costs by the relevant case of  $P \pm T^{\bar{r}}$ .

Let  $\theta = (\theta^+, \theta^-)$  be the nonnegative  $S$ -vector of long and short positions. There are  $S^+ = N \sum_{t=0}^T (T-t)k_t$  components in  $\theta^+$  and  $S^- = (N-L) \sum_{t=0}^T (T-t)k_t$  components in  $\theta^-$ .

In the definition of the feasible consumption set (1), redefine

$$N = S^+$$

$$N - L = S^-$$

$$k = K.$$

Then (1) still describes the feasible consumption set of (2), with the S-row vector  $P(T)$  corresponding to  $t = 0$  cash flows from the S investment opportunities and the  $K \times S$  matrix  $A(t)$  formed from the S investment opportunity columns.

With this reinterpretation, condition (3) is also still valid, and the calculations preceding Theorem 1 lead to the next theorem.

**THEOREM 5.** There are no arbitrage opportunities if and only if there exists a strictly-positive probability  $Q_j$  for each node  $j$  in the event-tree,  $j = 1, \dots, K$ , such that

- (i) the cost of a long investment opportunity initiated at some node  $j$  is underestimated by the conditional, expected discounted value of its future cash inflows.
- (ii) the revenue from a short investment opportunity initiated at node  $j$  is overestimated by the conditional, expected value of its future cash outflows.
- (iii) in both (i) and (ii), the discount rate  $r(j)$  depends on  $j$ , the node where the opportunity is initiated.

To illustrate the theorem and to finish its proof, examine investment opportunity A in Table III, which is initiated at node 2 (when  $t=1$ ). The steps leading to Theorem 1 show

$$[X_1(2) + T_1^+(2)]Q_2 \geq D_1(4)Q_4 + D_1(5)Q_5 + \sum_{j=6}^{11} [X_1(j) - t_1^+(j)]Q_j. \quad (20)$$

Divide both sides of (20) by the sum of all probabilities on its right-hand side (all probabilities attached to successor nodes of  $j=2$ ). Defining  $Q_2/(Q_4+Q_5+\sum_{j=6}^{11}Q_j) = 1+r(2)$ , rewrite (20) as

$$X_1(2) + T_1^+(2) \geq E_Q[D+X_1-t_1^+ | j=2]/1+r(2) \quad (21)$$

which is the inequality promised by (i). Similarly, for investment opportunity B, find

$$X_1(2) - T_1^-(2) \leq E_Q[D+X_1+t_1^- | j=2]/1+r(2) \quad (22)$$

as promised by (ii).

Of course, for any strategy initiated at node 0 (at  $t=0$ ), the conditional expectation is just the unconditional expectation with respect to the measure  $Q$ .

Generalizations of the earlier one-period results are trivial to establish. In particular, under the relatively standard assumptions employed herein, among the measures satisfying Theorem 5 are the minimax measures  $Q^*$ , minimizing the value function  $V(Q)$  of problem (9). The optimal multiplier vector in the dual problem (10) equals the  $S$ -dimensional vector  $\theta^*$  of optimal investment opportunity shares. The optimal consumption vector  $(c_0^*, c^*)$  solves the single constraint problem  $(P'_{Q^*})$ .

## VI. Conclusion

In the presence of transactions costs, no one will take simultaneous long and short positions in any single asset priced to eliminate arbitrage opportunities. This simple proposition is the key to unlocking the power of the duality approach to portfolio choice in the presence of transactions costs. Solutions to each trader's portfolio choice problem can then be interpreted as trader-specific Martingale (i.e., state price) measures governing the evolution of asset prices net of transactions costs paid. Several applications show that these

trader-specific measures play a role analogous to the measures already developed in the absence of transactions costs.

NOTES

<sup>1</sup>For an introduction to linear valuation operators and their uses, see Ross (1976, 1978) or Garman (1980). Some references to the duality approach in continuous time are Cox and Huang (1987a, b).

<sup>2</sup>See Dothan (forthcoming) or Huang and Litzenberger (1988) for the usual development without transactions costs.

<sup>3</sup>It is interesting to note that if third parties were not involved, that is, transactions costs were always paid from the buyer to the seller or vice versa, then  $w_i$  and  $u_i = 0$ ,  $i = 1, \dots, N$ , and  $Q$  would thus be a cum-transactions cost, equivalent Martingale measure.

<sup>4</sup>See McCormick (1976, p. 33-34) for another use of strict complementarity.

Figure 1:  
No Orthogonal Arbitrage Opportunities

Subsystem A

$$\begin{pmatrix} P_1 + T_1^+ \\ P_2 - T_2^- \end{pmatrix} = \begin{pmatrix} X_1(\omega_1) - t_1^+(\omega_1) & X_1(\omega_2) - t_1^+(\omega_2) & X_1(\omega_3) - t_1^+(\omega_3) \\ X_2(\omega_1) + t_2^-(\omega_1) & X_2(\omega_2) + t_2^-(\omega_2) & X_2(\omega_3) + t_2^-(\omega_3) \\ \text{"} & \text{"} & \text{"} \\ Z_1 & Z_2 & Z_3 \end{pmatrix} \begin{pmatrix} Q_1/1+r \\ Q_2/1+r \\ Q_3/1+r \end{pmatrix} + w_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Subsystem B

$$\begin{pmatrix} P_1 - T_1^- \\ P_2 + T_2^+ \end{pmatrix} = \begin{pmatrix} X_1(\omega_1) + t_1^-(\omega_1) & X_1(\omega_2) + t_1^-(\omega_2) & X_1(\omega_3) + t_1^-(\omega_3) \\ X_2(\omega_1) - t_2^+(\omega_1) & X_2(\omega_2) - t_2^+(\omega_2) & X_2(\omega_3) - t_2^+(\omega_3) \\ \text{"} & \text{"} & \text{"} \\ Z_1 & Z_2 & Z_3 \end{pmatrix} \begin{pmatrix} Q_1/1+r \\ Q_2/1+r \\ Q_3/1+r \end{pmatrix} + w_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

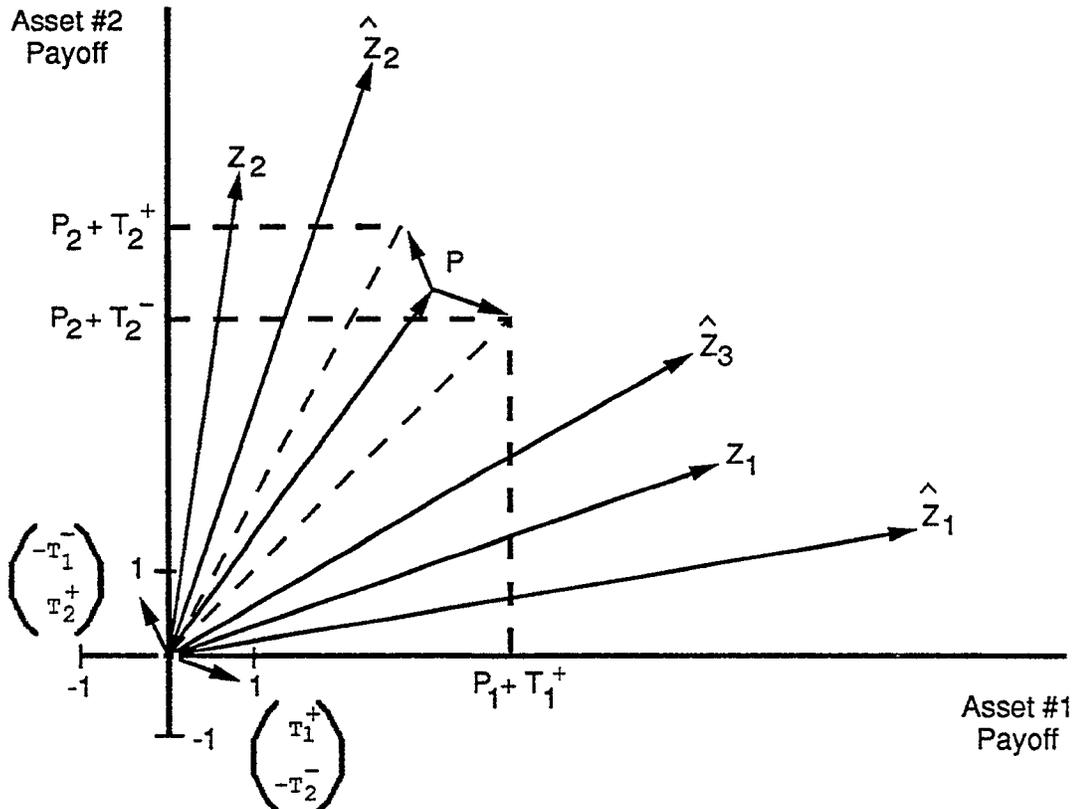
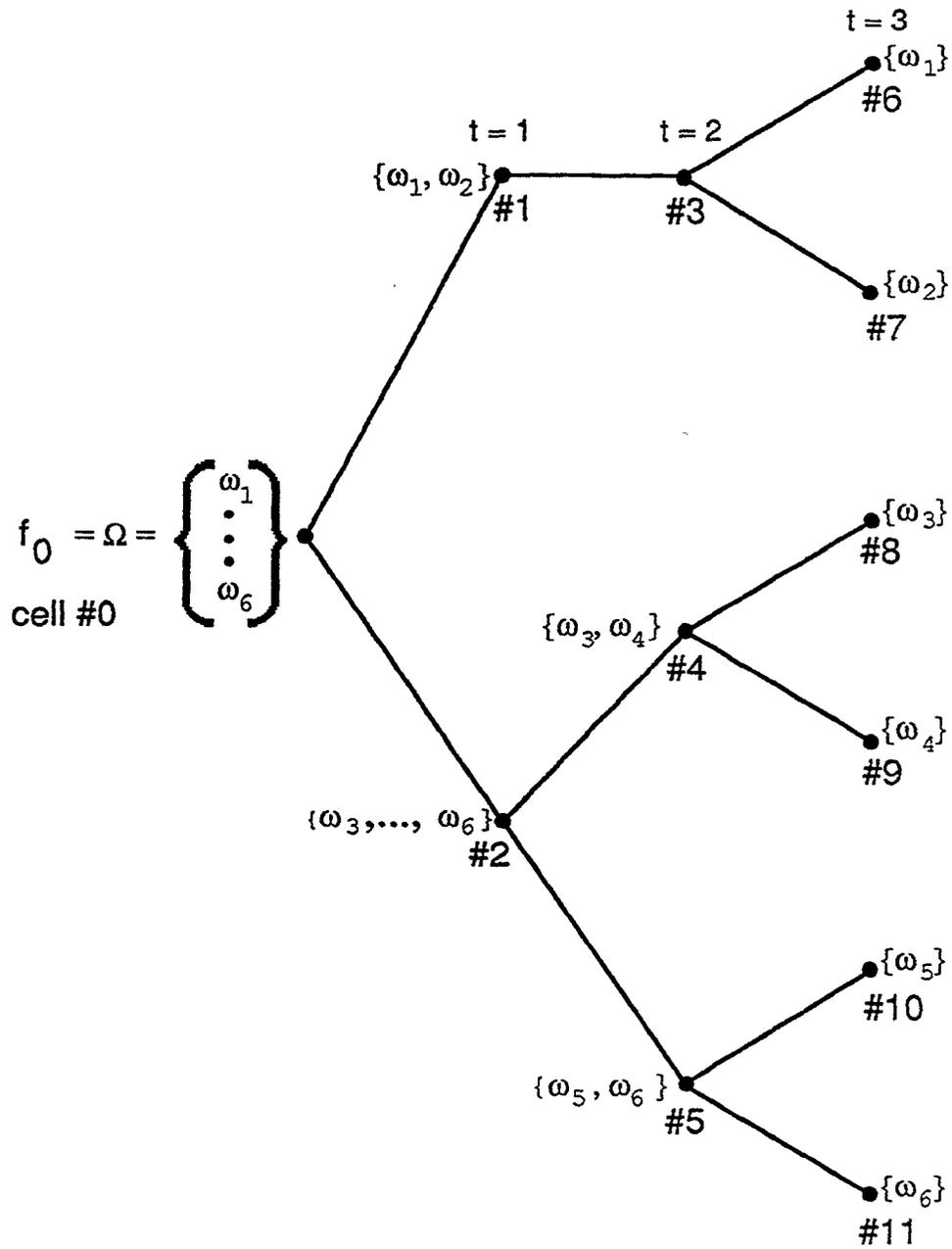


Figure 2:

An Event-Tree

(adapted from Dothan (forthcoming, Chapter 3))



$$f_0 = \{\omega_1, \dots, \omega_6\} \quad f_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \dots, \omega_6\}\}$$

$$f_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}\} \quad f_3 = \{\{\omega_1\}, \dots, \{\omega_6\}\}$$

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