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TECHNICAL APPENDIX: Sluggish responses of prices and inflation to monetary shocks in an inventory model of money demand*

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ABSTRACT

This technical appendix is designed to help the reader reproduce the results in our paper. Appendix A outlines the solution method when money growth is exogenous. Appendix B discusses the equilibrium dynamics of the model when nominal interest rates are exogenous. Appendix C discusses an algorithm for solving the model when nominal interest rates are exogenous.

*The paper, data, and codes are available at http://pages.stern.nyu.edu/~cedmond/. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

This technical appendix is designed to help the reader reproduce the results in our paper. The Matlab programs referred to in Small Caps font are available from the authors. Our programs make heavy use of the toolkit provided by Uhlig (1999) and, on occasion, make use of other functions available in Matlab's extra toolboxes (e.g., the optimization and statistics toolboxes).

Appendix A outlines the solution method when money growth is exogenous. Appendix B discusses the equilibrium dynamics of the model when nominal interest rates are exogenous. Appendix C discusses an algorithm for solving the model when nominal interest rates are exogenous.

Appendix A: Money growth exogenous

For convenience, we suppress dependence on histories. For any variable x, we denote the level of the variable at t by x_t, the steady-state level by $\bar{x}$, and the log deviation at t by $\hat{x}_t$. Therefore, $\hat{x}_t \equiv \log(x_t/\bar{x})$ and $x_t = \bar{x}\exp(\hat{x}_t)$ such that $100 \times \hat{x}_t$ is the approximate percentage deviation of x_t from its steady-state value.

The Matlab program which does most of the work when money growth is taken as exogenous is Money Exogenous. This program calls a number of function files, the purposes of which are described below.

A1. Non-stochastic steady-state

To begin with, we solve for the non-stochastic steady-state of the model. In the steady-state, all real variables are stationary and the price level is determined by:

$$\frac{P_t}{P_{t-1}} = \bar{\pi} = \bar{\mu},$$

where steady-state money growth $\bar{\mu}$ is an exogenous policy parameter. We then write the bank account constraints as:

$$\bar{c}(0) + \bar{z}(1) = \frac{\bar{z}(N) + \gamma\bar{y}}{\bar{\mu}} + \bar{x}, \tag{A1}$$

$$\bar{c}(s) + \bar{z}(s+1) = \frac{\bar{z}(s) + \gamma\bar{y}}{\bar{\mu}}, \qquad \text{for all } s = 1, 2, ..., N-2 \tag{A2}$$

$$\bar{c}(N-1) + \bar{z}(N) = \frac{\bar{z}(N-1) + \gamma\bar{y}}{\bar{\mu}}, \tag{A3}$$

where $\bar{c}(s)$ denotes steady-state real consumption, $\bar{z}(s)$ denotes steady-state end-of-period real balances, $\bar{x}$ denotes the steady-state real transfer between bank and brokerage accounts by a household who is active, γ is the paycheck parameter, $\bar{y}$ is the steady-state real aggregate endowment, and N denotes the number of periods between activity. Typically, we will have $\bar{z}(N) = 0$ and $\bar{z}(s) > 0$ for all $s < N$. In the exposition below, we impose this. In our actual computations, we check that solutions do have this form.

Steady-state transfers by the $s = 0$ households are given by:

$$\bar{x} = \left(1 - \frac{1}{\bar{\mu}}\right)\left\{\sum_{s=1}^{N-1} \bar{z}(s)\right\} + \left(1 - \frac{\gamma}{\bar{\mu}}\right) N\bar{y}. \tag{A4}$$

Since $\bar{z}(N) = 0$ and $\bar{z}(s) > 0$ for all $s < N$, there are $N-1$ binding Euler equations governing money holdings between periods. For $s = 0, 1, ..., N-2$, these Euler equations can be written:

$$\bar{c}(s+1) = \left(\frac{\beta}{\bar{\mu}}\right)^{1/\sigma} \bar{c}(s), \tag{A5}$$

where $1/\sigma > 0$ is the intertemporal elasticity of substitution and $\beta/\bar{\mu}$ is the steady-state real return to holding money between periods. Finding the non-stochastic steady-state involves solving $2N$ linear equations in $2N$ unknowns, specifically N consumptions $\bar{c}(s)$ for $s = 0, 1, ..., N-1$, plus $N-1$ real balances $\bar{z}(s)$ for $s = 1, ..., N-1$ and 1 transfer $\bar{x}$.

Given a specification of parameters, the function Steady State attempts to compute the non-stochastic steady-state by solving the $2N$ linear equations in $2N$ unknowns implied by (A1)-(A5). Notice that Steady State allows non-negativity constraints to be violated. A check function in the main program detects whether any non-negativity constraints have been violated. If this happens, another function, Resolve Kinks, is called and the steady-state is re-computed but this time allowing for the possibility that there are corner solutions.

Once a complete solution for the steady-state has been found, we can recover other interesting variables (such as $\bar{m}$ and $\bar{v}$) and then begin to set up the system of log-linear equations.

A2. System of log-linear equations

We start with the rate of inflation implied by the exchange equation:

$$\pi_t = \frac{m_{t-1}}{m_t}\mu_t = \left\{\frac{\frac{1}{N}\sum_{s=0}^{N-1}[z_{t-1}(s)] + y_{t-1}}{\frac{1}{N}\sum_{s=0}^{N-1}[z_t(s)] + y_t}\right\}\mu_t,$$

where $m_t \equiv M_t/P_t$ denotes aggregate real balances. Summing up the bank account constraints and using money-market clearing gives $m_t = \frac{1}{N}\sum_{s=0}^{N-1} z_t(s) + y_t$ as used on the right-hand side. Log-linearizing both sides of this expression, recognizing that $\bar{\pi} = \bar{\mu}$ and that steady-state velocity is $\bar{v} = \bar{y}/\bar{m}$, gives us the law of motion for inflation:

$$\hat{\pi}_t = \hat{\mu}_t - \frac{1}{N}\sum_{s=0}^{N-1} \frac{\bar{z}(s)}{\bar{m}}[\hat{z}_t(s) - \hat{z}_{t-1}(s)] - \bar{v}(\hat{y}_t - \hat{y}_{t-1}). \tag{A6}$$

Similarly, the log-linear bank account constraints corresponding to (A1)-(A3) are:

$$\bar{c}(0)\hat{c}_t(0) + \bar{z}(1)\hat{z}_t(1) = \gamma\frac{\bar{y}}{\bar{\mu}}\hat{y}_{t-1} - \gamma\frac{\bar{y}}{\bar{\mu}}\hat{\pi}_t + \bar{x}\hat{x}_t, \tag{A7}$$

$$\bar{c}(s)\hat{c}_t(s) + \bar{z}(s+1)\hat{z}_t(s+1) = \frac{\bar{z}(s)}{\bar{\mu}}\hat{z}_{t-1}(s) + \gamma\frac{\bar{y}}{\bar{\mu}}\hat{y}_{t-1} - \frac{\bar{z}(s)+\gamma\bar{y}}{\bar{\mu}}\hat{\pi}_t, \tag{A8}$$

$$\bar{c}(N-1)\hat{c}_t(N-1) = \frac{\bar{z}(N-1)}{\bar{\mu}}\hat{z}_{t-1}(N-1) + \gamma\frac{\bar{y}}{\bar{\mu}}\hat{y}_{t-1} - \frac{\bar{z}(N-1)+\gamma\bar{y}}{\bar{\mu}}\hat{\pi}_t, \tag{A9}$$

and the transfers by the active $s=0$ households are given by:

$$\bar{x}\hat{x}_t = \left(1-\frac{1}{\bar{\mu}}\right)\left\{\sum_{s=0}^{N-1}\bar{z}(s)\hat{z}_t(s)\right\} + \frac{1}{\bar{\mu}}\left\{\sum_{s=0}^{N-1}\bar{z}(s)\right\}\hat{\mu}_t + N\bar{y}\left(\hat{y}_t - \frac{\gamma}{\bar{\mu}}\hat{y}_{t-1} + \frac{\gamma}{\bar{\mu}}\hat{\pi}_t\right).$$

It is actually easier to drop the bank account constraint for $s=0$ and pin down $\hat{c}_t(0)$ with the log-linear resource constraint:

$$\frac{1}{N}\sum_{s=0}^{N-1}\bar{c}(s)\hat{c}_t(s) = \bar{y}\hat{y}_t. \tag{A10}$$

The Euler equations for money holdings can be written:

$$E_t\{\sigma[\hat{c}_{t+1}(s+1) - \hat{c}_t(s)] = E_t\{\hat{\pi}_{t+1}\}, \qquad \text{for } s = 0, 1, ..., N-2. \tag{A11}$$

In addition, we have the exogenous processes for the shocks $\hat{\mu}_t$ and $\hat{y}_t$. For the time being, assume that each of these follows a (possibly correlated) mean zero stationary homoscedastic first-order autoregressive process:

$$\hat{y}_{t+1} = \rho_y\hat{y}_t + \hat{\epsilon}_{y,t+1}, \qquad \rho_y \in [0,1), \tag{A12}$$

$$\hat{\mu}_{t+1} = \rho_\mu\hat{\mu}_t + \hat{\epsilon}_{\mu,t+1}, \qquad \rho_\mu \in [0,1), \tag{A13}$$

where for all t:

$$E_t\begin{bmatrix}\hat{\epsilon}_{y,t+1}\\ \hat{\epsilon}_{\mu,t+1}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad E_t\begin{bmatrix}\hat{\epsilon}^2_{y,t+1} & \hat{\epsilon}_{y,t+1}\hat{\epsilon}_{\mu,t+1}\\ \hat{\epsilon}_{y,t+1}\hat{\epsilon}_{\mu,t+1} & \hat{\epsilon}^2_{\mu,t+1}\end{bmatrix} = \begin{bmatrix}\sigma^2_y & \sigma_{y\mu}\\ \sigma_{y\mu} & \sigma^2_\mu\end{bmatrix}.$$

A3. Solving the model with money growth exogenous

We solve the model using the toolkit of Matlab programs written by Harald Uhlig (1999). To make use of these programs, we partition our variables into three categories: a vector of endogenous state-like variables, a vector of endogenous control-like variables, and a vector of exogenous state variables.

For the remainder of this Appendix, we suppress the endowment shocks to focus our attention on the policy innovations. (Both the money growth exogenous and the interest rate exogenous models are easily augmented with the additional equation governing the aggregate endowment process.) Also, we use the resource constraint (A10) instead of the transfer rule to reduce the number of endogenous variables by one — if needed, the path of transfers can always be computed once a solution to the rest of the model is in hand. Thus, define:

$$\hat{\mathbf{z}}_t \equiv \begin{bmatrix} \hat{z}_t(1) \\ \hat{z}_t(2) \\ \vdots \\ \hat{z}_t(N-1) \end{bmatrix}, \qquad \hat{\mathbf{c}}_t \equiv \begin{bmatrix} \hat{c}_t(0) \\ \hat{c}_t(1) \\ \vdots \\ \hat{c}_t(N-1) \end{bmatrix}, \qquad \text{and } \hat{\epsilon}_t \equiv \hat{\epsilon}_{\mu,t}.$$

Notice that $\dim(\hat{\mathbf{z}}_t) = N-1$ and $\dim(\hat{\mathbf{c}}_t) = N$. Given these definitions and the system of log-linear equations derived above, we can write out a system of the form:

$$\mathbf{0} = \mathbf{A}\hat{\mathbf{z}}_t + \mathbf{B}\hat{\mathbf{z}}_{t-1} + \mathbf{C}\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} + \mathbf{D}\hat{\mu}_t,$$

$$\mathbf{0} = E_t\left\{\mathbf{F}\hat{\mathbf{z}}_{t+1} + \mathbf{G}\hat{\mathbf{z}}_t + \mathbf{H}\hat{\mathbf{z}}_{t-1} + \mathbf{J}\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{\mathbf{c}}_{t+1} \end{bmatrix} + \mathbf{K}\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} + \mathbf{L}\hat{\mu}_{t+1} + \mathbf{M}\hat{\mu}_t\right\},$$

plus the law of motion for the exogenous money growth shocks. The coefficient matrices are constructed using the coefficients from the log-linear equations in (A6)-(A13). Note, for example, $\mathbf{A}$ is $(N+1)\times(N-1)$ and $\mathbf{F}$ is $(N-1)\times(N-1)$. The function file MATRICES MONEY constructs the matrices taking as given the structural parameters and the steady-state values of the endogenous variables.

We then try and solve for an equilibrium law of motion of the form:

$$\hat{\mathbf{z}}_t = \mathbf{P}\hat{\mathbf{z}}_{t-1} + \mathbf{Q}\hat{\mu}_t,$$

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} = \mathbf{R}\hat{\mathbf{z}}_{t-1} + \mathbf{S}\hat{\mu}_t.$$

In the main program, we invoke the command SOLVE which calls the programs written by Uhlig (1999) and attempts to solve for the matrices **P**, **Q**, **R**, and **S**. In the case where money is exogenous, we have a saddle-path solution. There are exactly as many stable eigenvalues as there are given initial conditions. Thus, we can solve for a unique equilibrium law of motion. We can then construct equilibrium processes for $\{\hat{\mathbf{z}}_t, \hat{\pi}_t, \hat{\mathbf{c}}_t\}$ given the shocks and solve for other variables, such as velocity and the nominal interest rate.

B. Equilibrium dynamics with exogenous interest rates

In this section, we prove a proposition about the model when nominal interest rates are exogenous. We posit a first-order autoregressive process for the nominal interest rate:

$$\hat{\imath}_{t+1} = \rho_i \hat{\imath}_t + \hat{\epsilon}_{i,t+1}, \qquad \rho_i \in [0, 1),$$

where, in a slight abuse of notation, the log deviation of the nominal interest rate is $\hat{\imath} \equiv \log[(1 + i_t)/(1 + \bar{\imath})]$ and where $1 + \bar{\imath} \equiv \bar{\mu}/\beta$ is the steady-state gross nominal interest rate. For the purposes of exposition, we shut down any shocks to the endowment process (thus, the only exogenous shocks come from the interest rate process which replaces the exogenous process for money growth). We then augment the system of stochastic difference equations with the Fisher relationship:

$$E_t\{\sigma[\hat{c}_{t+1}(0) - \hat{c}_t(0)]\} = \hat{r}_t \equiv \hat{\imath}_t - E_t\{\hat{\pi}_{t+1}\}, \tag{B1}$$

where $\hat{r}_t$ is the log deviation of the real interest rate.

The relevant equations of the model are the Fisher equation (B1) given above plus the resource constraint:

$$0 = \sum_{s=0}^{N-1} \bar{c}(s)\hat{c}_t(s), \tag{B2}$$

which allows us to drop the transfers $\hat{x}_t$ from the system; the bank account constraints:

$$\bar{z}(s+1)\hat{z}_t(s) + \bar{c}(s)\hat{c}_t(s) = \frac{\bar{z}(s)}{\bar{\pi}}\hat{z}_{t-1}(s) - \left(\frac{\bar{z}(s) + \gamma\bar{y}}{\bar{\pi}}\right)\hat{\pi}_t, \tag{B3}$$

for $s = 1, ..., N-1$, with the understanding that $\bar{z}(N) = 0$; the Euler equations for money holdings:

$$0 = E_t\{\sigma[\hat{c}_t(s) - \hat{c}_{t+1}(s+1)] - \hat{\pi}_{t+1}\}, \tag{B4}$$

for $s = 0, 1, ..., N-2$. We subtract the Fisher equation (B1) from the Euler equation for money holdings (B4) for $s = 0$ to get:

$$\hat{\imath}_t = E_t\{\sigma[\hat{c}_{t+1}(0) - \hat{c}_{t+1}(1)]\}, \tag{B5}$$

and we subtract (B4) for $s+1$ from (B4) for s to get:

$$0 = E_t\{[\hat{c}_t(s) - \hat{c}_t(s+1)] - [\hat{c}_{t+1}(s+1) - \hat{c}_{t+1}(s+2)]\}, \tag{B6}$$

for $s = 0, \ldots, N-3$.

We look for a solution of the form:

$$\hat{\mathbf{z}}_t = \mathbf{P}\hat{\mathbf{z}}_{t-1} + \mathbf{Q}\hat{\imath}_t,$$
$$\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} = \mathbf{R}\hat{\mathbf{z}}_{t-1} + \mathbf{S}\hat{\imath}_t,$$

and partition the $\mathbf{R}$ and $\mathbf{S}$ matrices as follows:

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} = \begin{bmatrix} \mathbf{R}_\pi \\ \mathbf{R}_c \end{bmatrix} \hat{\mathbf{z}}_{t-1} + \begin{bmatrix} \mathbf{S}_\pi \\ \mathbf{S}_c \end{bmatrix} \hat{\imath}_t.$$

And let:

$$\mathbf{R}_c = \begin{bmatrix} \mathbf{R}_c(0) \\ \mathbf{R}_c(1) \\ \vdots \\ \mathbf{R}_c(N-1) \end{bmatrix}, \qquad \mathbf{S}_c = \begin{bmatrix} \mathbf{S}_c(0) \\ \mathbf{S}_c(1) \\ \vdots \\ \mathbf{S}_c(N-1) \end{bmatrix}.$$

This allows us to write, for example:

$$\hat{c}_t(s) = \mathbf{R}_c(s)\hat{\mathbf{z}}_{t-1} + \mathbf{S}_c(s)\hat{\imath}_t,$$

where $\mathbf{R}_c(s)$ is the $1 \times (N-1)$ row vector associated with $\hat{c}_t(s)$.

Proposition. The eigenvalues of the matrix $\mathbf{P}$ are all equal to zero.

Proof. Write the Jordan canonical form of the matrix $\mathbf{P}$ as:

$$\mathbf{D} = \mathbf{V}^{-1}\mathbf{P}\mathbf{V}.$$

Since $\mathbf{P}$ is arbitrary, the columns of $\mathbf{V}$ are the eigenvectors and generalized eigenvectors of $\mathbf{P}$ and the Jordan form $\mathbf{D}$ is block diagonal. If the eigenvalues of $\mathbf{P}$ are distinct, then $\mathbf{D}$ is diagonal and the columns of $\mathbf{V}$ are the associated linearly independent eigenvectors. We are going to show that not only does $\mathbf{P}$ have no distinct eigenvalues ($\lambda(k) = \lambda$ for all $k = 1, ..., N-1$), but that they are also all zero ($\lambda = 0$).

To begin, define a rotation of $\hat{\mathbf{z}}_t$ by:

$$\hat{\zeta}_t \equiv \mathbf{V}^{-1}\hat{\mathbf{z}}_t,$$

so that:

$$\hat{\zeta}_t = \mathbf{D}\hat{\zeta}_{t-1} + \mathbf{V}^{-1}\mathbf{Q}\hat{\imath}_t,$$
$$\hat{c}_t(s) = \mathbf{R}_c(s)\mathbf{V}\hat{\zeta}_{t-1} + \mathbf{S}_c(s)\hat{\imath}_t,$$
$$\hat{\pi}_t = \mathbf{R}_\pi\mathbf{V}\hat{\zeta}_{t-1} + \mathbf{S}_\pi\hat{\imath}_t.$$

We can show that $\mathbf{P}$ has no distinct eigenvalues by showing that $\mathbf{D}$ cannot have any columns with only the diagonal element non-zero.

In the event that $\mathbf{D}$ has any columns that are all zeros except for a distinct non-zero eigenvalue $\lambda(k) \neq 0$ on the diagonal (k'th element of column k), then the initial condition $\hat{\zeta}_{t-1}(k) = 1$ and $\hat{\zeta}_{t-1}(j) = 0$ for $j \neq k$ implies $\hat{\zeta}_t(k) = \lambda(k)$ and $\hat{\zeta}_t(j) = 0$ so that $\hat{c}_t(s) = \mathbf{R}_c(s)\mathbf{V}(k)$, $\hat{\pi}_t = \mathbf{R}_\pi\mathbf{V}(k)$, $\hat{c}_{t+1}(s) = \mathbf{R}_c(s)\mathbf{V}(k)\lambda(k)$, and $\hat{\pi}_{t+1} = \mathbf{R}_\pi\mathbf{V}(k)\lambda(k)$, where $\mathbf{V}(k)$ denotes the k'th column vector of the matrix $\mathbf{V}$. We use these results to trace out the deterministic dynamics of the model (for which $\hat{\imath}_t = 0$).

Plugging $\hat{c}_t(s) = \mathbf{R}_c(s)\mathbf{V}(k)$ into the resource constraint (B2) gives:

$$0 = \sum_{s=0}^{N-1} \bar{c}(s)\mathbf{R}_c(s)\mathbf{V}(k). \tag{B7}$$

Similarly, plugging in for $\hat{c}_t(s)$ and $\hat{\pi}_t$ in the bank account constraints of equation (B3) gives:

$$\bar{z}(s+1)\mathbf{V}(s+1,k)\lambda(k) + \bar{c}(s)\mathbf{R}_c(s)\mathbf{V}(k) = \frac{\bar{z}(s)}{\bar{\pi}}\mathbf{V}(s,k) - \left(\frac{\bar{z}(s)+\gamma\bar{y}}{\bar{\pi}}\right)\mathbf{R}_\pi\mathbf{V}(i), \tag{B8}$$

with $\mathbf{V}(s,k)$ the scalar (s,k)'th element of the matrix $\mathbf{V}$. For the Fisher equation (B1) plugging in for $\hat{c}_t(0)$, $\hat{c}_{t+1}(0)$ and $\hat{\pi}_{t+1}$ leads to:

$$0 = \sigma\mathbf{R}_c(0)\mathbf{V}(k) - [\sigma\mathbf{R}_c(0) + \mathbf{R}_\pi]\mathbf{V}(k)\lambda(k), \tag{B9}$$

and for the money holdings Euler equation for $s = 0$:

$$0 = \sigma[\mathbf{R}_c(0) - \mathbf{R}_c(1)]\mathbf{V}(k)\lambda(k), \tag{B10}$$

and for $s = 0, ..., N-3$,

$$0 = [\mathbf{R}_c(s) - \mathbf{R}_c(s+1)]\mathbf{V}(k) - [\mathbf{R}_c(s+1) - \mathbf{R}_c(s+2)]\mathbf{V}(k)\lambda(k). \tag{B11}$$

From the fourth of these equations, (B10), we get

$$0 = [\mathbf{R}_c(0) - \mathbf{R}_c(1)]\mathbf{V}(k).$$

Using the fifth of these equations, (B11), for $s = 0, ..., N-3$ in order gives:

$$0 = [\mathbf{R}_c(s+1) - \mathbf{R}_c(s+2)]\mathbf{V}(k),$$

hence, for $s = 0, 1, ..., N-1$, we find $\mathbf{R}_c(s)\mathbf{V}(k)$ are all equal. Hence, from the first equation (B7) we get that they are all equal to zero. From the third of these equations, (B9), we then get that $\mathbf{R}_\pi \mathbf{V}(k) = 0$. From the second of these equations, (B8), for $s = N-1$ with $\bar{z}(N) = 0$ we get that $\mathbf{V}(N-1, k) = 0$. Iterating on these equations as s decreases to 1 gives $\mathbf{V}(s, k) = 0$ for $s = 1, \ldots, N-1$. Thus, $\mathbf{V}(k)$ is not an eigenvector.

Given that the matrix $\mathbf{D}$ cannot have any columns with all zeros except a non-zero element on the diagonal, and that (from above) the matrix $\mathbf{D}$ cannot be identically zero, we know that it has the Jordan form:

$$\underset{(N-1)\times(N-1)}{\mathbf{D}} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \lambda \end{bmatrix},$$

for some scalar λ. Since the first column is all zeros except for the λ on the diagonal, we have that $\lambda = 0$. Thus, the eigenvalues of the matrix $\mathbf{P}$ are all equal to zero. □

In short, $\mathbf{P}$ is a *nilpotent* matrix.

C. Solving the model with exogenous interest rates

For an arbitrary matrix $\mathbf{P}$ we can construct a Schur decomposition which characterizes the deterministic dynamics. The decomposition has the form:

$$\mathbf{P} = \mathbf{U}\mathbf{T}\mathbf{U}',$$

where $\mathbf{U}$ is unitary so $\mathbf{U}'\mathbf{U} = \mathbf{I}$ and $\mathbf{T}$ is upper triangular with the eigenvalues of $\mathbf{P}$ on the diagonal. We take here as given the proof that the eigenvalues of $\mathbf{P}$ are all equal to zero. Hence, the matrix $\mathbf{T}$ is strictly upper triangular with all of the elements on the main diagonal equal to zero.

Unfortunately, Matlab seems to suffer from some numerical accuracy problems; when we use its routines to decompose some arbitrary $\mathbf{P}$ matrix with all zero eigenvalues, it does not correctly find a "similar" $\mathbf{T}$ matrix that also has all zero eigenvalues. In this section, we provide a recursive algorithm that exploits the strict upper triangularity of $\mathbf{T}$ to find $\mathbf{P}$ exactly.

The equations of the model can again be written as in (B2)-(B6) and we again look for a solution of the form:

$$\hat{\mathbf{z}}_t = \mathbf{P}\hat{\mathbf{z}}_{t-1} + \mathbf{Q}\hat{\imath}_t,$$
$$\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} = \mathbf{R}\hat{\mathbf{z}}_{t-1} + \mathbf{S}\hat{\imath}_t,$$

where $\mathbf{P} = \mathbf{UTU}'$. If we define $\hat{\zeta}_t \equiv \mathbf{U}'\hat{\mathbf{z}}_t$, then we have:

$$\hat{\zeta}_t = \mathbf{T}\hat{\zeta}_{t-1} + \mathbf{U}'\mathbf{Q}\hat{\imath}_t,$$
$$\hat{\pi}_t = \mathbf{R}_\pi\mathbf{U}\hat{\zeta}_{t-1} + \mathbf{S}_\pi\hat{\imath}_t,$$
$$\hat{c}_t(s) = \mathbf{R}_c(s)\mathbf{U}\hat{\zeta}_{t-1} + \mathbf{S}_c(s)\hat{\imath}_t.$$

C1. Solving for the deterministic dynamics

The program Interest Simulations begins by computing the non-stochastic steady-state of the model just as for the exogenous money growth case. The program then calls a function, Matrices Interest, that constructs the basic coefficient matrices of the model; the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, ...$, etc, described above. We then solve for the model's deterministic dynamics by computing $\mathbf{U}$ and $\mathbf{T}$ and then backing out $\mathbf{P}$ and $\mathbf{R}$.

We proceed algorithmically, starting with the first column of $\mathbf{U}$. The program Interest Simulations calls the function Deterministic Dynamics which automates the procedure described below.

Since $\mathbf{T}$ is strictly upper triangular its elements satisfy $\mathbf{T}(k, j) = 0$ for $k \geq j$. Now start with the initial condition $\hat{\zeta}_{t-1}(1) = 1$ and $\hat{\zeta}_{t-1}(j) = 0$ for $j > 1$. This initial condition implies $\hat{\zeta}_t(1) = \mathbf{T}(1, 1) = 0$ and $\hat{\zeta}_t(j) = 0$ for $j > 1$. Hence $\hat{c}_t(s) = \mathbf{R}_c(s)\mathbf{U}(1)$, $\hat{\pi}_t = \mathbf{R}_\pi\mathbf{U}(1)$, $\hat{c}_{t+1}(s) = 0$, $\hat{z}_t(s) = 0$, and $\hat{\pi}_{t+1} = 0$, where $\mathbf{U}(1)$ is the first column of $\mathbf{U}$. Our equations for the deterministic dynamics parallel equations (B7)-(B11). Plugging $\hat{c}_t(s) = \mathbf{R}_c(s)\mathbf{U}(1)$ into the resource constraint:

$$0 = \sum_{s=0}^{N-1} \bar{c}(s)\mathbf{R}_c(s)\mathbf{U}(1). \tag{C1}$$

Similarly, for the cash flow constraints:

$$\bar{c}(s)\mathbf{R}_c(s)\mathbf{U}(1) = \frac{\bar{z}(s)}{\bar{\pi}}\mathbf{U}(s, 1) - \left(\frac{\bar{z}(s) + \gamma\bar{y}}{\bar{\pi}}\right)\mathbf{R}_\pi\mathbf{U}(1), \tag{C2}$$

and for the cash Euler equation for $s = 0$ we have:

$$0 = \sigma\mathbf{R}_c(0)\mathbf{U}(1), \tag{C3}$$

and for $s = 0, \ldots, N-3$ we have:

$$0 = [\mathbf{R}_c(s) - \mathbf{R}_c(s+1)]\mathbf{U}(1). \tag{C4}$$

The last two of these equations, (C3)-(C4), plus the resource constraint (C1) give $\mathbf{R}_c(s)\mathbf{U}(1) = 0$. Thus, (C2) implies that the elements of $\mathbf{U}(1)$ satisfy:

$$\mathbf{U}(s,1) \propto \frac{\bar{z}(s) + \gamma\bar{y}}{\bar{z}(s)},$$

for some constant of proportionality. This constant is pinned down by the fact that $\mathbf{U}$ is unitary. Specifically, since $\mathbf{U}$ is unitary:

$$\sum_{s=1}^{N-1} \mathbf{U}(s,1)^2 = 1.$$

Hence, for $s = 2, \ldots, N-1$ we have:

$$\mathbf{U}(s,1) = \left(\frac{\bar{z}(s) + \gamma\bar{y}}{\bar{z}(s)}\right)\left(\frac{\bar{z}(1)}{\bar{z}(1) + \gamma\bar{y}}\right)\mathbf{U}(1,1),$$

and

$$\mathbf{U}(1,1) = \left[1 + \sum_{s=2}^{N-1}\left(\frac{\bar{z}(s) + \gamma\bar{y}}{\bar{z}(s)}\right)^2\left(\frac{\bar{z}(1)}{\bar{z}(1) + \gamma\bar{y}}\right)^2\right]^{-1/2}.$$

After solving for all the elements of $\mathbf{U}(1)$ in this fashion, we can compute the implied value of $\mathbf{R}_\pi\mathbf{U}(1) \neq 0$.

With the first column of $\mathbf{U}$ solved for, we can proceed to the second. To do this, consider what happens when we start with the initial condition $\hat{\zeta}_{t-1}(2) = 1$ and $\hat{\zeta}_{t-1}(j) = 0$ for $j \neq 2$. Then, $\hat{\zeta}_t(1) = \mathbf{T}(1,2)$, $\hat{\zeta}_t(2) = \mathbf{T}(2,2) = 0$, and $\hat{\zeta}_t(j) = 0$. Hence, $\hat{c}_t(s) = \mathbf{R}_c(s)\mathbf{U}(2)$, $\hat{\pi}_t = \mathbf{R}_\pi\mathbf{U}(2)$, $\hat{z}_t(s) = \mathbf{U}(s,1)\mathbf{T}(1,2)$, $\hat{c}_{t+1}(s) = \mathbf{R}_c(s)\mathbf{U}(1)\mathbf{T}(1,2) = 0$ (from the result above that $\mathbf{R}_c(s)\mathbf{U}(1) = 0$), and $\hat{\pi}_{t+1} = \mathbf{R}_\pi\mathbf{U}(1)\mathbf{T}(1,2)$ (recall that $\mathbf{R}_\pi\mathbf{U}(1) \neq 0$ is computed above).

We can now run through the same sort of substitutions that gave us (C1)-(C4) to get the system of equations:

$$0 = \sum_{s=0}^{N-1} \bar{c}(s)\mathbf{R}_c(s)\mathbf{U}(2),$$

$$\bar{c}(s)\mathbf{R}_c(s)\mathbf{U}(2) + \bar{z}(s+1)\mathbf{U}(s+1,1)\mathbf{T}(1,2) = \frac{\bar{z}(s)}{\bar{\pi}}\mathbf{U}(s,2) - \left(\frac{\bar{z}(s) + \gamma\bar{y}}{\bar{\pi}}\right)\mathbf{R}_\pi\mathbf{U}(2),$$

$$0 = \sigma \mathbf{R}_c(0)\mathbf{U}(2) - \mathbf{R}_\pi \mathbf{U}(1)\mathbf{T}(1,2),$$

$$0 = [\mathbf{R}_c(s) - \mathbf{R}_c(s+1)]\mathbf{U}(2).$$

We will have to use properties of the unitary matrix $\mathbf{U}$ — that is, the fact $\mathbf{U}(2)'\mathbf{U}(2) = 1$ and the columns $\mathbf{U}(1)$ and $\mathbf{U}(2)$ are orthogonal, $\mathbf{U}(1)'\mathbf{U}(2) = 0$ — to solve out for the elements of $\mathbf{U}(2)$ *and* the element $\mathbf{T}(1,2)$ of the upper strictly triangular matrix $\mathbf{T}$. (When we were solving for the elements of $\mathbf{U}(1)$, the element $\mathbf{T}(1,1)$ was trivially zero, just as here the element $\mathbf{T}(2,2)$ is trivially zero.) Notice that every equation except $\mathbf{U}(2)'\mathbf{U}(2) = 1$ is linear.

With the first and second columns of $\mathbf{U}$ and the $(1,2)$ element of $\mathbf{T}$ solved for, we can now proceed to the third column of $\mathbf{U}$ and the $(1,3)$ and $(2,3)$ elements of $\mathbf{T}$. Notice that at each step, we will be adding one more unknown element of $\mathbf{T}$ but will also be adding one more orthogonality condition.

To see this pattern, let's work through the process one more time. Consider what happens when we start with the initial condition $\hat{\zeta}_{t-1}(3) = 1$ and $\hat{\zeta}_{t-1}(j) = 0$ for $j \neq 2$. Then, $\hat{\zeta}_t(1) = \mathbf{T}(1,3)$, $\hat{\zeta}_t(2) = \mathbf{T}(2,3)$, $\hat{\zeta}_t(3) = \mathbf{T}(3,3) = 0$, and $\hat{\zeta}_t(j) = 0$ otherwise. Hence $\hat{c}_t(s) = \mathbf{R}_c(s)\mathbf{U}(3)$, $\hat{\pi}_t = \mathbf{R}_\pi \mathbf{U}(3)$, $\hat{z}_t(s) = \mathbf{U}(s,1)\mathbf{T}(1,3) + \mathbf{U}(s,2)\mathbf{T}(2,3)$, $\hat{c}_{t+1}(s) = \mathbf{R}_c(s)\mathbf{U}(1)\mathbf{T}(1,3) + \mathbf{R}_c(s)\mathbf{U}(2)\mathbf{T}(2,3) = \mathbf{R}_c(s)\mathbf{U}(2)\mathbf{T}(2,3)$ (from the result above that $\mathbf{R}_c(s)\mathbf{U}(1) = 0$), and $\hat{\pi}_{t+1} = \mathbf{R}_\pi \mathbf{U}(1)\mathbf{T}(1,3) + \mathbf{R}_\pi \mathbf{U}(2)\mathbf{T}(2,3)$.

Paralleling the results above, our system of equations for the deterministic dynamics becomes:

$$0 = \sum_{s=0}^{N-1} \bar{c}(s)\mathbf{R}_c(s)\mathbf{U}(3), \tag{C5}$$

$$\begin{aligned} &\bar{c}(s)\mathbf{R}_c(s)\mathbf{U}(3) + \bar{z}(s+1)[\mathbf{U}(s+1,1)\mathbf{T}(1,3) + \mathbf{U}(s+1,2)\mathbf{T}(2,3)] \\ &= \frac{\bar{z}(s)}{\bar{\pi}}\mathbf{U}(s,3) - \left(\frac{\bar{z}(s) + \gamma\bar{y}}{\bar{\pi}}\right)\mathbf{R}_\pi \mathbf{U}(3), \end{aligned} \tag{C6}$$

$$0 = \sigma \mathbf{R}_c(0)\mathbf{U}(3) - \sigma \mathbf{R}_c(1)\mathbf{U}(2)\mathbf{T}(2,3) - \mathbf{R}_\pi[\mathbf{U}(1)\mathbf{T}(1,3) + \mathbf{U}(2)\mathbf{T}(2,3)], \tag{C7}$$

$$0 = [\mathbf{R}_c(s) - \mathbf{R}_c(s+1)]\mathbf{U}(3) - [\mathbf{R}_c(s+1) - \mathbf{R}_c(s+2)]\mathbf{U}(2)\mathbf{T}(2,3). \tag{C8}$$

Note that the interest rate Euler equation,

$$0 = [\mathbf{R}_c(0) - \mathbf{R}_c(1)]\mathbf{U}(2)\mathbf{T}(2,3),$$

is satisfied automatically (from the previous calculations for column $\mathbf{U}(2)$, we found that $0 = [\mathbf{R}_c(0) - \mathbf{R}_c(1)]\mathbf{U}(2)$), so we drop it. Again, we use the extra conditions $\mathbf{U}(3)'\mathbf{U}(3) = 1$ and $\mathbf{U}(1)'\mathbf{U}(3) = 0$ and $\mathbf{U}(2)'\mathbf{U}(3) = 0$ to pin down the $(1,3)$ and $(2,3)$ elements of $\mathbf{T}$.

Aside from equations of the form $\mathbf{U}(3)'\mathbf{U}(3) = 1$ we are always solving a system of linear homogeneous equations. For example, in the case of solving for the third column of $\mathbf{U}$ and the $(1,3)$ and $(2,3)$ elements of $\mathbf{T}$, we can write the problem as one of solving:

$$\mathcal{A}_3'\mathbf{w}_3 = 0,$$

where the unknowns are:

$$\mathbf{w}_3 = [\mathbf{R}_c(0)\mathbf{U}(3), \ldots, \mathbf{R}_c(N-1)\mathbf{U}(3), \mathbf{R}_\pi\mathbf{U}(3), \mathbf{T}(1,3), \mathbf{T}(2,3), \mathbf{U}(1,3), \ldots, \mathbf{U}(N-1,3)]',$$

and where the coefficient matrix $\mathcal{A}_3$ is constructed from the equations (C5)-(C8) and the orthogonality conditions. Note that the coefficient matrix makes use of the previous solutions for $\mathbf{U}(1)$, $\mathbf{U}(2)$, and the element $\mathbf{T}(1,2)$. Computing the (one-dimensional) null-space of the matrix $\mathcal{A}_3'$ [e.g., in Matlab, computing null($\mathcal{A}_3'$)] delivers a vector that satisfies $\mathcal{A}_3'\mathbf{w}_3 = 0$. We then scale this vector by using the condition $\mathbf{U}(3)'\mathbf{U}(3) = 1$ and move on to solve for $\mathbf{U}(4)$ and the $(1,4)$ and $(2,4)$ and $(3,4)$ elements of $\mathbf{T}$. Now we will face the problem of solving:

$$\mathcal{A}_4'\mathbf{w}_4 = 0,$$

where:

$$\mathbf{w}_4 = [\mathbf{R}_c(0)\mathbf{U}(4), \ldots, \mathbf{R}_c(N-1)\mathbf{U}(4), \mathbf{R}_\pi\mathbf{U}(3), \mathbf{T}(1,4), \mathbf{T}(2,4), \mathbf{T}(3,4), \mathbf{U}(1,4), \ldots, \mathbf{U}(N-1,4)]'.$$

The dimension of the unknown vector has increased by one, but there is one more equation (one more orthogonality condition) to satisfy too, so $\mathcal{A}_4$ is conformably larger. Of course, the solution to this null-space problem has to be scaled so that $\mathbf{U}(4)'\mathbf{U}(4) = 1$.

The function DETERMINISTIC DYNAMICS called by INTEREST SIMULATIONS carries out this procedure. For each $k = 1, ..., N-1$ the column $\mathbf{U}(k)$ and associated elements of $\mathbf{T}$ need to be found. The program constructs the appropriate $\mathcal{A}_k$ matrix using the equations for the deterministic dynamics and the orthogonality conditions and then solves the null-space problem and scales the solution vector $\mathbf{w}_k$ such that $\mathbf{U}(k)'\mathbf{U}(k) = 1$. Proceeding in the same manner $N-1$ times delivers a solution for $\mathbf{U}$ and $\mathbf{T}$ and the products $\mathbf{R}_c(s)\mathbf{U}(k)$, $\mathbf{R}_\pi\mathbf{U}(k)$. The desired $(N-1)\times(N-1)$ matrix is then $\mathbf{P} = \mathbf{UTU}'$. Since $\mathbf{U}$ is of full rank, the vectors $\mathbf{R}_c(s)$ and $\mathbf{R}_\pi$ can be recovered from the inverse of $\mathbf{U} = \mathbf{U}'$. This completes the solution for the deterministic dynamics. We then have to solve for the response of the model to shocks; that is, we have to solve for the additional matrices $\mathbf{Q}$ and $\mathbf{S}$.

C2. Solving for the response to interest rate shocks

The function DETERMINISTIC DYNAMICS returns the matrices $\mathbf{U}$ and $\mathbf{T}$ and the implied $\mathbf{P}$ and $\mathbf{R}$ given the structural parameters and non-stochastic steady-state of the model. Now, with the other coefficient matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, ...$, etc set up by the function MATRICES INTEREST, we can easily compute the additional matrices $\mathbf{Q}$ and $\mathbf{S}$. The program HARALD is a modification of the program written by Uhlig (1999) and uses the supplied matrix $\mathbf{P}$ to find $\mathbf{Q}$ and $\mathbf{S}$.

C3. Solving for the response to unexpected money growth shocks

In our model, there are many stochastic processes for money, all consistent with the same exogenously specified path for nominal interest rates in equilibrium. In the experiments that we carry out, we choose one of the many stochastic processes for the gross growth rate of the money supply that result in an equilibrium in which the short-term nominal interest rate follows a first-order autoregressive process similar to that estimated for the monthly Federal Funds Rate. The process for money growth that we choose has the property that a shock to the nominal interest rate, on impact, is associated with no movement in the current price level. In order to implement this, we solve for the responses of the endogenous variables to an unexpected money growth shock that leaves the path of the nominal interest rate unchanged. We then combine an appropriately scaled version of this money growth shock to our nominal interest rate shock to solve for the desired process for money growth.

To implement this procedure, we first denote by $\mathbf{Q}_i$ and $\mathbf{S}_i$ the response matrices found using the program HARALD. We are interested in finding a linear solution of the form:

$$\hat{\mathbf{z}}_t = \mathbf{P}\hat{\mathbf{z}}_{t-1} + \mathbf{Q}_i \hat{\imath}_t + \mathbf{Q}_\mu \hat{\epsilon}_{\mu,t},$$

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} = \mathbf{R}\hat{\mathbf{z}}_{t-1} + \mathbf{S}_i \hat{\imath}_t + \mathbf{S}_\mu \hat{\epsilon}_{\mu,t},$$

where $\hat{\epsilon}_{\mu,t}$ denotes the additional (IID) shock. The system of equations that must hold in equilibrium are:

$$\mathbf{0} = \mathbf{A}\hat{\mathbf{z}}_t + \mathbf{B}\hat{\mathbf{z}}_{t-1} + \mathbf{C}\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} + \mathbf{D}\hat{\imath}_t,$$

$$\mathbf{0} = E_t\left\{\mathbf{F}\hat{\mathbf{z}}_{t+1} + \mathbf{G}\hat{\mathbf{z}}_t + \mathbf{H}\hat{\mathbf{z}}_{t-1} + \mathbf{J}\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{\mathbf{c}}_{t+1} \end{bmatrix} + \mathbf{K}\begin{bmatrix} \hat{\pi}_t \\ \hat{\mathbf{c}}_t \end{bmatrix} + \mathbf{L}\hat{\imath}_{t+1} + \mathbf{M}\hat{\imath}_t\right\},$$

plus the law of motion for the exogenous interest rate shock. Plugging in the equilibrium laws of motion reduces this to:

$$\begin{aligned}
\mathbf{0} &= (\mathbf{AP}+\mathbf{B}+\mathbf{CR})\hat{\mathbf{z}}_{t-1} + (\mathbf{AQ}_i + \mathbf{CS}_i + \mathbf{D})\hat{\imath}_t + (\mathbf{AQ}_\mu + \mathbf{CS}_\mu)\hat{\epsilon}_{\mu,t}, \\
\mathbf{0} &= (\mathbf{FP}^2 + \mathbf{GP} + \mathbf{H} + \mathbf{JRP} + \mathbf{KR})\hat{\mathbf{z}}_{t-1} \\
&\quad + (\mathbf{FPQ}_i + \mathbf{FQ}_i\mathbf{N} + \mathbf{GQ}_i + \mathbf{JRQ}_i + \mathbf{JQ}_i\mathbf{N} + \mathbf{KS}_i + \mathbf{LN} + \mathbf{M})\hat{\imath}_t \\
&\quad + (\mathbf{FPQ}_\mu + \mathbf{GQ}_\mu + \mathbf{JRQ}_\mu + \mathbf{KS}_\mu)\hat{\epsilon}_{\mu,t},
\end{aligned}$$

where $\mathbf{N}$ is the coefficient matrix for the law of motion for the exogenous interest rate shock. Since these equalities always have to hold, we can write a matrix problem for the two remaining vectors of unknowns, $\mathbf{Q}_\mu$ and $\mathbf{S}_\mu$,

$$\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{FP}+\mathbf{G}+\mathbf{JR} & \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_\mu \\ \mathbf{S}_\mu \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{C9}$$

We can solve for the vector of unknowns by finding the (one-dimensional) null-space of the matrix of coefficients in (C9). This leaves us with a normalization to pin down the elements of $\mathbf{Q}_\mu$ and $\mathbf{S}_\mu$. We use the normalization that the response of money growth $\hat{\mu}_t$ to the shock $\hat{\epsilon}_{\mu,t}$ is one.

References

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