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Abstract

In the data sovereign default is always partial and varies in its duration. Debt levels during default episodes initially increase and do not experience reductions upon resolution. This paper presents a theory of sovereign default that replicates these properties, which are absent in standard sovereign default theory. Partial default is a flexible way to raise funds as the sovereign chooses its intensity and duration. Partial default is also costly because it amplifies debt crises as the defaulted debt accumulates and interest rate spreads increase. This theory is capable of rationalizing the large heterogeneity in partial default, its comovements with spreads, debt levels, and output, and the dynamics of debt during default episodes. In our theory, as in the data, debt grows during default episodes, and large defaults are longer, and associated with higher interest rate spreads, higher debt levels, and deeper recessions.

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1 Introduction

Understanding sovereign debt crises and defaults is central for emerging markets as default risk is a main driver of fluctuations in capital flows and economic activity. Sovereigns in these countries frequently miss payments on their debt, but almost always by only a fraction of the amount due. During these *partial defaults*, sovereigns continue to pay some of the debt, continue to borrow at higher-than-normal interest rates, and accumulate the defaulted debt as arrears until resolution. Some of these partial defaults take the form of protracted episodes associated with rising default and debt, and deeper recessions. The standard theory of sovereign default, as in the influential framework in Eaton and Gersovitz (1981), assumes that default is complete, rather than partial, and that it is followed by a period of exclusion without any borrowing, further default, or accumulation of arrears taking place. Default in that paradigm leads to a new start with reduced debt. In this paper, we propose a theory of partial default more in accord with the evidence, where partial default often leads to further defaults and to debt increases.

A central idea in our theory is that partial default is an alternative way to effectively borrow and inter-temporally transfer resources. As with standard borrowing through markets, partially defaulting raises current resources and increases future liabilities as most of the defaulted debt accumulates. Unlike standard borrowing, however, partial default does not have the acquiescence of the lenders and is associated with future resource costs. A main implication of our theory is that partial default is an amplifying force for debt crises. A country that misses payments will be in worse shape going forward because it will experience rising debt as the defaulted payments accumulate and any new borrowing occurs at high interest rates. This theory is capable of rationalizing the large heterogeneity in partial default, its comovements with economic outcomes, and rising debt during default episodes. In our theory, as in the data, large defaults are associated with higher interest rate spreads, higher debt levels, deeper recessions, and longer default episodes.

Our analysis puts at center stage partial default, which we define as the fraction in arrears of current debt payments due by sovereigns. We document the properties of partial default using data for 38 emerging markets since 1970. We find that sovereigns default often, with varying intensity and duration, missing their debt payments about one third of the time. During these events, often only a small fraction of the payment is missed; in about half of these events, less than 30% of the promised payments are missed. Default episodes vary in duration, with many lasting less than 2 years but some on occasion lasting much longer, over 30 years. Debt-to-output ratios feature a hump-shaped pattern during default episodes, and these episodes are typically not associated with a net reduction in debt. As a default episode starts, debt-to-output rises from 35% to 40%, in the middle of the episode debt reaches an even larger ratio of 47%, and debt falls to 36% towards the end of the default episode. We also find that partial default is systematically correlated with other outcomes of the debt crisis. During small defaults (those belonging in the bottom quartile of the distribution) sovereign governments miss on average 3% of payments, interest rate spreads are about 7%, debt-to-output ratios are about 37%, and output is at trend. During large defaults (those in the top quartile) in contrast, governments miss about 82% of their payments, face interest rate spreads of about 15%, debt-to-output ratios are 63%, and output is about 6% below trend. Large defaults are also associated with longer default episodes.[1]

Our framework consists of a sovereign government in a small open economy that borrows long-term bonds, can choose to partially default on its debt payments, and faces a stochastic stream of income. Partial default is a flexible yet expensive way to raise funds. Partial default is flexible because the government can choose when to start the default episode, the intensity of partial default every period, and when to end the default episode. The defaulted debt is accumulated at a rate that depends on a recovery factor parameter. This accumulated defaulted debt comes due in the future. Yet default is costly because it induces future resource costs that depend on the intensity of the default. The government can also raise resources by borrowing through markets at interest rates that compensate for potential default losses. Borrowing is always possible, even during default episodes. Expected default losses, however, are more elevated during default episodes, which increases interest rates and can deter borrowing altogether. This framework implies varying debt haircuts and maturity extensions that depend on the endogenous default episode length and intensity of the default.

The sovereign effectively faces a portfolio choice to intertemporally transfer resources and smooth consumption because it chooses how much to borrow and partially default every period. We characterize theoretically the trade-offs involved in these two decisions. The gains from borrowing are the increases in consumption net of the reduction in bond prices due to higher default, while its costs consist of the future coupon payments evaluated at future prices. Due to default risk, the gains from borrowing are capped by a Laffer Curve typical of sovereign default models. The benefits to partially defaulting increase linearly with the amount defaulted on and is not subject to a Laffer Curve, but is capped by the total level of debt due. The costs from partially defaulting include also the future payments from the accumulation of the defaulted debt, also evaluated at future prices, plus the future resource costs resulting from default.

[1] Reinhart and Rogoff (2009) and Cruces and Trebesch (2013) document that sovereign defaults throughout history have been partial in that creditors receive sizable debt recoveries after default episodes.

The portfolio shares of borrowing and partial defaulting result from weighing these marginal gains and costs. When income is high and debt is low borrowing is preferred to partial default because these are times when borrowing rates are low. As income falls and debt rises, borrowing rates increase and a mix of borrowing and partial default becomes optimal. When debt rises and income falls further, default is complete and borrowing collapses because borrowing rates are extremely high. Partial default increases borrowing rates because defaulted debt piles up as debt in arrears and future resources are diminished. The bond price schedule reflects these default decisions and shapes the dynamics of the model.

We estimate the model by targeting moments that summarize the empirical distribution of partial default and the behavior of interest rate spreads, debt, and output in emerging markets. The estimated parameters include the ones characterizing the default cost function as well as the debt recovery factor. We show that our over-identified model matches the target moments well and delivers the empirical distribution of debt and partial default, including the presence of many small defaults.

The parameterized model contains additional implications, consistent with the data, for the correlations of partial default with other variables, as well as for the length of default episodes, and debt haircuts and maturity extensions resulting from default episodes. In the model and the data, small defaults tend to be shorter and are associated with lower interest rate spreads, lower levels of debt, and smaller recessions, while large partial defaults are longer and associated with higher interest rate spreads, higher levels of debt, and deeper recessions. Our model matches the data in that it delivers a hump-shaped pattern for debt to output during episodes. During default episodes, debt continues to rise in the model, and default episodes do not result in a net reduction of the debt burden. Default episodes in our model also result in sizable debt haircuts and maturity extensions with magnitudes similar to those in the data.

To analyze the mechanisms behind the results, we study the equilibrium policy functions and the impulse responses to income shocks. When income falls, the bond price schedule tightens, making it more costly to roll over the debt. To alleviate the consumption decline, the government not only increases borrowing at higher interest rates but also partially defaults. The rise in borrowing and partial default increases future debt and creates the dynamic amplification of the shocks. As debt remains elevated, interest rates and partial default remain persistently high, even as the shock dissipates. These impulse responses show that recessions in our model have long-lasting effects on the functioning of international financial markets.

We also analyze the dynamics of default episodes. The severity of the debt crisis depends on the size and persistence of the recession and on the accumulation of debt from past borrowing and partial default. Default episodes generally start with a small partial default that occurs after a moderate downturn when debt is high enough. These small defaults are resolved quickly if the recession is temporary. When the recession is deeper and more persistent, however, the small but rising partial defaults amplify the debt crisis by inducing a rapid increase in debt at increasingly higher interest rates. Small partial defaults coupled with persistent recessions are the culprit of prolonged debt crises. The episode ends when output recovers sufficiently to repay the accumulated debt. Larger debt crises require stronger output recoveries for the resolution of default as they feature larger accumulated debt from past borrowing and partial defaults.

Default episodes are associated with debt haircuts and maturity extensions that depend on the intensity of partial default and also on the length of the episode. Longer episodes have larger haircuts because the default discount is compounded with repeated partial defaults. The model features maturity extensions because defaults occur on payments due, which are inherently short term, and the new liabilities are in the form of long-term bonds. The path and intensity of partial default also shape maturity extensions because the overall maturity of the defaulted debt depends on whether larger defaults occur early or late during the episode.

We perform two counterfactual experiments that relate to discussions around resolution mechanisms for sovereign defaults. In the first counterfactual, we eliminate the possibility of borrowing during default episodes and argue that, in our model, this policy is similar to adding more stringent pari passu clauses to the bond contracts. This application is motivated by the fact that in the baseline model borrowing during default episodes implies differential haircuts to lenders because bonds issued later in the episode experience fewer periods with partial default compared to bonds issued earlier. Such differential treatment violates pari passu clauses. In the second counterfactual, we decrease the debt recovery factor on defaulted debt. This counterfactual implements higher debt relief policies within our model, as in the Highly Indebted Poor Countries initiative proposed by the International Monetary Fund and the World Bank. Our analysis of these counterfactuals suggests that pari passu clauses lead to lower default frequency, shorter default episodes, and smaller debt haircuts from defaults. Debt relief initiatives that increase haircuts, reduce the incidence of large defaults, reduce debt levels, but also decrease debt sustainability.

Finally, we contrast our partial default model directly to the reference full default model of Arellano (2008) and Chatterjee and Eyigungor (2012). We highlight two striking differences across models. First, the reference full default model is silent about the behavior of debt and spreads during periods of default, while our model accounts for the substantial observed variation and comovements in these variables. Second, the reference full default model features a fresh start with complete debt discharge after default and a consumption and borrowing burst as soon as the default episode ends. In contrast, our model matches the data in that, without a fresh start, debt does not decline in default episodes.

Related Literature. This paper introduces the study of partial default in the literature on sovereign debt and embraces the lack of commitment inherent in sovereign contracts. In addressing new aspects of the evidence, it departs in essential ways from work in the tradition of Eaton and Gersovitz (1981). In the standard short-term debt model of Aguiar and Gopinath (2006) or Arellano (2008), default can only be on the total amount of debt, is followed by a period of exclusion, and ends in a full discharge of debt.[2] In these papers, a default episode, spanning the moment of default and the specified moment of redemption, precludes any continuation of debt repayments on any coupons, further borrowing, or accumulation of arrears. During default episodes, the sanction of complete exclusion from financial markets is assumed and rationalized by assumed commitment from lenders to collude in not lending to the sovereign. By design, such theory cannot study the partial default episodes that we measure and focus here.

Following the work on debt restructuring, as in Bulow and Rogoff (1989), an active recent strand of research explores the implications of less blunt resolutions of defaults in quantitative sovereign debt models. Yue (2010) and Benjamin and Wright (2013) develop models for debt renegotiations for resolving defaults. Yue (2010) models renegotiation via Nash bargaining, while Benjamin and Wright (2013) model renegotiation with a dynamic alternating offers framework. Recent work in Mihalache (2017) and Dvorkin et al. (2018) extend the renegotiation framework to long-term bonds and endogenous maturity choice. Here, as in our work, the length of the default episode, the debt haircut, and the maturity extensions after default are endogenous. Default episodes in these models end with haircuts and delays that depend on bargaining forces, and in the work of Mihalache (2017) and Dvorkin et al. (2018), they also end with an extension in the maturity of debt. Importantly in these papers, just as in the earlier papers and in contrast to our work and actual data, default episodes are an impasse state without debt repayments, borrowing, or accumulation of the defaulted debt and lead to a new start with reduced debt.

In some recent work, Gordon and Guerron-Quintana (2018) and Erce and Mallucci (2018) also

[2] These features are also true of extensions of the model to long-term debt, including Chatterjee and Eyigungor (2012) or Hatchondo, Martinez, and Sosa-Padilla (2016).

study heterogeneity in defaults. Gordon and Guerron-Quintana (2018) consider a model in which when the sovereign defaults, it does so only on long-term debt coupons due in the present, and not on the future coupons. Erce and Mallucci (2018) consider selective defaults on one-period bonds based on whether the debt is issued domestically or abroad. We note that, in contrast with our model and the data, in these two papers the defaulted debt dissipates and default episodes are associated with reductions in the value of debt.

Our work is also related to the literature on private defaultable debt and personal bankruptcy. As in the literature of sovereign debt, most work, as in Chatterjee et al. (2007) and Livshits, MacGee, and Tertilt (2007), has focused on full defaults and private bankruptcy. In this context the assumption that default is a discrete action upon which debts are discharged coincide with much of bankruptcy law, where debt is formally discharged. Recently, however, analyzing partial defaults is attracting more attention because defaults outside formal bankruptcy procedures are substantial, as documented by Dawsey and Ausubel (2004). In Mateos-Planas and Seccia (2014), households default partially on their debts giving rise to incomplete consumption insurance in an environment with a complete set of securities. Herkenhoff and Ohanian (2015) share with us the feature that borrowing continues after default, as they model foreclosures with a long process where lenders effectively finance the borrower after payments are stopped but before the house is lost.

The rest of the paper is organized as follows. Section 2 documents the evidence. Section 3 describes the model, defines the equilibrium, and discusses its main properties. Section 4 maps the model to data, discusses the quantitative implications of the model against the evidence, and analyzes impulse-response functions and default episodes in the model. Section 5 conducts counterfactual analysis of various policies discussed. Section 6 compares the implication of our model to the standard sovereign default model. Section 7 concludes. We also include appendices with data description, computational details, construction of measures for haircuts and extensions, and details of how we implement the reference standard model of total default.

2 The Empirical Properties of Sovereign Defaults

In this section we use panel data for emerging markets and document the properties of sovereign defaults. We first document that governments default with varying intensity and duration and that many defaults are small and short. We also document that default is systematically correlated with outcomes of the debt crisis: large defaults are associated with higher interest rate spreads, larger debt levels, deeper recessions, and longer default episodes. Finally, we document that default episodes do not lead to a decrease in debt.

2.1 Data

We use a panel dataset for 38 emerging markets from 1970 to 2010. The sample of countries consists of those from the J.P. Morgan Emerging Markets Bond Index (EMBI+). We use public debt statistics from the World Development Indicators (WDI) and public bond spreads from the Global Financial Database. From the WDI, we construct measures for *debt* defined as the ratio of total government debt public and publicly guaranteed (PPG) to output, and *output* defined as the detrended log of real output, detrended with a linear country-specific trend. From the Global Financial Database, we get annual series for the EMBI+ government bond spreads, which we use as our measure of *spreads*.

We also use information on debt in arrears to construct the panel series for default intensity, which we label as *partial default*. For each year and country, we define *arrears* as the sum of PPG interest and principal in arrears. We define *debt due* as the sum of PPG debt service and arrears. We define partial default as the ratio of the arrears to the debt due for each year and country. This variable is our measure of default intensity:

$$\text{Partial Default} = \frac{\text{Arrears}}{\text{Arrears} + \text{Debt Service}}. \tag{1}$$

Our definition of partial default essentially measures the fraction of payments missed. With the time series measure of partial default for many countries, we also study the duration of default episodes. We measure the duration of default episodes, which we label as *default episode length*, as the number of consecutive years that a country has positive values for partial default. Appendix A contains the list of countries and all the variables in more detail.

2.2 Empirical Findings

We document the distributions of partial default and default episode length and their comovement with interest rate spreads, debt-to-output ratios, and output.

We start by describing time series of partial default for two emerging market countries with a history of sovereign default: Argentina and Russia. In Figure 1 we plot the time series of partial default from 1970 to 2010 for these countries. The figure shows that default varies in intensity and is always partial, ranging from small levels of less than 10%, as was the case in the late 2000s for Russia, to high levels of more than 90%, as in the case of Argentina in the early 2000s. In terms of default episode length, Argentina experienced two episodes with lengths equal to 10 and 9 years, and Russia experienced one episode of a length equal to 20 years.

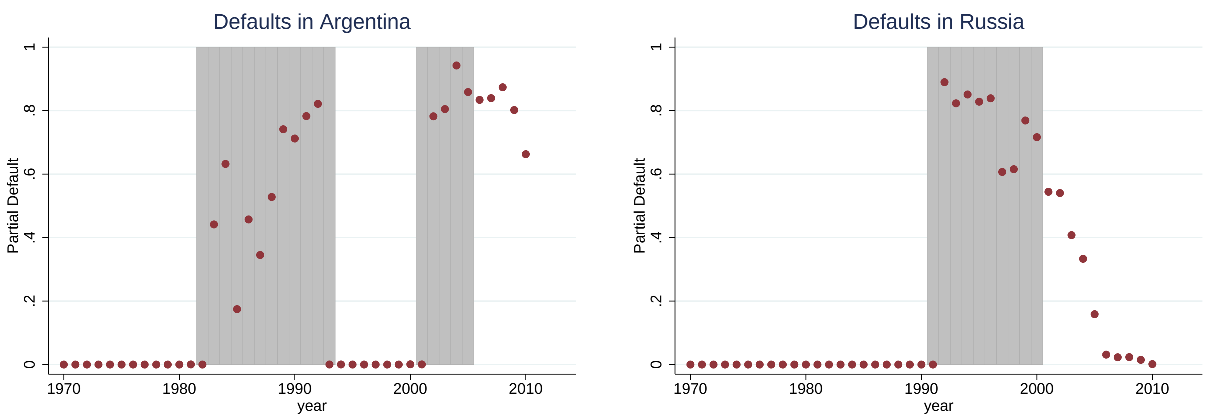


Figure 1: Time Series for Partial Default

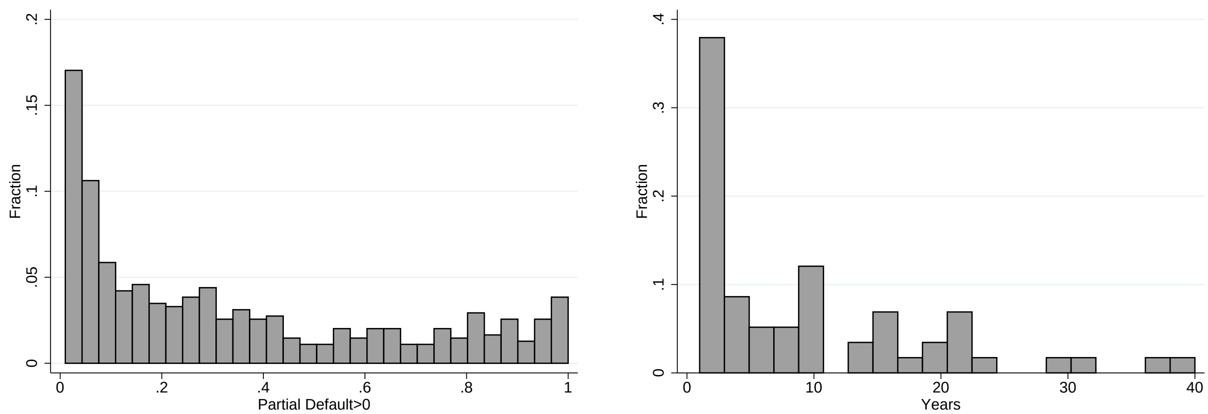
Although our partial default measure is distinct from the popular Standard and Poor's (S&P) binary definition of default, it correlates with it.[3] The shaded areas in Figure 1 are the years S&P classifies Argentina and Russia as being under default. The figure shows that having positive partial default is correlated with having the S&P default flag.

We now study the properties of partial default for the 38 emerging markets. Sovereigns partially default often, and defaults vary in intensity and duration. The frequency of positive partial default in the panel dataset is 35%. The varying intensity of partial default across years and countries is illustrated in Figure 2a. Here we plot the histogram of partial default (conditional on positive partial default) for panel data, the year×country series of the 38 emerging markets. The histogram

[3] S&P records a sovereign as being in default if it has failed to meet any payments on the due date and removes the default flag after it settles a payment with creditors.

shows that countries partially default at varying degrees covering the full range. Sovereigns often default on a small amount of debt due; about 37% of the time, partial default is less than 20%.

Default episodes also vary in duration. In Figure 2b we plot the histogram of default episode length for the 58 default episodes in the dataset. Most of the default episodes are short-lived; almost 40% of the episodes last less than 2 years. The histogram has a long right tail, as few episodes last more than 30 years. The distribution of the default episode length in our dataset is similar to the one documented in Benjamin and Wright (2013).



(a) Partial Default

(b) Default Episode Length

Figure 2: Partial Default and Default Episode Length

In Table 1 we summarize the distributions of partial default and default episode length. Emerging markets often have positive partial default, the average intensity of the default is modest, yet the variation is large. The mean partial default conditional on positive partial default is also 35% with a standard deviation of 16%. The mean length of the default episode is equal to 9 years, but a large fraction of the defaults are short. As the table shows, the fraction of default episodes that last less than 2 years is 38%.

The table also reports the dynamics of debt to output during default episodes. Debt to output feature on average a hump-shaped pattern during default episodes, and default episodes do not lead to reductions in debt. In the period before the episode starts, the ratio of debt to output is 35%. The default episode starts with a higher debt-to-output ratio of 40%. During the default episode, the debt continues to rise and is equal to 47% in the middle of the episode.[4] Debt decreases toward the end of the episode and equals 36% in the period after the episode ends. Our

[4]We define the middle of the episode as the total length of the episode divided by 2 rounded to the nearest integer.

findings are consistent with those in Benjamin and Wright (2013) who, using a different dataset, find that debt is not smaller following the end of the default episode than it was prior to the episode.

Table 1: Partial Default and Default Episodes in Percentages

Partial Default Frequency $\mid > 0$	
Mean	35
Standard deviation	16
Default Episodes	
Episode length (years)	9
Fraction of short episodes ($\leq$ 2 years)	38
Debt During Episode	
Before episode	35
Beginning of episode	40
Middle of episode	47
After episode	36

Next we document the comovement of partial default with spreads, debt to output, output, and episode length. Table 2 shows that partial default is positively correlated with spreads and debt to output, with correlations equal to 25% and 34%, and negatively correlated with output by -12%. The correlation between partial default and default episode length is computed across the 58 default episodes where partial default is the mean value for the episode. Partial default is positively correlated with episode length, with a correlation value of 55%; longer episodes are more intense.

To illustrate the magnitude of the comovement of partial default with interest rate spreads, debt to output, and output, we divide the panel dataset into four bins based on the levels of partial default and report for each bin the mean of the variables of interest. We report the means of partial default, spreads, debt-to-output ratios and output across the partial default bins in Table 3. The no default bin consists of the observations with zero partial default. We partition the observations with positive partial default into three groups. The small partial default bin

Table 2: Correlations with Partial Default

Spreads	.25
Debt to output	.34
Output	-.12
Episode length	.55

contains the observations below the 25th percentile; the medium partial default bin contains the observations between the 25th and 75th percentiles; and the large partial default bin contains the observations above the 75th percentile.

The means of partial default across the groups display a large variation. Small defaults on average have only 3% of their payments in default, medium defaults have 27% of their payments in default, and large defaults have an average of 82% of their payments in default.

Table 3: Partial Default, Spreads, Debt, and Output

	No default	Partial default > 0		
Means (%)		Small $(0 - 25\%)$	Medium $(25 - 75\%)$	Large $(25 - 75\%)$
Partial default	0	3	27	82
Spreads	4	7	7	15
Debt to output	26	37	49	63
Output	1	0	-3	-6

Table 3 shows that spreads, debt to output, and output have sizable differences as partial default varies. Spreads in periods of no default are on average 4%. During small and medium defaults, spreads rise modestly to an average of 7%. During large defaults, however, spreads more than double and are on average 15%. Debt to output in periods of no default is on average 26% and is higher in periods when sovereigns partially default. Debt to output rises monotonically to 37%, 49%, and 63% in periods of small, medium, and large defaults. The higher debt-to-output ratios during partial default run counter to the benchmark narrative that governments default to reduce their debt burden. The increase in debt during default occurs because defaulted debt is accumulated as debt in arrears and because governments continue to borrow new loans while partial default is positive. Finally, output in periods of no default is on average 1% above trend. Output deteriorates as default rises and reaches -6% below trend during large defaults.[5]

[5]These comovements are related to results in Cruces and Trebesch (2013) and Tomz and Wright (2013). Cruces and Trebesch (2013) find that spreads are higher following default episodes with large haircuts during renegotiations. Tomz and Wright (2013) document that output tends to be lower during default episodes.

3 The Model

Our environment consists of a small open economy with a stochastic endowment stream, that borrows long-term bonds, and that can choose to partially default on the debt due. The debt defaulted on accumulates as arrears and partial default imposes future resource costs to the economy that are increasing in the intensity of the default. Borrowing rates reflect default risk and compensate creditors for expected losses. We discuss the details of the model environment (Section 3.1) and its recursive formulation (Section 3.2). We then characterize the model equilibrium (Section 3.3) and develop empirical measures for haircuts and maturity extensions during default episodes in the model economy (Section 3.4).

3.1 Model Environment

The sovereign receives each period a stochastic endowment z_t that follows a Markov process with transition probabilities $\pi(z_{t+1}, z_t)$. The sovereign discounts the future at rate β and maximizes expected utility over consumption sequences, c_t, with preferences given by

$$E\left\{\sum_{t=0}^{\infty}\beta^t\ u(c_t)\right\}. \tag{2}$$

The actual income of the sovereign, y_t, is the endowment z_t net of any costs associated with default.

The sovereign trades long-term bonds with international lenders. Long-term bonds are perpetuity contracts with coupon payments that decay at rate δ, as in Hatchondo and Martinez (2009). The contract specifies a price q_t and a face value b_t such that the sovereign receives $q_t b_t$ units of goods in period t and promises to pay, conditional on not defaulting, $\delta^{n-1}\ b_t$ units of goods in every future period $t+n$. This specification allows for long maturity debt within a tractable structure, as a single state variable will incorporate all past borrowing.

Each period, the borrower has total payments due, a_t, which consist of all the obligations from past borrowing and defaulted debt. It chooses which fraction of these obligations to partially default on, d_t, and new borrowing, b_t. The sovereign can also buy back debt at its market price by choosing negative borrowing $b_t < 0$. The bond price is a function $q(a_{t+1}, d_t, z_t)$ that depends on total debt due the following period a_{t+1}, partial default today d_t, and the endowment

z_t because these variables determine future defaults. Consumption is constrained by income less total debt service due net of default, $a_t(1-d_t)$, plus new borrowing

$$c_t = y_t - a_t\ (1-d_t) + q(a_{t+1}, d_t, z_t)\ b_t. \tag{3}$$

Default carries a direct resource cost that is increasing in the intensity of the default and depends on the shock, such that income in the period after default is $y_{t+1} = z_{t+1}\Psi(d_t, z_{t+1}) \leq z_{t+1}$.

A factor κ of the debt defaulted on, $d_t a_t$, becomes future debt obligations. Such an amount is annuitized into perpetuities with decay rate δ so that $(1-\delta)$ of $\kappa d_t a_t$ is added to next period's obligations a_{t+1}. The total debt obligations due next period a_{t+1} include the accumulation of the defaulted debt $(1-\delta)\kappa d_t a_t$, the coupon payments due from the long-term legacy debt δa_t, and the new borrowing b_t. Formally,

$$a_{t+1} = \delta\ a_t + (1-\delta)\ \kappa\ d_t\ a_t + b_t. \tag{4}$$

Since κ represents the fraction of the defaulted value today that is carried over as an obligation in the next period, we refer to κ as the *debt recovery factor*. This recovery factor controls the haircut from default under the assumption that no further defaults occur in the future.[6] As we explore in detail below, actual haircuts constructed from empirical measures for default episodes do not only depend on κ because episodes in general last for several periods and vary in their intensity, all of which affect haircuts.

The sovereign borrows from many identical competitive risk neutral international lenders. A lender purchases sovereign bonds by issuing securities at the risk-free world gross interest rate R. In each subsequent period, the claims of the lender against the sovereign consist of the promised coupons from the initial bond and, potentially, any accumulated defaulted debt that results from a partial default on the bond.

The sovereign chooses borrowing and partial default to maximize utility taking as given the bond price function. Free entry among international lenders determines bond prices by driving expected profits from lending to zero, taking as given the sovereign's future partial default and borrowing decisions. Since bonds issued in different periods are perfectly substitutable and part of the total

[6] This nominal haircut can be shown to be $1 - \frac{(1-\delta)\kappa}{1+r-\delta}$. Note that when $\kappa = (1+r-\delta)/(1-\delta) > 1$, the nominal haircut is zero as defaulted coupons accumulate at the risk free rate, and that when $\kappa = 1$ defaulted debt is accumulated at no discount on its face value but the nominal haircut is still positive because defaulted coupons do not carry an interest.

face value of the debt a_{t+1}, the price of new bonds is also the price of existing bonds in the secondary market.

3.2 Recursive Formulation

We focus on recursive Markov equilibria and represent the infinite horizon decision problem of the sovereign borrower as a recursive dynamic programming problem, and the lenders' value also as a recursive functional equation. The state vector of the model consists of three variables $\{a, y, z\}$: a is total debt due, y is the income of the economy, and z is the endowment shock, which we need to keep track of due to the Markovian structure of shocks. In this context, the bond price is a function, $q(a', d, z)$, of what is known at the time of borrowing about the state tomorrow that will determine partial default and future borrowing. This state consists of the debt due tomorrow a' (we are now switching to standard recursive notation where primes denote future values), the partial default decision today d because of its effect on output tomorrow, and today's endowment shock z because it helps predict tomorrow's shock.

Borrower The recursive problem of the borrower with state $\{a, y, z\}$ is to choose new borrowing b, partial default d, and consumption c to maximize its value

$$V(a, y, z) = \max_{b,d,c} \{u(c) + \beta \sum_{z'} \pi(z', z) V(a', y', z')\}, \tag{5}$$

subject to the budget constraint that indicates that consumption is income minus net debt repayment

$$c = y - (1 - d)\ a + q(a', d, z)\ b, \tag{6}$$

the law of motion for the evolution of debt due that incorporates defaulted debt and new borrowing

$$a' = \delta a + (1 - \delta)\kappa\ d\ a + b, \tag{7}$$

the evolution of income which depends on partial default and on the shock

$$y' = z'\ \Psi(d, z'), \tag{8}$$

and the constraint that default cannot exceed the debt due and is weakly positive

$$0 \leq d \leq 1. \tag{9}$$

This problem determines the optimal borrowing and partial default policy functions $b(a, y, z)$ and $d(a, y, z)$, and the evolution of the debt due $a'(a, y, z)$, as well as the consumption function $c(a, y, z)$.

Lenders There are many identical competitive risk neutral lenders that discount time at rate $1/R$. The value of one unit of debt $H(a, y, z)$ equals the expected discounted stream of payments, and contains the amount paid today plus the expected discounted continuation value such that

$$\begin{aligned} H(a, y, z) = [1 - d(a, y, z)] + \frac{1}{R} \; &[\delta + (1 - \delta) \; \kappa \; d(a, y, z)] \\ &\sum_{z'} \pi(z', z) \; H\left(a'(a, y, z), z'\Psi[d(a, y, z), z'], z'\right). \end{aligned} \tag{10}$$

This expression reflects explicitely that partial default $d(a, y, z)$ reduces the value for lenders today and increases it tomorrow, as the defaulted debt accumulates and comes due in the future. Partial default also reduces expected tomorrow's output through the Ψ function. An additional channel, implicit in the equation, arises through the evolution of debt due $a'(a, y, z)$ which affects the future value $H(a', y', z')$: everything else equal, as seen in (7), both higher partial default and additional borrowing increase total debt due tomorrow which tends to reduce the value of debt.

Equilibrium We now define the equilibrium for this economy. The equilibrium entails the sovereign solving its problem understanding how its behavior, and the behavior of its future selves, affect the prices it faces, and that competition among lenders leads to zero profits in expected value on loans. Formally:

Definition 1 *A recursive Markov equilibrium consists of (i) the borrower's decision rules for borrowing $b(a, y, z)$, partial default $d(a, y, z)$, and consumption $c(a, y, z)$ which induce rules of debt due $a'(a, y, z)$, (ii) the value of existing debt $H(a, y, z)$, and (iii) the bond price function $q(a', d, z)$ such that*

1. *Taking as given the bond price function $q(a', d, z)$, the borrower's decision rules satisfy the*

borrower's optimization problem in eq. (5) subject to (6) to (9).

2. *Taking as given borrowers' behavior, the value of debt $H(a', d, z)$ satisfies its recursive formulation described in eq. (10).*

3. *Bond prices $q(a', d, z)$ yield expected zero profits to lenders such that*

$$q(a', d, z) = \frac{1}{R} \sum_{z'} \pi(z', z) \; H\left(a', z'\Psi(d, z'), z'\right). \tag{11}$$

3.3 Characterization of Equilibrium

In this section we discuss the factors weighing in the sovereign's decisions to partially default and borrow. We also examine the price of bonds, an important factor in those decisions, by spelling out the links between future decisions to borrow and partially default and the shape of the pricing function.

Borrowing and Partial Default To understand the trade-offs involved in the choice of borrowing b and partial default d, we analyze the optimality conditions for the sovereign. In these derivations, we follow an heuristic approach and assume that the price function $q(z, a', d)$, the value function $v(z, a, y)$, and the default cost function $\Psi(d, z')$ are differentiable.[7] For notational convenience, we also define the bond price as a direct function of the decisions b and d and the state a using the law of motion of debt such that $Q(z, b, d, a) \equiv q(z, \delta a + b + (1-\delta)\kappa \; d \; a, d)$, where the derivates of the price function correspond to $Q_b = q_{a'}$ and $Q_d = q_{a'}(1-\delta)\kappa a + q_d$.

Using compact notation, the first-order conditions for an interior solution for the problem in eq. (5) with respect to b and d are:

$$b: \quad u_c(c) \;\Big[\underbrace{q + Q_b b}_{\text{borrowing gain}} \Big] \; = \; \beta \; E\{u_c(c') \; \underbrace{\Lambda(d', q')}_{\text{debt burden}} \}, \tag{12}$$

$$d: \quad u_c(c) \;\Big[\underbrace{a + Q_d b}_{\text{partial default gain}} \Big] = \; \beta \; E\{u_c(c') \; [\underbrace{(1-\delta) \; \kappa \; a \; \Lambda(d', q')}_{\text{debt burden}} \; - \; \underbrace{z'\Psi_d}_{\text{default cost}} \;]\}, \tag{13}$$

[7]We know from the work of Clausen and Strub (2016) that in the short term debt sovereign default problem there are states where the functions involved are not differentiable. Still, it is convenient to describe the trade-offs using derivatives which are valid in the region where these functions are differentiable.

where the term $\Lambda(d', q')$ captures the reduction in resources tomorrow from a one unit increase in a' that results from increasing b and d, which reads

$$\Lambda(d', q') \equiv \underbrace{1 - d'}_{\text{tomorrow's repayment}} + \underbrace{[\delta + (1-\delta)\ \kappa\ d']\ q'}_{\text{further repayments}}.$$

These two optimality equations illustrate how the borrower can transfer future resources to the present by borrowing b or by defaulting d. They equate the marginal gain in utility from borrowing or from partially defaulting to the marginal reduction in utility to repaying the future debt burden and default costs implied by each type of action. Effectively b and d are different ways to alter the debt position, each one with its own costs and benefits. Borrowing may mean higher interest rates while partial default leads to a future output loss.

The left hand side (LHS) terms in square brackets in eq. (12) and eq. (13) capture the gains from borrowing and defaulting, which depend on the bond price function. The right hand side (RHS) terms of both expressions illustrate the costs of repaying the current and future coupons as well as the accumulated defaulted debt, which are evaluated at the price q'. The costs also incorporate the resource costs from default.

Consider first the optimality condition for b in eq. (12) which is similar to that arising in many dynamic sovereign default models. The LHS contains the marginal benefit from borrowing one unit of b which increases consumption by q but is discounted by the decrease in the price with more borrowing, $q_b < 0$. In our model, as it is typical in models of sovereign default, the resources raised by borrowing are capped by a Laffer curve; at the top of the Laffer curve the marginal benefit of borrowing is zero. Such Laffer curve arises because default risk limits the possibility of inter-temporally transferring resources with loans.

The RHS of eq. (12) is the discounted marginal cost from borrowing which contains the terms in $\Lambda(d', q')$ corresponding to repaying back the next period's coupon as well as refinancing the future coupons and the defaulted debt in arrears. The debt burden Λ includes the fraction of the coupon payment tomorrow that is paid back $(1 - d')$, and the continuation amount which includes the surviving fraction of the long term debt δ and the fraction not paid but accumulated $(1-\delta)\kappa d'$ evaluated at tomorrow's debt price q'. The accumulation of the defaulted debt allows the borrower to create long-term debt regardless of the duration of the debt, including when $\delta = 0$ and the stock of debt is short-term.

Consider now the optimality conditions for partial default d in eq. (13). This condition contains the trade-off for partially defaulting on the debt, accumulating the defaulted debt, and experiencing the default costs. The LHS of this equation shows the marginal gain from defaulting fraction d as consumption is increased by a but this benefit is discounted by the fact that the price q of new debt b falls as d decreases. This fall in the bond price, described by $Q_d = (1-\delta)\kappa a q_{a'} + q_d$, is due to both a direct effect $q_d < 0$ and, via $q_{a'} < 0$, due to the increase in a' induced by d. Raising resources through partial default does not have the acquiescence of lenders and these resources are not capped by a Laffer curve but by the total level of debt due a. The costs of partially defaulting are on the RHS. One component arises because the debt defaulted on is annuitized and accumulates at the recovery rate κ. The resulting debt burden from partially defaulting, after annuitization, $(1-\delta)\kappa a$, is also given by $\Lambda(d', q')$. Partial default also carries an additional cost in terms of reduced resources. Such default cost is encoded in $\Psi_d < 0$. Note that partial default tends to be more advantageous for larger levels of debt a as the direct benefit of defaulting are increasing in a while the direct cost of defaulting is $-\Psi_d$ which is independent of a.

By combining eqs. (12) and (13), we derive the following condition that equates the expected returns of borrowing and partial default weighted by marginal utility as follows

$$R^b \equiv E\left\{u'_c \, \frac{\Lambda(d', q')}{q + Q_b b}\right\} = E\left\{u'_c \frac{(1-\delta)\kappa}{1 + Q_d b/a}\left[\Lambda(d', q') + \frac{z'}{(1-\delta)\kappa a}(-\Psi_d)\right]\right\} \equiv R^d, \qquad (14)$$

where R^d and R^b are the expected returns to be paid out from borrowing and partially defaulting respectively weighted by marginal utility.

The implication from eq. (13) to eq. (14) is that those weighted expected returns (discounted) from borrowing and partial default are equated to current marginal utility, $\beta^{-1}u_c(c) = R^b = R^d$, when these two choices are positive and interior. These conditions also imply that when partial default is zero, $d = 0$, then it must be that $R^b \leq R^d$ so that the rate of return to be paid from borrowing is lower than that from partially defaulting.

The shape of the price function and the default costs are the determinants of the optimal portfolio of b and d. A positive b is an attractive choice when the price q is high and not too steep, such that $|q_{a'}|$ is small, which means low and insensitive future default probabilities. Here b is attractive because the marginal increase in consumption with b is high, and any future default costs are minimized. A positive d becomes attractive only when q is low and steep. Here paying the default costs, encoded in $\Psi_d < 0$, may be worth doing since it allows to increase consumption today at a rate of 1. The extra costs from d in terms of the price of new loans can be eliminated by setting new loans b close to zero. The attractiveness of d also depends on the accumulation factor κ.

Partial default is most attractive when arrears accumulate at a low κ rate.

Condition (14) also highlights that the shape of the penalty function Ψ and, in particular, its derivative $(-\Psi_d)$ are important for when default will be partial, total, or zero. As we will see in the quantitative section, when $(-\Psi_d)$ is increasing with d, default is most likely partial because it becomes increasingly costly, so it is optimal to stop defaulting before $d = 1$.

Bond Price Function We have seen how the sovereign's optimality conditions contain the bond price function q and the response of equilibrium prices to further indebtedness or additional default through the derivatives $q_{a'}$ and q_d. By substitution of function H from (10) into the equilibrium condition (11), we can write the pricing function q recursively as

$$\begin{aligned} q(a', d, z) = \frac{1}{R} E \{ & [1 - d(a', z'\Psi(d, z'), z')] + [\delta + (1-\delta)\kappa d(a', z'\Psi(d, z'), z')] \\ & q\left(\delta a' + b(a', z'\Psi(d, z'), z') + (1-\delta)\kappa d(a', z'\Psi(d, z'), z') a',\ d'(a', z'\Psi(d, z'), z'),\ z'\right)\}, \end{aligned} \tag{15}$$

or, more compactly,

$$q = \frac{1}{R}\ E\{(1 - d') + [\delta + (1-\delta)\kappa\ d']\ q'\} = \frac{1}{R}\ E\{\Lambda(d', q')\}. \tag{16}$$

The price responses, as expressed in the derivatives $q_{a'}$ and q_d, are themselves functions of equilibrium behavior, albeit in future periods. Sovereign default environments display time inconsistency in that current decision makers, if they could commit, would make choices for future actions that are different from the actual equilibrium future actions. Time inconsistency transforms the equilibrium decision rule from the solution of a mere maximization problem into the result of a game of the sovereign against its future selves, while the recursive representation of equilibrium restricts it to be Markov Perfect.[8] We can use results in Mateos-Planas and Ríos-Rull (2015) to derive the expressions for the derivatives of the price function q.

[8]See Krusell and Smith (2003) for a discussion of these issues. Mateos-Planas and Ríos-Rull (2015) characterize the equilibrium of sovereign default economies with the aid of Generalized Euler Equations (GEE), effectively substituting the derivatives of prices with future decision rules and their derivatives, which the current sovereign takes as given.

The derivative of q with respect to future debt due a' is

$$
\begin{aligned}
q_{a'} = \frac{1}{R} E \Bigg\{ & - d'_a [1 - (1-\delta)\kappa\ q'] && \text{direct loss from not paying} \\
& + [\delta + (1-\delta)\kappa\ d'] && \text{continuation amount gets diluted because more future debt} \\
& \Bigg\{ \beta\ \frac{E\{u''_c\ \Lambda(d'', q'')\} - q'\ u'_c}{b'\ u'_c}\ [\delta + (1-\delta)\kappa\ d' + b'_a] && \text{carries over and induces further borrowing} \\
& + d'_a\ \left[\beta \frac{E\{u''_c\ [\Lambda(d'', q'')\ (1-\delta)\kappa a' - z''\Psi'_d]\} - a'\ u'_c}{b'\ u'_c} \right] && \text{and more future partial default} \Bigg\} \Bigg\}
\end{aligned}
$$

and the derivative with respect to partial default d reads

$$
\begin{aligned}
q_d = \frac{1}{R} E \Bigg\{ & - d'_y\ \Psi_d z' [1 - (1-\delta)\kappa\ q'] && \text{lower output tomorrow yields more default} \\
& + [\delta + (1-\delta)\kappa\ d'] && \text{continuation amount gets diluted because} \\
& \Bigg\{ \beta\ \frac{E\{u''_c\ \Lambda(d'', q'')\} - q'\ u'_c}{b'\ u'_c}\ b'_y\ \Psi_d\ z' + && \text{less output tomorrow induces further borrowing} \\
& \left[\beta \frac{E\{u''_c\ [\Lambda(d'', q'')\ (1-\delta)\kappa a' - z''\Psi'_d]\} - a'\ u'_c}{b'\ u'_c} \right] d'_y \Psi_d z' && \text{and more future partial default} \Bigg\} \Bigg\}.
\end{aligned}
$$

The derivatives of the bond price function with respect to today's choices for borrowing and partial default, a' and d, depend on future policy rules for borrowing, partial default, and consumption. We see that the derivative of the price with respect to tomorrow's debt due a' has both a direct negative effect, that arises from a higher likelihood of a loss from not paying, and a continuation effect, the so called dilution effect.[9] The continuation amount, $(\delta + (1-\delta)\kappa\ d')$, is diluted because higher future debt induces further borrowing, $(\delta + (1-\delta)\kappa\ d' + b'_a)$, which is weighted by the continuation value of the debt represented in the marginal utilities, and leads to more future partial default, d'_a, which generates further future output costs. The derivative of the price with respect to partial default d also has a direct effect from the output cost that induces higher partial default tomorrow, and the dilution effect on the continuation amount $(\delta + (1-\delta)\kappa\ d')$. This dilution occurs because less output tomorrow induces more borrowing tomorrow, $(b'_y \Psi_d z')$, and higher partial default, $(d'_y \Psi_d z')$, both of which reduce the continuation value of the debt. The marginal rates of substitution affect the size of the dilution effects because they reflect the

[9]See Hatchondo, Martinez, and Sosa-Padilla (2016) for a quantitative study of dilution in the standard complete default model.

responses of the future continuation bond prices via the optimality conditions of the sovereign's future selves.

3.4 Default Episodes: Length, Haircut, Maturity Extensions

Below we compare our model to data on the duration of default episodes as well as haircuts and maturity extensions during these episodes. In this section, we develop expressions for these variables that are consistent with empirical measures.

In our model the duration, debt haircuts, and maturity extensions are endogenous and depend on when the sovereign chooses to start defaulting on the debt, when to exit by repaying the accumulated debt in arrears, and the path of partial default and borrowing during the episode. We define a *default episode* as a sequence of periods of time with consecutive positive partial default, which is the same definition we use on the data. The default episode *length* is the number of periods with positive partial default.

We follow the methodology in Benjamin and Wright (2013) and Cruces and Trebesch (2013) to compute haircuts and maturity extensions of default episodes. This methodology compares present values of streams of old defaulted debt instruments, hereafter DD, to new restructured debt instrument. We discount the streams at the risk free rate and define duration using the standard Macaulay concept.[10] To develop the expressions for haircuts and maturity extensions, consider a default episode that starts in period 1 and ends in period N, such that $d_{N+1} = 0$. The face values of the defaulted debt during the episode are given by $\{d_0 a_0, d_1 a_1, d_2 a_2, ...d_N a_N\}$. The present value of the of old defaulted debt instruments $value(DD)$ and its duration, $dur(DD)$, are

$$\text{value}(DD) = \sum_{t=1}^{N} \frac{d_t a_t}{(1+r)^{t-1}}, \qquad \text{dur}(DD) = \frac{1}{\text{value}(DD)} \sum_{t=1}^{N} t \frac{d_t a_t}{(1+r)^{t-1}}.$$

The new restructured debt instruments, hereafter ND, correspond to the arrears that get paid during the default episode and the terminal value of these arrears. The face values of the debt due

[10] Macaulay debt duration is the weighted average of the time of each coupon payment, with weights equal to the fraction of the bond's value on each payment. The duration of a default-free perpetuity bonds with decay δ is

$$\text{duration} = \frac{1}{q} \sum_{n=1} \left(n \frac{\delta^{n-1}}{(1+r)^n} \right) = \frac{1}{q} \frac{1+r}{(1+r-\delta)^2} = \frac{1+r}{1+r-\delta}.$$

where the discount price is $q = \frac{1}{1+r-\delta}$.

$\{a_2, a_3, ...a_{N+1}\}$ contain both the coupons from legacy debt that have not been defaulted on as well as the arrears that accumulate as new debt instruments $\kappa d_t a_t(1-\delta)$. This latter component of arrears can be expressed recursively as

$$a_t^{ND} = (1-\delta)\kappa d_{t-1} a_{t-1} + \delta a_{t-1}^{ND}$$

for $t = 2, ..., N+1$. The new debt instruments in each period correspond to the arrears paid during the default episode net of partial default, $(1-d_t)a_t^{ND}$ plus the value of the terminal coupon a_{N+1}^{ND}. The present value of the new debt instruments is given by

$$\text{value}(ND) = \sum_{t=2}^{N} \frac{(1-d_t)a_t^{ND}}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N}\frac{1+r}{1+r-\delta}$$

while their duration is

$$\text{dur}(ND) = \frac{1}{\text{value}(ND)} \sum_{t=2}^{N} t \; \frac{(1-d_t)a_t^{ND}}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N}\left(N\frac{1+r}{1+r-\delta} + \frac{1}{(1-\frac{\delta}{1+r})^2}\right).$$

We define the *haircut* as one minus the ratio of the value of the new debt instruments ND to the defaulted debt instruments DD, and the *maturity extension* as the difference in duration between these two instruments

$$\text{haircut} = 1 - \frac{\text{value}(ND)}{\text{value}(DD)}, \tag{17}$$

$$\text{maturity extension} = \text{dur}(ND) - \text{dur}(DD). \tag{18}$$

The debt haircut and maturity extensions depend in our model on the path and intensity of partial default during the episode, on the length of the episode, and on the recovery factor κ. Larger partial defaults and longer episodes are associated with larger debt haircuts.

4 Quantitative Results

We now study the quantitative properties of our model and compare them to the data on sovereign defaults in emerging markets from Section 2. We map the model to the data and illustrate its mechanics by analyzing the resulting decision rules and impulse response functions. We then confront the implications of our model for the distribution of partial default and the properties of default episodes against the data.

4.1 Specification and Parameterization

Functional Forms. The utility function is $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. We pose a 10 state Tauchen (1986)-approximation of an autoregressive process for $\log z$ with autocorrelation ρ and Normal innovations with standard deviation σ_η.

Recall that output depends on the shock and on default costs in such a way that output next period is $y' = z'\Psi(z', d)$. Following the quantitative sovereign default literature (Arellano (2008) and Chatterjee and Eyigungor (2012)), the output costs of default are assumed to be increasing and convex in the shock. Specifically, we assume that default costs are only realized for z above a threshold z^* and are then linearly increasing in z with slope ϕ_1. We also assume that these costs are increasing and convex in partial default, d, with slope parameter ϕ_0 and curvature parameter γ. The functional form we consider is

$$\Psi(z', d) = (1 - \phi_0 d^\gamma)(1 - \hat{\phi}_1(z' - z^*)), \tag{19}$$

where $\hat{\phi}_1 = \phi_1$ if $d > 0$ and $z' > z^*$, and zero otherwise.[11]

We use existing studies to assign some parameters and target moments of the model economy to specify the remaining ones. We choose parameters so the variables in the model are measured at a quarterly frequency.

Parameters Specified Ex-ante. We assigned directly standard values for risk aversion $\sigma = 2$ and the risk-free interest rate $r = 1\%$. The decay parameter δ is set so that the duration of the debt equals 5 years, which is similar to the average duration of debt in emerging markets. The bottom panel of Table 4 displays these choices.

Parameters from Matching Moments. The remaining eight parameters,

$$\Theta = \{\phi_0, \phi_1, \gamma, z^*, \beta, \kappa, \rho, \sigma_\eta\},$$

[11] This specification extends that in Arellano (2008) and Chatterjee and Eyigungor (2012) with an intensive margin of default. See our discussion around the standard reference model in Appendix D.

are estimated by minimizing the sum of the proportional square residuals of ten moments from emerging markets data.[12] The first four moments that we target summarize the empirical distribution of partial default. These targets are the frequency, mean and standard deviation of partial default, and the mean of small defaults as reported in Tables 1 and 3. Four additional moments concern the properties of debt and interest rate spreads, and they are the mean debt-to-output ratio, the standard deviations of debt to output and of sovereign spreads, and the correlation of spreads with output. These eight moments are the means across the 38 emerging countries in Section 2 (for example, the target correlation of output and spreads is the mean correlation across these countries). Finally, we also include as targets the standard deviation and persistence of output in Argentina as reported in Arellano (2008).

We solve our model with global methods and outline the algorithm to compute the model in Appendix B. To compare the model moments to the data targets, we simulate the model and aggregate the quarterly model time series to an annual frequency. We simulate the model for 750,000 periods and discard the initial 10% of observations. This long simulation approximates the limiting distribution across the states $\{z, a, y\}$. We construct the series for the ratio of the value of debt to yearly output as $\bar{q}a_t/4y_t$, where $\bar{q} = 1/(1-\delta+r)$ is the risk-free price of coupon a_t. Given the series for quarterly price of bonds q_t, we construct the annualized spreads from the bond price as

$$s_t = (1/q_t + \delta)^4 - (1+r)^4. \tag{20}$$

The standard deviation and persistence of output in the simulation are directly compared to the corresponding quarterly data moments. The remaining eight data moments related to debt, partial default, and spreads are based on observations at annual frequency. Hence, we annualize the model series for debt service due, defaulted debt, and output by summing over the observations from the corresponding four quarters, and for spreads by averaging over those four quarters. The total value of debt can then be calculated using the risk-free price in the same way as it was done for the quarterly series, and annualized partial default is the ratio of annual defaulted debt over annual debt due. Table 4 displays the parameter values.

Discussion of Estimates The threshold parameter z^* is such that default costs are positive when the shock is above 93% of the mean. The value of ϕ_0 says that the loss of output due to the intensity of default can be 4% at most, when default is total. As we will see below these parameters produce very minor default costs. The debt recovery factor κ implies a 7% default

[12] The weights on the residuals are uniform across moments, except for the frequency and mean level of partial default which receive double weight.

Table 4: Parameter Values

Estimated Parameters		
Default costs	$\phi_0 = 0.04$	$\phi_1 = 0.206$
	$\gamma = 1.621$	$z^* = 0.933 \times \bar{z}$
Discount factor	$\beta = 0.987$	
Debt recovery factor	$\kappa = 0.926$	
Shock process	$\rho = 0.928$	$\sigma_\eta = 0.028$
Assigned Parameters		
Risk aversion	$\sigma = 2$	
Risk-free rate (quarterly)	$r = 0.01$	
Decay parameter (quarterly)	$\delta = 0.96$	

discount in the face value of coupon payments defaulted.

The eight parameters have consequences for all of the ten targeted moments. However, our experience in the estimation process makes us associate some of the parameters closely to certain moments. For instance, the parameters controlling the z shock are primarily related to the persistence and volatility of output. The persistence and volatility for productivity have to be slightly smaller than those for the targeted output process. In a similar vain, the discount rate β is closely connected to the debt-to-output ratio. The two parameters describing the fixed component of the penalty for default, ϕ_1 and z^*, have an impact on the frequency of defaults and the size of defaults at the low quantile. A larger z^*, which waives the fixed cost at low shock levels, raises the frequency of default and reduces the size of partial defaults. A larger ϕ_1 works in the opposite direction; a bigger cost discourages very small partial defaults. The parameters γ and ϕ_0, describing the penalty for default that depends on its intensity, mainly affect the size of partial default and its volatility, as well as the correlation of spreads with output. A larger γ decreases the size and volatility of partial default and increases the correlation of spreads with output. A larger ϕ_0, in contrast, increases the size and volatility of partial default yet also increases the correlation of spreads with output. Finally, the parameter that controls the accumulation of defaulted debt κ increases the volatilities of spreads and debt and decreases the correlation of spreads with output.

4.2 Moments in Model and Data

Table 5 shows moments of the model and the data. In the model and the data, partial default is positive about one-third of the time. On average, default amounts to one-third of the debt repayment due. Partial default, conditional on being positive, is volatile and features many small defaults in both the model and the data. In the model, the standard deviation of partial default is 24%, and the average partial default of small defaults is 7%. These moments are moderately higher than the data counterparts of 16% and 3%.

Debt is high and volatile in both the model and the data. In the model, the mean of debt to output is 36% with a standard deviation of 18%. These moments are close to the data counterparts of 37% and 21%. In terms of the interest rate spreads, the model produces a volatility similar to the data and a spread that is negatively correlated with output, with a correlation comparable to the empirical counterpart. The model also matches well the resulting series for output in terms of its persistence and standard deviation.

Overall, we think that the fit of the model is very good. Note that while the average percentage residual is 23% of the value of the target this is mainly the result of the poor match of the average size of small defaults that has a residual more than 100%. We found that small changes in the parameter values yield large changes to this moment, which makes it hard to target it with precision. Without this moment, average residuals are just 11%.

Table 5: Moments in Model and Data

	Data	Model
Target Moments		
Partial default frequency (in %)	35	34
Partial default mean (in %)	35	31
Partial default st. dev. (in %)	16	24
Small ($0-25\%$) partial defaults (in %)	3	7
Debt-to-output ratio (in %) mean	37	36
Debt-to-output ratio (in %) st. dev.	21	18
Interest Rate Spread (in %) st. dev.	3.6	3.9
Correlation (output, spread)	-.30	-.32
Output persistence	0.93	0.93
Output (in %) st. dev.	0.08	0.08
Residuals		
Average Residual (as % of target)		22.9
Residuals Excluding Small Defaults (as % of target)		11.2
Other Moments		
Debt service to output (in %) mean	7	7
Corr. (debt, spread)	.24	.47
Spread (in %) mean	5.3	1.2

Table 5 also reports some additional untargeted moments from the model and compares them with the data. The model generates a mean level of debt due relative to output of 7%, which is equal to the average in the data.[13] As shown in Chatterjee and Eyigungor (2012), models with long-term debt, as our model, are able to reproduce both debt stocks and debt flow payments. The model generates a positive correlation between debt and spreads, which is also a feature of the data. Higher debt increases default probabilities in both the model and the data.

In terms of mean spreads, the model predicts a mean spread of 1.2%, which is lower than the average in the data of 5.3%. The discrepancy arises partly because we are matching default frequencies and partial default, which determine the expected losses for creditors, and are modeling creditors as risk neutral. The wedge between spreads and expected default losses is related to the credit spread puzzle in corporate bonds where default losses cannot explain the observed spreads, as explored in Chen, Collin-Dufresne, and Goldstein (2009). Longstaff et al. (2011) find that a similar wedge arises for sovereign spreads and estimate, for example, that about 64% of sovereign spreads can be attributed to a single global factor that contains large risk premia. The spread more directly comparable to our model's corresponds to the remaining proportion, or 1.9% (5.3% $\times$0.36), which is much closer to the model's prediction.

4.3 Decision Rules and Impulse Responses

In this subsection we study the model's decision rules and impulse responses for partial default, interest rate spreads, borrowing, and consumption.

Decision Rules Figure 3a plots the default decision $d(z, a, y)$ as a function of the level of debt due a for two levels of the shock $z = \{z_3, z_5\}$, a low value and the median. We scale the total debt due by the risk-free discount price $1/(1 + r - \delta)$ relative to mean output. The level for current output in the decision rules equals the level of the shock $y = z$. The figure shows that partial default increases with the level of debt due and decreases with the level of output. When debt is small enough, partial default is zero; as debt increases, partial default increases; and when debt is large enough, default is total.[14] Comparing the two curves also shows how partial default

[13] We measure debt due as the sum of debt service paid and arrears.

[14] Note that for the low shock, the change in default is continuous since it does not carry a fixed cost. For the larger shock level, default presents a discontinuity where partial default jumps to a positive value, reflecting the fixed cost from defaulting in this case.

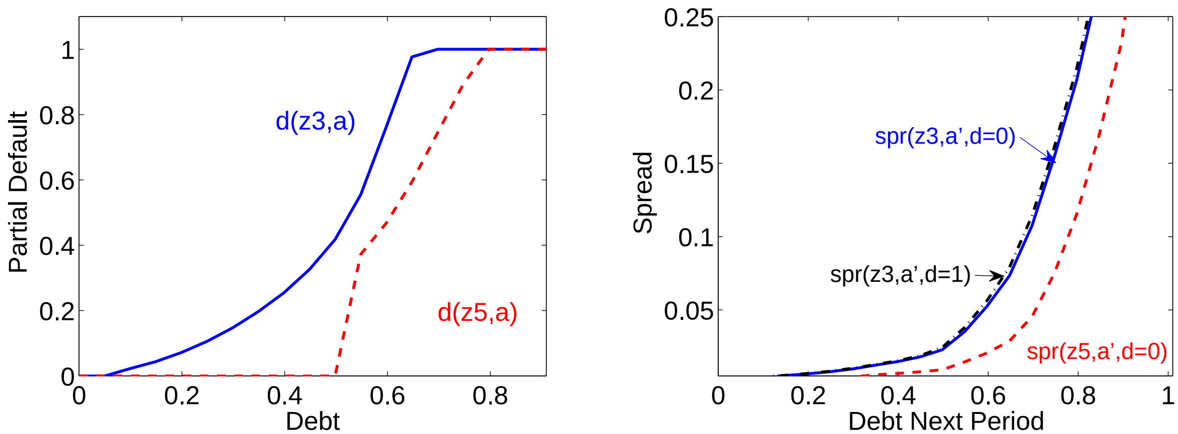
decreases monotonically with output, given debt due.

Our model generalizes the results from models of full default that feature default being more likely with high debt and low output (Arellano (2008)) to the case of partial default. In our model, not only is positive partial default more likely with high debt and low output, but also the intensity of default increases with high debt and low output.

Consider now the spread schedule $s(z, a', d)$, constructed from bond prices as in (20). In Figure 3b, we plot the spread schedule as a function of next period's debt due a (also scaled by the risk-free discount price and relative to mean output) for the same two levels of the shocks z as above, and with partial default equal to zero $d = 0$. We also plot an additional curve in the low output state for positive default, specifically $d = 1$, in order to gauge the impact of defaulting on spreads coming solely from the future output loss. With long-term debt and accumulation of arrears, future debt due a' depends not only on borrowing b but also on today's debt due a and default d, with $a' = \delta a + b + (1 - \delta)\kappa da$. The spread schedules with $d = 0$ and $d = 1$ in the figure have different underlying legacy debt δa and borrowing b because we condition on the same debt due tomorrow a' across schedules.

The figure shows that spreads increase as future debt due increases. Low levels of future debt due have zero spreads because predicted partial default is low. The increase in spreads as a function of future debt due is fairly smooth in our model because, as discussed above, default intensity varies gradually. The figure also shows that spreads are higher when output is low today, as this indicates a higher likelihood of low output in the future and hence of higher partial default. Finally, the spread schedule also depends on the partial default decision today d, the reason being that higher d induces a larger output loss next period, and hence the possibility of higher partial default in the future. The direct output loss from partial default in the future, however, only has a modest impact on spreads which is an indication of the small estimated output costs from default.

The default policy rule suggests a direct impact of debt on default intensity, which tends to impart a positive correlation between these two variables. From the spread function, since the influence of default via the penalty is weak, the correlation between default intensity and spread must reflect the two common factors, output shocks and debt, driving a positive comovement between the two variables. These policy rules, however, do not account for the feedback effect of default on the accumulation of debt and hence spreads, which, as we describe below, is important for the time series comovements.



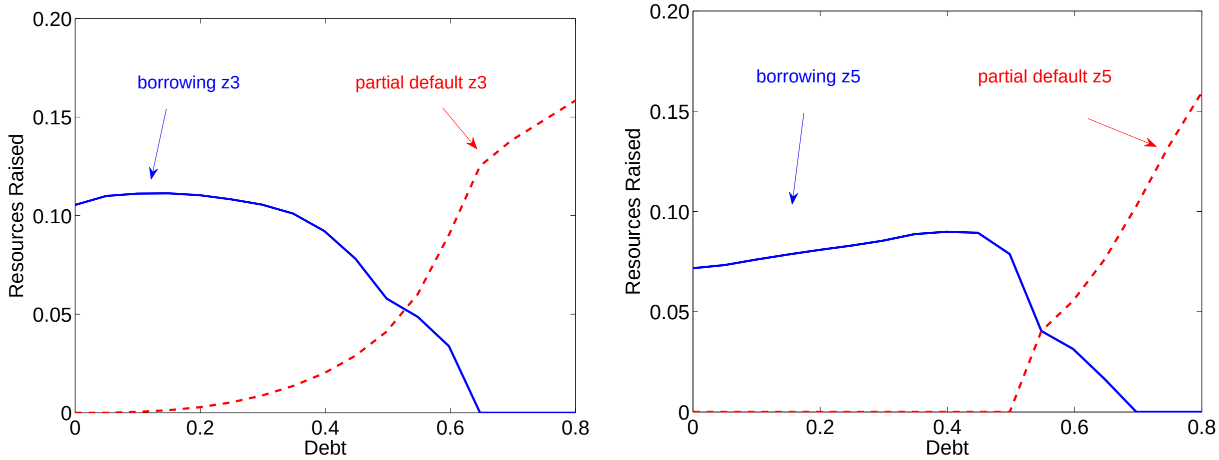
(a) Default Decision Rule

(b) Spread Schedule

Figure 3: Partial Default and Spreads

We consider next the policy rules for borrowing. As discussed above, in our model the country can transfer future resources to the present by borrowing or by partially defaulting. The optimal portfolio mix for borrowing and default changes with debt and income. Figure 4 plots the total resources raised with borrowing $q(z, a', d,)b(z, a, y)$ and with partial default $d(z, a, y)a$ (relative to mean output) as a function of debt due a for the shocks $z = z_3$ and $z = z_5$ with $y = z$. When debt is low, the country uses only borrowing to raise external resources. As debt rises, the portfolio shifts toward partial default, with borrowing declining sharply and partial default rising toward 100%. The reason behind the portfolio switch is the shape of the spread schedules encoded in $q(z, a', d)$. As debt due a increases, the interest rate spread for additional borrowing b increases because default risk is higher with higher total debt due tomorrow $a' = \delta a + b$. The states with high a are associated with severely restricted credit access, which makes it optimal for the country to use default as a way to raise resources. The policy rules for the optimal portfolio mix suggest that during default episodes, borrowing and default will move in opposite directions as long as debt accumulates within those episodes.

Figure 4 also shows that high z shocks are associated with less borrowing when current debt is less than about 50% of output. For higher levels of debt, however, this pattern is reversed and borrowing is higher for high levels of the shock. Such procyclical borrowing is typical in sovereign default models because of the more lenient price schedules in booms.



Low Shock High Shock

Figure 4: Portfolio of Borrowing and Default

4.3.1 Impulse Responses

We now analyze the equilibrium time series of partial default and spreads, as well as borrowing, total debt due, and output, in response to shocks. We construct impulse response functions in our nonlinear model as follows. We simulate $1,000,000$ paths from the model for $1,000$ periods. From periods 1 to 500 the aggregate shocks follow their underlying Markov chains. In period 501 we reduce the value of productivity z in each simulation. From period 501 on the aggregate shocks follow the conditional Markov chains. The impulse responses plot the average, across the $1,000,000$ paths, of the variables from period 500 to 600. The time series in this impulse responses are at a quarterly frequency.

Figure 5 contains impulse responses for a small (solid lines) and a large (dashed lines) shock in period 0. The small shock is generated by reducing the values of z one grid index, which is a decline of 4.3% or about half of the standard deviation, and the large shock is generated by a reduction of two grid indexes, which is a decline of 8.6%. The top left panel of the figure plots the paths for the shocks z in percentage deviations. The paths for output essentially mirror the paths for productivity as the impact of default costs is not large.[15] The right top panel shows the path for consumption, which on impact falls less than output but stays depressed longer than output

[15] Output depends not only on the shock but also on partial default $y' = z'\Psi(z', d)$, but in these impulse response functions output is extremely close to the shock. The main deviation is in period 1 with output falling 0.1% more than the shock. These small differences also show how our estimated output costs of default are quite low.

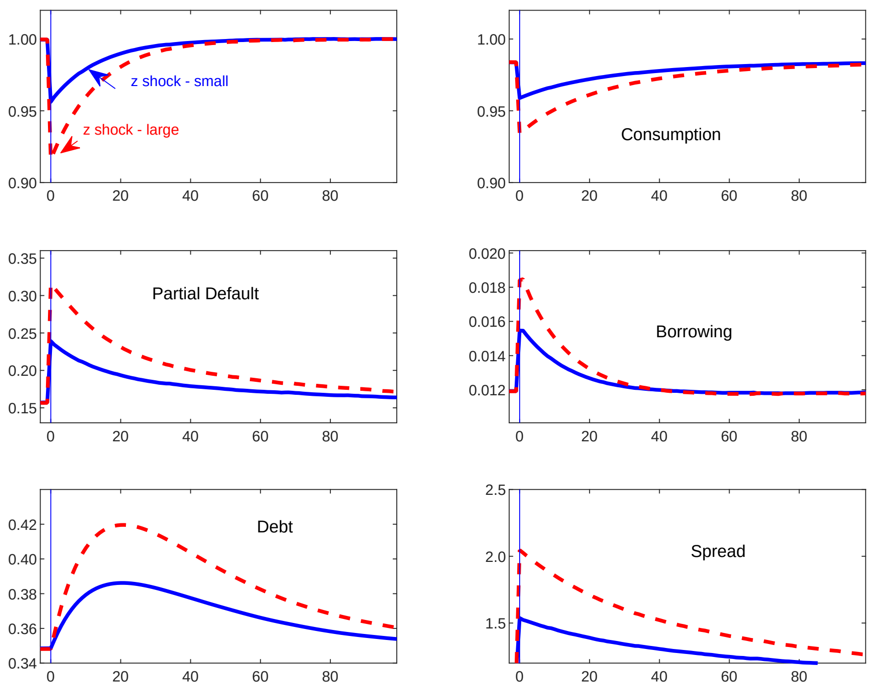


Figure 5: Impulse Response Functions

does. Default risk restricts the ability of the economy to smooth consumption, which leads to a large decline on impact. Consumption remains depressed because debt accumulates and remains persistently high.

The middle panels of Figure 5 plot the paths for the portfolio choices of partial default d and borrowing b. Partial default increases by about 8% on impact and features a response that is more persistent than the shock. By period 40, the shock has largely recovered, while partial default remains above trend. Borrowing in the figure is scaled by the default-free price $\bar{q} = 1/(1 + r - \delta)$ to convert it to a value and reported relative to mean output. Borrowing also increases on impact, but the effect is much less persistent than the effect on partial default.

The bottom left panel plots the response for debt due a, also scaled by $\bar{q}$ relative to mean output, that results from the choices of partial default d and borrowing b. The stock of debt features a hump shape, increasing for the first 20 periods after the shock and then slowly decreasing. The rise in partial default and borrowing both increase total debt due; partial default delays the repayment of debt through the accumulation of arrears, and borrowing directly increases the total debt due. The impulse response of total debt is slow and quite persistent. Debt remains elevated for much longer after the shock has returned to its mean.

Finally, the bottom right panel in the figure plots the path for the interest rate spread. Spreads rise on impact about 60 basis points and decrease slowly. Spreads rise because, as discussed in Section 4.3, partial default increases with low shocks and higher debt due. The effects on spreads are also very persistent. In period 40, for example, spreads continue to be 25 basis points higher than before the shock.

The impulse response functions to a larger negative shock are shown in Figure 5 with dashed lines. The responses of the variables have a larger magnitude and are more persistent than the responses to the smaller shock. To see this higher persistence, consider the paths for total debt. By period 60, the small and larger shocks have both recovered, yet debt is about 2 percentage points higher after the larger negative shock. The more persistent debt dynamics introduce more persistence in all the remaining variables.[16]

These impulse responses illustrate the propagation and amplification mechanisms in our model. Adverse shocks tighten spread schedules, making it more costly to roll over the debt. Partial

[16] The impulse response functions to positive shocks are almost exactly the mirror image of the impulse response functions to negative shocks. Only the dynamics of spreads are asymmetric. Booms feature smaller reductions in spreads.

default increases to alleviate the consumption decline arising from low output and tight financial conditions. New borrowing expands moderately despite spreads being high also to support consumption. The rise in partial default and borrowing both increase the future debt due and create a dynamic amplification of shocks. As debt stays elevated, partial default and spreads remain persistently high even as the shock returns to its mean. Moreover, the larger the shock the stronger the propagation and amplification. Recessions in our model have long-lasting effects on the functioning of international financial markets as debt levels rise with the accumulation of defaulted debt.

Having described the impulse response functions, we now turn to discuss the implications for the comovement of the variables of interest. As for the comovements of the intensity of partial default, the amplification effect of default on debt accumulation strengthens the correlation with debt. The tight comovements of partial default with both output and spreads should be reflected in large correlations with the anticipated sign. Regarding default episodes, the amplification in debt accumulation via default may very well imply that episodes of default end with levels of debt that are larger than they were at the start, and that episodes characterized by higher default will tend to last longer. That borrowing returns to trend faster than default suggests that during default episodes, with sustained default and increased debt, credit may become increasingly restricted.

4.4 Default Intensity

We now compare the quantitative implications of our model against the data with regard to default intensity. We show that the model can replicate the empirical comovement of partial default with spreads, debt, and output documented for emerging markets in Section 2. We construct these statistics from the long simulation with annualized series.

We first analyze the mean spread, debt to output, and output across bins based on the intensity of default. We partition the limiting distribution based on partial default into four bins. The no-default bin corresponds to the observations with zero partial default (recall that they amount to 65% of the observations in the data and 64% in the model). The bins labeled small, medium, and large correspond to periods with values for positive partial default of sizes (0-25%), (25%-75%), and (75%-100%). In Table 6 we report the averages of partial default, spreads, debt to output, and output across these default bins in both the data and the model.

The distributions of partial default in the model and data have a wide range, with zero, small,

Table 6: Partial Default, Spreads, Debt, and Output

Means (%)	No default	Partial default > 0 Small $(0-25\%)$	Medium $(25-75\%)$	Large $(75-100\%)$
			Data	
Partial default	0	3	27	82
Debt to output	26	37	49	63
Output	1	0	-3	-6
Spreads	0	3	3	11
			Model	
Partial default	0	7	27	64
Debt to output	28	40	49	68
Output	4	-4	-8	-12
Spreads	0	1	1	7

medium, and large partial defaults, although the distribution is a bit narrower in the model relative to the data. As Table 6 reports, the model and data have many observations that feature positive but small defaults as reflected by the mean of partial default for small defaults of 7% and 3% for the model and the data respectively. This moment was a target for the specification of parameters and was already reported in Table 5. The model and data also have many observations with large defaults, as reflected by the mean of partial defaults for large defaults of 64% and 82% in the model and data, respectively. The model produces medium defaults of 27%, the same size as in the data.

The table shows that debt to output monotonically increases with partial default intensity in the model, ranging from 40% to 68%, similar to the data which ranges from 37% to 63%. Output in the model also decreases monotonically with partial default in both the model and the data, although this relation is stronger in the model.

In the table we report the difference in the mean spread across the four groups relative to the no-default group. As explained above, our model produces a lower mean spread; the comovement of spreads with partial default, however, mirrors the data. In both the model and the data, spreads are narrower during smaller defaults. Small and medium defaults have somewhat narrow spreads in both the model and the data. The spread difference for these groups is 1% and 3% in the model and data, respectively. Large defaults have larger spreads, with a spread difference equal to 7% and 11% in the model and data, respectively.

We analyze these comovements also with correlations of spreads, debt, and output with partial default, conditional on positive partial default. Table 7 shows that both in our model and the data, periods of large partial default are associated with high spreads, high debt, and low output. Correlations in the model have the same sign as in the data, although they are somewhat stronger.

Table 7: Comovement with Partial Default

	Data	Model
Correlation with partial default		
Spread	25	59
Debt to output	44	72
Output	-12	-70

The comovements of the spreads, debt to output, and output with partial default reflect the response of these variables to shocks and the dynamics of debt illustrated with the decision rules and the impulse response functions analyzed in Section 4.3.

4.5 Default Episodes

As we have documented, countries go in and out of partial default recurrently. Our model has implications not only for the conditions that lead countries to partially default but also for the properties of default episodes. Recall that we define the default episode in the model the same way as we did for the data, that is, as a sequence of periods of uninterrupted positive partial default, $d > 0$, that are preceded and followed by at least one period with zero default, $d = 0$. In this section, we explore the model's quantitative implications for the length and dynamics of default episodes, and the resulting haircuts and maturity extensions.

In the model, the length of the default episode depends on the length of the recession and the degree of debt accumulation during the recession. The economy enters into partial default when hit by a large enough low shock, given its level of debt. As long as the economy continues in recession, it accumulates debt and remains in the default episode. When output recovers sufficiently, the economy exits partial default and the default episode ends. In this framework, longer default episodes arise from longer recessions with larger accumulation of debt.

To illustrate these mechanics, we plot the average paths for the variables of interest during default episodes. We consider paths for short default episodes lasting 3 years and long default episodes lasting 10 years. In particular, we simulate the economy for 1,000,000 periods and sort default episodes based on their length. We then draw the means of the shock, output, debt due, borrowing, partial default, and spreads for short and long default episodes. The resulting paths are shown in Figure 6.

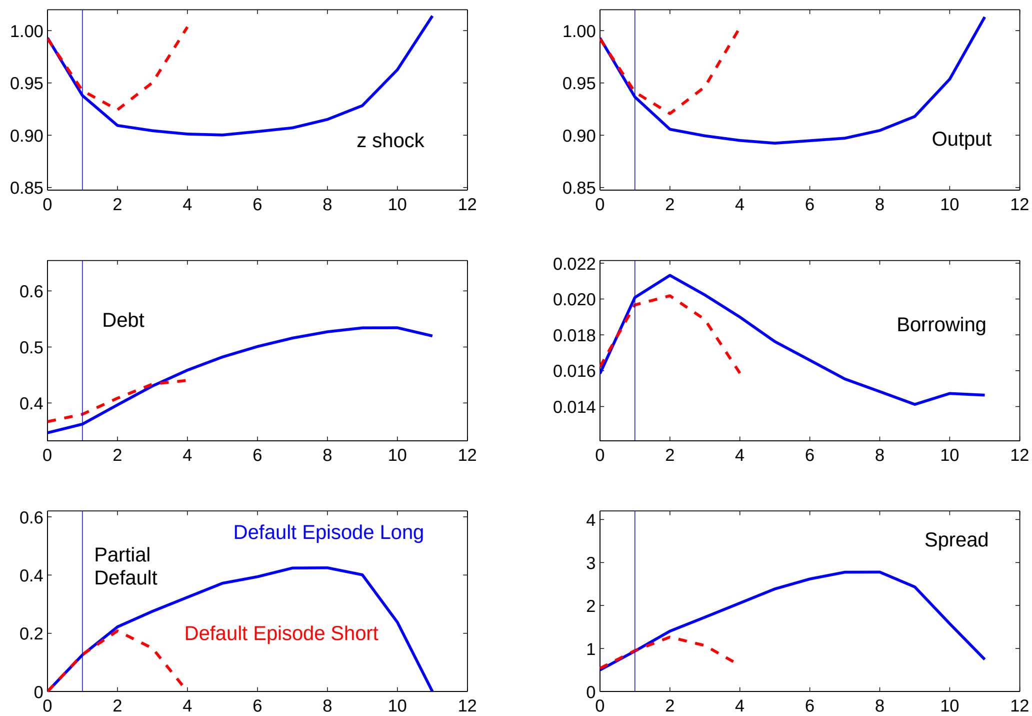


Figure 6: Default Episodes

The short and long default episodes start in period 1 of Figure 6 when output is about 6% below its mean. During short default episodes, output falls for one more period and then recovers. During the long default episode, output remains depressed until period 8 before it recovers. Longer episodes also feature larger declines in output than shorter default episodes. The episodes end when output recovers sufficiently. The level of output that leads to an end of the episode, however, is higher in the long than in the short episode.

Debt accumulates during the episode mainly because partial default increases and the defaulted debt accumulates as new borrowing is small and falls after the initial periods. Spreads also increase during the episode. In longer episodes, debt and spreads increase by more. The long episode requires higher output upon exit precisely because the long episode leads to larger accumulated debt.

In the model, the dynamics of output and debt during default episodes are consistent with the empirical findings of Benjamin and Wright (2013) who study default episodes in emerging markets. They find that default episodes are more likely to start when output is below trend and that they end when output has returned to trend. They also find that debt upon exiting the default episode is no smaller than it was prior to the episode, and in many cases it is strictly larger. The dynamics of output and debt that our model generates during default episodes are consistent with these findings.

We now compare the properties of default episode length in the model to our cross-country data. Table 8 shows that on average default episodes last several years, in both the model and the data, although in the model they are shorter and equal to 5 years compared to 9 years in the data. The model generates a large fraction of short default episodes. About 44% of defaults in the model last less than 2 years, close to the data fraction of 38%. The model and data feature large heterogeneity in default episode length, with a very similar coefficient of variation of around 1.

Table 8: Default Episodes

	Data	Model
Properties of Episodes		
Mean episode length (years)	9	5
Percentage of short episodes (≤ 2)	38	45
Coefficient of variation for episode length	1.1	1.2
Haircut (%)	36	30
Maturity extension	6	5
Debt in episode (%)		
Before	35	37
Beginning	40	40
Middle	47	45
After	36	44
Correlations (%)		
Correlation (length, haircut)	62	91
Correlation (length, partial default)	54	74

In interpreting these results, it is important to note that our model abstracts from coordination failures among lenders, which is the leading theory for the length of default episodes, as in Benjamin and Wright (2013). These authors show that coordination failures can account quantitatively for large delays in resolving default episodes. Their framework, however, is unable to generate the skewness of the length distribution that contains a large fraction of short default episodes. We view our theory of default episode length as complementary to the one based on coordination failures.

In terms of debt dynamics, in the second section of Table 8, we compare the evolution of debt to output during default episodes in the model to the one in the data. Our model features the hump-shaped evolution of debt during default episodes present in the data. In the model, debt to output in the period before the default episode is equal to 37%. At the start of the default episode, debt rises to 40% and continues to grow. In the middle of the episode, debt is 45%, and toward the end of the episode, it falls modestly to 44%. In our model, as in the data, default episodes do not lead to reductions in debt, and debt ratios continue to rise during the default episode.

We also compute in our model default episodes the haircuts that lenders experience and the maturity extensions, as well as the the correlation between haircuts and episode length. We compare our model statistics with empirical estimates of haircuts and maturity extensions in Meyer, Reinhart, and Trebesch (2018) and in Fang, Schumacher, and Trebesch (2016), and estimates of the comovement of haircuts and default episode length in Benjamin and Wright (2013).[17] These authors find that haircuts after default episodes are 36% on average and that resolutions of default episodes involve debt exchanges with maturity extensions that increase the duration of the debt by 5.5 years. As shown in Table 8, our model also predicts sizable haircuts and maturity extensions during default episodes. Haircuts and maturity extensions in the model are on average 30% and 5.4 years, respectively, which are close to the estimates for the data. Finally, in terms of correlations, the bottom of Table 8 shows that in our model longer default episodes feature larger haircuts and higher levels of partial default, just as in the data. The magnitudes of the correlations in the model of are a bit larger than their data counterpart.

5 Counterfactuals

In this section we compare the implications of our model to two alternative environments that represent policies aimed at improving the resolution of defaults and debt sustainability. The first counterfactual (Section 5.1) considers an environment where the country does not have access to international markets when default is positive. As we will see, such an environment is similar to adding more stringent pari passu clauses to the bond contracts in our model. The second

[17] We use these external sources for these estimates as our dataset does not contain enough information to compute the implied haircuts and maturity extensions given that it lacks the entire cash-flow structure of the debt.

counterfactual (Section 5.2) considers the case of higher debt relief from default which results from a parameterization with a lower value of the recovery factor κ. We use this counterfactual as a way to study the debt relief initiative for Highly Indebted Poor Countries. Our results suggest that pari passu clauses lead to lower default frequency, shorter default episodes, and lower haircuts from defaults. Debt relief policies, in contrast, increase haircuts, lower debt-to-output ratios, and reduce the incidence of large defaults.

5.1 No Market Access during Default (Pari Passu)

An important feature of our model that differs from many sovereign default models is that the economy continues to have access to financial markets during default episodes. As we have seen, bond markets are endogenously somewhat restricted in our model because of elevated default risk during periods with positive partial default, but modest levels of new bonds continue to be issued. In this subsection, we explore the effects of financial market access during defaults by comparing our model to a modified framework in which new borrowing during periods of partial default carries an additional cost that effectively shuts down market access during default.

Recall that in our model all the coupon payments due from past issuances are combined into a single state variable a. Payments for these coupons are treated equally across issuances as the partial default choice d is applied to the entire sum of coupons a. Nevertheless, empirical measures of haircuts as derived in equation (17) will differ across vintages of bonds issued at different points in time during a default episode. For example, haircuts on bonds issued later in the default episode are smaller than the haircuts on the original legacy debt the economy held at the beginning of the episode. The reason is that bonds issued later in the episode experience fewer periods with positive partial default compared to bonds issued earlier.[18] Such instances of differential treatment across bonds during default episodes are classic violations of pari passu clauses in bond markets, which require that all creditors are treated equally.[19]

In this context, the comparative statics we consider in this section that shuts down new borrowing during default episodes ameliorates pari pasu concerns. In fact, we can directly interpret the

[18] Average haircuts in the benchmark model during default episodes of 5 years are, for example, 25% for the legacy debt outstanding at the beginning of the episode and 10% for bond issues during the last year of the episode.

[19] As explained in Olivares-Caminal (2013), bonds with pari passu clauses, sometimes called most favored creditor clauses, stipulate that "during default episodes, if subsequent settlements have better terms, those terms will also be extended to the previously exchanged bonds."

implicit additional costs for borrowing during default episodes as costs from additional pari passu clauses and potential litigation during default.

In the first two columns of Table 9 we compare the results from an economy with no market access during default to our benchmark model. The effects on the distribution of partial default are sizable. Without market access, partial defaults are less frequent and more homogeneous. The default frequency drops from 34% to 14%, and small and large defaults are largely eliminated, reflected by a drop in the standard deviation of partial default by a third. These changes in partial default are reflected in smaller and less volatile interest rate spreads.[20] Default episodes become slightly shorter for two reasons. First, not having access to international markets adds extra costs from defaults which encourage faster exit from default episodes. Second, debt increases less during default episodes than in the benchmark because no new bond issuances occur, for example, debt to output increases only about 5% during default episodes that last 10 years, compared to the increase of close to 20% for the benchmark model (as seen in Figure 6). With less debt accumulated during default episodes, the economy can exit default faster. Shorter episodes with no market access also lead to smaller haircuts and somewhat smaller maturity extensions.

This experiment suggests that pari passu clauses that remove market access during default episodes could have sizable implications for defaults, making them less frequent, more homogeneous, and shorter.

5.2 HIPC Initiative: Larger Debt Relief

Another important feature of our model that differs from the sovereign default literature is that the defaulted debt does not dissipate after default, but instead accumulates, with a fraction κ of the defaulted debt coming due in the future. We now turn to explore the implications of lowering the recovery factor κ from 0.926 in the benchmark to 0.85, which essentially doubles the debt relief on the face value of debt relative to the baseline.

Since 1996, the International Monetary Fund and the World Bank have promoted the Heavily Indebted Poor Countries (HIPC) debt relief initiative. Countries that qualify for this program experience debt relief from multilateral institutions and bilateral creditors. The purpose of this

[20] Although the economy is not issuing any new loans during periods of positive partial default in this comparative static, the debt accumulates as defaulted debt during the episode and carries a corresponding spread priced in secondary markets. The measure of spreads we use for this comparative static are these secondary market spreads.

Table 9: Properties of Alternative Environments (in %)

	Benchmark	Pari Passu (no mkt access)	High Debt Relief (lower κ)	Full Default Reference
		Partial Default $\mid > 0$		
Partial default frequency	34	14	33	2
Partial default mean	31	26	28	100
Partial default st. dev.	24	16	22	0
Small partial defaults	7	11	7	100
Large partial defaults	64	46	58	100
		Default Episodes		
Mean episode length	5	4	5	5
Fraction of short episodes (≤ 2)	45	45	45	20
Haircut	30	26	36	100
Maturity extension	5.4	5.3	5.4	–
Corr. (length, partial default)	74	60	77	0
Corr. (length, haircut)	91	91	94	0
		Overall First and Second Moments		
Spread mean	1.2	0.3	1.2	3.4
Spread st. dev.	3.9	0.5	1.8	2.0
Debt-to-output mean	36	34	28	33
Debt-to-output st. dev.	18	14	15	15
Corr. (partial default, spread)	59	83	83	–
Corr. (partial default, debt)	72	51	71	–
Corr. (partial default, output)	-70	-54	-70	–
Corr. (spread, output)	-32	-57	-55	-26
Corr. (spread, debt)	47	52	61	44

initiative is to bring debt to sustainable levels and free-up resources for spending. While many countries that have qualified for the HIPC initiative have levels of output per capita below those in emerging markets, a few of the countries in the initiative are quite similar to emerging markets.[21] Honduras, Bolivia, and Guyana, for example, that are part of the HIPC initiative, have very similar output per capita, ranging from \$5,000 to \$9,000 PPP U.S. dollars, to Nigeria, El Salvador, and Philippines which are emerging markets that are not part of the initiative. In this context, we use our model to evaluate the effects of larger debt relief for these borderline countries.

The third column of Table 9 shows the results of the experiment where we lower κ while keeping all other parameters at their benchmark values. Lowering the recovery factor κ reduces the size

[21] Most of the HIPC countries such as Burundi, Niger, Mozambique, Sierra Leona, Madagascar, Togo, Guinea Bissau, and Burkina Faso have output per capita of less than \$2,000 PPP U.S. dollars in 2018.

of defaults, especially of the large partial defaults, reduces the volatility of spreads, increases haircuts, and lowers debt-to-output ratios. The other properties of partial default, spreads, and debt, however, experience only moderate changes. With lower κ, partial default is somewhat lower on average and less volatile, and large partial defaults are reduced from 64% to 58%, haircuts increase from 30% to 36%, and average debt-to-output ratios are reduced from 36% to 28%. With higher haircuts, borrowing becomes more restricted because creditors adjust the bond price schedule to avoid losses. The volatility of sovereign spreads is reduced by half and the correlations of spreads with partial default, output, and debt are strengthened.

This counterfactual shows that debt relief initiatives can be useful to eliminate large defaults and lower the volatility of interest rate spreads but do not change much mean spreads or mean partial default. These results illustrate that high debt relief lowers debt sustainability levels because creditors respond with tighter bond price schedules leaving equilibrium mean spreads and partial default unchanged. In practice, the HIPC initiative has had mixed results in terms of alleviating the debt burden of countries. Our counterfactual illustrates that general equilibrium forces respond to these policies and somewhat offset them.

6 Fresh Start: Full Default Reference Sovereign Debt Model

The reference model of debt and default in the literature, e.g., Arellano (2008) or Chatterjee and Eyigungor (2012), is one where defaults affect all of the debt payments, both present and future. We modify our model to obtain a version of this reference standard model as follows. We impose that default can only be total, upon which the entire stock of debt vanishes, and the country is temporarily excluded from financial markets. With probability λ the country re-enters financial markets holding zero debt, a fresh start. A default episode here is defined to contain the period when the sovereign defaults and the periods when it remains excluded from credit markets subsequently. The duration of the episode is therefore exogenous.

For a proper comparison between the two models, we will not be using the same parameter values, but will instead consider the two models as alternative lenses through which to look at the same data. Consequently, we proceed to estimate this version of the standard model using some of the same targets as in Section 4. We estimate the discount factor, the parameters controlling the output costs from default, and the re-entry probability λ so that the model delivers the best approximation to these moments: average level of debt over output, standard deviation of spreads, correlation of spreads with output, and mean default episode length. Of course with complete default, we cannot include as targets the mean and standard deviations of the default size. The details are in Appendix D.

The last column in Table 9 presents the results from the reference model. Default here is very infrequent, only 2% of the time, in contrast with the benchmark frequency of 34%, and when default occurs, it is on 100% of the debt, which makes the model silent on the correlations of partial default with other variables. The model can deliver similar moments for spreads and debt to output, yet default episodes are impasse states, characterized by financial exclusion and exogenous duration.

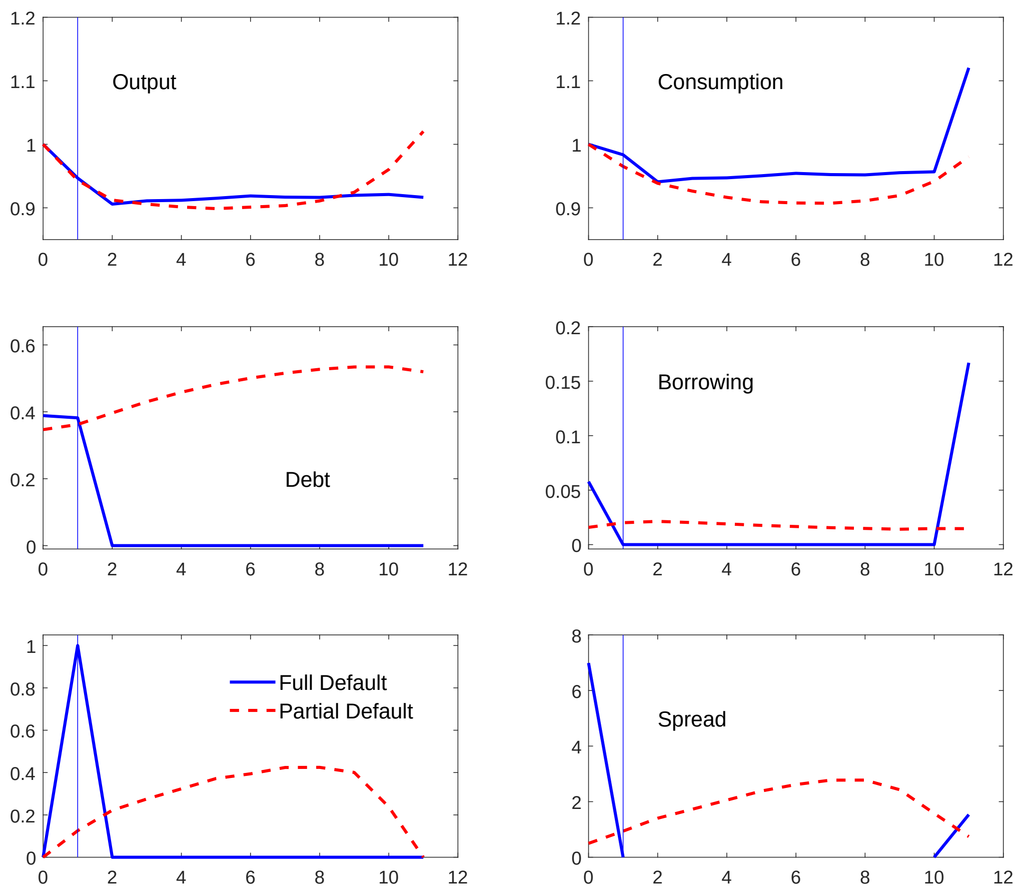


Figure 7: Default Episodes in Reference Full Default Model

We contrast default episodes in the full default model to our partial default model. Figure 7 plots the paths for the variables of interest for default episodes lasting 10 years. Although the length of default episodes of 5 years on average is exogenous in the full default model, we choose realized paths with exactly 10 years of exclusion after default, and average the variables across these realizations.

The top two panels in the figure show the paths of output and consumption relative to period 0. Output in both models is depressed during default episodes, but for different reasons. Output in our model is depressed because of selection effects, as the country chooses to remain in default only during recessions, while in the full default model, output is depressed because of default costs. Consumption is also depressed in both models during default episodes, although the decline is larger in our model as the country continues to repay some of the debt. In period 11, the first period after the end of the default episode, the full default model displays a large increase in consumption of about 15%. In contrast, consumption in our model after default increases quite mildly, and is only close to trend. This burst in consumption following a default episode in the full default model arises because the country experiences a fresh start. The middle panels in the figure illustrate the paths for total debt relative to output and borrowing. In the full default model, debt is eliminated after default, no borrowing occurs during the episode, and the country borrows heavily in period 11, following its fresh start. Such sharp surge in borrowing after the episode is the reason for the consumption boom. The bottom two panels show partial default and spreads. In the full default model, these default episodes are impasse states with no dynamics in debt, partial default, or spreads.

The standard model is at odds with the data in several dimensions. As we have described, major shortcomings of the full default model have to do with its extreme assumptions on the default size and its consequences. Additional discrepancies with data are the sharp increases in consumption and borrowing upon recovering access to financial markets at the end of a default episode.

7 Conclusion

Based on the evidence for sovereign debt crises in emerging markets, we have provided a new model of sovereign debt, where partial default acts as an alternative form of borrowing and is a powerful amplifying mechanism for debt crises. The framework contains empirical implications that fit the data for the intensity of partial default, the length of default episodes, and the debt haircuts and maturity extensions from default episodes. The model also delivers empirically consistent implications for the distribution of partial default, the correlations of partial default with spreads, debt, and output, and the path of debt during default episodes.

We use the framework to study the consequences of pari pasu clauses and debt relief policies, and find that pari passu clauses make default less likely and shorter, while debt relief policies reduce the incidence or large defaults but reduce debt sustainability.

This paper has presented a theory that modifies the standard paradigm of default in most of sovereign default models. In standard theory, default is the culmination of the crisis and is followed by an impasse period with no debt repayment or borrowing, after which the economy reemerges making a fresh start with reduced debt. The focus of that theory has been on the dynamics prior to the default and has not been designed to study life during default episodes. Emerging economies, however, spend more than a third of the time in default episodes, and hence abstracting from these periods misses a large part of the dynamics. In our theory, we have integrated the dynamics prior to default with the dynamics occurring during default episodes by recognizing that partial default may lead to more debt and more default. Our framework embraces the non-commitment environment inherent in sovereign debt markets; the same lack of commitment that underpins sovereign default also rules out automatic debt relief after default. Accumulated defaulted debt leads to higher interest rate spreads, more debt, and longer and deeper debt crises.

Our quantitative and empirical findings also suggest that existing default resolution mechanisms are not effective at reducing the debt burden for emerging markets experiencing defaults. In practice, default has led to more default and countries exit default episodes with as much debt as they enter with. We think that a salient oversight for debt restructuring efforts, led by multinational organizations, international creditors, and also the academic literature, is abstracting from the dynamics of debt during default episodes. We view our framework as a potentially useful one to analyze debt restructuring mechanisms because it incorporates the important debt dynamics during defaults.

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Appendix

A Data sources and variables

Our sample of countries consists of all 38 emerging countries from the J.P. Morgan Emerging Market Bond Index (EMBI+). The countries in the sample are: Argentina, Bulgaria, Belize, Brazil, Chile ,Dominican Republic, Ecuador, Gabon, Ghana, Indonesia, Iraq, Jamaica, Morocco, Mexico, Nigeria, Pakistan, Panama, Peru, Philippines, Poland, Russian Federation, El Salvador, Serbia, Trinidad and Tobago, Turkey, Ukraine, Uruguay, Venezuela, Vietnam, South Africa, China, Colombia, Egypt, Hungary, Korea, Sri Lanka, Malaysia, and Tunisia. The data is annual and comes from the World Development Indicators (WDI) and the Global Financial Database (GFD). The data span 1970 to 2010, and the time series coverage for each variable is based on availability. The spread data from GFD is the country EMBI(+) spread. From the WDI dataset we construct the following variables:

- Partial default is the ratio of arrears public and publicly guaranteed (PPG) relative to the sum of debt service PPG and arrears PPG. The observations for positive partial default contain those with partial default greater than .1%.
- Debt to output is that ratio of debt PPG to gross national income.
- Output is the linearly detrended log of gross national income with country specific trends.
- Default episode is the window of time with consecutive periods with positive partial default after and before periods with zero partial default.

B Computation

We describe computation aspects for the model with stochastic recovery factor κ. We have used this slight generalization of the benchmark model to verify the robustness of the results.

Model with stochastic κ. In this case, κ can take on values κ_i in some discrete set with an iid probability π_{κ_i}. Since this extension does not affect the definition of the state, the model is as set out in section 3 except for a few simple details that we include for completeness. The evolution of debt payments for the sovereign includes κ_i so it becomes

$$a' = \delta a + (1-\delta)\kappa_i d a + b. \tag{B.1}$$

The value to the lender evaluates continuation values over the distribution of κ so that

$$\begin{aligned} H(z,a,y) &= 1 - d(z,a,y) \\ &+ \frac{1}{R}\sum_i \pi_{\kappa_i}\left[\delta + (1-\delta)\kappa_i d(z,a,y)\right] \sum_{z'} \pi(z',z) H(z', a'(z,a,y,\kappa_i), y'(z',z,a,y)) \end{aligned} \tag{B.2}$$

where $a'(z,a,y,\kappa_i) = \delta a + (1-\delta)\kappa_i d(z,a,y) a + b(z,a,y)$.

The expected value conditional on a', that is after the realization of the recovery shock κ_i, gives an ex-post price

$$\tilde{q}(z,a',d) = \frac{1}{R}\ E_z[H(z',a',y')]. \tag{B.3}$$

The relevant price is the ex-ante value over the distribution of κ_i:

$$q(z,b,d,a) = \sum_i \pi_{\kappa_i} \tilde{q}(z, \delta a + b + (1-\delta)\kappa_i d a, d).$$

This is now the price showing in the sovereign's constraint which becomes

$$c = y - a(1-d) + q(z,b,d,a)\ b \tag{B.4}$$

Grids and interpolation. We define grids for the state (z,a,y), and the decisions (b,d). Continuation values V and H need to be evaluated on real values for y' and a' off the grid. Also we represent borrowing and default decisions, b and d, that are continuous (at least piece-wise). Therefore, in order to obtain q, $\tilde{q}$ must be evaluated at continuous values a'. Likewise, q will have to be evaluated on continuous values for b and d. To achieve this, we have considered a mixture of interpolation schemes, including linear and cubic splines. Since outcomes are robust, the results reported are based on the following choice: bilinear interpolation over (y,a) for V and H; bilinear interpolation over (b,d) for q; linear interpolation over a' for $\tilde{q}$.

Discontinuity of default. The default decision rule $D(z,a,y)$ may not be continuous everywhere. One reason is that in some states there is a fixed component to the cost of defaulting, i.e., a discrete drop in Ψ as d becomes strictly positive. There, default will jump discontinuously. Treating $d(z, a, y)$ as continuous in (y, a) is inappropriate, and may produce misleading outcomes: Interpolation near the discontinuity may appear to yield some default where there is no default, or some partial default where there should be full default. To deal with this situation, we introduce a discrete decision variable for the extensive margin, whether to default or not at all, $d_{01} \in \{0, 1\}$. The agent's decision is broken down as follows. There is a value to defaulting

$$V^D(z, a, y) = \max_{b,d} \left\{ u(c) + \beta \sum_j \pi_{\kappa_j} \sum_{z'} \pi(z', z) V(z', a', y') \right\}$$

subject to the constraints spelled out in the main text. This yields continuous decision rules $d^D(.)$ and $b^D(.)$. And there is a value to not-defaulting

$$V^{ND}(z, a, y) = \max_{b} \left\{ u(c) + \beta \sum_{z'} \pi(z', z) V(z', \delta a + b, z') \right\},$$

which yields $b^{ND}(.)$. The ex-ante value

$$V(z, a, y) = \max\{V^D(z, A, z), V^{ND}(z, a, y)\},$$

yielding the extensive-margin discrete default decision $d_{01}(z, a, y)$. The enveloping intensive-margin default rule is

$$d(z, a, y) = \begin{cases} 0 & d_{01}(z, a, y) = 0 \\ d^D(z, a, y) & d_{01}(z, a, y) = 1 \end{cases}$$

The discontinuity of default will cause H to be discontinuous where the discrete decision $d_{01}(...)$ switches. Two sections are defined accordingly, one for the default states D and another for no-default states ND. In a defaulting state

$$H^D(z, a, y) = (1 - d(z, a, y)) + \frac{1}{R} \sum_j \pi_{\kappa_j} \left(\delta + (1 - \delta)\kappa_j d(z, a, y)\right) \sum_{z'} \pi(z', z) H(z', a', y'),$$

while in a no-defaulting state

$$H^{ND}(z, a, y) = 1 + \frac{1}{R}\delta \sum_{z'} \pi(z', z) H(z', a', y'),$$

where $a' = \delta\ a\ + b(z,a,y)\ +\ (1-\delta)\kappa_j\ d(z,a,y)a$, $y' = z'\ \Psi(d(z,a,y),z')$. So H is:

$$H(z,a,y) = d_{01}(z,a,y)H^D(z,a,y) + (1 - d_{01}(z,a,y))H^{ND}(z,a,y).$$

Continuation values. In the agent's decision problem, one issue is how to evaluate the continuation values of the envelope function V at points outside of the 2-dim grid defined over (y,a). Such a point would be inside a box defined by the four grid points, say $(y_j, y_{j+1}, a_i, a_{i+1})$. If $d_{01}(z', a_i, y_{j+1}) = 1$, defaulting also must be the option in all the 4 points, and V obtains by interpolating V^D between the four grid points. If $d_{01}(z', a_{i+1}, y_j) = 0$, no-defaulting also must be the option in all the 4 points, and V obtains by interpolating V^{ND} between the four grid points. In the other cases, where the default extensive-margin discrete choice is not the same in all points of the box, interpolate V^D and V^{ND} separately, and pick the value that dominates. For the price function,

$$\tilde{q}(z,a',d) = \frac{1}{R}\ \sum_{z'} \Gamma(z',z)H((z',A',z'+\epsilon')\Psi(d,z'))$$

we have the continuation value of H which has to be evaluated at points outside of the 2-dim grid defined over (y,A). We proceed similarly as above, defining a 2-dimensional box and interpolating either H^D or H^{ND}.

Optimization. In the default case $d_{01} = 1$, over the two dimensions b and d, use the minimization routine PRAXIS based on the principal axis method (Richard Brent; FORTRAN90 version by John Burkardt. URL: `http://people.sc.fsu.edu/~jburkardt/f_src/praxis/praxis.html`). For the no-default case $d_{01} = 0$, over the single dimension b, we use Golden Section search, with the bracketing using Fortran90 intrinsic procedure `maxloc` on the grid. The algorithm for our moment-matching exercise is based on the software BOBYQA, authored by Michael J. D. Powell, that minimizes the sum of squares of the target moments with bound constraints, by combining the trust region method and the Levenberg-Marquardt method.

Algorithm. The solution is found by iterating backwards starting from some terminal conditions. The objects that need initialisation are as follows. The algorithm starts from the outcomes of a finite economy which means the following terminal values: $H = H^D = 0$, $H^{ND} = 1$, $v^D = u(y)$, $v^{ND} < v^D$, and $d_{01} = 1$. The algorithm in each iteration solves the following sequence: updating of debt prices; agent decision; lender values. Specifically, initialise H^D, H^{ND}, H, d_{01}, v^D, v^{ND} and iterate over the following updating sequence:

- Prices
  - $H^D, H^{ND}, d_{01} \mapsto \tilde{q}$
  - $\tilde{q} \mapsto q$
- Sovereign:
  - $q, v^D, v^{ND}, d_{01} \mapsto v^D, b^D, d^D, d$
  - $q, v^D, v^{ND}, d_{01} \mapsto v^{ND}, b^{ND}$
  - $v^D, v^{ND}, b^D, b^{ND}, d^D \mapsto d_{01}, b, d$
- Lender's value:
  - $H, b^D, d^D \mapsto H^D$
  - $H, b^{ND} \mapsto H^{ND}$
  - $H^D, H^{ND}, d_{01} \mapsto H$

Convergence. Regarding convergence, one clear challenge is the feedback between the agent's decisions d and b and the contract valuation H and therefore prices q. The practical procedure must involve a first long run of iterations, possibly followed by some damping. We have found that damping is not necessary for our main results.

Convexity and multiple local optima. The value function (as a function of debt a) may become eventually flat, so it must feature a convex section. This is because when default approaches 100% the penalty cannot increase further even if debts increase. The convex section can create multiplicity and discontinuous decisions.This may be relevant in equilibrium because, given that debt is long term, there is positive recovery even at high-default positions so debt prices do not fall fast enough to prevent this type of situations. Nonetheless, in our calculations these problematic regions appear to lie outside of the ergodic set and pose no difficulty.

This research has utilised Queen Mary's Apocrita HPC facility http://doi.org/10.5281/zenodo.438045.

C Haircut and maturity extensions

Remember coupon payments are such that: $a_{t+1} = \delta a_t + b_t + (1-\delta)\kappa d_t a_t$.

Without default risk. It is useful to start with the case without default risk. The value of the payments to a bond that pays one unit 1 today is:

$$val = \sum_{t=1}^{\infty} \left(\frac{\delta}{1+r} \right)^{t-1} = \frac{1+r}{1+r-\delta}$$

Its duration measured by the Macaulay duration:

$$dur \equiv \frac{1}{val} \sum_{t=1}^{\infty} t \left(\frac{\delta}{1+r} \right)^{t-1} = \frac{\sum_{t=1}^{\infty} t \left(\frac{\delta}{1+r} \right)^{t-1}}{\sum_{t=1}^{\infty} \left(\frac{\delta}{1+r} \right)^{t-1}} = \frac{\frac{1}{(1-\frac{\delta}{1+r})^2}}{\frac{1+r}{1+r-\delta}} = \frac{1+r}{1+r-\delta},$$

where the second equality follows from the fact that $\sum tx^{t-1} = 1/(1-x)^2$.

Old debt instruments. The present value and the duration of the of old defaulted debt instruments DD from time zero are

$$valDD = \sum_{t=1}^{N} \frac{d_t a_t}{(1+r)^{t-1}}, \qquad durDD = \frac{1}{valDD} \sum_{t=1}^{N} t \frac{d_t a_t}{(1+r)^{t-1}}.$$

New debt instruments. Regarding the new debt instruments, in each period t there is new coupons due arising from the defaults occurred in the preceding periods $t = 1, 1, 3, ..., t-1$ within the episode we denote a_t^{ND} for "new debt". In other words,

$$a_t^{ND} \equiv (1-\delta)\kappa[d_{t-1}a_{t-1} + \delta d_{t-2}a_{t-2} + \delta^2 d_{t-3}a_{t-3} + ... + \delta^{t-2}d_1 a_1].$$

Note this can be constructed recursively as $a_t^{ND} = (1-\delta)\kappa d_{t-1}a_{t-1} + \delta a_{t-1}^{ND}$ for $t = 2, ..., N+1$ with $a_1^{ND} = 0$. To calculate the current value of these new bonds at t, we subtract the part of a_t^{ND} that is defaulted at t, so it is $a_t^{ND}(1-d_t)$. The present value is calculated as

$$\begin{aligned} valND &= \sum_{t=2}^{N} \frac{a_t^{ND}(1-d_t)}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N} \sum_{t=1}^{\infty} \frac{\delta^{t-1}}{(1+r)^{t-1}} \\ &= \sum_{t=2}^{N} \frac{(1-d_t)a_t^{ND}}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N} \frac{1+r}{1+r-\delta} \end{aligned}$$

The duration of the new debt instruments is therefore

$$
\begin{aligned}
durND &= \frac{1}{valND}\left[\sum_{t=2}^{N}\frac{a_t^{ND}(1-d_t)}{(1+r)^{t-1}}t + \frac{a_{N+1}^{ND}}{(1+r)^N}\sum_{t=1}^{\infty}\frac{\delta^{t-1}}{(1+r)^{t-1}}(N+t)\right] \\
&= \frac{1}{valND}\left[\sum_{t=2}^{N}\frac{a_t^{ND}(1-d_t)}{(1+r)^{t-1}}t + \frac{a_{N+1}^{ND}}{(1+r)^N}\left(N\sum_{t=1}^{\infty}\left(\frac{\delta}{1+r}\right)^{t-1} + \sum_{t=1}^{\infty}\left(\frac{\delta}{1+r}\right)^{t-1}t\right)\right] \\
&= \frac{1}{valND}\left[\sum_{t=2}^{N}t\frac{(1-d_t)a_t^{ND}}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N}\left(N\frac{1+r}{1+r-\delta} + \frac{1}{(1-\frac{\delta}{1+r})^2}\right)\right]
\end{aligned}
$$

Haircuts for different vintages. Here we consider haircuts on different debt vintages within an episode. Consider an episode with $d_t > 0$ over $t = 1, ..., N$. Denote by $a_t^L(j)$ the coupon due in period $t = 2, ..., N+1$ of new (i.e., not-yet-defaulted) debt whose first payment was due in period j, for $j = 1, ..., N+1$. We can refer to j as the vintage of this new debt. It is a fact that $a_t^L(j) > 0$ only if $t \geq j$. More specifically, we have:

$$
a_t^L(j) = \begin{cases} \delta^{t-j}a_j & \text{if } j = 1,\ t = 2, 3, ..., N \\ \delta^{t-j}b_{j-1} & \text{if } j = 2, ..., N+1,\ t = j, j+1, ..., N+1 \end{cases}
$$

The value of defaulted debt of vintage $j = 1, ..., N$:

$$
valDD^L(j) = \sum_{t=2}^{N}\frac{d_t a_t^L(j)}{(1+r)^{t-1}}.
$$

The default on new debt generates new restructured debt corresponding to each vintage, $a_t^{ND,L}(j)$, which can be calculated recursively

$$
a_t^{ND,L}(j) = (1-\delta)\kappa d_{t-1}a_{t-1}^L(j) + \delta a_{t-1}^{ND,L}(j)
$$

for $t = 2, 3, ..., N+1$ and for $j = 1, ..., N+1$, with $a_1^{ND,L} = 0$. The value, after adjusting for current default,

$$
valND^L(j) = \sum_{t=2}^{N}\frac{a_t^{ND,L}(j)(1-d_t)}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND,L}(j)}{(1+r)^N}\frac{1+r}{1+r-\delta}
$$

Let $a_t^{ND,L} = \sum_j a_t^{ND,L}(j)$. Going back to total arrears a^{ND}, we know $a_t = a_t^{ND} + a_t^L$, so

$$a_t^{ND} = (1-\delta)\kappa d_{t-1}(a_{t-1}^{ND} + a_{t-1}^L) + \delta a_{t-1}^{ND},$$

which makes clear that the difference between a_t^{ND} and $a_t^{ND,L}$ is that the latter does not include coupons defaulted in previous periods. With $valND^L(j)$ and $valDD^L(j)$ we can calculate haircuts for the different vintages j.

D Details of the reference model

This reference model can be seen as a special case of our model that obtains after imposing certain restrictions: Default is a binary choice, either zero or total, $d \in \{0,1\}$, so there is no intensive margin in the default decision; defaulted payments are not recoverable and are fully discounted, i.e., $\kappa = 0$; furthermore, the act of defaulting today cancels the entire bond and, when in default, the sovereign is excluded from world markets and cannot therefore borrow or save, which dispenses with the joint portfolio choice of borrowing and defaulting.

We speak of a default episode as including periods when the sovereign remains excluded from markets following a default decision. Specifically, following a decision to default, the sovereign enters default status with $d = 1$, and only leaves that status with some exogenous reentry probability λ. Unlike in our benchmark model, in this reference model default episodes are therefore characterised by passive autarky and their duration is entirely exogenous. We model the penalty for being in default status in the same way as in Equation (8) of our benchmark model, that is as a loss of output in the following period.

The above assumptions deliver a reference model that is comparable to Arellano (2008) with short-term debt $\delta = 0$, and to Chatterjee and Eyigungor (2012) with long-term debt $\delta > 0$.[22] For consistency with the setting of our benchmark model, we use long-term debt.

Given the binary default choice in this reference model, we need to specifiy a value function for the default status v^D and one for the clean status v^{ND}, with the optimal value consisting of

[22] Except that they draw the distinction between coupon payments and maturity of bonds, a minor distinction for our present purpose. Another slight difference is that those papers assume that the output loss is contemporaneous with default.

the envelope $\max\{v^D, v^{ND}\}$.[23] It is known that the computation of this model with long-term debt would generally face issues of convergence and non-monotonicity in the decision rule for borrowing. We find these issues are due to the non-convexities arising from a poor approximation to the pricing of debt. In the spirit of Chatterjee and Eyigungor (2012), we introduce stochastic iid shocks to output with the purpose of smoothing out the slope of the price of debt with respect to the level of debt.[24] More specifically, the expression for income in Equation (8) is now replaced by $y_{t+1} = (z_{t+1} + e_{t+1})\Psi(d_t, z_{t+1})$ where, besides the usual persistent Markov shock z_{t+1}, the new added shock e_{t+1} follows a zero-mean Normal distribution with standard deviation σ_e.

For the numerical implementation, the model is set up to an annual frequency. Accordingly, the interest rate $r = 0.04$, and the maturity parameter $\delta = 0.8341$ in order to match the duration of debt of 5.05 years of our benchmark model. The reentry probability $\lambda = 0.20$ implies a length of default episodes of 5 years, similar to the endogenous length in our benchmark model. σ is taken unchanged from Table 4, and $\kappa = 0$ by assumption.

For income, we aim at a persistence of 0.74 and standard deviation of about 0.07.[25] These targeted moments will be mainly influenced by the persistence and volatility, ρ_z and σ_η, of the process z, and by the volatility σ_e of the new iid shock e. We choose a technical $\sigma_e = 0.05$ which generally works to deal with the non-convexities. Since this shock raises the volatility of the income process and lowers its persistence, we need to adjust the parameters ρ_z and σ_η to correct for that. Our choice $\rho_z = 0.85$ and $\sigma_\eta = 0.04$ results in an autocorrelation of output of 0.66 and a standard deviation of 0.10 which we see as a good compromise given the tension in matching the two moments. Finally, the number of points of the Markov chain approximating the process for z has been doubled from 10 to 20 since this appears to help overcome the numerical difficulties discussed.

The remaining parameters are the discount rate β and the parameters of the penalty function $\Psi(z', d)$ in (19), γ, ϕ_0, ϕ_1, and z^*. The parameter γ is irrelevant since default can only be 100%, and ϕ_0 can be set to zero in keeping with the specification typical of existing literature.[26] We are thus left with three parameters β, ϕ_1 and z^* which we pin down to match the average level of debt over output, the standard deviation of spreads, and the correlation of spreads with output,

[23]As explained in Appendix B, we are already making this distinction in the the benchmark model.

[24]As it transpires from Appendix B, the intensive margin in the (partial) default decision in our benchmark model appears to overcome this type difficulty without need of resorting to such additional shocks.

[25]The moments at quarterly frequency in Table 5 have been converted to annual frequency moments so that persistence is 0.93^4 and the standard deviation becomes $0.08 \times \sum_{j=0}^{2\times(4-1)} 0.93^j$. See, for instance, Jorda and Marcellino (2004).

[26]At this point, it is worth noting that the cost function in Chatterjee and Eyigungor (2012) is the special case

targets that have also been used for our benchmark model. The resulting calibrated parameters are $\beta = 0.8731$, $\phi_1 = 1.55077$ and $z^* = 0.8 \times \bar{z}$. Table 10 below shows the targeted moments in the data and in the calibrated reference model. The one visible discrepancy is in the underestimation of the standard deviation of the spreads.

Table 10: Reference Model - Calibration Moments

	Data	Model
Debt-to-output mean	0.35	0.33
Spread standard deviation	0.036	0.020
Correlation output-spread	-0.29	-0.26

of our specification (19) when $\phi_0 = 0$. In their papern, given the shock z, output is

$$\text{output} = \begin{cases} z & \gamma_0 + \gamma_1 z < 0 \\ z(1 - \gamma_0 - \gamma_1 z) & \text{otherwise} \end{cases}$$

Defining the cost threshold as $z^* \equiv -\gamma_0/\gamma_1$, the term in the second line becomes $z(1 + \gamma_1 z^* - \gamma_1 z)$. This is identical to output in our benchmark model $z\Psi(z, d)$ according to (19) once we substitute $\phi_1 = \gamma_1$.