

Federal Reserve Bank of Minneapolis  
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## DEBT CONSTRAINED ASSET MARKETS

Timothy J. Kehoe and David K. Levine\*

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### ABSTRACT

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We develop a theory of general equilibrium with endogenous debt limits in the form of individual rationality constraints similar to those in the dynamic consistency literature. If an agent defaults on a contract, he can be excluded from future contingent claims markets trading and can have his assets seized. He cannot be excluded from spot markets trading, however, and he has some private endowments that cannot be seized. All information is publicly held and common knowledge, and there is a complete set of contingent claims markets. Since there is complete information, an agent cannot enter into a contract in which he would have an incentive to default in some state. In general there is only partial insurance: variations in consumption may be imperfectly correlated across agents; interest rates may be lower than they would be without constraints; and equilibria may be Pareto ranked.

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## 1. Introduction

It is well known that changes in individual consumption are imperfectly correlated with those in aggregate consumption. One way to model this phenomenon is to impose constraints on the collection of debts. The goal of this paper is to provide a theoretical foundation for such models. Our theory considers an infinite horizon general equilibrium model in which endogenous debt limits have the form of individual rationality constraints similar to those in the dynamic consistency literature. If an agent defaults on a contract, he can be excluded from future contingent claims markets trading and can have his assets seized. He cannot be excluded from spot market trading, however, and he has some private endowments that cannot be seized. All information is publicly held and common knowledge, and there is a complete set of contingent claims markets. Since there is complete information, an agent cannot enter into a contract in which he would have an incentive to default in some state. In general there is only partial insurance: variations in consumption may be imperfectly correlated across agents; interest rates may be lower than they would be without constraints; and equilibria may be Pareto ranked.

We draw a distinction between economies in which the aggregate social endowment exceeds the sum of the individual private endowments and those in which the two are the same. In the former case, we say that the economy has positive assets. The interpretation is that private endowments represent things that cannot be seized in order to collect debts; the remaining part of the social endowment represents assets that can be seized. Our point of view is similar to that taken in the study of the time consistency of government debt by Chari, Kehoe, and Prescott (1987) and by Chari and Kehoe (1988). Here we focus on the extent to which private debt and insurance markets can be enforced by the threat of exclusion from intertemporal and interstate trade. This limits attention to allocations that are not only socially feasible, but that are individually rational in the sense that,

at each time and in each state, each agent receives at least the present value of utility he could receive by trade on spot markets alone.

We examine both allocations that are efficient relative to the individual rationality constraints, and competitive equilibria in which the budget set reflects these constraints as well as the usual income constraints. We then relate these ideas to each other: In the case of what we call identically homothetic preferences, an allocation is constrained efficient if and only if it can be decentralized as a constrained competitive equilibrium with transfer payments. An important special case is when there is a single good.

The case in which preferences are not identically homothetic is more complicated. We show that the technical role played by the notion of efficiency in the analogues of the two welfare theorems is assumed here by a technical concept that we call conditional efficiency. We also show by means of two examples that conditional efficiency is neither necessary nor sufficient for constrained efficiency. We then use technical results analogous to the two welfare theorems, which relate conditionally efficient allocations to equilibria, to argue that these two examples serve as counter-examples to the two welfare theorems in their usual form.

Examining efficient allocations, we find a kind of "Folk theorem," most clearly related to that in Friedman (1971) and Fudenberg and Maskin (1986): for discount factors close enough to one, first best allocations may be sustained. We show by example that, even when the first best is not attainable, intertemporal and interstate trade is still possible. Consequently, this is a legitimate model of partial insurance. The story of limited intersectoral insurance is an attractive one: A successful sector will not pay an unsuccessful one unless sufficient future compensation is offered. This limits the amount of insurance that can be provided, however. If too much debt is incurred, it will not be repaid.

There are alternative explanations of the imperfect correlation between changes in individual and in aggregate consumption. Within the Arrow-Debreu framework, state dependent preferences are a possible explanation. It is also true that if there are many goods, individual consumption may be imperfectly correlated with aggregate consumption merely because of price fluctuation in spot markets and income effects on spot demand. Neither of these explanations seem particularly plausible in light of the fact that the decline of entire industries and regions seem to have a permanent effect on the welfare of individual workers: Why are the unsuccessful sectors not paid off by the successful ones? It is, of course, possible to explain the facts by means of the incomplete market theory introduced by Radner (1972), but this begs the question: Why are some markets closed? With the notable exception of Allen and Gale (1988), there has been little research on this question.

One approach to explaining imperfect insurance relies on private information problems, either in the form of moral hazard or of adverse selection. Green (1987), Hammond (1987), and Banerjee and Newman (1988) have explored models of this type, and there are some general theorems in the finite horizon case due to Prescott and Townsend (1984a,b). These private information models are complicated to begin with, and Prescott and Townsend (1984a,b) argue that to decentralize equilibria it may be necessary to introduce price lotteries. Such schemes may, however, be difficult to enforce in practice. From the empirical point of view, it is unclear that this explanation really comes to grips with the problem of the absence of intersectoral insurance. Would there really be an important moral hazard problem, for example, if line workers (who are heavily monitored anyway) received payments contingent on the overall economic success of their industry or region?

Yet another framework that leads to partial insurance is the overlapping generations model with uncertainty. Since one can argue that in practice a major problem is that of young workers (who are more mobile and enter more successful sectors) not insuring older workers (who may be stuck in a declining sector), this is a potentially attractive explanation. But it flies in the face of

substantial evidence of the importance of bequest motives (see, for example, Darby 1979 and Kotlikoff and Summers 1981), and Barro's (1974) observation that with a bequest motive overlapping generations behave as a single infinitely-lived family.

The notion of debt constraints is not new in economics. Many empirical researchers, including Hayashi (1985) and Zeldes (1989), have found evidence of debt constraints in individual consumption patterns. As a theoretical issue it has been studied primarily in monetary theory where money is the only asset and the inability to sell it short creates a debt constraint. With the exceptions of Townsend (1987) and Levine (1989), who consider a private information interpretation, the lack of a richer menu of assets in this type of model is not explained.

The line we take here is most heavily influenced by Prescott and Townsend (1984a,b) and the related work of Atkeson and Lucas (1991). Rather than focus on incentive compatibility constraints as they do, we focus on individual rationality constraints. The constraints that we impose are similar in spirit to those in the international debt literature: Schechtman and Escudero (1977) and Manuelli (1986), for example, allow countries to borrow only as much as they have an incentive to pay back, no matter what the realization of uncertainty. To simplify matters, we assume that it is possible to exclude traders from intertemporal trade, but not from spot markets. This is consistent with the observations that it is relatively easy to deny credit, somewhat more difficult to seize assets, and extraordinarily costly to tax spot trade in order to collect debts. We should point out that it is important that it is possible to prevent debtors in default from making loans and to seize their assets: Bulow and Rogoff (1989) show that merely denying credit is not a sufficient threat to create a loan market.

Our theory has interesting empirical implications. Unlike the incomplete markets model, there is no limit here on the types of contingent claims that can be traded. Rather, the amount of contingent claims of a particular type that can be sold are limited by the extent to which the debt will

be honored. One consequence is that, as in the overlapping generations model and debt constrained monetary models, real interest rates tend to be lower than the subjective discount factor: Borrowers are constrained, lenders are not. To induce lenders to offer a low quantity of loans, they are offered a low return.

In summary, we present a relatively simple theory for the case of identically homothetic preferences, a condition that is always satisfied when there is just a single good. The first and second welfare theorems hold, and there are second best equilibria in which intertemporal markets are active. This theory has the potential to explain the existing lack of correlation between private and aggregate consumption. The picture when there are income effects is more disturbing. Here we have the beginnings of a theory of mechanism design where it is impossible to prevent certain types of trade. This places a potentially intolerable burden on the price system: Not only must prices serve to facilitate trade, but they must keep individuals from renegeing on contracts. As a result, both welfare theorems may fail.

## 2. The Model

Both time and uncertainty are discrete. All information is publicly held and common knowledge. Information at time  $t = 1, 2, \dots$ , about current and future conditions is indexed by the *state*  $\eta_t$ . There are  $k$  different states. Information states form a Markov chain. The transition probability from  $\eta$  to  $\eta'$  is  $\pi(\eta' | \eta)$ . The probability distribution over states is fixed by the transition probabilities induced by a historically given initial state  $\eta_0$ . A (*state*) *history*  $s = (\eta_1, \eta_2, \dots, \eta_t)$  lists the states that have occurred up to some date  $t$ . We let  $t(s)$  denote the length of the history  $s$  and let  $S$  denote the countable set of all possible histories. The Markov transition probabilities induce a probability distribution over  $S$ :

$$\pi_s = \pi(\eta_t | \eta_{t-1}) \pi(\eta_{t-1} | \eta_{t-2}) \cdots \pi(\eta_1 | \eta_0).$$

To simplify notation, we use  $\eta_s$  to denote  $\eta_{t(s)}$ , the last state in the history  $s$ , and we write  $\sigma \succeq s$  if  $s = (\eta_1, \dots, \eta_t)$  and  $\sigma = (\eta_1, \dots, \eta_t, \eta_{t+1}, \dots, \eta_\tau)$ , that is, if  $\sigma$  is a logically possible continuation of  $s$ .

The economy has  $m$  agents and  $n$  goods in each history. Let  $x_{js}^i \geq 0$  be the consumption of good  $j$  by agent  $i$  in history  $s$ . Deleting a subscript or superscript yields a vector:  $x_s^i$  is the consumption vector by agent  $i$  in history  $s$ ,  $x^i$  is the consumption plan for all state histories, and  $x$  is the consumption plan for all agents, that is, an allocation. If  $x_s = x(\eta_s)$ , that is, if the allocation at  $s$  depends only on the current state, we say that the allocation is stationary.

The preferences of agent  $i$  are given by the von Neumann-Morgenstern utility function

$$U_i(x^i) = (1-\delta) \sum_{s \in S} \delta^{t(s)-1} \pi_s u_i(x_s^i, \eta_s).$$

Notice that  $\delta$ , the subjective discount factor, is the same for all agents.

(A.1)  $u_i(\cdot, \eta)$  is continuous, concave, strictly quasi-concave, and strictly monotonically increasing.

For  $x_s^i > 0$ ,  $u_i(x_s^i, \eta)$  is continuously differentiable.

There are two types of endowments in this economy, private endowments and assets. The private endowment of agent  $i$  in history  $s$  depends only on the current state and is denoted  $\bar{x}^i(\eta_s)$ . We assume that private endowments are strictly positive:

(A.2)  $\bar{x}^i(\eta) > 0$ ,  $i = 1, \dots, m$ ;  $\eta = 1, \dots, k$ .

(We adopt the convention that  $>$  for vectors means strictly greater in every component.) The aggregate social endowment in state  $\eta$  is denoted  $\bar{x}(\eta)$ ,

(A.3)  $\bar{x}(\eta) \geq \sum_{i=1}^m \bar{x}^i(\eta)$ ,  $\eta = 1, \dots, k$ .

Notice that we do not assume equality in (A.3): the social total may strictly exceed the sum of private endowments. The interpretation is that the private endowment  $\bar{x}^i(\eta)$  represents goods and services, such as labor, that cannot be physically disassociated from the agent. Let  $w(\eta) = \bar{x}(\eta) - \sum_{i=1}^m \bar{x}^i(\eta)$ ; these are assets, such as land, that are available for consumption but can change hands. Since we assume that assets can be seized for nonpayment of debts but private endowments cannot, this distinction is important in considering individual rationality constraints. If strict equality holds in (A.3), we say that the economy has no assets. If, on the other hand, for every state  $\eta = 1, \dots, k$ ,  $w(\eta) > 0$ , we say that the economy has positive assets.

The assumptions that  $\bar{x}^i(\eta) > 0$  and, if there are positive assets, that  $w(\eta) > 0$  are, unfortunately, not compatible with thinking of private endowments and assets as being composed of distinct goods. These assumptions are overly strong, however. They are made only to ensure that the consumer's income is strictly positive in any quasi-equilibrium and, in the case of positive assets, that the value of his assets is strictly positive in any quasi-equilibrium. It is trivial to replace these assumptions with any of a number of assumptions about monotonicity of utility and irreducibility of the economy and obtain the same results.

Our basic supposition is that private contracts can neither prohibit agents from engaging in trades in spot markets nor tax agents' endowments. They can, however, exclude agents from engaging in intertemporal transactions and tax agents' assets. This means that agents can be denied credit and that any assets held by agents can be attached for the payment of past debts. We could interpret this supposition as a reasonable representation of the way in which actual asset markets work: Although it is possible in principle to garnish wages to collect past debts, it is not very practical to do so. Debtors' prisons no longer exist, and, indeed, modern bankruptcy laws make it possible for agents to preserve some assets even in the face of bankruptcy.

Within the context of this paper, however, it is preferable to interpret this supposition solely in terms of the physical structure of the environment: There is a continuum of identical agents of each type  $i$ . On one hand, trades on spot markets are anonymous, and private endowments cannot be physically disassociated from agents. On the other hand, agents must identify themselves to make contracts and to collect on them in contingent claims markets. Therefore, creditors can seize the assets of debtors who default on their debts and keep track of any future attempts of these debtors to enter contingent claims markets. Furthermore, these debtors would never find it possible to pay off their creditors in order to be allowed renewed access to contingent claims markets: Since there is complete information about what would happen at every state history and every conceivable type of asset is allowed, the possibility of such a mutually advantageous transaction would already have been part of the original contingent claims contract between the two.

To describe the individual rationality constraints, we must add to the model the notion of spot prices. Let  $p_{js} \geq 0$  denote the spot price of good  $j$  at history  $s$ . We denote by  $p$  the infinite vector of spot prices. Letting  $y_s^i$  denote the income of agent  $i$  in history  $s$ , we define the *indirect utility function*  $v_i(p_s, y_s^i, \eta_s)$  as the solution to

$$\max u_i(x_s^i, \eta_s)$$

subject to

$$p_s \cdot x_s^i \leq y_s^i.$$

We say that the allocation-price pair  $(x, p)$  is *spot market supporting* if

$$(SS) \quad u_i(x_s^i, \eta_s) = v_i(p_s, p_s \cdot x_s^i, \eta_s)$$

for each state history  $s$  and agent  $i$ . In other words, no agent has any incentive to recontract in spot markets.

An allocation-price pair  $(x, p)$  is said to be *(interim) individually rational* if, for each state history  $s$  and agent  $i$ ,

$$(IR) \quad (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} U_i(x_{\sigma}^i, \eta_{\sigma}) \geq (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} V_i(p_{\sigma}, p_{\sigma} \cdot \bar{x}^i(\eta_{\sigma}), \eta_{\sigma}).$$

This condition requires that the utility of the allocation  $x$  be at least as high as that which can be attained in spot markets alone.

It is worth pointing out that our assumptions of no intertemporal production and additively separable preferences play important simplifying roles here. The right-hand side of the individual rationality condition (IR) is relatively simple because it depends only on private endowments (and prices) in the state history  $\sigma$ . If preferences were not separable, the optimal plan upon withdrawing from the economy would depend upon consumption prior to that time. Similarly, if private intertemporal production were possible, plans between different periods would again be interdependent. In either case the ability of private individuals to substitute between periods independent of the economy at large would enable them to provide a degree of self-insurance to substitute for insurance provided by markets. The effect would be to strengthen the individual rationality constraint; that is, fewer social plans would be admissible in a world in which self-insurance is available to individuals who elect not to participate in the economy. In this sense, the assumptions of no intertemporal production and additively separable preferences should be seen as an extreme case, and indeed the least favorable one for our theory.

The assumption that all agents have the same discount factor is a similar extreme case. If the economy consisted of two class of consumers, for example, one more impatient than the other, then the impatient consumers would want to borrow from the patient consumers and to then default. Like the assumptions of additively separable preferences and no production, the assumption of

identical discount factors gives the least possible role for debt constraints to make a difference on equilibrium outcomes. That we have a rich theory even under these extreme assumptions indicates that our specification of debt constraints would have even more impact on equilibria in economies where these assumptions are relaxed.

An allocation is *socially feasible* if for each state history  $s$

$$(SF) \quad \sum_{i=1}^m x_s^i \leq \bar{x}(\eta_s).$$

An allocation-price pair  $(x,p)$  is said to be *admissible* if it satisfies spot market supporting, individual rationality, and social feasibility. An allocation-price pair is *efficient* if it is admissible and cannot be dominated by any other admissible allocation-price pair. It is *conditionally efficient* (that is, conditional on a given vector of spot prices  $p$ ) if it is admissible and it cannot be Pareto dominated by any allocation  $\bar{x}$  such that  $(\bar{x},p)$  satisfies  $\sum_{i=1}^m p_s \cdot \bar{x}_s^i \leq p_s \cdot \bar{x}(\eta_s)$  and (IR). Conditional efficiency is stronger than efficiency in the sense that  $\bar{x}$  need not be admissible because  $(\bar{x},p)$  may violate (SS), or indeed (SF). On the other hand, it is weaker than efficiency in the sense that an allocation individually rational at prices other than  $p$  might Pareto dominate  $x$ . Finally, an allocation is *first best* if it cannot be Pareto dominated by any other socially feasible allocation.

It is easy to show that conditional efficiency is equivalent to another concept of efficiency: We say that  $(x,p)$  is *financially efficient* if it is admissible and there exists no allocation  $\bar{y}$  of incomes to consumers in each state such that, for each  $i$  and  $s$ ,

$$\sum_{i=1}^m \bar{y}_s^i \leq p_s \cdot \bar{x}(\eta_s)$$

$$(1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} v_i(p_{\sigma}, \bar{y}_{\sigma}^i, \eta_{\sigma}) \geq (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} v_i(p_{\sigma}, p_{\sigma} \cdot \bar{x}^i(\eta_{\sigma}), \eta_{\sigma})$$

$$(1-\delta) \sum_{s \in S} \delta^{t(s)-1} \pi_s v_i(p_s, \bar{y}_s^i, \eta_s) \geq U_i(x^i)$$

with at least one of the last inequalities being strict. In other words, there is no Pareto superior reallocation of incomes in each state that satisfies individually rationality.

Although efficiency is the obvious notion of constrained efficiency for this economy, it is the notion of conditional efficiency that plays the more important role, for example, in the analogues of the two welfare theorems. (For another model in which an alternative notion usurps the position of the obvious efficiency criterion, see Grossman 1977.) If there is only one good in each state history, however, there are no spot markets, so efficiency and conditional efficiency are the same. Another situation in which efficiency and conditional efficiency coincide is when the state utility functions  $u_i$  are *identically homothetic*. This means that there are  $k$  homogeneous of degree one utility functions  $u(\cdot, \eta)$ ,  $\eta = 1, \dots, k$ , that satisfy (A.1) and  $m \times k$  monotonically increasing functions  $g_i(\cdot, \eta)$  such that

$$u_i(\cdot, \eta) = g_i(u(\cdot, \eta), \eta), \quad i = 1, \dots, m$$

and  $u_i(\cdot, \eta)$  satisfies (A.1). Notice that this condition does not in any sense reduce the model to a representative consumer model. As we shall see, however, it does effectively reduce it to a one-good model.

The distinction between efficiency and conditional efficiency arises because of the role played by spot prices in the admissibility concept. One way to avoid the problems caused by this distinction would be to deny defaulters access to spot markets. This would replace  $v_i(p_\sigma, p_\sigma \cdot \bar{x}^i(\eta_\sigma), \eta_\sigma)$  in the rationality constraints with  $u_i(\bar{x}^i(\eta_\sigma), \eta_\sigma)$ ; spot prices would not show up. This would involve a very different set of assumptions about the environment and the nature of the enforcement technology.

A consumption plan  $x^i$  may be thought of as a map from  $S$  to  $\mathbb{R}^n$ , and can be viewed as an element of  $\mathbb{R}^\infty$ . Restricting traders to bounded consumption plans puts  $x^i$  in  $\ell_\infty$ . (Recall that  $\ell_\infty$

is the Banach space of sequences  $(y_1, y_2, \dots)$  with  $|y_i|$  bounded and  $\|y\| = \sup |y_i|$ ; see, for example, Dunford and Schwartz 1957.) Intertemporal prices  $q$  are then an element of the dual of this space,  $\ell_\infty^*$ , the space of bounded linear functions on  $\ell_\infty$ . (Recall that, although  $\ell_\infty^*$  contains  $\ell_1$ , the Banach space of sequences  $(y_1, y_2, \dots)$  with  $\sum_{i=1}^\infty |y_i|$  bounded and  $\|y\| = \sum_{i=1}^\infty |y_i|$ , it unfortunately also contains functions not in  $\ell_1$ .) Letting  $e_{js}$  be the plan consisting of consuming one unit of good  $j$  in history  $s$  and zero of all other goods in all other histories, we define  $q_{js} = q(e_{js})$  and  $q_s = (q_{1s}, \dots, q_{ns})$ .

A *constrained transfer equilibrium* is a triple  $(x, p, q)$  with the allocation  $x \in \ell_\infty$ , spot prices  $p \in \mathbb{R}^\infty$ , and intertemporal prices  $q \in \ell_\infty^*$  such that the allocation is socially feasible (SF); it exhausts the value of the social endowment,

$$(E.1) \quad \sum_{i=1}^I q(x^i) = q(\bar{x});$$

each agent's allocation  $x^i$  maximizes utility subject to a budget constraint and interim individual rationality constraints,

$$(E.2) \quad \max U_i(z^i)$$

subject to

$$q(z^i) \leq q(x^i)$$

$$(1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_\sigma u_i(z_\sigma^i, \eta_\sigma) \geq (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_\sigma v_i(p_\sigma, p_\sigma \cdot \bar{x}(\eta_\sigma), \eta_\sigma), \quad s \in S;$$

and the spot prices  $p$  are consistent with the intertemporal prices  $q$  in the sense that

$$(E.3) \quad x, z \in \ell_\infty, \quad \text{and } p_s \cdot x_s = p_s \cdot z_s \quad \text{for all } s \in S \text{ imply } q(x) = q(z).$$

Notice that this last condition implies that  $p_s$  and  $q_s$  must be proportional if  $q_s \neq 0$ .  $p_s = 0$  is impossible because of (A.1), (SS), and (SF). The interpretation of equilibria is that each agent is

constrained by his individual rationality constraint: no one would agree to trade with him in violation of this constraint, since they know that the corresponding loan would never be repaid.

A particularly important type of constrained transfer equilibrium is a constrained ownership equilibrium for a given initial asset distribution. An *initial asset distribution* assigns each agent in each state a vector of goods  $w^i(\eta) \geq 0$ , such that

$$\sum_{i=1} (\bar{x}^i(\eta) + w^i(\eta)) = \bar{x}(\eta).$$

If  $w^i(\eta) > 0$  for all  $\eta = 1, \dots, k$ , we say that agent  $i$  has positive assets. A *constrained ownership equilibrium* is a constrained transfer equilibrium  $(x, p, q)$  such that  $q(x^i) = q(\bar{x}^i + w^i)$ .

To understand the import of these definitions, it is useful to review the several (mutually equivalent) definitions of a transfer payments equilibrium in the Arrow-Debreu setting. One definition is as we have given: a socially feasible allocation  $x$  and prices  $q$  such that  $x^i$  is utility maximizing given the income  $q(x^i)$ . Alternatively, the definition may be made in terms of unit of account transfer payments: a vector of lump sum subsidies  $\sigma^i$ , a socially feasible allocation  $x$ , and a price  $q$  such that  $x^i$  is utility maximizing given the income  $q(\bar{x}^i + w^i) + \sigma^i$ . Even in our constrained setting, these two definitions are equivalent, since we can define  $\sigma^i = q(x^i) - q(\bar{x}^i + w^i)$ , or vice versa. In the Arrow-Debreu setting, there is yet a third equivalent definition, and that is to make the transfer in terms of goods; that is, a transfer payments equilibrium may be defined as a socially feasible endowment  $\bar{x}$ , a socially feasible allocation  $x$ , and prices  $q$  such that  $x^i$  is utility maximizing given the income  $q(\bar{x}^i)$ . (The real subsidy is  $\bar{x}^i - \bar{x}^i$ .) If we wish, we may take  $\bar{x} = x$ . In our setting, such a definition is not well defined or equivalent. It is not well defined, because we must specify two endowments: private endowments and assets. If we allow private endowments to be redistributed, this will change the individual rationality constraint, and in general the set of allocations that can be supported as equilibria. Moreover, this violates the spirit of the idea that private

endowments cannot be taken from the person to whom they adhere. If we allow only the redistribution of assets, there may not be enough assets to redistribute to generate all transfer payments equilibria. This may be seen most easily in case there are no assets ( $w^i = 0$ ). If we define a transfer equilibrium by means of real transfers of assets, then a transfer payments equilibrium (by this definition) is the same as an ownership equilibrium. But if we allow unit of account transfers the two definitions are not equivalent, and it is not true that when there are no assets that the set of transfer equilibria (by the original definition) are the same as the ownership equilibria: if the individual rationality constraints do not bind at all (see the example of efficient transfer equilibria below), then it will be possible to make unit of account transfers that change the allocation of resources to favor particular individuals.

Since it makes a difference in this setting whether transfers in terms of goods or in terms of the unit of account, we ought to indicate why the unit of account definition is the right one. If we wish, we may regard the transfer either as government enforced, or as an entry fee for participating in the economy. In the former case, the government police powers are limited in that it cannot violate individual rationality constraints. In neither case is there implicit any restriction that transfers be in terms of goods. The initial unit of account lump sum tax is simply a debt that must be paid by the individual in question. In equilibrium it is possible to pay the debt, and doing so is preferable to not joining the economy. Like any other debt in our framework, this debt is denoted in units of account. Notice, incidentally, that another interpretation of a transfer equilibrium is simply as an equilibrium of an economy in which there is preexisting debt. The set of constrained transfer equilibria simply characterizes the types of preexisting debt that can potentially be repaid in an equilibrium.

### 3. A Cyclic Economy

Characterizing equilibria is difficult in our general framework. In this section we explore the properties of a deterministic economy with two states that alternate over time. Although the equilibrium that we study is neither a first-best equilibrium nor a barter equilibrium, where intertemporal markets do not operate, its allocation is stationary. Furthermore, although there are no assets, the equilibrium is conditionally efficient. Because there is only one good, it is also efficient. Perhaps the most interesting feature of this economy is that the interest rate is lower in the constrained equilibrium than it is in the unconstrained equilibrium.

Example 1. There are three states:  $\eta_0 = 3$ ,  $\pi(1|3) = \pi(2|3) = 1/2$ , and  $\pi(1|2) = \pi(2|1) = 1$ . In other words, there is a 50-50 chance of the state being 1 or 2 in the first period, and after that they alternate. There are two agents and one good. Each agent has the same utility function, which does not depend on that state,

$$u_i(x, \eta) = \log x, \quad i = 1, 2; \eta = 1, 2.$$

It is the private endowment that varies with the state,  $\bar{x}^1(1) = \bar{x}^2(2) = \bar{x}_1$  and  $\bar{x}^1(2) = \bar{x}^2(1) = \bar{x}_2$ .

There are no assets.

If there were no individual rationality constraints, we could solve for the unique equilibrium, which is also the unique symmetric Pareto optimum. It is

$$q_{jt} = \delta^{t-1}, \quad j = 1, 2; t = 1, 2, \dots$$

$$x_{jt}^i = (\bar{x}_1 + \bar{x}_2)/2, \quad i = 1, 2; j = 1, 2; t = 1, 2, \dots$$

Notice that the constant rate of interest is  $q_{jt}/q_{jt+1} - 1 = 1/\delta - 1$ .

When we add the individual rationality constraints, the problem faced by agent  $i$  becomes

$$\max (1/2)(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \log x_{1t}^i + (1/2)(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \log x_{2t}^i$$

subject to

$$\sum_{t=1}^{\infty} (q_{1t}x_{1t}^i + q_{2t}x_{2t}^i) \leq \sum_{t=1}^{\infty} (q_{1t}\bar{x}_{1t}^i + q_{2t}\bar{x}_{2t}^i)$$

$$(1-\delta) \sum_{r=t}^{\infty} \delta^{r-t} \log x_{jr}^i \geq (1-\delta) \sum_{r=t}^{\infty} \delta^{r-t} \log \bar{x}_{jr}^i, \quad j = 1, 2; t = 1, 2, \dots$$

$$x_{jt}^i \geq 0.$$

The first-order conditions are

$$(1-\delta)\delta^{t-1} \frac{1}{2x_{jt}^i} - \lambda_i q_{jt} + (1-\delta) \left[ \sum_{r=1}^t \delta^{t-r} \mu_{jr}^i \right] \frac{1}{x_{jt}^i} = 0, \quad j = 1, 2; t = 1, 2, \dots$$

Notice that the number of Lagrange multipliers  $\mu_{jt}^i$  in these conditions gets larger as  $t$  gets larger;  $x_{jt}^i$  shows up in more and more rationality constraints.

For this example to be interesting it must be the case that the unconstrained equilibrium violates the individual rationality constraints. Suppose that  $(\bar{x}_1, \bar{x}_2) = (15, 4)$  and  $\delta = 1/2$ . Then the utility of agent  $i$  in state  $i$  is  $\log(19/2)$  in the unconstrained equilibrium. Suppose that  $\eta_1 = i$  so that  $\bar{x}_i = 15$  then the utility of agent  $i$  in autarky is  $(2/3) \log 15 + (1/3) \log 4$ . This exceeds the utility in the unconstrained equilibrium since  $15^2 4 > (19/2)^3$ . Consequently, the unconstrained equilibrium violates the individual rationality constraints.

Consider, however, the allocation  $x^1(1) = x^2(2) = 10$ ,  $x^1(2) = x^2(1) = 9$ . The utility of agent  $i$  in state  $i$  is  $(2/3) \log 10 + (1/3) \log 9$ , which equals the utility in autarky since  $10^2 9 = 15^2 4$ .

The individual rationality constraints of agent  $i$  in state  $j$ ,  $j \neq i$ , are obviously satisfied since the agent has no incentive to renege on the contract when it is his turn to collect.

Notice that since the rationality constraints only bind every other period, the multipliers  $\mu_{jt}^i$  can be nonzero only every other period. Let  $q_t = (5/9)^{t-1}$ ,  $t = 1, 2, \dots$ , and

$$\sum_{\tau=1}^t 2^{\tau-t-1} \mu_{j\tau}^i = (10/9)^t - 1, \quad t = 1, 3, 5, \dots$$

if  $i = j$ . If  $i \neq j$ , the formula holds for  $t = 2, 4, 6, \dots$ . The allocation then satisfies the first order conditions when  $\lambda_1 = \lambda_2 = 1/9$  and satisfies the budget constraints. Consequently, it is an equilibrium. Notice that the constant interest rate in this equilibrium is  $9/5 - 1 = 4/5$ , which is less than the unconstrained equilibrium interest rate  $2 - 1 = 1$ .

If the two consumers are patient enough, then the threat of exclusion from intertemporal trade is enough to enforce the first-best allocation as an equilibrium. We can calculate the value of  $\delta$  for which the unconstrained equilibrium satisfies the individual rationality constraints:

$$\log(19/2) = (1-\bar{\delta}) \sum_{t=1}^{\infty} \bar{\delta}^{2t-2} \log 15 + (1-\bar{\delta}) \sum_{t=1}^{\infty} \bar{\delta}^{2t-1} \log 4.$$

It is

$$\bar{\delta} = \frac{\log 15 + \log 2 - \log 19}{\log 19 - \log 4 - \log 2} = 0.52805.$$

For all  $\delta$  such that  $\bar{\delta} \leq \delta < 1$ , the unconstrained equilibrium is the unique equilibrium of the debt constrained economy.

#### 4. Efficiency

Our goal is to show that efficient allocation-price pairs exist and describe in a rough way what they are like. For discount factors close enough to one, a kind of "Folk theorem" shows that

there are spot prices such that some first best allocations are interim individually rational. The proofs of the following two results can be found in the appendix.

Proposition 1. Efficient allocation-price pairs exist.

Proposition 2. Suppose that the Markov chain  $\pi$  has a single ergodic class and no transient states. Suppose further either (i) that every agent has positive assets or (ii) that  $u_i$  is strictly concave. Then there is a discount factor  $\bar{\delta} < 1$  and an allocation-price pair  $(x,p)$  with  $x$  stationary such that for all  $\delta$ ,  $\bar{\delta} \leq \delta < 1$ ,  $(x,p)$  is efficient, conditionally efficient, and first-best.

Recall that we have assumed that individuals are infinite-lived. What role does this play in the analysis? It is not hard to show that Proposition 2 continues to hold in an economy with finite-lived agents and strictly positive assets, provided that, in addition to having a discount factor sufficiently close to one, agents are sufficiently long-lived. With no assets and strictly concave utility, however, Proposition 2 will not hold with finite lifetimes: in the final period of life no debts will be honored, so no lending or borrowing will take place in the previous period, and so forth. In other words, and more generally, if there are no assets, nonbarter equilibria in this model unravel in the finite horizon in a way similar the way that equilibria in finite horizon monetary models do, and for similar reasons. On the other hand, with even a small amount of assets, this will not happen, in much the same way that a small amount of government backing of the money supply will prevent finite horizon unraveling in monetary models. (Notice, however, that in such models as the overlapping generations model with money it is the time horizon for the economy as a whole that is important, here it is the length of an agent's life.) In this sense the infinite life assumption should not be viewed as crucial, since the case of positive assets is the most interesting one.

We now turn our attention to the relation between our concepts conditional efficiency and efficiency. The next result, whose proof can be found in the appendix, provides an interesting case where the two concepts are the same.

**Proposition 3.** If utility functions are identically homothetic, then an allocation-price pair is efficient if and only if it is conditionally efficient.

If utility functions are not identically homothetic, then there may be efficient allocation price pairs that are not conditionally efficient, as the next example demonstrates.

**Example 2.** There are four states. The initial state is  $\eta_0 = 1$ . Conditional on 1 there is a 50–50 chance of either 2 or 3;  $\pi(2|1) = \pi(3|1) = 1/2$ . Conditional on 2 or 3, state 4 occurs and is absorbing;  $\pi(4|2) = \pi(4|3) = \pi(4|4) = 1$ . In states 2 and 4 there is one good. In state 4 there are no assets. This effectively reduces the economy to a one-period, two-state economy because nothing can happen in state 4, and state 1 is unreachable. (That this example violates the assumption of the same number of goods in every state is, of course, inconsequential; we could modify easily our assumptions or include an additional good in states 2 and 4.)

Each consumer has preferences

$$U_i(x) = (1/2)a^i x_2^i - (1/2)(40)(b_1^i (x_{13}^i)^{-4} + b_2^i (x_{23}^i)^{-4}),$$

where  $(a^1, b_1^1, b_2^1) = (1, 1, 243)$  and  $(a^2, b_1^2, b_2^2) = (2, 243, 1)$ . In state 2 each consumer has a private endowment of 1, and the social endowment is 4. State 3 has two goods, as illustrated in the Edgeworth box in Figure 1. Consumer 1 has the private endowment  $\bar{x}_1(3) = (1, 12)$ , and consumer 2 has  $\bar{x}_2(3) = (11, 1)$ . The social endowment is  $(13, 13)$ . Corresponding to the endowment point  $E_1$ ,

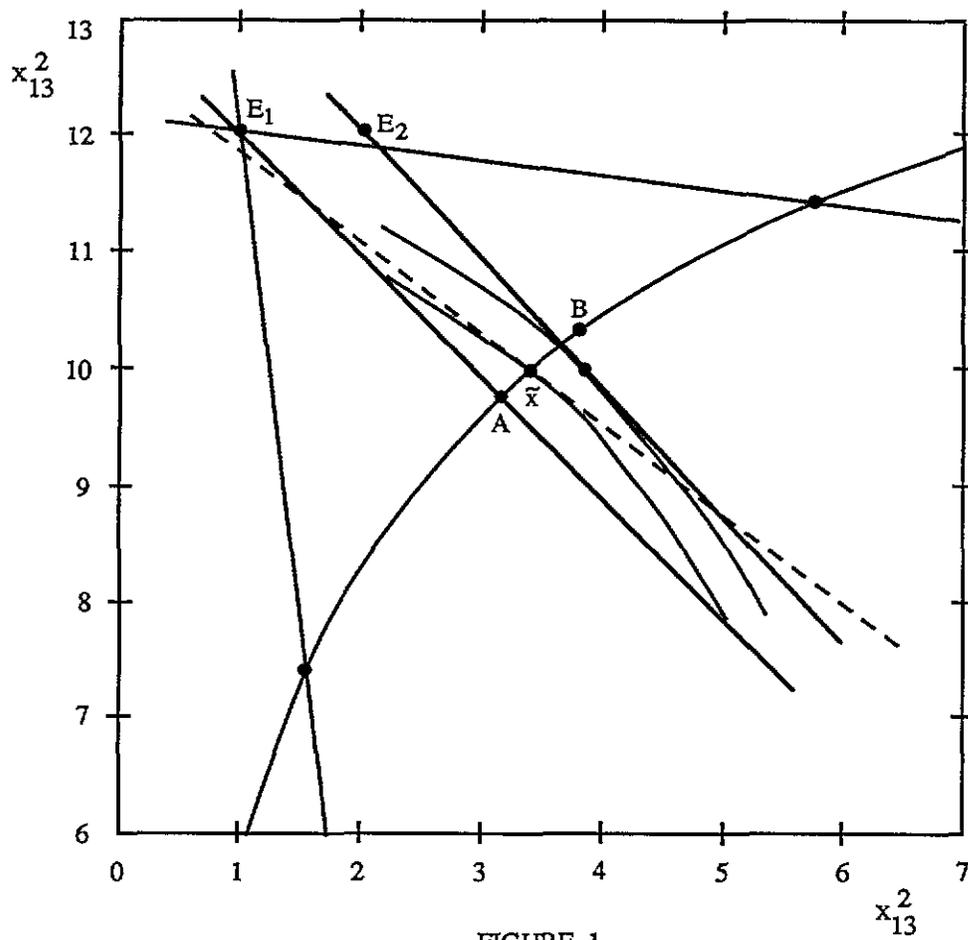


FIGURE 1

which assigns the assets (1,0) to consumer 2, there are three static competitive equilibria, as illustrated in Figure 1.

Consider now the allocation-price pair where  $x_2^1 = x_2^2 = 2$  and  $(x_3^1, x_3^2) = ((3.25, 9.75), (9.75, 3.25))$  together with the spot prices  $p_3 = (1,1)$ . We want to show that this allocation-price pair, which corresponds to point A in Figure 1, is efficient but not conditionally efficient. Let us first argue that this allocation-price pair is not conditionally efficient: The consumers' preferences have been constructed so that some movement up the contract curve in state 3 along with agent 1 paying agent 2 in state 2 is Pareto improving. In fact, for every point on the segment AB there exists a trade from agent 1 to agent 2 in state 2 so that both consumers are better off; here B is (3.96586, 10.37420). Let  $(\bar{x}_3^1, \bar{x}_3^2)$  be a point in this segment, for example,  $(\bar{x}_3^1, \bar{x}_3^2) = ((3.5, 9.98780), (9.5, 3.01220))$ . If we reallocate in state 2 so that  $\bar{x}_2^1 = 1.8$ ,  $\bar{x}_2^2 = 2.2$ , then it is easy to check that both consumers are better off. Since  $\bar{x}$  is socially feasible, it satisfies  $\sum_{i=1}^2 p_s \cdot \bar{x}_s^i \leq p_s \cdot \bar{x}(\eta_s)$ ,  $s = 2, 3$ . The allocation-price pair  $(\bar{x}, p)$  also satisfies the individual rationality constraint (IR): Neither consumer would want to default in state 2 since each receives more than the private endowment  $\bar{x}^i(2) = 1$ ,  $i = 1, 2$ . Consumer 1 would not want to default in state 3 since  $\bar{x}_3^1$  yields higher utility than does  $x_3^1$ , which is the best that he can do when faced with spot prices  $p_3 = (1,1)$  and income  $p_3 \cdot \bar{x}^1(3)$ . Consumer 2 would not want to default either, however, since then he would only have income  $p_3 \cdot \bar{x}^2(3)$  because he would lose his assets; this corresponds to point  $E_2$  in the figure. The best that consumer 2 can do with this income, (9,3), yields a lower level of utility than does  $\bar{x}_3^2$ .

The allocation-price pair  $(x, p)$  is efficient, however: Since there is no possibility of enforcing any intertemporal contracts after the first period, the only admissible allocation-price pairs correspond to static equilibria in the Edgeworth box for some endowment point between  $E_1$  and  $E_2$ . None of these allocations can Pareto dominate  $x$ : as we move up the contract curve, the relative spot

prices move in the wrong direction, as shown in Figure 1. The possibility of Pareto improving trades is exhausted long before we arrive at the next equilibrium.

Conversely, there are examples of economies with allocation-price pairs that are conditionally efficient but not efficient.

**Example 3.** There are two goods per period, two agents, and two equally probable i.i.d. states. There are no assets and a total endowment of two units of each good each period. Each agent has a fixed endowment of 1 of good 2. The endowment of good 1 is random and either 0 or 2 depending on the state: The endowment is either  $(\bar{x}^1(1), \bar{x}^2(1)) = ((0,1), (2,1))$  in state 1 or the reverse in state 2. Preferences are identical and do not depend on the state,  $u(x_1^i, x_2^i, \eta) = \log(x_1^i - \alpha) + x_2^i$ ,  $\alpha < 1$ . See Figure 2. The symmetric first best point A has  $(x_{1a}^1, x_{2a}^1) = (x_{1a}^2, x_{2a}^2) = (1,1)$  and spot prices  $p_a = (1, 1 - \alpha)$ . Corresponding to the endowment  $E_1$  in state 1 is the static competitive equilibrium B with  $(x_{1b}^1, x_{2b}^1) = (2 - x_{1b}^2, 2 - x_{2b}^2) = (1 - \alpha, 0)$  and spot prices  $p_b = (1, 1 - \alpha/2)$ .

The barter allocation switches randomly between B and its reflection depending on the state. We now argue that there is a choice of  $\delta$  such that the barter allocation-price pair is conditionally efficient. It is easy to show that, since the spot prices are constant, if there is an allocation that Pareto dominates the barter allocation and satisfies  $\sum_{i=1}^2 p_b \cdot x_s^i \leq p_b \cdot \bar{x}$  and the individual rationality constraints, then there is such an allocation that is symmetric and stationary. Let  $z$  be both the transfer from consumer 2 to consumer 1 if state 1 occurs and the transfers from consumer 1 to consumer 2 if state 2 occurs. For the corresponding allocation to satisfy the individual rationality constraints at spot prices  $p_b$ , it must be the case that

$$\begin{aligned} (1 - \delta)v(p_b, p_b \cdot \bar{x}^1(2) - z) + (\delta/2)(v(p_b, p_b \cdot \bar{x}^1(1) + z) + v(p_b, p_b \cdot \bar{x}^1(2) - z)) \\ \geq (1 - \delta)v(p_b, p_b \cdot \bar{x}(2)) + (\delta/2)(v(p_b, p_b \cdot \bar{x}(1)) + v(p_b, p_b \cdot \bar{x}(2))) \end{aligned}$$

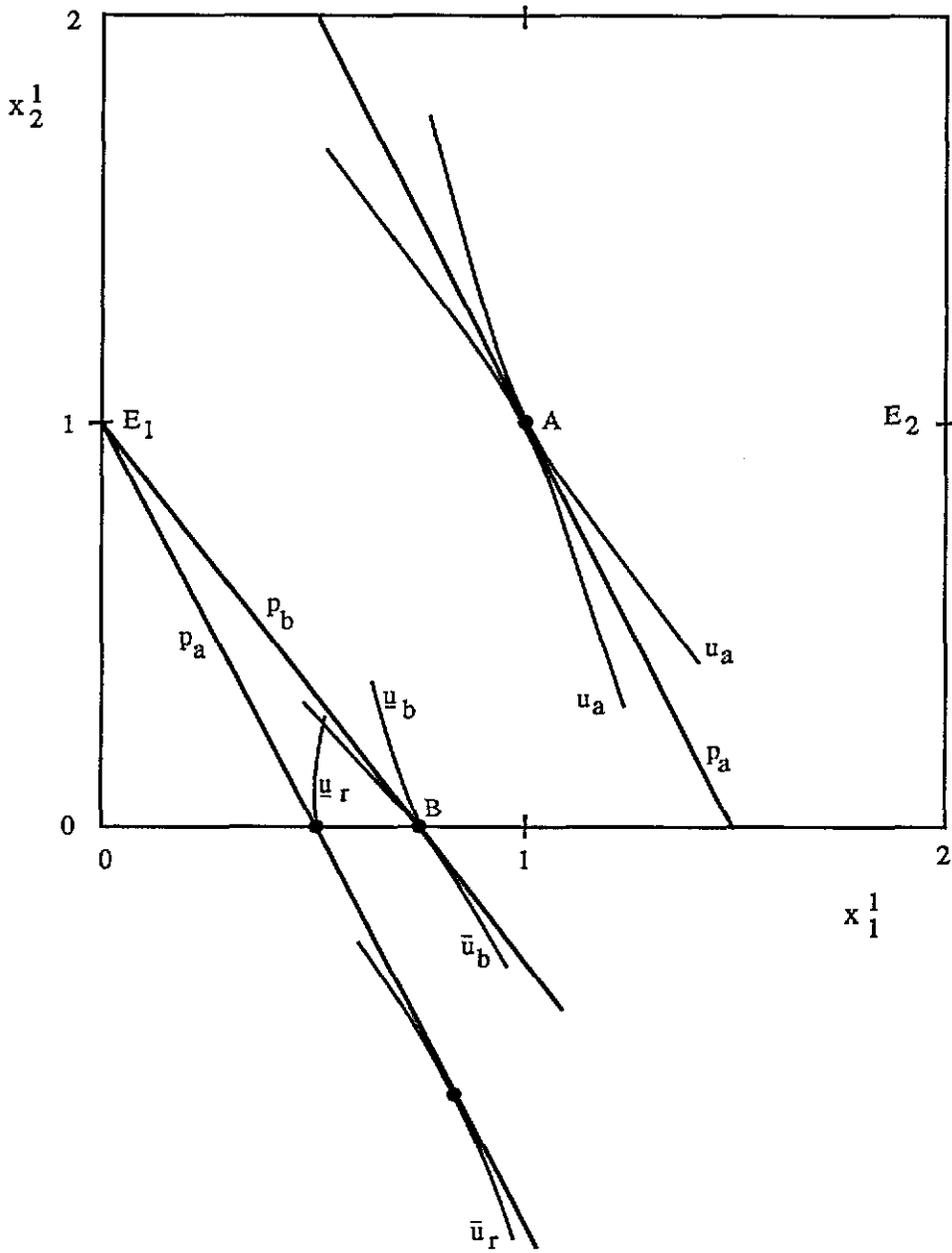


FIGURE 2

$$(1-\delta)v(p_b, p_b \cdot \bar{x}^1(1) + z) + (\delta/2)(v(p_b, p_b \cdot \bar{x}^1(1) + z) + v(p_b, p_b \cdot \bar{x}^1(2) - z)) \\ \geq (1-\delta)v(p_b, p_b \cdot \bar{x}(1)) + (\delta/2)(v(p_b, p_b \cdot \bar{x}(1)) + v(p_b, p_b \cdot \bar{x}^1(2))).$$

The concavity of utility implies that the second of these inequalities, which says that the transfer  $z$  must increase utility if the bad state occurs, holds only if  $z \geq 0$ . It is easy to show that we can always choose  $\delta > 0$  small enough so that  $z = 0$  is the only solution to these inequalities. Indeed, in our example the first inequality becomes, for  $z$  small enough,

$$(1-\delta/2) \left[ \log \left[ \frac{2-\alpha}{2} \right] + 2 - \frac{2z}{2-\alpha} \right] + (\delta/2) \log \left[ \frac{2-3\alpha+2z}{2} \right] \\ \geq (1-\delta/2) \left[ \log \left[ \frac{2-\alpha}{2} \right] + 2 \right] + (\delta/2) \log \left[ \frac{2-3\alpha}{2} \right].$$

The concavity of the utility function now implies that, if there is no improvement in utility for a small  $z$ , then there can be no improvement. Differentiating the left-hand side of this inequality with respect to  $z$ , we find that  $\delta \geq (2-3\alpha)/(2-2\alpha)$  is necessary for an improvement in welfare. Consequently, if

$$0 < \delta < \frac{2-3\alpha}{2-\alpha},$$

then the barter allocation-price pair  $B$  is conditionally efficient.

We now argue that we can choose  $\alpha$  so that it satisfies the above condition and so that the first best allocation-price pair  $A$  is also conditionally efficient. For  $A$  to satisfy the individual rationality constraints at spot prices  $p_a$ , it must be the case that

$$u(1,1) \geq (1-\delta)v(p_a, p_a \cdot \bar{x}^1(2)) + (\delta/2)(v(p_a, p_a \cdot \bar{x}^1(1)) + v(p_a, p_a \cdot \bar{x}^1(2))).$$

(That utility at  $A$  is greater than the expected discounted value of utility in the bad state is obvious.)

In our example, this inequality becomes

$$\log(1-\alpha) + 1 \geq (1-\delta/2) \left[ \log(1-\alpha) + \frac{2-\alpha}{1-\alpha} \right] + (\delta/2) \log(1-2\alpha)$$

$$\frac{\delta}{2} \left[ \log \frac{1-\alpha}{1-2\alpha} + \frac{2-\alpha}{1-\alpha} \right] \geq \frac{1}{1-\alpha}.$$

Notice that for any value of  $\delta > 0$  this inequality is satisfied for  $\alpha$  close enough to 0.5. In fact, it is easy to check that, if  $\delta = 0.5$  and  $\alpha = 0.499$ , both barter and the first best are conditionally efficient. Since the first best allocation Pareto dominates the barter allocation, however, only the first best allocation-price pair is efficient.

To keep the calculations simple, this example, has relied on a simple utility function that bounds consumption of the first good away from zero and allows corner solutions. Neither of these features is necessary, however, to produce this sort of example. All that we need is a nonhomothetic utility function such that, first, the relative spot price of the second good is higher at barter than it is at the first best and, second, the best utility obtainable in the bad state when faced with the spot prices of the first best allocation-price pair is very low. The reader should compare Figures 2 and 3 to see how such examples can be constructed.

## 5. Equilibrium

We now turn to constrained transfer equilibria. After describing prices in a rough way, we consider the decentralizability of conditionally efficient allocations. In an economy with positive assets, this is always possible: We show that analogs of the first and second welfare theorem both hold. Example 3 shows that it may be possible to Pareto-rank conditionally efficient allocations, however, so the welfare theorems do not rule out the possibility of coordination failure. In the case of strictly positive assets, we are also able to show that ownership equilibria exist.

The proof of the next result is completely standard.

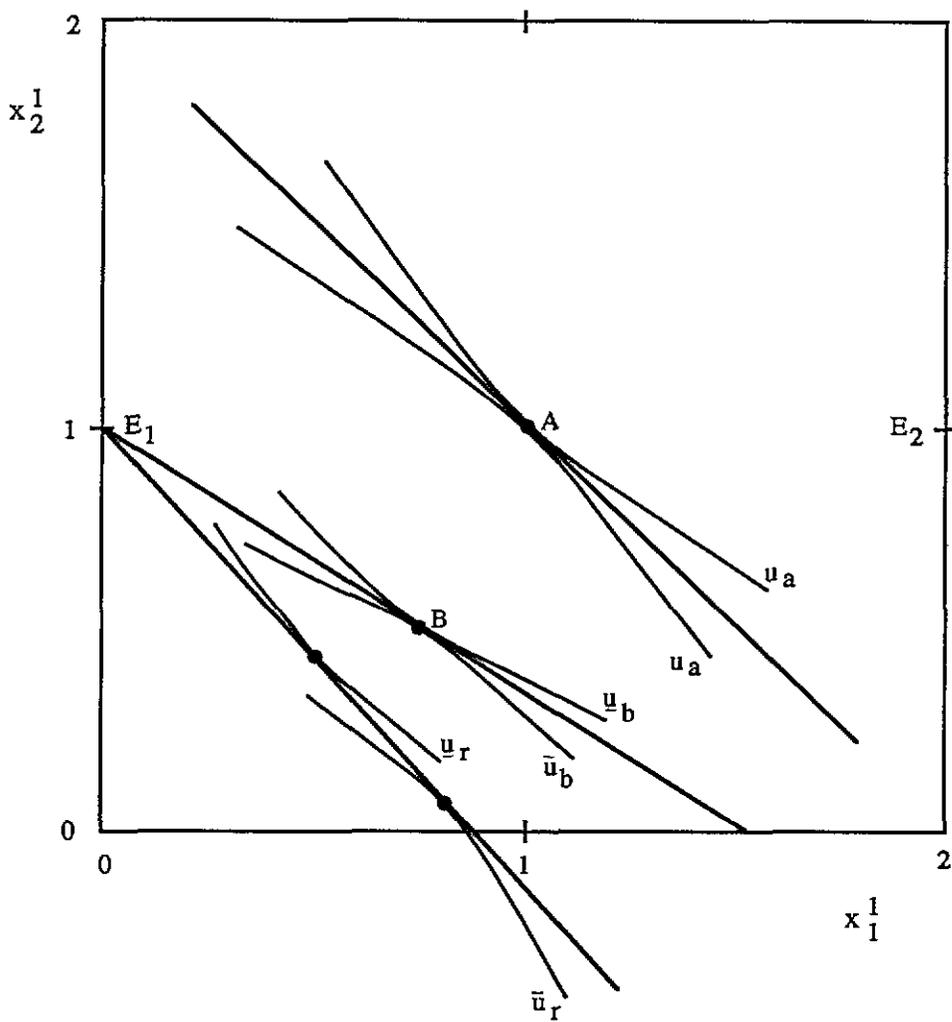


FIGURE 3

**Proposition 4 (First welfare theorem).** If  $(x,p,q)$  is a constrained transfer equilibrium, then  $(x,p)$  is conditionally efficient.

Since this result says that no allocation-price pair that is not conditionally efficient can be made into a constrained transfer equilibrium, Example 2 illustrates a failure of the usual second welfare theorem. There the efficient allocation-price pair that is not conditionally efficient cannot be decentralized as a constrained transfer equilibrium.

To prove the analogue of the second welfare theorem for conditionally efficient allocation-price pairs, we define the concept of a quasi-equilibrium: A *(constrained) quasi-equilibrium* is a triple  $(x,p,q)$  with  $q \neq 0$  that potentially differs from a transfer equilibrium in that we require only that, if  $z^i$  is individually rational and  $U_i(z^i) \geq U_i(x^i)$ , then  $q(z^i) \geq q(x^i)$ . We do not require that, if  $U_i(z^i) > U_i(x^i)$ , then  $q(z^i) > q(x^i)$ . Naturally, every transfer equilibrium is a quasi-equilibrium. The following lemma, whose proof can be found in the appendix, says that if every agent has a cheaper point in his constraint set, then a quasi-equilibrium is a transfer equilibrium (compare with Debreu 1962).

**Lemma 1.** Suppose that  $(x,p,q)$  is a quasi-equilibrium and that, some agent  $i$ , there exists  $\bar{x}^i$  that is individually rational and satisfies  $q(\bar{x}^i) < q(x^i)$ . Then  $z^i$  individually rational and  $U_i(z^i) > U_i(x^i)$  imply  $q(z^i) > q(x^i)$ . Furthermore,  $q_s \neq 0$  for any  $s \in S$ .

We now examine the structure of prices in quasi-equilibria. Denote the set of prices  $q \in \ell_{\infty}^*$  such that  $q(x^i) = q(z^i)$  if  $x^i$  and  $z^i$  are the same except in a finite number of components as  $fa$ . Recall that  $fa$  is the closed linear subspace of finitely additive measures in  $\ell_{\infty}^*$  and that  $\ell_{\infty}^* = \ell_1 + fa$ . We write  $q^1$  and  $q^f$  for the respective components of  $q \in \ell_{\infty}^*$ . For  $x^i \in \ell_{\infty}$ , we say that  $x^i \geq 0$  if  $x_s^i \geq 0$  for all  $s \in S$ ;  $\ell_{\infty+}$  is the set of all nonnegative elements of  $\ell_{\infty}$ . If  $x^i$  is interior to  $\ell_{\infty+}$ ,

we say that  $x^i > 0$ . If  $q \in \ell_\infty^*$  and  $q(x^i) \geq 0$  for all  $x^i \in \ell_{\infty+}$ , we say that  $q \geq 0$ . If, in addition,  $q(x^i) > 0$  for all  $x^i \in \ell_{\infty+}$  such that  $x^i \neq 0$ , we say that  $q > 0$ . For a proof of the following result, see the appendix.

**Lemma 2.** If  $(x, p, q)$  is a quasi-equilibrium, then  $q^l, q^f, q \geq 0$ .

We would like the finitely additive part of prices,  $q^f$ , to be zero. The following result, whose proof is presented in the appendix, shows that, at least when assets are strictly positive, this must be the case. See Bewley (1972) and Prescott and Lucas (1972) for related results.

**Lemma 3.** Suppose that there are positive assets and that  $(x, p, q)$  is a quasi-equilibrium. Then  $q^f(w) = 0$ . Furthermore, there exists  $b > 1$  such that, for all  $T$ ,

$$\sum_{t(s)=T} q_s \cdot w(\eta_s) \leq (b-1)^{-1} b^{2-T} \sum_{t(s)=1} q_s \cdot w(\eta_s).$$

That is, the value of assets in all histories in period  $T$  is bounded by the value of assets in all histories in the first period multiplied by a scalar that decreases exponentially in  $T$ .

**Proposition 5 (Second welfare theorem).** Suppose that  $(x, p)$  is conditionally efficient. Then there exist prices  $q \in \ell_\infty^*$  such that  $(x, p, q)$  is a constrained quasi-equilibrium.

**Proof.** Normalize  $p_s$  to lie in the unit simplex, and consider the one-good economy with preferences.

$$V_i(y^i) = (1-\delta) \sum_{s \in S} \delta^{t(s)-1} \pi_s v_i(p_s, y_s^i, \eta_s)$$

where  $v_i$  is the indirect utility function defined previously. The consumption sets are

$$Y^i = \left\{ y^i \in \ell_\infty \mid (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_\sigma v_i(p_\sigma, y_\sigma^i, \eta_\sigma) \geq (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_\sigma v_i(p_\sigma, p_\sigma \cdot \bar{x}^i(n_\sigma), n_\sigma) \right\},$$

which are convex and closed and have nonempty interior in  $\ell_\infty$ . The social feasibility condition is

$$\sum_{i=1}^m y_s^i \leq p_s \cdot \bar{x}(\eta_s), \quad s \in S.$$

The conditional efficiency of  $(x,p)$  implies that the allocation  $y$  with  $y_s^i = p_s \cdot x_s^i$  is efficient in this artificial one-good economy. A standard argument, due to Debreu (1954), says that there exist prices  $r \in \ell_\infty^*$  such that the allocation  $y$  and the prices  $r$  are an unconstrained quasi-equilibrium of the artificial economy. For any consumption plan in the original economy  $z^i$ , we define  $y(z^i) = p_s \cdot z_s^i$ . We define  $q \in \ell_\infty^*$  as

$$q(z^i) = r(y(z^i)), \quad \text{all } z^i \in \ell_\infty. \quad \square$$

In the usual welfare theory, the second welfare theorem is typically strengthened by assuming that the allocation to be supported assigns every agent positive consumption of some good. This ensures that every agent has a cheaper point in his consumption set and, hence, that the quasi-equilibrium is in fact an equilibrium. Here the analogous assumption is that the allocation-price pair assigns every agent a consumption plan that is larger in every component than the barter consumption plan. This assumption cannot be satisfied if there are no assets and may not be satisfied even if there are.

In one interpretation of the usual welfare theory, it is not the set of equilibria but rather the set of efficient quasi-equilibria that plays the central role. The first welfare theorem says that this set includes all equilibria. The second welfare theorem says that it is exactly the same as the set of efficient allocations. What we have shown here is that the same situation holds with debt constrained asset markets, except that the concept of conditional efficiency replaces the standard concept of efficiency. Our examples in Section 4 show that conditional efficiency and efficiency are quite distinct concepts, so this is a significant departure from the usual theory.

As noted above, Example 2 is a counterexample to the usual second welfare theorem: If the efficient allocation-price pair in that example could be decentralized as a transfer equilibrium, then it would be conditionally efficient by Proposition 5, but it is not. Similarly, Proposition 6 combined with Example 3 illustrates a failure of the usual first welfare theorem. There the allocation-price pair that is conditionally efficient but not efficient can be decentralized as a constrained transfer equilibrium: Although Lemmas 1 and 3 do not apply directly, we can nevertheless exploit the stationarity of the allocation-price pair to directly compute supporting prices in  $\ell_1$  such that each agent is at a utility maximum subject to his constraints. In addition, the first best allocation, which is also conditionally efficient, can also be supported as a transfer equilibria. Consequently, Example 3 represents a failure of the usual first welfare theorem in the strong sense that there are two equilibria that can be Pareto ranked.

## 6. Existence of Equilibrium

As well as characterizing equilibria in this type model we are interested in computing them. One approach is to compute equilibria of a truncated version of the model. This approach to computation is closely related to proofs of existence that prove existence for a truncated model then take limits (see, for example, Bewley 1972 and Wilson 1981). Unfortunately, even here there are problems that do not come up in models without debt constraints.

A T-truncated constrained transfer equilibrium is a triple  $(x, p, q)$  with the allocation  $x \in \ell_\infty$ ,  $p \in R^\infty$ , and  $q \in \ell_1$  such that the allocation is socially feasible, it exhausts the value of the social endowment, and each agent's allocation maximizes T-truncated utility subject to the budget constraint and T-truncated interim individual rationality constraints:

$$\max(1 - \delta) \sum_{t(s) \leq T} \delta^{t(s)-1} \pi_s u_i(z_s^i, \eta_s)$$

subject to

$$q(z^i) \leq q(x^i)$$

$$(1-\delta) \sum_{\sigma \geq s}^{t(\sigma) \leq T} \delta^{t(\sigma)-t(s)} \pi_{\sigma} u_i(z_{\sigma}^i, \eta_{\sigma})$$

$$\geq (1-\delta) \sum_{\sigma \geq s}^{t(\sigma) \leq T} \delta^{t(\sigma)-t(s)} \pi_{\sigma} v_i(p_{\sigma}, p_{\sigma} \cdot \bar{x}(\eta_{\sigma}), \eta_{\sigma}), \quad t(s) \leq T.$$

As before, we require that, if  $p_s \cdot x_s = p_s \cdot z_s$  for all  $s \in S$ , then  $q(x) = q(z)$ . Notice that we can set  $q_s = 0$  for  $t(s) > T$  since  $x_s$  has no value in consumption and no role in interim individual rationality. We define the notion of a  $T$ -truncated constrained ownership equilibrium in which  $q(x^i) = q(\bar{x}^i + w^i)$  similarly.

**Lemma 4.** If every agent has positive assets, a  $T$ -truncated constrained ownership equilibrium exists.

**Proof.** Individual optimization is a finite horizon problem since  $q_s = 0$  for  $t(s) > T$ . Define excess demand as solution subject to additional constraint that  $z_s^i \leq 2\bar{x}(\eta_s)$ . For any  $p$  let  $x_s^i(p)$  maximize  $u_i(z_s^i, \eta_s)$  subject to  $p_s \cdot z_s^i \leq p_s \cdot \bar{x}(\eta_s)$ . Then the plan  $x_s^i(p) + (1/2)\bar{w}^i$  strictly satisfies all constraints. Familiar arguments now imply that aggregate excess demand is a convex-valued, upper-hemi-continuous correspondence and that an equilibrium exists.  $\square$

**Proposition 6.** If every agent has positive assets, then a constrained ownership equilibrium exists.

**Proof.** Lemma 4 yields a sequence of truncated ownership equilibria  $(x^T, p^T, q^T)$ . We can normalize  $q^T$  so that  $\|\sum_{t(s)=1} q_s^T(w(\eta_s))\| = 1$  and normalize  $p^T$  so that  $p_s^T$  lies in the unit simplex. The same argument that provides the exponential bound on  $q$  in Lemma 3 can be used to find a similar bound on  $q^T$  where the constant  $b$  is independent of  $T$ . This implies that the closure of  $\{q^T \in \ell_1 \mid (x^T, p^T, q^T) \text{ is a truncated ownership equilibrium}\}$  is compact in  $\ell_1$  (see, for example, Dunford and Schwartz 1957, p. 338). Since  $\{x^T \in \ell_{\infty} \mid (x^T, p^T, q^T) \text{ is a truncated ownership equilibrium}\}$  is

bounded in  $\ell_\infty$ , by the Alaoglu Theorem, its closure is compact in the weak\* topology on  $\ell_\infty$ . Consequently, there is a subsequence  $(x^T, p^T, q^T)$ , which we can take to be indexed in the same way as the original  $(x^T, p^T, q^T)$ , so that  $x^T$  converges to  $x$  in the weak\* topology,  $p^T$  converges to  $p$  in the product topology, and  $q^T$  converges to  $q$  in the norm topology. Since  $x^T$  is bounded,  $q^T(x^{iT})$  converges to  $q(x^i)$ . Therefore,  $q(x^i) = q(\bar{x}^i + w^i)$ . Moreover, since  $x^T$  converges to  $x$  in the weak\* topology, it converges pointwise to  $x$ . Therefore,  $x$  is socially feasible. Since  $p^T$  also converges pointwise to  $p$ ,  $(x, p)$  satisfies (SS). In addition, because of discounting,  $(x, p)$  also satisfies (IR). Finally, the convergence of  $q^T$  to  $q$  in the norm topology and the convergence of  $p^T$  to  $p$  in the product topology assure us that  $q_s$  and  $p_s$  are proportional.

What we must show is that  $x^i$  is really optimal given constraints induced by  $q$ . Suppose instead that  $\hat{x}^i$  is better. Construct  $\bar{x}^i$  by selling  $\epsilon > 0$  of assets and using half the proceeds to purchase  $2\bar{x}(\eta_s)$  in all states  $s$  with  $t(s) \geq \bar{T}$  and  $\epsilon'$  of all goods in all states. For fixed  $\epsilon$ , we can obviously balance the budget by making  $\bar{T}$  large enough and  $\epsilon'$  small enough. Notice that for  $\epsilon$  small enough all the constraints in the consumer's problem are (uniformly) strictly satisfied. Let  $\bar{x}^{iT}$  be the truncated version of this plan. It follows, since  $q^T$  converges to  $q$  in the norm topology, that, for  $T$  large enough,  $\bar{x}^{iT}$  satisfies the budget constraint. Moreover, individual rationality is clearly satisfied for  $t(s) \geq \bar{T}$ . Moreover, for some  $\tilde{T}$ ,  $\delta^T$  is very small. Then take  $T$  so large that  $p_s^T$  is nearly  $p_s$  for  $t(s) \leq \bar{T} + \tilde{T}$  and so that  $T \geq \bar{T} + \tilde{T}$ . For  $t(s) < \bar{T}$ , individual rationality must be satisfied because they are strictly satisfied at  $q$ .  $\square$

## 7. Efficiency Versus Equilibrium

With more than one good per period and utility functions that are not identically homothetic, transfer equilibria may fail to be constrained efficient allocation-price pairs and vice-versa. If we accept the logic that any admissible allocation-price pair can be implemented by a central credit

agency, then the agency may wish to use a nonmarket rationing scheme to carry out intertemporal trade. This leaves open the possibility that a small coalition may meet secretly to defeat the credit agency by writing private contracts. If this is possible, then not all admissible allocation-price pairs can be implemented.

There is also a sense in which the definition of equilibrium is too strong. We should again imagine a central credit agency enforcing the individual rationality constraints by completely excluding defaulters from intertemporal trade. While the central credit agency by assumption cannot carry out more extreme punishments for default, however, there is no reason it should not use less extreme punishments. This leads us to define the idea of an equilibrium with partial exclusion. We argue that equilibria with partial exclusion are essentially the same as the allocation-price pairs that cannot be blocked by small coalitions.

From an economic point of view, the most significant element of this theory is that equilibria with partial exclusion may be either better or worse than equilibria with full exclusion. Even if it is possible to exclude defaulters permanently from intertemporal trade, it may not be desirable to do so. Since it is a feature of modern economies that punishments for default are typically less than the maximum possible and indeed, do not fully exclude individuals from intertemporal trade, debt constrained asset markets with income effects provide a possible explanation. (Private information problems that lead to a positive probability of default in equilibrium provide another.)

A *transfer equilibrium with partial exclusion* is a triple  $(x,p,q)$  such that  $x$  is socially feasible (SF),  $(x,p)$  satisfies spot market supporting (SS),  $(x,q)$  exhausts the value of social endowment (E.1), there exist constants  $U_g^i$  such that  $x^i$  maximizes utility subject to the budget constraint and individual rationality constraints

$$\max u(z^i)$$

subject to

$$q(z^i) \leq q(x^i)$$

$$(1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} u_i(z_{\sigma}^i, \eta_{\sigma}) \geq U_s^i, \quad s \in S,$$

and the spot prices  $p$  are consistent with intertemporal prices in the sense that (E.3) holds. This definition is the same as that of a transfer equilibrium with complete exclusion except that we replace the individual rationality constraints with the *strong individual rationality constraints* (with respect to given constants  $U_s^i$ )

$$(SIR) \quad (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} u_i(z_{\sigma}^i, \eta_{\sigma}) \geq U_s^i.$$

Notice the constants  $U_s^i$  must also satisfy these constraints when  $z^i = x^i$  since otherwise  $x$  itself would violate (SIR). Notice too that, if  $(x, p, q)$  is a transfer equilibrium with complete exclusion that we can replace the constants  $U_s^i$  with constants  $\tilde{U}_s^i$  that satisfy

$$\tilde{U}_s^i = (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} v_i(p_{\sigma}, P_{\sigma} \cdot \bar{x}^i(\eta_{\sigma}), \eta_{\sigma}).$$

This obviously implies that every transfer equilibrium with complete exclusion is a transfer equilibrium with partial exclusion.

How do we interpret the constants  $U_s^i$  that define the individual rationality constraints (SIR)? If default takes place, current utility is the indirect utility from the endowment. In future periods, the penalty for default must be paid. This can have the form of exclusion from intertemporal trade for some random number of periods. No such penalty can leave the agent with less utility than the indirect utility from his endowment in each state. If he is not excluded from trade at all, however, he would receive the same utility as in the original allocation  $x$ . In this case, where there is no penalty at all for default, the average present value of default exceeds that of  $x$  as long as the agent was making (rather than receiving) payments in the initial period. Clearly then, we can construct

any penalty utility in between no penalty and complete exclusion simply by randomizing between complete exclusion and no penalty at all. The argument of the previous paragraph shows that the broadest range of possible equilibrium allocations is obtained when the penalty exactly equals the one time gain from default: any less, and  $x$  itself will violate the constraints; any more, and certain allocation price pairs may be ruled out by permitting seemingly beneficial trades.

Our analogues of the first and second welfare theorem now apply directly to transfer equilibrium with partial exclusion (or the corresponding quasi-equilibrium in the case of the second welfare theorem). We say that an allocation-price pair is *weakly conditionally efficient* (with respect to given constants  $U_j^i$ ) if potential blocking allocations must satisfy (SIR) rather than (IR). Indeed, if we ask how a social planner might prevent a blocking allocation, we see that he has the option of not enforcing certain private contracts. In other words, a blocking allocation must have the feature that agents would not want to give it up and resume the original plan, that is, would have to satisfy (SIR). There is no reason, however, to consider only blocking by the coalition of all agents. If no coalition can block an admissible allocation-price pair subject to feasibility for the coalition and (SIR), then we say that it is *strongly admissible*.

Just as the proof of the first welfare theorem may be easily strengthened to show that every competitive equilibrium is in the core, here it may be strengthened to show that every equilibrium with partial exclusion is strongly admissible. Since every strongly admissible allocation-price pair is weakly conditionally efficient, it follows from the second welfare theorem that every strongly admissible allocation-price pair may be supported as a quasi-equilibrium with partial default. Observe, however, that the conditions under which a quasi-equilibrium is an equilibrium are no longer useful.

What do we gain or lose from weakening the notion of equilibrium in this way? If we reexamine Example 2, we discover that the efficient allocation that is not conditionally efficient is

weakly conditionally efficient. The only reason that agent 1 can trade with agent 2 in state 3 is because there are assets, so agent 2's individual rationality constraint is not binding. We could make the strong individual rationality constraint  $\bar{U}_3^2$  be the utility yielded by the original allocation  $x$ , however, so any movement up the contract curve would violate 2's strong individual rationality constraint. This example shows how it may be desirable not to use the greatest possible punishment in competitive equilibrium: the efficient allocation-price pair that is not conditionally efficient is weakly conditionally efficient and hence can be implemented as a transfer equilibrium with partial exclusion.

If we consider the identically homothetic case, we see that, since the first and second welfare theorems have their usual interpretations, there is a loss from weakening the notion of equilibrium. Let us reexamine Example 1 and consider stationary allocations. The agent with the large endowment then receives average present value utility  $(2/3) \log(15-z) + (1/3) \log(4+z)$ , where  $z$  is the constant amount traded from the high endowment agent to the low endowment agent. Starting at  $z = 0$ , this function initially increases until it reaches a maximum at  $z = 7/3$ , then it decreases until at  $z = 5$  it has the same value as at  $z = 0$ . Consequently at  $z = 5$  the individual rationality constraint binds, and this is the constrained equilibrium that we have already calculated. On the other hand, any  $z$  greater or equal  $7/3$  and less or equal  $5$  corresponds to a stationary equilibrium with partial exclusion: in each case additional trade would lower the high endowment agent's utility below that in the proposed allocation, and so it violates the individual rationality constraints. Notice that any  $z$  less than  $7/3$  does not correspond to an equilibrium with partial exclusion: in this case the additional trade increases both agents' utility and so cannot violate the constraints.

The benefit of allowing partial exclusion is that it may be possible to implement socially desirable allocations that cannot be implemented without partial exclusion. The cost is that there are many more undesirable equilibria, and the system may wind up at one of these "by accident."

There is one question left unanswered in this discussion: Can every efficient allocation be implemented as an equilibrium with partial exclusion? Or more weakly, Is every efficient allocation strongly admissible? We conjecture that the answer is no in general, but the difficulty of actually computing an efficient allocation that is not first best, not identically homothetic, and has an infinite horizon has prevented us from constructing a counterexample.

One important case in which the answer is yes is in the case of finite horizon models such as Example 2. Let us declare that a model is finite horizon if there is an absorbing state in which there are no assets that is reached in a fixed finite time with probability one. Notice that, if there are assets, the truncated equilibrium existence theorem, Lemma 4, guarantees there is an ownership equilibrium, while the proof of Proposition 6 shows that the equilibrium correspondence is upper-hemi-continuous as the time horizon goes to infinity. On the other hand, any infinite horizon equilibrium can be realized as a finite horizon equilibrium by having assets of the correct utility in the terminal state. Moreover, the welfare theorems apply directly. The major difference is that in the finite horizon case, since there are no assets and no uncertainty after a finite time, there is no intertemporal trade after this time. If there are no assets even in the initial periods, then there is no intertemporal trade at all. This follows from the fact that there is no trade in the terminal state and backwards induction. This argument also shows that any admissible allocation is weakly conditionally efficient: since the strong individual rationality constraints are defined by the admissible allocation itself, there are (thinking of this allocation as the endowment) no assets, and consequently no additional trade beyond that implicit in the allocation. In this sense we see that equilibrium with partial exclusion can support essentially all admissible allocations.

## Appendix

Proof of Proposition 1. For each state history  $s$ , let  $\alpha_s$  be a vector of utility weights, one for each agent, lying in the unit simplex in  $\mathbb{R}^m$ . (A.1) implies that, for each  $\alpha_s$ , there is a closed, convex set of allocations in  $\mathbb{R}^{m \times n}$  that maximize

$$\sum_{i=1}^m \alpha_s^i u_i(x_s^i, \eta_s)$$

subject to social feasibility constraints, that is, allocations that are efficient at  $\eta_s$  relative to the weights  $\alpha_s$ . (A.1) implies that there is a unique vector of Lagrange multipliers  $p_s \in \mathbb{R}^n$  associated with the social feasibility constraints. We can normalize  $p_s$  to lie in the unit simplex in  $\mathbb{R}^n$  (although it is then no longer a vector of Lagrange multipliers). In this way, each infinite vector  $\alpha$  gives rise to a set of allocation-price pairs  $(x,p)$ . By construction, any such pair satisfies (SS) since it is efficient within each state history.

Consider  $\alpha$ ,  $x$ , and  $p$  each as vectors in  $\mathbb{R}^\infty$  in the product topology. The set of  $\alpha \in \mathbb{R}^\infty$  such that  $\alpha_s$  is in the unit simplex for each  $s \in S$  is clearly compact. Furthermore, it is easy to show that the correspondence  $F: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty \times \mathbb{R}^\infty$  defined by the rule  $F(\alpha) = (x(\alpha), p(\alpha))$  is upper-hemi-continuous. It is also easy to show that the set  $X$  of allocation-price pairs  $(x,p)$  that are socially feasible and satisfy (IR) is a closed set. Since  $F$  is upper-hemi-continuous,  $F^{-1}(X)$  is closed and therefore compact. In fact,  $F^{-1}(X)$  is also nonempty. Fix an initial asset distribution and, for each  $\eta$ , calculate a single-period competitive equilibrium  $(x(\eta), p(\eta))$  and associated welfare weights  $\alpha(\eta)$ . Then the infinite vector  $(x,p)$  with  $x_s = x(\eta_s)$ ,  $p_s = p(\eta_s)$ ,  $s \in S$ , satisfies  $F(\alpha) = (x,p)$  for  $\alpha_s = \alpha(\eta_s)$ . Since  $x^i(\eta_s)$  is in a single period competitive allocation for prices  $p(\eta_s)$  where the income of agent  $i$  is

$$p(\eta_s) \cdot (x^i(\eta_s) + w^i(\eta_s)) \geq p(\eta_s) \cdot \bar{x}^i(\eta_s),$$

it follows that

$$u_i(x^i(\eta_s), \eta_s) \geq v_i(p(\eta_s), p(\eta_s) \cdot \bar{x}^i(\eta_s), \eta_s).$$

Multiplying by the probability  $\pi_s$  and the appropriate power of the discount factor  $\delta$  and adding up yields (IR). In other words, if there is never a single-period gain from defaulting on contingent claims contracts, there can never be an intertemporal gain from doing so.

We now choose  $x$ ,  $p$ , and  $\alpha$  to solve

$$\max \sum_{i=1}^m U_i(x^i)$$

subject to

$$(x, p) \in F(\alpha)$$

$$\alpha \in F^{-1}(X).$$

Since the objective function is continuous and the constraint set compact, it follows that a maximum exists. The corresponding  $(x, p)$  is an efficient allocation-price pair.  $\square$

**Proof of Proposition 2.** Suppose that  $x$  is first best. Then  $x_s$  must be Pareto efficient in the ordinary sense for the static economy at  $\eta_s$ . Notice that this does not depend on  $\delta$ . Let  $p_s$  be the efficiency prices corresponding to  $x_s$ . Clearly,  $(x, p)$  satisfies (SS). If it also satisfies (IR), then it is both efficient and conditionally efficient. It therefore suffices to find a first best allocation  $x$  that together with efficiency prices  $p$  satisfy (IR) for all  $\delta$  near enough to one.

Since  $\pi$  has a single ergodic class and no transient states, it has a unique stationary distribution  $\pi^*(\eta) > 0$ . Consider the static economy with  $m$  agents and  $n \times k$  goods with preferences

$$\bar{u}_i(x^i) = \sum_{\eta=1}^k \pi^*(\eta) u_i(x^i(\eta), \eta)$$

and social endowment  $\bar{x}(\eta)$ . (Here  $x^i(\eta)$  denotes the quantity of good  $j$  consumed by agent  $i$  in state  $\eta$ .) If  $\bar{x}$  is Pareto efficient for this static economy, then the stationary allocation  $\bar{x}_s = \bar{x}(\eta_s)$  is first best in the dynamic economy.

Let  $\bar{x}$  be a competitive equilibrium of this static economy with endowments  $\bar{x}^i + w^i$ . The corresponding efficiency prices are  $\bar{p}$ . Then  $\bar{x}$  is Pareto efficient and

$$\sum_{\eta=1}^k \pi^*(\eta) u_i(\bar{x}^i(\eta), \eta) \geq \sum_{\eta=1}^k \pi^*(\eta) v_i[\bar{p}(\eta), \bar{p}(\eta) \cdot (\bar{x}^i(\eta) + w^i(\eta)), \eta]$$

since the ordinary budget constraint allows trade on spot markets. Since  $\bar{p}(\eta) \neq 0$  and  $w^i(\eta) \geq 0$ ,

$$\sum_{\eta=1}^k \pi^*(\eta) u_i(\bar{x}^i(\eta), \eta) \geq \sum_{\eta=1}^k \pi^*(\eta) v_i(\bar{p}(\eta), \bar{p}(\eta) \cdot \bar{x}^i(\eta), \eta).$$

If each agent has positive assets, this inequality is strict. We first consider this case. Since the probability that  $\eta_s = \eta$  conditional on  $\eta_0$  converges exponentially to  $\pi^*(\eta)$ , as  $\delta$  approaches one,

$$(1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_\sigma u_i(\bar{x}^i(\eta_\sigma), \eta_\sigma) \rightarrow \sum_{\eta=1}^k \pi^*(\eta) u_i(\bar{x}^i(\eta), \eta).$$

Similarly,

$$(1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_\sigma v_i(\bar{p}^i(\eta_\sigma), \bar{p}(\eta_\sigma) \cdot \bar{x}^i(\eta_\sigma), \eta_\sigma) \rightarrow \sum_{i=1}^k \pi^*(\eta) v_i(\bar{p}(\eta), \bar{p}(\eta) \cdot \bar{x}^i(\eta), \eta).$$

Since the rate of convergence depends only on  $\eta_\sigma$ , which can take on only finitely many values, convergence in both cases is uniform, and we conclude that for  $\delta$  close enough to one,

$$(1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_\sigma u_i(\bar{x}^i(\eta_\sigma), \eta_\sigma) > (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_\sigma v_i(\bar{p}(\eta_\sigma), \bar{p}(\eta_\sigma) \cdot \bar{x}(\eta_\sigma), \eta_\sigma)$$

which is (IR).

Consider now the case where

$$\sum_{\eta=1}^k \pi^*(\eta) u_i(\bar{x}^i(\eta), \eta) = \sum_{\eta=1}^k \pi^*(\eta) v_i(\bar{p}(\eta), \bar{p}(\eta) \cdot \bar{x}^i(\eta), \eta),$$

but where  $u_i$  is strictly concave. The strict concavity of  $u_i$  implies that

$$u_i(\bar{x}^i(\eta), \eta) = v_i(\bar{p}(\eta), \bar{p}(\eta) \cdot \bar{x}^i(\eta), \eta).$$

Letting  $\eta$  be the appropriate  $\eta_{\sigma}$ , and multiplying by  $\delta^{t(\sigma)-t(s)} \pi_{\sigma}$ , and adding up yields

$$(1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} u_i(\bar{x}^i(\eta_{\sigma}), \eta_{\sigma}) = (1-\delta) \sum_{\sigma \geq s} \delta^{t(\sigma)-t(s)} \pi_{\sigma} v_i(p(\eta_{\sigma}), p(\eta_{\sigma}) \cdot \bar{x}^i(\eta_{\sigma}), \eta_{\sigma})$$

which again is (IR), this time with exact equality.  $\square$

Proof of Proposition 3. Since utility is identically homothetic and differentiable, a vector of spot prices such that (SS) is satisfied is proportional to  $p_s = Du(\bar{x}(\eta_s), \eta_s)$ . If we normalize  $p_s$  to lie on the unit simplex in  $R^n$ , then it is unique. In other words, all admissible allocations have the same spot prices.

To see the conditional efficiency implies efficiency, suppose that  $(x, p)$  is conditionally efficient but dominated by an admissible allocation  $(\bar{x}, \bar{p})$ . Since  $(\bar{x}, \bar{p})$  satisfies (SS),  $\bar{p} = p$  and  $(x, p)$  is not conditionally efficient. To see the converse, that efficiency implies conditional efficiency, suppose that  $(x, p)$  is efficient but dominated by  $(\bar{x}, p)$  that is socially feasible and satisfies (IR) but violates (SS). Then for each  $s$  we can find  $\hat{x}_s$  that is efficient at  $\eta_s$  and dominates  $\bar{x}_s$ . Consequently,  $(\hat{x}, p)$  satisfies (SS), and since  $(\bar{x}, p)$  satisfies (IR) so does  $(\hat{x}, p)$ . This implies that  $(\hat{x}, p)$  is admissible and dominates  $(x, p)$ , which contradicts the efficiency of  $(x, p)$ .  $\square$

Proof of Lemma 1. Assume not, that  $z^i$  is individually rational,  $U_i(z^i) > U_i(x^i)$ , and  $q(z^i) = q(x^i)$ .

The concavity of utility implies that, for  $0 < \lambda < 1$ ,  $\lambda \bar{x}^i + (1-\lambda)z^i$  is individually rational. As

$\lambda \rightarrow 0$ ,  $U_i(\lambda \bar{x}^i + (1-\lambda)z^i) \rightarrow U_i(z^i) > U_i(x^i)$  since  $U_i$  is continuous in the  $\ell_\infty$  topology. For all  $0 < \lambda < 1$ , however  $q(\lambda \bar{x}^i + (1-\lambda)z^i) = \lambda q(\bar{x}^i) + (1-\lambda)q(z^i) < q(x^i)$ , a contradiction.

Let  $e_{js}$  be the consumption plan that places unit weight on consumption of good  $j$  in state  $s$  and zero weight elsewhere, and let  $e_s = \sum_{j=1}^n e_{js}$ . The monotonicity of  $u_i$  implies that  $U_i(x^i + e_s) > U_i(x^i)$  and that  $x^i + e_s$  is individually rational. Consequently, if there exists  $\bar{x}^i$  individually rational such that  $q(\bar{x}^i) < q(x^i)$ , then  $q(x^i + e_s) = q(x^i) + q(e_s) > q(x^i)$ . Therefore,  $q(e_s) > 0$ , which implies  $q_s \neq 0$ .  $\square$

**Proof of Lemma 2.** Suppose that  $q^1 < 0$ . Then, for some  $e_{js}$ ,  $q^1(e_{js}) = q(e_{js}) = q_{js} < 0$ . Consequently,  $q^1(x^i + e_{js}) < q^1(x^i)$ , which implies  $q(x^i + e_{js}) < q(x^i)$ . Since  $u_i$  is monotonically increasing, however,  $U_i(x^i + e_{js}) \geq U_i(x^i)$  and  $x^i + e_{js}$  individually rational. This contradicts  $(x, p, q)$  being a quasi-equilibrium.

Suppose now that  $q^f(\bar{x}^i) < 0$  for some  $\bar{x}^i \geq 0$ . For  $T > 0$  consider the consumption plan  $\bar{x}^{iT}$ ,  $\bar{x}_s^{iT} = 0$  for  $t(s) \leq T$  and  $\bar{x}_s^{iT} = \bar{x}_s^i$  for  $t(s) > T$ . Then  $q^f(\bar{x}^{iT}) = q^f(\bar{x}^i)$  since  $\bar{x}^{iT}$  and  $\bar{x}^i$  differ only in a finite number of components. As  $T \rightarrow \infty$ , however,  $q^1(\bar{x}^{iT}) \rightarrow 0$ , which implies that  $q(x^i + \bar{x}^{iT}) < q(x^i)$  for some  $T$ . Once again,  $U_i(x^i + \bar{x}^{iT}) \geq U_i(x^i)$  and  $x^i + \bar{x}^{iT}$  individually rational contradict  $(x, p, q)$  being a quasi-equilibrium.  $\square$

**Proof of Lemma 3.** Let  $\bar{x}^i \in \ell_\infty$  be such that  $u_i(\bar{x}^i, \eta_s) = v_i(p_s, p_s \cdot \bar{x}^i(\eta_s), \eta_s)$  for all  $s \in S$ . Define  $\bar{x}^{is}$  to be such that  $\bar{x}_\sigma^{is} = \bar{x}_\sigma^i$  for  $\sigma \geq s$  and  $\bar{x}_\sigma^{is} = x_\sigma^i$  otherwise. Similarly define  $\bar{x}^{is}$  to be such that  $\bar{x}_\sigma^{is} = \bar{x}_\sigma^i(\eta_\sigma)$  for  $\sigma \geq s$  and  $\bar{x}_\sigma^{is} = x_\sigma^i$  otherwise. Notice that  $p_\sigma \cdot \bar{x}_\sigma^{is} = p_\sigma \cdot \bar{x}_\sigma^i$  for all  $s \in S$  implies that  $q(\bar{x}^{is}) = q(\bar{x}^i)$ . Now define  $w_\sigma^s = w(\eta_\sigma)$  for  $\sigma \geq s$  and  $w_\sigma^s = 0$  otherwise. For some agent  $i$

$$q(x^i - \bar{x}^{is}) = q(x^i - \bar{x}^i) \geq q(w^s)/m = Q \geq 0.$$

Define  $\omega_\sigma^s = w(\eta_\sigma)$  for  $\sigma = s$  and  $\omega_\sigma^s = 0$  otherwise. Consider the consumption plan

$$\hat{x}^i = \bar{x}^{is} + (Q/(2q(\omega^s)))\omega^s.$$

Notice the  $\hat{x}^i$  costs strictly less than  $x^i$  and that  $\hat{x}^i$  satisfies the individual rationality constraints since it provides more consumption than  $\bar{x}^{is}$ , which is individually rational. Consequently, since  $(x, p, q)$  is a quasi-equilibrium,  $U_i(\hat{x}^i) \leq U_i(x^i)$ .

The set of allocations that are socially feasible and yield higher utility than the private endowment in every state is compact. The monotonicity and differentiability of utility therefore implies that there exists some  $\epsilon > 0$  such that, for all  $z_s$  that are socially feasible and yield higher utility than the private endowment and all  $\lambda > 0$ ,

$$u_i(z_s^i + \lambda w(\eta_s), \eta_s) \geq u_i(z_s^i, \eta_s) + \lambda \epsilon.$$

Setting  $z^i = \hat{x}^i$  and  $\lambda = Q/(2q(\omega^s))$ , we can multiply by probabilities and sum across states to yield

$$\begin{aligned} U_i(\hat{x}^i) - U_i(x^i) &\geq U_i(\bar{x}^i) + \epsilon Q/(2q(\omega^s)) - U_i(x^i) \\ &\geq U_i(\bar{x}^i) - U_i(\bar{x}) + \epsilon Q/(2q(\omega^s)). \end{aligned}$$

Consequently,

$$\epsilon Q \leq 2q(\omega^s)(U_i(\bar{x}) - U_i(\bar{x}^i))$$

or, defining  $B = 2m(U_i(\bar{x}) - U_i(\bar{x}^i))/\epsilon$ ,

$$q(w^s) \leq Bq_s \cdot w(\eta_s).$$

In other words, the value of assets in all histories that follow  $s$  is bounded by a multiple of the value of assets at  $s$ .

Summing histories over  $t(s) = T$ , we obtain

$$B \sum_{t(s)=T} q_s \cdot w(\eta_s) \geq \sum_{t(s)=T} q(w^s) = \sum_{t(s) \geq T} q_s \cdot w(\eta_s) + q^f(w).$$

Since  $q \in \ell_\infty^*$ ,  $\sum_{t(s) \geq T} q_s \cdot w(\eta_s) \geq \sum_{t(s)=T} q_s \cdot w(\eta_s)$  approaches zero as  $T$  approaches infinity. Consequently,  $q^f(w) = 0$ .

To prove the second half of the lemma, define  $v_T = \sum_{t(s)=T} q_s \cdot w(\eta_s)$ . We can sum the above inequality over histories  $t(s) = t$  to find

$$Bv_t \geq \sum_{r=t}^{\infty} v_r.$$

Consequently,

$$v_t \geq (B-1)^{-1} \sum_{r=t+1}^{\infty} v_r \geq (B-1)^{-1} \sum_{r=t+1}^T v_r.$$

We can now calculate recursively,

$$\begin{aligned} v_1 &\geq (B-1)^{-1} \left[ v_2 + \sum_{t=3}^T v_t \right] \geq (B-1)^{-1} ((B-1)^{-1} + 1) \sum_{t=3}^T v_t \\ &\geq (B-1)^{-1} ((B-1) + 1)^2 \sum_{t=4}^T v_t \geq (B-1)^{-1} ((B-1)^{-1} + 1)^{T-2} v_T. \end{aligned}$$

Setting  $b = (B-1)^{-1} + 1$  now yields the desired inequality.  $\square$

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