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COMPUTATIONAL ALGORITHMS FOR SOLVING VARIANTS OF FUERST'S MODEL

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I. The Model

Household preferences are given by

$$(1) \qquad E_0\sum_{t=0}^{\infty}(\beta^*)^t u(C_t,L_t),$$

where C_t denotes consumption of the market produced good and L_t denotes hours worked supplied to firms ($0 \le L_t \le 1$). Later, we will work with the following parametric form for U:

$$(2) \qquad u(C_t,L_t) = \left[C_t^{(1-\gamma)}(1-L_t)^\gamma\right]^\psi/\psi,$$

where $0 < \gamma < 1$, $\gamma\psi < 1$, $(1-\gamma)\psi < 1$ are required for u to have positive derivatives in C and (1-L). The last two conditions are required for u to be concave in C for fixed L and concave in (1-L) for fixed C, respectively.

Firms operate the following production technology

$$(3) \qquad f^*(K_t,z_tH_t) = K_t^\alpha(z_tH_t)^{(1-\alpha)} + (1-\delta^*)K_t.$$

Here, K_t denotes the beginning-of-period t stock of capital, H_t denotes hours of work ($0 \le H_t \le 1$) hired by the firm. Also, $\delta*$ denotes the rate of depreciation on a unit of capital. Finally, z_t is the state of technology:

$$(4) \qquad z_t = \exp(\mu t+\theta_t).$$

Here, μ is the growth rate of z_t and θ_t is a stationary stochastic process with mean θ.

The household is composed of four members who meet at the beginning of each period and pool their resources at that time. The four members are: a worker, a shopper, a firm, and a financial intermediary. The shopper faces the following cash-in-advance constraint:

$$(5) \qquad M_t - N_t \ge P_tC_t$$

where M_t denotes the household's beginning-of-period-t money holdings and N_t denotes money sent by the household to the financial intermediary at the beginning of period t. Finally, P_t denotes the period t price level. The decision about the magnitude of N_t—hence, the division of M_t between the shopper and the intermediary—must be made by the household prior to observing the current period's realization of θ_t and X_t. The variable X_t is defined next.

The firm faces the following cash-in-advance constraint:

$$(6) \quad N_t + X_t \geq P_t(K_{t+1} - (1-\delta^*)K_t) + W_tH_t.$$

Here, $N_t + X_t$ denotes cash loaned to the firm by the intermediary. The intermediary obtains N_t from the household and X_t by a "helicopter drop" from the government. Also, W_t denotes the money wage.

The economy-wide resource constraint is given by

$$(7) \quad C_t + K_{t+1} = f^*(K_t, z_tH_t).$$

Condition (7) is the goods market clearing condition. In equilibrium, (7) must be satisfied. Another equilibrium condition is

$$(8) \quad L_t = H_t.$$

Equation (8) is the labor market clearing condition.

Cash balances that the household begins period t + 1 with are determined by

$$(9) \quad M_{t+1} = M_t + X_t + W_tL_t + P_tf^*(K_t, z_tH_t) - W_tH_t - P_tK_{t+1} - P_tC_t.$$

The money market clearing condition is that $M_t = M_t^s$, the per capita money stock.

II. Removing Technology Growth

The first step in analysis is to convert this to a nongrowing economy. Accordingly, define the following transformed variables

$$
(10)\quad \begin{aligned} &c_t = C_t \exp(-\mu t), \quad k_{t+1} = K_{t+1} \exp(-\mu t), \quad \bar{M}_t = M_t \exp(-\mu t), \\ &\bar{N}_t = N_t \exp(-\mu t), \quad \bar{W}_t = W_t \exp(-\mu t), \quad \bar{X}_t = X_t \exp(-\mu t). \end{aligned}
$$

In this notation, (1) is proportional to

$$
(11)\quad \sum_{t=0}^{\infty} \beta^t u(C_t, L_t)
$$

where $\beta = \beta^* \exp[(1-\gamma)\psi\mu]$, and the function $u(\bullet,\bullet)$ is defined in (2). Also, the production function is

$$
(12)\quad f(k_t, H_t, \theta_t) = \exp(-\alpha\mu) k_t^{\alpha} [\exp(\theta_t) H_t]^{(1-\alpha)} + (1-\delta)k_t,
$$

where

$$
(13)\quad \delta = 1 - (1-\delta^*) \exp(-\mu).
$$

The cash-in-advance constraints are now

$$
(14)\quad \bar{M}_t - \bar{N}_t \geq P_t c_t
$$

$$
(15)\quad \bar{N}_t + \bar{X}_t \geq P_t(k_{t+1} - (1-\delta)k_t) + \bar{W}_t H_t.
$$

The resource constraint is

$$
(16)\quad c_t + k_{t+1} = f(k_t, H_t, \theta_t).
$$

Finally, the household's money accumulation equation is given by

$$
(17)\quad \bar{M}_{t+1} \exp(\mu) = \bar{M}_t + \bar{X}_t + \bar{W}_t L_t + P_t f(k_t, \exp(\theta_t) H_t) - \bar{W}_t H_t - P_t k_{t+1} - P_t c_t.
$$

III. A Scaled Version of the Model

Recall M_t^a denotes the per-capita stock of money at the beginning of time t. Also, let $\overline{M}_t^a = \exp(-\mu t) \times M_t^a$. We scale by $\overline{M}_t^a$ as follows

$$(18)\quad m_t = \overline{M}_t/\overline{M}_t^a, \quad n_t = \overline{N}_t/\overline{M}_t^a, \quad x_t = \overline{X}_t/\overline{M}_t^a, \quad p_t = P_t/\overline{M}_t^a, \quad w_t = \overline{W}_t/\overline{M}_t^a.$$

With this change of variable, (17) becomes

$$(19)\quad m_{t+1} = \frac{m_t + x_t + w_t L_t + p_t f(k_t,\ \exp(\theta_t)H_t) - w_t H_t - p_t k_{t+1} - p_t c_t}{(1+x_t)}.$$

Similarly, the cash-in-advance constraints become

$$(20)\quad m_t - n_t \geq p_t c_t$$

$$(21)\quad n_t + x_t \geq p_t(k_{t+1} - (1-\delta)k_t) + w_t H_t.$$

The utility function and resource constraint are still given by (11) and (16), respectively.

IV. Household First Order Conditions with Binding Cash Constraints

We suppose that (20)-(21) are met as a strict equality. Whether this restriction is binding can be checked ex post. In addition, we discretize the exogenous shocks as follows

$$(22)\quad \begin{array}{l} \theta_t \in \{\theta^1, \ldots, \theta^{n_\theta}\} \\ x_t \in \{x^1, \ldots, x^{n_x}\}. \end{array}$$

Let s_t denote the date t realization of θ_t and x_t. Then s_t can take on precisely $N = n_\theta \cdot n_x$ values. We adopt the following convention

$$
\begin{aligned}
s=1 &\rightarrow (\theta_t, x_t) = (\theta^1, x^1) \\
s=2 &\rightarrow (\theta_t, x_t) = (\theta^1, x^2) \\
&\vdots \\
s=n_x &\rightarrow (\theta_t, x_t) = (\theta^1, x^{n_x}) \qquad (23) \\
s=n_x+1 &\rightarrow (\theta_t, x_t) = (\theta^2, x^1) \\
&\vdots \\
s=N &\rightarrow (\theta_t, x_t) = (\theta^{n_\theta}, x^{n_x}).
\end{aligned}
$$

Let $\theta(s_i)$ denote the value of θ_t associated with $s = i$ for $i = 1, \ldots, N$. Similarly, $x(s_i)$ denotes the value of x_t associated with $s = i$. We assume that s is a realization from a Markov chain with

(24) $\pi_{ij} = \text{Prob}[s_t = j \mid s_{t-1} = i]$.

At the beginning of the period, the household observes k and s—the beginning-of-period capital stock and the realization of the *previous* period's exogenous shocks. The current period realization of the technology shock and monetary injection are observed after n is chosen, but before the other variables are selected. Let κ denote the aggregate per capita capital stock and, with one exception, let primes (′) denote next period's value of a variable. The exceptional case, s′, denotes the current period's realization of s. The household takes $p(\kappa, s, s')$ and $w(\kappa, s, s')$ as given and solves the following problem

(25) $$J(m, s, k, \kappa) = \max_{n \in (0,m)} \sum_{s'} \pi_{ss'} \left\{ \max_{L,k'} u(c, L) + \beta J(m', s', k', \kappa') \right\}$$

subject to

$$m' = \frac{m + x(s') + wL + pf(k,H,s') - wH - pk' - pc}{1 + x(s')}$$

$$(26) \quad c = \frac{m - n}{p}$$

$$H = \frac{n + x(s') - p(k' - (1-\delta)k)}{w}.$$

Here, the binding cash-in-advance constraints have been used to eliminate c and H as decision variables.

The first order condition for n is

$$(27) \quad \sum_{s'} \pi_{ss'} \left\{ -u_c\left(\frac{m-n}{p},\ L\right) \frac{1}{p} + \beta J_1(m',s',k',\kappa') \ \frac{p(\kappa,s,s')}{w(\kappa,s,s')} \ \frac{f_H(k,H,s')}{1 + x(s')} \right\} = 0$$

where

$$(28) \quad f_H(k,H,s') = \frac{\partial}{\partial H} f(k,H,s').$$

In words, equation (27) says the following. Increase n by a unit and this reduces current consumption by the random amount, $1/p(\kappa,s,s')$, depending on the realization of s'. The cost of this, in utility terms, as of the beginning of the period, is $\Sigma_{s'}\pi_{ss'}u_c(m-n/p,\ L)/p$. By increasing n, funds are redirected to the firm (via the financial intermediary), which uses them to hire 1/w units of labor (see the third constraint in (26)). This produces a cash inflow to the firm in the amount $(p/w)f_H$ during the period. This results in $(p/w)f_H/(1+x)$ units of cash being available to the household at the beginning of next period. The marginal utility value of a dollar next period is $J_1(m',s',k',\kappa')$. This explains the other terms in (27).

Next, consider the first order condition for labor supply, L:

$$(29) \quad u_L(c,L) + \beta J_1(m',s',k',\kappa') \ \frac{w(\kappa,s,s')}{1 + x(s')} = 0.$$

This condition just says that the marginal utility cost of an increase in work effort ($-u_L(c,L)$) has to equal the marginal benefit of the extra cash generated.

Finally, consider the first order condition for k'

$$
\begin{aligned}
(30)\qquad & \beta J_3(m',s',k',\kappa') \\
& - \beta J_1(m',s',k',\kappa')[p(\kappa,s,s')]^2 f_H(k,H,s')/[(1+x(s'))w(\kappa,s,s')] = 0.
\end{aligned}
$$

This says that the marginal benefit of extra $k'(\beta J_3)$ must be equated to the cost. The cost is determined as follows. Given the third constraint in (26), the extra k' is financed by a decrease in H in the amount p/w. This produces a fall in cash revenues in the current period of $(pf_H)p/w$. Cash available at the beginning of next period then falls by $[(pf_H)p/w]/(1+x)$. Finally, the current utility value of one dollar next period is βJ_1.

The following envelope results are easily established

$$
(31)\qquad J_1(m,s,k,\kappa) = \sum_{s'} \pi_{ss'} u_c\left(\frac{m-n}{p},\ L\right) \frac{1}{p(\kappa,s,s')}
$$

$$
(32)\qquad J_3(m,s,k,\kappa) = \sum_{s'} \pi_{ss'} \beta J_1(m',k',\kappa')p(\kappa,s,s')f_k(k,H,s')/(1+x(s')).
$$

Substitute (32) into (30) to get

$$
\begin{aligned}
(33)\qquad & 0 = \sum_{s'} \pi_{s's''} \beta J_1(m'',k'',\kappa'')p(\kappa',s',s'')f_k(k',H',s'')/(1+x(s'')) \\
& - J_1(m',s',k',\kappa')[p(\kappa,s,s')]^2 f_H(k,H,s')/[(1+x(s'))w(\kappa,s,s')].
\end{aligned}
$$

The first order conditions for the household problem are given by (27), (29), and (33) with J_1 being defined in (31).

V. General Equilibrium

In equilibrium, households maximize (satisfy the first order conditions derived in the previous section) and labor, commodity, capital, and money markets clear, i.e., $L = H$, $k = \kappa$, $c + k' = f(k,H,s')$, $m = m' = 1$. We use the latter conditions to develop expressions for p and w and to reduce the number of unknowns in our first order conditions, (27), (29), and (33). First, imposing the strict equality in (20) and clearing in the commodity and money markets implies

$$(34)\quad p(k,s,s') = \frac{1 - n(k,s)}{f[k,L(k,s,s'),\theta(s')] - k'(k,s,s')}.$$

Here, we express n as a function of k, s. From the point of view of the household it is actually a function of κ, k, m, s. However, here and below we make use of the money and capital market equilibrium by setting $m = 1$ and $\kappa = k$.

Using the third constraint in (26), imposing labor market equilibrium and rewriting yields

$$(35)\quad w(k,s,s') = \frac{n(k,s) + x(s') - p(k,s,s')(k'(k,s,s') - (1-\delta)k)}{L(k,s,s')}.$$

The notation makes clear that equilibrium k′ and L depend on k, s, s′.

It is convenient to define the following functions:

$$(36)\quad y(k,s,s') = f(k,L(k,s,s'),\theta(s')) = k^{\alpha}[\exp(\theta(s'))L(k,s,s')]^{(1-\alpha)} + (1-\delta)k$$

$$(37)\quad f_H(k,s,s') = f_2(k,L(k,s,s'),\theta(s'))$$

$$(38)\quad f_K(k,s,s') = f_1(k,L(k,s,s'),\theta(s'))$$

$$(39)\quad c(k,s,s') = y(k,s,s') - k'(k,s,s')$$

$$(40)\quad J_m(k,s) = \sum_{s'} \pi_{ss'} u_1(c(k,s,s'),L(k,s,s'))\ \frac{1}{p(k,s,s')}$$

(41) $u_c(k,s,s') = u_1(c(k,s,s'),L(k,s,s'))$

(42) $u_L(k,s,s') = u_2(c(k,s,s'),L(k,s,s'))$.

Then, the household's first order conditions, after imposing all market clearing conditions, are the following. For fixed k, s:

$$(43)\quad 0 = \sum_{s'} \pi_{ss'} \left\{ -\frac{u_c(k,s,s')}{p(k,s,s')} + \beta J_m(k'(k,s,s'),s') \, \frac{p(k,s,s')}{w(k,s,s')} \, \frac{f_H(k,s,s')}{1+x(s')} \right\}$$

$$(44)\quad u_L(k,s,s') + \beta J_m(k'(k,s,s'),s') \, \frac{w(k,s,s')}{1+x(s')} = 0,$$

$$(45)\quad \begin{aligned} & J_m(k'(k,s,s'),s') \, \frac{[p(k,s,s')]^2}{1+x(s')} \, \frac{1}{w(k,s,s')} \, f_H(k,s,s') \\ & = \sum_{s''} \pi_{s's''} \beta J_m\{k'[k'(k,s,s'),s'],s''\} \, \frac{p[k'(k,s,s'),s',s''] f_k[k'(k,s,s'),s',s'']}{1+x(s'')}, \end{aligned}$$

$s' = 1, \ldots, N$. In (45) s'' denote next period's realization of s.

It is convenient to substitute out for $\beta J_m/(1+x)$ in (45) from (44). Recall, $\beta J_m(k'(k,s,s'),s')/(1 + x(s'))$ is the value of obtaining an extra unit of cash in the current period, after s' is realized. That is, one extra unit of cash in the current period gives rise to $1/(1 + x(s'))$ extra units of cash at the beginning of the next period, and the marginal value of cash at the beginning of the next period is valued at $\beta J_m(k'(k,s,s'),s')$. According to (44), this value equals $-u_L(k,s,s')/w(k,s,s')$ in equilibrium. Using this, our previous first order condition, (45), reduces to

$$(46)\qquad 0 = \left[\frac{-u_L(k,s,s')}{w(k,s,s')}\right]\left\{\frac{[p(k,s,s')]^2}{w(k,s,s')}\, f_H(k,s,s')\right\}$$
$$-\sum_{s''}\pi_{s's''}\beta\left[\frac{-u_L[k'(k,s,s'),s',s'']}{w[k'(k,s,s'),s',s'']}\right]p[k'(k,s,s'),s',s'']f_k[k'(k,s,s'),s',s''].$$

It is useful to express (46) in words. An increase in k' by one unit can be financed by decreasing hours worked by p/w, which causes cash revenues to drop by $p^2 f_H/w$. The first object in square brackets in (46) translates this cash reduction into utility terms. The term after the minus sign in (46) reflects the benefit associated with the increase in k'. The term, pf_k, gives the increased cash flow resulting from the extra capital, while the term in large square brackets translates this into utility terms. Thus, (46) says that investment will be undertaken at a level where the marginal costs equal the expected marginal benefits of a further increase in investment.

VI. *Solving the Model Using Judd's Collocation Method*

It is convenient to simplify (43) by substituting for βJ_m from (44):

$$(47)\qquad 0 = \sum_{s'}\pi_{ss'}\left\{\frac{u_c(k,s,s')}{p(k,s,s')} + \frac{u_L(k,s,s')p(k,s,s')f_H(k,s,s')}{[w(k,s,s')]^2}\right\}.$$

We seek functions, $n(k,s)$, $k'(k,s,s')$, and $L(k',s,s')$ which satisfy (44), (46), and (47). Given these three functions, other functions of interest can be found using (34)–(42).

We cannot hope to obtain $L(k,s,s')$, $n(k,s)$, and $k'(k,s,s')$ exactly since in general they are infinite-dimensional objects. Thus, we confine ourselves to a space of functions, $n(k,s;\Psi_n)$, $k'(k,s,s';\Psi_k)$, and $L(k,s,s';\Psi_L)$, where Ψ_n, Ψ_k, and Ψ_L are finite dimensional parameter vectors. The decision rule parameters to be selected are Ψ, where

$$(48) \quad \Psi = \begin{pmatrix} \Psi_n \\ \Psi_k \\ \Psi_L \end{pmatrix}.$$

In the following, we describe one of the solution strategies outlined in Judd (1990). In particular, we select Ψ so that (44), (46), and (47) are satisfied exactly for several k, s. Since the exact solution satisfies (44), (46), and (47) for all possible (k,s), we hope this method of selecting Ψ produces decision rules with acceptable accuracy properties.

To see this solution strategy, first define the following functions:

$$(49) \quad y(k,s,s';\psi_L) = k^{\alpha}[\exp(\theta(s'))L(k,s,s';\psi_L)]^{(1-\alpha)} + (1-\delta)k$$

$$(50) \quad f_H(k,s,s';\psi_L) = f_2[k,L(k,s,s';\psi_L),\theta(s')]$$

$$(51) \quad f_L(k,s,s';\psi_L) = f_1[k,L(k,s,s';\psi_L),\theta(s')]$$

$$(52) \quad p(k,s,s';\Psi) = \frac{1 - n(k,s;\Psi_n)}{f\left[k,L(k,s,s';\Psi_L),\theta(s')\right] - k'(k,s,s';\Psi_k)}$$

$$(53) \quad w(k,s,s';\Psi) = \frac{n(k,s;\Psi_n) + x(s') - p(k,s,s';\Psi)\left(k'(k,s,s';\Psi_k) - (1-\delta)k\right)}{L(k,s,s';\Psi_L)}$$

$$(54) \quad c(k,s,s';\psi) = y(k,s,s';\psi_L) - k'(k,s,s';\psi_k)$$

$$(55) \quad u_c(k,s,s';\psi) = u_1(c(k,s,s';\psi),L(k,s,s';\psi_L))$$

$$(56) \quad u_L(k,s,s';\psi) = u_2(c(k,s,s';s),L(k,s,s';\psi_L)),$$

for $s' = 1, \ldots, N$, and

$$(57) \quad J_m(k,s;\Psi) = \sum_{s'} \pi_{ss'} \frac{u_c(k,s,s';\Psi)}{p(k,s,s';\Psi)}.$$

Next, define the function $g(k,s;\Psi)$ which maps Ψ into R^{2N+1}:

$$(58)\quad g_i(k,s;\Psi) = u_L(k,s,s'=i;\Psi) + \beta J_m(k'(k,s,s'=i;\Psi_k),s'=i)\ \frac{w(k,s,s'=i;\Psi)}{1 + x(s'=i)}$$

for $i = 1, \ldots, N$. Also,

$$g_{N+i}(k,s;\Psi) = \frac{u_L(k,s,s'=i;\Psi)}{w(k,s,s'=i;\Psi)}\left\{\frac{[p(k,s,s'=i;\Psi)]^2}{w(k,s,s'=i;\Psi)}\ f_H(k,s,s'=i;\Psi)\right\}$$

$$(59)\qquad - \sum_{s''=1}^{N} \pi_{s'=i,s''}\beta\left[\frac{u_L[k'(k,s,s'=i),s'=i,s'';\Psi]}{w[k'(k,s,s'=i)s'=i,s'';\Psi]}\right]$$

$$\times\, p[k'(k,s,s'=i;\Psi),s'=i,s'';\Psi]f_k[k'(k,s,s'=i;\Psi),s'=i,s'';\Psi],$$

$i = 1, \ldots, N$.

$$g_{2N+1}(k,s;\Psi) = \sum_{s'=1}^{N} \pi_{ss'}\left\{\frac{u_c(k,s,s';\Psi)}{p(k,s,s';\Psi)} + \frac{u_L(k,s,s';\Psi)p(k,s,s';\Psi)f_H(k,s,s';\Psi)}{[w(k,s,s';\Psi)]^2}\right\}.$$

Let

$$(60)\quad g(k,s;\Psi) = \begin{bmatrix} g_1(k,s;\Psi) \\ \vdots \\ g_{2N+1}(k,s;\Psi)\end{bmatrix}.$$

Now, if there were some value of Ψ, say Ψ^*, such that the resulting decision rules were exact, then $g(k,s;\Psi^*) = 0$ for each $s \in \{1,\ldots,N\}$ and for all $k > 0$. In general there will be no such value for Ψ since Ψ contains a finite number of parameters, while $g(k,s;\Psi)$ for $k > 0$ and $s \in \{1,\ldots,N\}$ represents an uncountable number of equations. One strategy for selecting a value for Ψ is to set $g(k,s;\Psi) = 0$ for $s = 1, \ldots, N$, and for a finite collection of values of k, $\{k_1,\ldots,k_{n_k}\}$. Define

$$(61)\quad g(\Psi) = \begin{bmatrix} g(k_1,s-1;\Psi) \\ g(k_1,s-2;\Psi) \\ \vdots \\ g(k_1,s-N;\psi) \\ g(k_2,s-1;\Psi) \\ g(k_2,s-2;\Psi) \\ \vdots \\ g(k_{n_k},s-N;\Psi) \end{bmatrix}.$$

Note that (61) is $N = n_k N(2N+1)$ equations. Judd's collocation method works with Ψ vectors which contain N elements. Thus, g maps R^N into R^N. Our computation strategy is to find $\hat{\Psi}$ such that

$$(62)\quad g(\hat{\Psi}) = 0.$$

A standard FORTRAN program for solving nonlinear equations can be used to find $\hat{\Psi}$.

The function g is completely defined once we describe a method for selecting $k_1, \ldots, k_{n_k}$, and a set of parametric functions, $k'(k,s,s';\Psi_k)$, $L(k,s,s';\Psi_L)$. We follow Judd in making n, k', and L Chebyshev polynomial functions of k for fixed s, s'. In particular, let $T_i(z)$ be the i^{th} Chebyshev polynomial in z, for $i = 0, 1, 2, \ldots$. (See *Numerical Recipes in C*, p. 158, for a definition of T_i.) These polynomials map the interval [-1,1] into itself. Thus, let $z(k)$ be a map from $k \in \{k_1,\ldots,k_{n_k}\}$ to [-1,1]. Such a map is defined in equation 5.6.10 on p. 160 of *Numerical Recipes in C*.

Consider $k'(k,s,s';\Psi_k)$ first. Then, for each fixed s, s', define

$$(63)\quad k'(k,s,s';\Psi_k) = \sum_{i=1}^{n_k} a_i^{s,s'}(\Psi_k) T_{i-1}(z(k)).$$

We assume there are $N^2 \times n_k$ elements in Ψ_k, exactly the same as the number of coefficients, $a_i^{s,s'}$, $i = 1, \ldots, n_k$, s, $s' = 1, \ldots, N$. For each i, s, s', $a_i^{s,s'}(\Psi_k)$ selects an element of Ψ_k.

The vector Ψ_L also contains $N^2 \times n_k$ elements and $L(k,s,s';\Psi_L)$ is defined analogous to (63). In particular,

$$(64) \quad L(k,s,s';\Psi_L) = \sum_{i=1}^{n_k} b_i^{s,s'}(\Psi_L)T_{i-1}(z(k)) \text{ for } s,\ s' = 1, \ldots, N.$$

The function $n(k,s;\Psi_n)$ is defined in the obvious way, given (63)-(64).

Finally, we select $k_1 \leq k_2 \leq \ldots \leq k_{n_k}$ as follows. First, compute $z_\ell = \cos(\pi(\ell-½)/n_k)$ for $k = 1, \ldots, n_k$. Then, $k_i = z^{-1}(z_i)$, $i = 1, \ldots, n_k$. The boundaries, k_1 and k_{n_k}, of this set should satisfy

$$(65) \quad \max_{\substack{s,s' \in \{1,\ldots,N\} \\ k \in \{k_1,\ldots,k_{n_k}\}}} k'(k,s,s';\Psi_k) \leq k_{n_k}.$$

$$(66) \quad \min_{\substack{s,s' \in \{1,\ldots,N\} \\ k \in \{k_1,\ldots,k_{n_k}\}}} k'(k,s,s';\Psi_k) \geq k_1.$$

Ensuring that (65) and (66) are satisfied may require some experimentation.

This computational strategy is presented to provide a basis for discussion only. It may actually be too computationally burdensome to implement. For example, suppose there are three values for each of the exogenous shocks, so that $N = 9$. Then, suppose $n_k = 10$. In this case $N = 1,710$, and (62) represents a problem of solving 1,710 equations in 1,710 unknowns! Some alternatives are probably necessary.

One way to reduce the dimension of the problem dramatically would be to assume that s is iid. I think that in this case, k' and L are functions of k and s' only and n is a function of k. In this case, the dimension of the problem drops to $n_k(2N+1)$, or 190 in our example. This is manageable using Judd's collocation method. However, the iid case is not very interesting. Empirically, it is strongly counterfactual. From a theoretical standpoint it is also not very interesting since there is a strong presumption that results are not robust to adding serial correlation. For example, a persistent monetary shock may—by driving up next period's price level a lot—have very little impact on employment.

An alternative possibility would be to eliminate Ψ_n from the parameter vector and let n(k,s) be determined implicitly by (47), given $k'(k,s,s';\Psi_k)$ and $L(k,s,s';\Psi_L)$. This would reduce the dimension of the problem, (62). It is important to note, though, that evaluating the appropriately modified version of (60) would require solving (47) not just for n(k,s), but also for $n[k'(k,s,s';\Psi_k),s']$, $s' = 1, \ldots, N$. The latter are required for βJ_m in (58) and next period's w in (59).

Two alternative computational strategies include the Bizer–Judd–Coleman (B–J–C) strategy, which is described in the next section, and the linear quadratic approximation method.

VIII. The Bizer-Judd-Coleman Algorithm

A problem with solving a set of equations like (62) using standard algorithms is that the computational burden rises roughly with the square of the number of equations. The Bizer–Judd–Coleman algorithm can also be used to solve (62), but its computational burden only rises linearly in the number of equations. Given the large number of equations in our problem it makes sense to consider B–J–C.

The B–J–C algorithm computes $\hat{\Psi}$ in (62) as the limit of a sequence, Ψ^0, Ψ^1, Ψ^2, ..., where Ψ^0 is an initial guess. The method generates Ψ^r, $r = 1, 2, \ldots$, recursively by using (44), (46), and (47) to define a mapping, G, from $R^{n_k N(2N+1)}$ into itself:

(67) $\Psi^r = G(\Psi^{r-1})$, $r = 1, 2, 3, \ldots$.

Then, $\hat{\Psi} = \lim_{r\to\infty}\Psi^r$.

The mapping G is defined as follows. Let Ψ^{r-1} be given, and fix k, s. Let $L(k,s,s';\Psi_L^{r-1})$, $n(k,s;\Psi_n^{r-1})$, and $k'(k,s,s';\Psi_k^{r-1})$ determine hours worked, money supplied to financial intermediaries and investment in periods *after* the present period in (44), (46), and (47). Then, (44) and (46) for $s' = 1, \ldots, N$, and (47) define 2N + 1 equations in the 2N + 1 unknowns: n(k,s) and L(k,s,s'), k'(k,s,s') for $s' = 1, \ldots, N$. Because of restriction (47), these equations must be solved jointly using a nonlinear, multi-equation solver. This can be done for each $k \in \{k_1, \ldots, k_{n_k}\}$ and $s = \{1, \ldots, N\}$. Thus, values for k'(k,s,s'), n(k,s), and L(k,s,s') are obtained at each of the $n_k \times N^2$ discrete points (k,s,s'). It is then trivial to fit $n_k - 1^{th}$ order Chebychev polynomials to these points in order to get Ψ^r.

This method for finding $\hat{\Psi}$ in (62) replaces an algorithm which is order $[n_k N(2N+1)]^2$ (Judd's) by one that is order N^2. This does not mean that B-J-C will dominate the collocation method for a given finite N, only that this must be so for N sufficiently large.

As Judd and others have pointed out, the B-J-C algorithm can be quite slow. One way to speed it up replaces the operator G by $\tilde{G}$:

(68) $\Psi^r = \tilde{G}(\Psi^{r-1}) \equiv \Psi^{r-1} + [I - G'(\Psi^{r-1})]^{-1}[G(\Psi^{r-1}) - \Psi^{r-1}]$.

Here, $G'(\Psi^{r-1})$ is the matrix of derivatives of G with respect to Ψ, evaluated at $\Psi = \Psi^{r-1}$. In practice the matrix inverse in (68) can be hard to compute, since the number of operations increase with the square of the dimension of G. However, the truncated inverse, $I + G'(\Psi^{r-1}) + [G'(\Psi^{r-1})]^2 + \ldots + [G'(\Psi^{r-1})]^p$ may be relatively straightforward to compute for small values of p, say p = 10 or 20. A potential problem with (68) is that the computational cost of matrix inversion may increase with the square of the matrix's dimension. Applying this to the B-J-C algorithm may defeat its advantage relative to Judd's collocation method.

IX. L-Q Approximate Decision Rules

Both Judd's collocation method and the BJC algorithm require an initial guess for the decision rule parameters, Ψ. A good way to get this is to compute decision rules which solve linearized versions of the problem's Euler equations, (44), (46), and (47). The linearization procedure takes a Taylor series expansion about the nonstochastic, steady-state values of the variables. We first discuss this algorithm conditional on having steady states. We then turn to the steady state formulas.

A. The L-Q Approximation

It is convenient to revert to the time subscript notation. Thus,

$$(69)\quad p_t = \frac{1 - n_t}{f(k_t, L_t, \theta_t) - k_{t+1}} = p(k_t, k_{t+1}, n_t, L_t, \theta_t)$$

$$(70)\quad w_t = \frac{n_t + x_t - p_t(k_{t+1} - (1-\delta)k_t)}{L_t} = w(k_t, k_{t+1}, n_t, L_t, \theta_t, x_t).$$

The Euler equation (47), can be written

$$(71)\quad E[q(k_t, k_{t+1}, L_t, n_t, s_t) \mid s_{t-1}, k_t] = 0$$

where

$$(72)\quad q(k_t,k_{t+1},L_t,n_t,s_t) = \frac{u_c[f(k_t,L_t,\theta_t) - k_{t+1},L_t]}{p(k_t,k_{t+1},n_t,L_t,\theta_t)} + \frac{u_L[f(k_t,L_t,\theta_t) - k_{t+1},L_t]}{[w(k_t,k_{t+1},n_t,L_t,\theta_t,x_t)]^2} \times p(k_t,k_{t+1},n_t,L_t,\theta_t) f_L(k_t,L_t,\theta_t).$$

The variable s_{t-1} enters the conditioning set in (71), reflecting the possibility that the distribution of s_t may depend on the realization of s_{t-1}.

Write equations (44), after substituting for J_m from (40), as follows

$$(73)\quad E[Q(k_t,k_{t+1},k_{t+2},L_t,L_{t+1},n_t,n_{t+1},s_t,s_{t+1})|k_t,s_{t-1},s_t] = 0$$

where

$$(74)\quad Q(k_t,k_{t+1},k_{t+2},L_t,L_{t+1},n_t,n_{t+1},s_t,s_{t+1}) = u_L[f(k_t,L_t,\theta_t) - k_{t+1},L_t] + \beta \frac{u_c[f(k_{t+1},L_{t+1},\theta_{t+1}) - k_{t+2},L_{t+1}]}{p(k_{t+1},k_{t+2},n_{t+1},L_{t+1},\theta_{t+1})} \times \frac{w(k_t,k_{t+1},n_t,L_t,\theta_t,x_t)}{(1 + x_t)}.$$

Finally, write equation (46) as follows

$$(75)\quad E[W(k_t,k_{t+1},k_{t+2},L_t,L_{t+1},n_t,n_{t+1},s_t,s_{t+1})|k_t,s_{t-1},s_t] = 0$$

where

$$(76)\quad W(k_t,k_{t+1},k_{t+2},L_t,L_{t+1},n_t,n_{t+1},s_t,s_{t+1}) = u_L[f(k_t,L_t,\theta_t) - k_{t+1},L_t] \left[\frac{p(k_t,k_{t+1},n_t,L_t,\theta_t)}{w(k_t,k_{t+1},n_t,L_t,\theta_t,x_t)}\right]^2 f_L(k_t,L_t,\theta_t) - \beta \frac{u_L[f(k_{t+1},L_{t+1},\theta_{t+1}) - k_{t+2},L_{t+1}]}{w(k_{t+1},k_{t+2},n_{t+1},L_{t+1},\theta_{t+1},x_{t+1})} p(k_{t+1},k_{t+2},n_{t+1},L_{t+1},\theta_{t+1}) f_k(k_{t+1},L_{t+1},\theta_{t+1}).$$

The variable s_{t-1} enters the conditioning set in (73) and (75) reflecting that n_t is selected after k_t, s_{t-1} is observed, so that given k_t, s_{t-1} determines n_t.

We seek linear decision rules

$$(77)\quad k_{t+1} - k^s = k^1(k_t-k^s) + k^2(s_{t-1}-s) + k^3(s_t-s)$$

(78) $L_t - L^* = L^1(k_t-k^*) + L^2(s_{t-1}-s) + L^3(s_t-s)$

(79) $n_t - n^* = n^1(k_t-k^*) + n^2(s_{t-1}-s)$

which solve a linearized version of (71), (73), and (75). Here, k^*, L^*, n^*, and s denote the nonstochastic steady-state values of k_t, L_t, n_t, and s_t. Also, k^1, L^1, n^1 are scalars, while k^2, k^3, L^2, L^3, n^2, are each 1×2 vectors.

We assume s_t has the following representation

$$s_t = \gamma + As_{t-1} + \epsilon_t,$$

so that the nonstochastic steady-state value of s_t is $s = (I-A)^{-1}\gamma$. Here, ϵ_t is white noise, uncorrelated with s_{t-1}. We now think of s_t as

(80) $s_t = \begin{pmatrix}\theta_t \\ x_t\end{pmatrix}$.

Let the linearized versions of q, Q, and W be

(81) $\tilde{q}(k_t,k_{t+1},L_t,n_t,s_t) = \tilde{q}_1(k_t-k^*) + \tilde{q}_2(k_{t+1}-k^*) + \tilde{q}_3(L_t-L^*) + \tilde{q}_4(n_t-n^*) + \tilde{q}_5(s_t-s)$,

where $\tilde{q}_i$, $i = 1, 2, 3$ are scalars and $\tilde{q}_5$ is a 1×2 vector. Also,

$$\begin{aligned}\tilde{Q}(k_t,k_{t+1},k_{t+2},L_t,L_{t+1},n_t,n_{t+1},s_t,s_{t+1}) &= \tilde{Q}_1(k_t-k^*) \\ &+ \tilde{Q}_2(k_{t+1}-k^*) + \tilde{Q}_3(k_{t+2}-k^*) + \tilde{Q}_4(L_t-L^*) + \tilde{Q}_5(L_{t+1}-L^*) \\ &+ \tilde{Q}_6(n_t-n^*) + \tilde{Q}_7(n_{t+1}-n^*) + \tilde{Q}_8(s_t-s) + \tilde{Q}_9(s_{t+1}-s).\end{aligned} \quad (82)$$

Here, $\tilde{Q}_i$, $i = 1, \ldots, 7$ are scalars and $\tilde{Q}_8$, $\tilde{Q}_9$ are 1×2 vectors. Finally,

$$\begin{aligned}\tilde{W}(k_t,k_{t+1},k_{t+2},L_t,L_{t+1},n_t,n_{t+1},s_t,s_{t+1}) &= \tilde{W}_1(k_t-k^*) + \tilde{W}_2(k_{t+1}-k^*) + \tilde{W}_3(k_{t+2}-k^*) \\ &+ \tilde{W}_4(L_t-L^*) + \tilde{W}_5(L_{t+1}-L^*) + \tilde{W}_6(n_t-n^*) + \tilde{W}_7(n_{t+1}-n^*) + \tilde{W}_8(s_t-s) + \tilde{W}_9(s_{t+1}-s).\end{aligned} \quad (83)$$

Here, $\tilde{W}_i$ are scalars for $i = 1, \ldots, 7$, and 1×2 vectors for $i = 8, 9$. Given a set of model parameter values, the steady-state quantities and $\tilde{q}_i$, $i = 1, \ldots, 5$, $(\tilde{Q}_i, \tilde{W}_i)$, $i = 1, \ldots, 9$ are straightforward to compute. Formulas are given in the Appendix.

We will use a method of undetermined coefficients to find the 13 decision rule parameters, k^1, k^2, k^3, L^1, L^2, L^3, n^1, n^2 (k^2, k^3, L^2, L^3, n^2 each contain two parameters.) Applying the expectation in (71), making use of (77)-(79), and replacing q by $\tilde{q}$,

$$
\begin{aligned}
&E\left[\tilde{q}(k_t, k_{t+1}, L_t, n_t, s_t) \,|\, s_{t-1}, k_t\right] = \tilde{q}_1(k_t - k^s) + \tilde{q}_2\left[k^1(k_t - k^s) + k^2(s_{t-1} - s) + k^3 A(s_{t-1} - s)\right] \\
&\quad + \tilde{q}_3\left[L^1(k_t - k^s) + L^2(s_{t-1} - s) + L^3 A(s_{t-1} - s)\right] + \tilde{q}_4\left[n^1(k_t - k^s) + n^2(s_{t-1} - s)\right] \\
&\quad + \tilde{q}_5 A(s_{t-1} - s) \\
&= [\tilde{q}_1 + \tilde{q}_2 k^1 + \tilde{q}_3 L^1 + \tilde{q}_4 n^1](k_t - k^s) \\
&\quad + \left[\tilde{q}_2(k^2 + k^3 A) + \tilde{q}_3(L^2 + L^3 A) + \tilde{q}_4 n^2 + \tilde{q}_5 A\right](s_{t-1} - s) = 0.
\end{aligned} \tag{84}
$$

Doing the same for (82), we get

$$
\begin{aligned}
&E\left[\tilde{Q}(k_t, k_{t+1}, k_{t+2}, L_t, L_{t+1}, n_t, n_{t+1}, s_t, s_{t+1}) \,|\, k_t, s_{t-1}, s_t\right] = \tilde{Q}_1(k_t - k^s) \\
&\quad + \tilde{Q}_2\left[k^1(k_t - k^s) + k^2(s_{t-1} - s) + k^3(s_t - s)\right] \\
&\quad + \tilde{Q}_3\left[(k^1)^2(k_t - k^s) + k^1 k^2(s_{t-1} - s) + (k^1 k^3 + k^2 + k^3 A)(s_t - s)\right] \\
&\quad + \tilde{Q}_4\left[L^1(k_t - k^s) + L^2(s_{t-1} - s) + L^3(s_t - s)\right] \\
&\quad + \tilde{Q}_5\left[L^1 k^1(k_t - k^s) + L^1 k^2(s_{t-1} - s) + (L^1 k^3 + L^2 + L^3 A)(s_t - s)\right] \\
&\quad + \tilde{Q}_6\left[n^1(k_t - k^s) + n^2(s_{t-1} - s)\right]
\end{aligned}
$$

(85)

$$+ \tilde{Q}_7\Big[n^1k^1(k_t-k^s) + n^1k^2(s_{t-1}-s) + (n^1k^3+n^2)(s_t-s)\Big] + \tilde{Q}_8(s_t-s) + \tilde{Q}_9A(s_t-s)$$

$$= \Big[\tilde{Q}_1+\tilde{Q}_2k^1+\tilde{Q}_3(k^1)^2+\tilde{Q}_4L^1+\tilde{Q}_5L^1k^1+\tilde{Q}_6n^1+\tilde{Q}_7n^1k^1\Big](k_t-k^s)$$

$$+ \Big[\tilde{Q}_2k^2+\tilde{Q}_3k^1k^2+\tilde{Q}_4L^2+\tilde{Q}_5L^1k^2+\tilde{Q}_6n^2+\tilde{Q}_7n^1k^2\Big](s_{t-1}-s)$$

$$+ \Big[\tilde{Q}_2k^3 + \tilde{Q}_3(k^1k^3+k^2+k^3A) + \tilde{Q}_4L^3 + \tilde{Q}_5(L^1k^3+L^2+L^3A)$$

$$+ \tilde{Q}_7(n^1k^3+n^2) + \tilde{Q}_8 + \tilde{Q}_9A\Big](s_t-s) = 0.$$

$$E\Big[\tilde{W}(k_t,k_{t+1},k_{t+2},L_t,L_{t+1},n_t,n_{t+1},s_t,s_{t+1})\,|\,k_t,s_{t-1},s_t\Big]$$

(86)

$$= \Big[\tilde{W}_1 + \tilde{W}_2k^1 + \tilde{W}_3(k^1)^2 + \tilde{W}_4L^1 + \tilde{W}_5L^1k^1 + \tilde{W}_6n^1 + \tilde{W}_7n^1k^1\Big](k_t-k^s)$$

$$+ \Big[\tilde{W}_2k^2 + \tilde{W}_3k^1k^2 + \tilde{W}_4L^2 + \tilde{W}_5L^1k^2 + \tilde{W}_6n^2 + \tilde{W}_7n^1k^2\Big](s_{t-1}-s)$$

$$+ \Big[\tilde{W}_2k^3 + \tilde{W}_3(k^1k^3+k^2+k^3A) + \tilde{W}_4L^3 + \tilde{W}_5(L^1k^3+L^2+L^3A)$$

$$+ \tilde{W}_7(n^1k^3+n^2) + \tilde{W}_8 + \tilde{W}_9A\Big](s_t-s) = 0.$$

The requirement that (84)–(86) be zero provides us with 13 equations in the 13 unknowns;

(87) $\tilde{q}_1 + \tilde{q}_2k^1 + \tilde{q}_3L^1 + \tilde{q}_4n^1 = 0$

(88) $\tilde{q}_2(k^2+k^3A) + \tilde{q}_3(L^2+L^3A) + \tilde{q}_4n^2 + \tilde{q}_5A = (0,0)$

(89) $\tilde{Q}_1 + \tilde{Q}_2k^1 + \tilde{Q}_3(k^1)^2 + \tilde{Q}_4L^1 + \tilde{Q}_5L^1k^1 + \tilde{Q}_6n^1 + \tilde{Q}_7n^1k^1 = 0$

(90) $\tilde{Q}_2k^2 + \tilde{Q}_3k^1k^2 + \tilde{Q}_4L^2 + \tilde{Q}_5L^1k^2 + \tilde{Q}_6n^2 + \tilde{Q}_7n^1k^2 = (0,0)$

(91) $\tilde{Q}_2k^3 + \tilde{Q}_3(k^1k^3+k^2+k^3A) + \tilde{Q}_4L^3 + \tilde{Q}_5(L^1k^3+L^2+L^3A) + \tilde{Q}_7(n^1k^3+n^2) + \tilde{Q}_8 + \tilde{Q}_9A = (0,0)$

(92) $\tilde{W}_1 + \tilde{W}_2k^1 + \tilde{W}_3(k^1)^2 + \tilde{W}_4L^1 + \tilde{W}_5L^1k^1 + \tilde{W}_6n^1 + \tilde{W}_7n^1k^1 = 0$

(93) $\tilde{W}_2k^2 + \tilde{W}_3k^1k^2 + \tilde{W}_4L^2 + \tilde{W}_5L^1k^2 + \tilde{W}_6n^2 + \tilde{W}_7n^1k^2 = (0,0)$

(94) $\tilde{W}_2k^3 + \tilde{W}_3(k^1k^3+k^2+k^3A) + \tilde{W}_4L^3 + \tilde{W}_5(L^1k^3+L^2+L^3A) + \tilde{W}_7(n^1k^3+n^2) + \tilde{W}_8 + \tilde{W}_9A = (0,0).$

Here, (0,0) represents a 1×2 vector of zeros.

Equations (87)–(94) may be solved recursively. First, note that only k^1, L^1, and n^1 appear in (87), (89), and (92). Thus, there is a hope that these parameters can be computed using these equations. To do this, first substitute out for n^1 in (89) and (92) using (87). Then substitute out for L^1 in (92) using (89). The resulting version of (92) is a single nonlinear equation in the single unknown, k^1. Solve this for $0 \le k^1 \le \beta^{-1/2}$ using numerical methods. (It would be interesting to check whether $(k^1\beta)^{-1}$ solves this equation if k^1 does.)

We proceed now to find k^2, L^2, n^2, k^3, L^3. We treat k^1, n^1, L^1 as known numbers from here on. Solve (90) for n^2:

(95) $$n^2 = -\tilde{Q}_6^{-1}[\tilde{Q}_2+\tilde{Q}_3k^1+\tilde{Q}_5L^1+\tilde{Q}_7n^1]k^2 - \tilde{Q}_6^{-1}\tilde{Q}_4L^2.$$

Substitute this into (93) and solve that for L^2

(96) $$L^2 = (\tilde{W}_6\tilde{Q}_6^{-1}\tilde{Q}_4-\tilde{W}_4)^{-1}\Big[(\tilde{W}_2+\tilde{W}_3k^1+\tilde{W}_5L^1+\tilde{W}_7n^1) - \tilde{W}_6\tilde{Q}_6^{-1}(\tilde{Q}_2+\tilde{Q}_3k^1+\tilde{Q}_5L^1+\tilde{Q}_7n^1)\Big]k^2 = Dk^2,$$

say, where D is a scalar. Substitute (96) into (95) to get

(97) $$n^2 = Bk^2$$

where

(98) $$B = -\tilde{Q}_6^{-1}[\tilde{Q}_2+\tilde{Q}_3k^1+\tilde{Q}_5L^1+\tilde{Q}_7n^1] - \tilde{Q}_6^{-1}\tilde{Q}_4D,$$

a scalar.

Substitute (96) and (97) into (88) to get

(99) $$\tilde{q}_2(k^2+k^3A) + \tilde{q}_3(Dk^2+L^3A) + \tilde{q}_4Bk^2 + \tilde{q}_5A = 0.$$

Solve this for k^2 to get

(100) $$k^2 = B_0 + k^3B_1 + L^3B_2,$$

where

$$B_0 = -[\tilde{q}_2+\tilde{q}_3D+\tilde{q}_4B]^{-1}\tilde{q}_5A$$

(101) $$B_1 = -[\tilde{q}_2+\tilde{q}_3D+\tilde{q}_4B]^{-1}A\tilde{q}_2$$

$$B_2 = -[\tilde{q}_2+\tilde{q}_3D+\tilde{q}_4B]^{-1}A\tilde{q}_3.$$

Here, B_0 is a 1×2 vector, while B_1, B_2 are 2×2 matrices.

The remaining equations are (91) and (94), and these can be used to determine k^3 and L^3. Substitute (96), (97), and (100) into (91) to get,

(102) $$A_0 + k^3A_1 + L^3A_2 = (0,0)$$

where

(103) $$A_0 = (\tilde{Q}_3+\tilde{Q}_5D+\tilde{Q}_7B)B_0 + \tilde{Q}_8+\tilde{Q}_9A$$

(104) $$A_1 = (\tilde{Q}_2+\tilde{Q}_3k^1+\tilde{Q}_5L^1+\tilde{Q}_7n^1)I_2 + (\tilde{Q}_3+\tilde{Q}_5D+\tilde{Q}_7B)B_1 + \tilde{Q}_3A$$

(105) $$A_2 = (\tilde{Q}_3+\tilde{Q}_5D+\tilde{Q}_7B)B_2 + \tilde{Q}_5A + \tilde{Q}_4I_2,$$

and I_2 is the 2×2 identity matrix. Here A_0 is a 1×2 vector and A_1, A_2 are 2×2 matrices.

Substitute (96), (97), and (100) into (94) to get

(106) $$z_0 + k^3z_1 + L^3z_2 = (0,0)$$

where

(107) $$z_0 = (\tilde{W}_3+\tilde{W}_5D+\tilde{W}_7B)B_0 + \tilde{W}_8 + \tilde{W}_9A$$

(108) $$z_1 = (\tilde{W}_2+\tilde{W}_3k^1+\tilde{W}_5L^1+\tilde{W}_7n^1)I_2 + (\tilde{W}_3+\tilde{W}_5D+\tilde{W}_7B)B_1 + \tilde{W}_3A$$

(109) $$z_2 = (\tilde{W}_3+\tilde{W}_5D+\tilde{W}_7B)B_2 + \tilde{W}_5A + \tilde{W}_4I_2,$$

where z_0 is a 1×2 vector and z_1, z_2 are 2×2 matrices.

Solving (102) for L^3 get

(110) $L^3 = -(A_0+k^3A_1)A_2^{-1}$.

Substituting this into (106) and solving for k^3, get

(111) $k^3 = (z_0-A_0A_2^{-1}z_2)(A_1A_2^{-1}z_2-z_1)^{-1}$.

We now have all the objects sought. First, get k^1, L^1, n^1 as discussed after equation (94). Then, get k^3, L^3, k^2, n^2, L^2 recursively using (111), (110), (100), (97), and (96).

B. Steady States

Consider labor demand. Suppose the household increases the amount of labor hired and finances this by shifting funds from consumption. The cost of this is $u_c(C_t,L_t)/P_t$. (Here, we work with untransformed variables.) The benefit is as follows. The extra cash can be used to hire $1/W_t$ workers, who generate $(P_t/W_t)f_L^*(k_t,z_tL_t)$ extra money receipts. (Here, we use the equilibrium condition, $L = H$.) This money can be spent in the next period, producing $\beta^*u_c(C_{t+1},L_{t+1})/P_{t+1}$ units of utility. Thus, the first order condition is

(112) $u_c(C_t,L_t)/P_t = \left[\beta^*u_c(C_{t+1},L_{t+1})/P_{t+1}\right](P_t/W_t)f_L^*(K_t,z_tL_t)$.

In steady state, $C_t = \exp(\mu t)c^s$ (see (10)), so that

(113) $\dfrac{u_c(C_t,L_t)}{\beta^*u_c(C_{t+1},L_{t+1})} = \dfrac{\exp\{\mu[1-(1-\gamma)\psi]\}}{\beta^*} = (1+r)$,

where the utility function is given in (2). Expression (113) defines the steady-state real rate of interest, which we denote by $1 + r$. The variable p_t settles to a constant p^*, so that in steady state,

$$(114)\quad P_t = (1+x)^t \exp(-\mu t) p^*.$$

(See equation (18) for the definition of p_t.) Similarly, w_t settles to w^* in steady state, so that (10) and (18) imply

$$(115)\quad W_t = (1+x)^t w^*,$$

in steady state. Finally, note that

$$(116)\quad f_L^*(K_t, z_t L_t) = f_L(k^*, L^*, \theta)\ \exp(\mu t),$$

in steady state. (See (3) and (12) for definition of f* and f.)

Substitute (113)-(116) into (112) and scale both sides by $\exp(-\mu t)$ to get

$$(117)\quad f_L = \frac{w^*}{p^*}\ \frac{1 + x}{\exp(\mu)}\ (1+r) = \frac{w^*}{p^*}\ (1+R)$$

where

$$(118)\quad f_L = \exp(-\alpha\mu)(1-\alpha)(k^*/L^*)^\alpha\ \exp[(1-\alpha)\theta].$$

The expression to the right of w^*/p^* in (117) is the standard Fisherian formula for the nominal rate of interest, since $(1+x)\ \exp(-\mu)$ is the rate of inflation. Note that $f_L > w^*/p^*$ for x large enough. That is, firms operate at a point where the (scaled) marginal product of labor exceeds the (similarly scaled) real wage. This reflects that their receipts have to be held in the form of cash for a while, during which time they lose value. Under these circumstances, firms' revenues must be high enough to compensate them not just for the usual labor and capital costs, but also for their losses due to inflation. Equation (117) is firms' labor demand equation.

Now consider households' labor supply decision. The household could decide to increment work effort a bit, at a cost, $-u_L(C_t,L_t)$. The benefit is that it earns a wage, W_t, which it can spend on W_t/P_{t+1} goods in period t + 1, for a utility benefit $\beta^*[u_c(C_{t+1},L_{t+1})/P_{t+1}]W_t$. Equating costs and benefits,

$$-u_L(C_t,L_t) = \beta^*[u_c(C_{t+1},L_{t+1})/P_{t+1}]W_t,$$

or,

$$(119)\quad \frac{-u_L(C_t,L_t)}{u_c(C_{t+1},L_{t+1})} = \beta^* \frac{W_t}{P_{t+1}}.$$

Now, in steady-state,

$$(120)\quad \frac{-u_L(C_t,L_t)}{u_c(C_{t+1},L_{t+1})} = \frac{-u_L(c^*,L^*)}{u_c(c^*,L^*)} \exp(\mu t) \exp\{\mu[1-(1-\gamma)\psi]\}.$$

Substituting (114), (115), and (120) into (119) and scaling both sides by $\exp(-\mu t)$,

$$(121)\quad \frac{-u_L(c^*,L^*)}{u_c(c^*,L^*)} = \frac{w^*}{p^*}\frac{1}{1+R}.$$

According to (121), when households equate the (scaled) marginal rate of substitution between leisure and consumption to the (scaled) return to working, they recognize that when x is large enough (so that R is large), the real wage overstates the return to working. This is because they have to hold the cash receipts from working, during which time they lose value. Combining (117) and (121), one gets

$$(122)\quad -\frac{u_L}{u_c} = \left(\frac{1}{1+R}\right)^2 f_L .$$

Now consider the typical household's capital investment decision. Suppose the household employs less of the labor input at date t and uses the proceeds to buy one unit of extra capital, K_{t+1}. This requires reducing employment by P_t/W_t units and results in a reduction of current period revenues of $[P_t^2/W_t]f^*_{L,t}$. This in turn results in reduced consumption in period t + 1, so the total cost to the household is $[\beta^* u_c(C_{t+1},L_{t+1})/P_{t+1}]\,[P_t^2/W_t]\,f^*_{L,t}$. On the benefit side, the extra unit of capital in t + 1 generates $P_{t+1}f^*_{K,t+1}$ extra units of revenue, which can be applied to extra consumption in t + 2. Balance of these costs and benefits implies

$$(123)\quad \left[\beta^* u_c(c_{t+1},L_{t+1})\ \frac{1}{P_{t+1}}\right][P_t^2/W_t]\,f^*_{L,t} = P_{t+1}f^*_{K,t+1}(\beta^*)^2 u_c(C_{t+2},L_{t+2})/P_{t+2},$$

or,

$$(124)\quad \frac{u_c(C_{t+1},L_{t+1})}{\beta^* u_c(C_{t+2},L_{t+2})}\ \frac{P_t}{P_{t+1}}\ \frac{P_t}{W_t}\ f^*_L(K_t,z_tL_t) = \frac{P_{t+1}}{P_{t+2}}\ f^*_K(K_t,z_tL_t)\,.$$

Substituting (113)–(116) into (124), we get that, in steady state:

$$(125)\quad \frac{\exp\{\mu\,[1-(1-\gamma)\,\psi]\}}{\beta^*}\ \frac{p^*}{w^*}\ f_L = f^*_K$$

where

$$(126)\quad f^*_K = \alpha\,[\exp(\theta+\mu)\,L^*/k^*]^{(1-\alpha)} + 1-\delta^*.$$

Substituting for w^*/p^* from (117) into (125),

$$(127)\quad (1+r)(1+R) = f_K^*.$$

Note from (122) and (127) that steady-state allocative efficiency (i.e., $-u_L/u_c = f_L$ and $f_K^* = u_c(C_t,L_t)/\beta^* u_c(C_{t+1},L_{t+1})$) is achieved by following the "Friedman rule" and setting R = 0. This requires setting $x = \beta^* \exp[\mu(1-\gamma)] - 1$

We find k^* and L^* by solving (122) and (127). Write these out using our functional forms,

$$(128)\quad \frac{\gamma}{1-\gamma}\,\frac{c^*}{1-L^*} = \left(\frac{1}{1+R}\right)^2 (1-\alpha)\,\exp[(1-\alpha)\theta - \alpha\mu]\,(k^*/L^*)^{\alpha}.$$

$$(129)\quad (1+r)(1+R) = \alpha[\exp(\theta+\mu)L^*/k^*]^{(1-\alpha)} + 1 - \delta^*.$$

To find L^* and k^*, first use (129) to solve for $\nu = k^*/L^*$. Then, use (128) to find L^* given ν. In (128), replace c^* by $L^*(\exp[(1-\alpha)\theta - \alpha\mu]\nu^{\alpha} - \delta\nu)$.

Using k^* and n^*, we can compute p^*, w^*, and n^* as follows. First, according to (69),

$$(130)\quad p^* = \frac{1-n^*}{c^*}.$$

According to (70),

$$(131)\quad p^*\left(\delta k^* + \frac{w^*}{p^*}L^*\right) = n^* + x.$$

Substitute (117) and (130) into (131) and rearrange, to get

$$(132)\quad n^* = \left\{1 + \frac{\delta k^* + f_L L^*/(1+R)}{c^*}\right\}^{-1} \left\{\frac{\delta k^* + f_L L^*/(1+R)}{c^*} - x\right\}.$$

It is important to verify the conditions $w^s \geq 0$, $p^s \geq 0$, $c^s \geq 0$, $0 \leq n^s \leq 1$, $0 \leq L^s \leq 1$, $k^s > 0$.

X. *Perturbations on Fuerst's Model*

In this section, we consider two perturbations on the model. We describe suitably modified linearization procedures for obtaining approximate solutions to these models. In the first perturbation, k_{t+1} is selected prior to observing s_t. The second perturbation modifies Fuerst's model by allowing n_t to be set based on observing s_t.

A. *Fuerst's Model with Investment Insensitive to Monetary Shock*

The linearized decision rule in this case is

$$(133)\quad k_{t+1} - k^s = k^1(k_t - k^s) + k^2(s_{t-1} - s).$$

Also, the capital first order condition must be modified to reflect that s_t is not in the information set at the time that k_{t+1} is selected. Thus, (75) becomes

$$(134)\quad E[W(k_t, k_{t+1}, k_{t+2}, L_t, L_{t+1}, n_t, n_{t+1}, s_t, s_{t+1}) | k_t, s_{t-1}] = 0.$$

Working through the logic in Section IX.A., we find that to determine the 11 unknowns, k^1, k^2, L^1, L^2, L^3, n^1, n^2, we must solve (87)-(92) with $k^3 = 0$. In addition, the sum of the expression to the left of the equality in (93) and the product of expression to the left of the equality in (94) with A must equal (0,0). That is, instead of (93) and (94), we require

$$(135)\quad \begin{aligned} &\tilde{W}_2 k^2 + \tilde{W}_3 k^1 k^2 + \tilde{W}_4 L^2 + \tilde{W}_5 L^1 k^2 + \tilde{W}_6 n^2 + \tilde{W}_7 n^1 k^2 \\ &\quad + \left[\tilde{W}_3 k^2 + \tilde{W}_4 L^3 + \tilde{W}_5 (L^2 + L^3 A) + \tilde{W}_7 n^2 + \tilde{W}_8 + \tilde{W}_9 A\right] A = (0,0). \end{aligned}$$

To determine k^1, n^1, L^1, simply execute the same calculations described after equation (94). This leaves k^2, L^2, L^3, and n^2 to be determined by equations (88), (90), (91), and (135). This is a problem of solving 8 equations in 8 unknowns. The easiest way to solve these is to set them up in matrix algebra form. In particular, let

(136) $\beta = (k^2, L^2, L^3, n^2)$.

Here, β is a 1×8 vector. Also, let

$$(137)\quad X = \begin{bmatrix} \tilde{q}_2 I & (\tilde{Q}_2+\tilde{Q}_3k^1+\tilde{Q}_5L^1+\tilde{Q}_7n^1)I & \tilde{Q}_3 I & (\tilde{W}_2+\tilde{W}_3k^1+\tilde{W}_5L^1+\tilde{W}_7n^1)I + \tilde{W}_3A \\ \tilde{q}_3 I & \tilde{Q}_4 I & \tilde{Q}_5 I & \tilde{W}_4 I + \tilde{W}_5 A \\ \tilde{q}_3 A & \underset{\sim}{0} & \tilde{Q}_4 I + \tilde{Q}_5 A & (\tilde{W}_4 I + \tilde{W}_5 A)A \\ \tilde{q}_4 I & \tilde{Q}_6 I & \tilde{Q}_7 I & (\tilde{W}_6 I + \tilde{W}_7 A) \end{bmatrix}$$

and

$$(138)\quad Y = \left[\tilde{q}_5 A, \underset{\sim}{0}_{1,2}, \tilde{Q}_8 + \tilde{Q}_9 A, (\tilde{W}_8 + \tilde{W}_9 A)A\right].$$

In (137), X is an 8×8 matrix, I denotes the 2×2 identity matrix, and $\underset{\sim}{0}$ denotes the 2×2 matrix of zeros. In (138), Y is a 1×8 vector and $\underset{\sim}{0}_{1,2}$ denotes a 1×2 vector of zeros. Then, (88), (90), (91), and (135), in matrix notation is

$$(139)\quad \beta X + Y = \underset{\sim}{0}_{1,8},$$

which is trivial to solve for β. In (139), $\underset{\sim}{0}_{1,8}$ denotes a 1×8 vector of zeros.

B. Fuerst's Model with Full Contemporaneous Information

In this version of the model, we seek decision rules which feed back on $s_t - s$ only:

(140) $k_{t+1} - k^s = k^1(k_t - k^s) + k^3(s_t - s)$

(141) $L_t - L^s = L^1(k_t - k^s) + L^3(s_t - s)$

(142) $n_t - n^s = n^1(k_t - k^s) + n^3(s_t - s)$.

The first order conditions are:

(143) $E[q(k_t, k_{t+1}, L_t, n_t, s_t) | k_t, s_t] = 0$

(144) $E[Q(k_t, k_{t+1}, k_{t+2}, L_t, L_{t+1}, n_t, n_{t+1}, s_t, s_{t+1}) | k_t, s_t] = 0$

(145) $E[W(k_t, k_{t+1}, k_{t+2}, L_t, L_{t+1}, n_t, n_{t+1}, s_t, s_{t+1}) | k_t, s_t] = 0$.

Applying a suitably modified version of the calculations in (84), get

$$\begin{aligned}
&E\left[\tilde{q}(k_t, k_{t+1}, L_t, n_t, s_t) | s_t, k_t\right] = \tilde{q}_1(k_t - k^s) + \tilde{q}_2\left[k^1(k_t - k^s) + k^3(s_t - s)\right] \\
&\quad + \tilde{q}_3\left[L^1(k_t - k^s) + L^3(s_t - s)\right] + \tilde{q}_4\left[n^1(k_t - k^3) + n^3(s_t - s)\right] + \tilde{q}_5(s_t - s) \\
&= [\tilde{q}_1 + \tilde{q}_2 k^1 + \tilde{q}_3 L^1 + \tilde{q}_4 n^1](k_t - k^s) + [\tilde{q}_2 k^3 + \tilde{q}_3 L^3 + \tilde{q}_4 n^3 + \tilde{q}_5](s_t - s) = 0.
\end{aligned} \tag{146}$$

Doing the same for (85), we get

$$\begin{aligned}
&E\left[\tilde{Q}(k_t, k_{t+1}, k_{t+2}, L_t, L_{t+1}, n_t, n_{t+1}, s_t, s_{t+1}) | k_t, s_t\right] = \tilde{Q}_1(k_t - k^s) + \tilde{Q}_2\left[k^1(k_t - k^s) + k^3(s_t - s)\right] \\
&\quad + \tilde{Q}_3\left[(k^1)^2(k_t - k^s) + (k^1k^3 + k^3A)(s_t - s)\right] + \tilde{Q}_4\left[L^1(k_t - k^s) + L^3(s_t - s)\right] \\
&\quad + \tilde{Q}_5\left[L^1k^1(k_t - k^s) + (L^1k^3 + L^3A)(s_t - s)\right] + \tilde{Q}_6\left[n^1(k_t - k^s) + n^3(s_t - s)\right] \\
&\quad + \tilde{Q}_7\left[n^1k^1(k_t - k^s) + (n^1k^3 + n^3A)(s_t - s)\right] + \tilde{Q}_8(s_t - s) + \tilde{Q}_9A(s_t - s) \\
&= \left[\tilde{Q}_1 + \tilde{Q}_2k^1 + \tilde{Q}_3(k^1)^2 + \tilde{Q}_4L^1 + \tilde{Q}_5L^1k^1 + \tilde{Q}_6n^1 + \tilde{Q}_7n^1k^1\right](k_t - k^s) \\
&\quad + \left[\tilde{Q}_2k^3 + \tilde{Q}_3(k^1k^3 + k^3A) + \tilde{Q}_4L^3 + \tilde{Q}_5(L^1k^3 + L^3A) + \tilde{Q}_6n^3 + \tilde{Q}_7(n^1k^3 + n^3A) + \tilde{Q}_8 + \tilde{Q}_9A\right](s_t - s) = 0.
\end{aligned} \tag{147}$$

Thus, the relevant equations for determining k^1, k^3, n^1, n^3, L^1, and L^3 are:

(148) $\tilde{q}_1 + \tilde{q}_2k^1 + \tilde{q}_3L^1 + \tilde{q}_4n^1 = 0$

(149) $\tilde{q}_2k^3 + \tilde{q}_3L^3 + \tilde{q}_4n^3 + \tilde{q}_5 = (0,0)$

(150) $\tilde{Q}_1 + \tilde{Q}_2k^1 + \tilde{Q}_3(k^1)^2 + \tilde{Q}_4L^1 + \tilde{Q}_5L^1k^1 + \tilde{Q}_6n^1 + \tilde{Q}_7n^1k^1 = 0$

(151) $\tilde{Q}_2k^3 + \tilde{Q}_3(k^1k^3+k^3A) + \tilde{Q}_4L^3 + \tilde{Q}_5(L^1k^3+L^3A) + \tilde{Q}_6n^3 + \tilde{Q}_7(n^1k^3+n^3A) + \tilde{Q}_8 + \tilde{Q}_9A = (0,0)$

(152) $\tilde{W}_1 + \tilde{W}_2k^1 + \tilde{W}_3(k^1)^2 + \tilde{W}_4L^1 + \tilde{W}_5L^1k^1 + \tilde{W}_6n^1 + \tilde{W}_7n^1k^1 = 0$

(153) $\tilde{W}_2k^3 + \tilde{W}_3(k^1k^3+k^3A) + \tilde{W}_4L^3 + \tilde{W}_5(L^1k^3+L^3A) + \tilde{W}_6n^3 + \tilde{W}_7(n^1k^3+n^3A) + \tilde{W}_8 + \tilde{W}_9A = (0,0)$.

Note that only k^1, L^1, and n^1 appear in (148), (150), and (152). Moreover, these are precisely the same equations that were used to solve for k^1, L^1, and n^1 in Fuerst's model, and so the algorithm described after (87)-(94) can be used here too. The remaining unknowns are k^3, L^3, and n^3, which can be solved using (149), (151), and (153). We express these equations in matrix notation. Let β be the 1×6 vector of unknowns:

(154) $\beta = (k^3\, L^3\, n^3)$.

Let X be the following 6×6 matrix:

(155) $$X = \begin{bmatrix} \tilde{q}_2I & (\tilde{Q}_2+\tilde{Q}_3k^1+\tilde{Q}_5L^1+\tilde{Q}_7n^1)I + \tilde{Q}_3A & (\tilde{W}_2+\tilde{W}_3k^1+\tilde{W}_5L^1+\tilde{W}_7n^1)I + \tilde{W}_3A \\ \tilde{q}_3I & \tilde{Q}_4I + \tilde{Q}_5A & \tilde{W}_4I + \tilde{W}_5A \\ \tilde{q}_4I & (\tilde{Q}_6I+\tilde{Q}_7A) & \tilde{W}_6I + \tilde{W}_7A \end{bmatrix},$$

where, as before, I denotes the 2×2 identity matrix. Also, Y denotes the 1×6 element vector:

(156) $Y = [\tilde{q}_5 \;\; \tilde{Q}_8 + \tilde{Q}_9A \;\; \tilde{W}_8 + \tilde{W}_9A]$.

Then, β is the solution to

(157) $\beta X + Y = 0$.

XI. Some Numerical Results

Our three variants of Fuerst's model were solved using the linearization procedure discussed in Sections IX and X. In each case, we set, after rounding, $\mu = 0.004$, $A_{11} = 0.95$, $A_{21} = A_{12} = 0$, $\beta = 1.03^{-0.25}$, $\theta = 1$, $x = 0.07/4$, $\gamma = 2.99$, $\alpha = 0.34$. We think of the time period in the model as one quarter. We are particularly interested in the employment response to a monetary shock and computed the following elasticity:

$$(158)\quad \eta \equiv \frac{L_1^3}{L^s}.$$

Here, $L^3 = (L_1^3\ L_2^3)$ and L^3 is defined in (78). In words, η is the contemporaneous percent change in employment associated with a one percentage point change in the growth rate of money. In (158), η is the elasticity in a neighborhood of nonstochastic steady-state.

We also computed the contemporaneous impact of a change in the money growth rate on the nominal interest rate. To do this, we used the following relations:

$$(159)\quad p_t = p(k_t, k_{t+1}, n_t, L_t, \theta_t) \equiv \frac{1 - n_t}{f(k_t, L_t, \theta_t) - k_{t+1}}$$

$$(160)\quad w_t = w(k_t, k_{t+1}, n_t, L_t, \theta_t, x_t, p_t) \equiv \frac{n_t + x_t - p_t\big(k_{t+1} - (1-\delta)k_t\big)}{L_t}$$

$$(161)\quad k_{t+1} - k^s = k^1(k_t - k^s) + k^2(s_{t-1} - s) + k^3(s_t - s)$$

$$(162)\quad L_t - L^s = L^1(k_t - k^s) + L^2(s_{t-1} - s) + L^3(s_t - s)$$

$$(163)\quad n_t - n^s = n^1(k_t - k^s) + n^2(s_{t-1} - s) + n^3(s_t - s).$$

Equations (159), (160) are just (69) and (70). Also, it is convenient to write k^3, L^3, and n^3 in (161)–(163) as follows:

$$
\begin{aligned}
& k^3 = (k_1^3 \;\; k_2^3) \\
(164) \quad & L^3 = (L_1^3 \;\; L_2^3) \\
& n^3 = (n_1^3 \;\; n_2^3).
\end{aligned}
$$

Below, we call the "Fuerst model" the one that is emphasized in Sections I–IX of the paper. In that model, $n^3 = 0$. The "sluggish capital model" is the one described in Section X.A. In that model, $k^3 = n^3 = 0$. Finally, the "cash-in-advance model" is the one described in Section X.B. In that model, $k^2 = n^2 = L^2 = 0$. It is convenient to write k_2^3, L_2^3, and n_2^3 as follows:

$$
\begin{aligned}
& k_x = k_2^3 \\
(165) \quad & L_x = L_2^3 \\
& n_x = n_2^3.
\end{aligned}
$$

Here, k_x, L_x, and n_x describe the contemporaneous impact on investment, hours worked, and cash loans to financial intermediaries, respectively, of a unit perturbation in x in nonstochastic steady-state.

The final relation which we use to compute the contemporaneous nominal interest impact of a money growth shock is

$$(166) \quad R_t = f_L(k_t, L_t, \theta_t) p_t / w_t,$$

which holds in equilibrium. To see why R_t is a gross money rate of return, note that the denominator in (166) measures an input of dollars, while the numerator measures the consequent marginal output of dollars.

The total impact of a perturbation in x_t on R_t is the sum of the direct and indirect effects in equations (159)–(164) and (166). Evaluating the total derivative, dR_t/dx_t, in steady state, we get

$$(167) \quad R_x = \frac{dR}{dx} = f_{LL} \frac{p}{w} L_x + \frac{f_L}{w} \frac{dp}{dx} - \frac{f_L}{w^2} p \frac{dw}{dx},$$

where

(168) $\frac{dp}{dx} = p_{k'}k'_x + p_n n_x + p_L L_x$

(169) $\frac{dw}{dx} = w_{k'}k'_x + w_n n_x + w_L L_x + w_x$,

and formulas for $p_{k'}$, p_n, p_L, $w_{k'}$, w_n, w_L, w_x, and w_p are given in the Appendix. Also, L_x, k'_x, and n_x are defined in (164).

Results for the three variants of the model and for alternative parameterizations are given in Table 1.

Table 1:

Computational Results*

Parameter Values			Model					
			Fuerst		Sluggish Capital		Cash-in-Advance	
ψ	A_{22}	δ	η	R_x	η	R_x	η	R_x
0	0	1	.424	-.938	1.88	-4.16	0	0
0	.81	1	-.602	-.726	.734	-2.89	-1.25	.714
0	0	.02	-.032	-.013	1.42	-2.99	0	0
-1.3	0	1	.189	-.929	1.92	-4.198	0	0
-4	0	1	-.172	-.927	2.023	-4.29	0	0
0	.81	.02	-2.19	.193	-.988	-2.28	-.214	.298
0	.5	.02	.198	.111	.677	-2.77	-.701	.163
1.3	.81	.02	-2.36	.621	-1.27	-2.17	-3.40	.192
-1.3	.81	.02	-2.22	-.016	-.882	-2.33	-1.67	.331
-4	.81	.02	-2.41	-2.80	-.784	-2.39	-1.25	.357

*A_{22} is the autocorrelation of the money growth rate, x_t, δ is the rate of depreciation on capital, defined in (13), and ψ is a curvature parameter in the utility function, see (2). For the remaining parameter values underlying the computations, see the text.

Appendix

The Coefficients of the Linearized Euler Equations

Our utility and production functions are:

$$u(c,L) = [c^{(1-\gamma)}(1-L)^{\gamma}]^{\psi}/\psi$$

$$f(k,L,\theta) = \exp(-\alpha\mu)k^{\alpha}[\exp(\theta)L]^{(1-\alpha)} + (1-\delta)k.$$

In Section IX we discussed how to compute n^s, k^s, and L^s, the steady-state values of n_t, k_t, and L_t. Then, we need the following objects:

$$f_k = \alpha \exp(-\alpha\mu)[\exp(\theta)L^s/k^s]^{(1-\alpha)} + 1 - \delta$$

$$f_L = (1-\alpha) \exp(-\alpha\mu) \exp[(1-\alpha)\theta](k^s/L^s)^{\alpha}$$

$$f_\theta = (1-\alpha) \exp(-\alpha\mu)k^{\alpha}[\exp(\theta)L^s]^{(1-\alpha)}$$

$$f_{L\theta} = (1-\alpha)f_L$$

$$f_{k\theta} = \alpha(1-\alpha) \exp(-\alpha\mu)[\exp(\theta)L^s/k^s]^{(1-\alpha)}$$

$$f_{Lk} = \alpha f_L/k^s$$

$$f_{kk} = -(1-\alpha)\alpha \exp(-\alpha\mu)(k^s)^{\alpha-2}[\exp(\theta)L^s]^{(1-\alpha)}$$

$$f_{LL} = -\alpha f_L/L^s.$$

Also,

$$c^s = \exp(-\alpha\mu)(k^s)^{\alpha}[\exp(\theta)L^s]^{(1-\alpha)} - \delta k^s$$

$$u_c = (1-\gamma)(c^s)^{[(1-\gamma)\psi-1]}(1-L^s)^{\gamma\psi}$$

$$u_L = -\gamma(c^s)^{(1-\gamma)\psi}(1-L^s)^{(\gamma\psi-1)}$$

$$u_{cc} = [(1-\gamma)\psi - 1]u_c/c^s$$

$$u_{cL} = -\gamma\psi u_c/(1-L^s)$$

$$u_{LL} = -(\gamma\psi-1)u_L/(1-L^s).$$

Finally, the steady-state prices and their derivatives are given by

$$p = \frac{1 - n^*}{c^*}$$

$$p_L = -pf_L/c^*$$

$$p_k = \frac{-(1-n^*)f_k}{(c^*)^2}$$

$$p_{k'} = \frac{1 - n^*}{(c^*)^2}$$

$$p_n = -\frac{1}{c^*}$$

$$p_\theta = \frac{-(1-n^*)f_\theta}{(c^*)^2}$$

$$w = \frac{n^* + x - p\delta k^*}{L^*}$$

$$w_p = -\frac{\delta k^*}{L^*}$$

$$w_L = -\frac{w}{L^*} + w_p p_L$$

$$w_k = \frac{p(1-\delta)}{L^*} + w_p p_k$$

$$w_{k'} = \frac{-p}{L^*} + w_p p_{k'}$$

$$w_\theta = -\frac{k^* \delta p_\theta}{L^*}$$

$$w_n = \frac{1}{L^*} + w_p p_n$$

$$w_x = \frac{1}{L^*}.$$

Here, it has been convenient to think of w_t in (70) as a function, $w(k_t, k_{t+1}, n_t, L_t, x_t, p_t)$

$$\tilde{q}_1 = \frac{u_{cc}f_k}{p} - \frac{u_c}{p^2}\, p_k + \left[\frac{u_{Lc}f_k}{w^2} - \frac{2u_L}{w^3}\, w_k\right] pf_L + \frac{u_L}{w^2}\, [p_k f_L + pf_{Lk}]$$

$$\tilde{q}_2 = \frac{-u_{cc}}{p} - \frac{u_c}{p^2}\, p_{k'} + \left\{\frac{-u_{Lc}}{w^2} - \frac{2u_L}{w^3}\, w_{k'}\right\} pf_L + \frac{u_L}{w^2}\, p_{k'} f_L$$

$$\tilde{q}_3 = \frac{u_{cc}f_L + u_{cL}}{p} - \frac{u_c}{p^2}\, p_L + \left[\frac{u_{Lc}f_L + u_{LL}}{w^2} - \frac{2u_L}{w^3}\, w_L\right] pf_L + \frac{u_L}{w^2}\, [pf_{LL} + p_L f_L]$$

$$\tilde{q}_4 = \frac{-u_c}{p^2}\, p_n - \frac{2u_L}{w^3}\, w_n pf_L + \frac{u_L}{w^2}\, p_n f_L$$

$$\tilde{q}_5 = \left\{\frac{u_{cc}f_\theta}{p} - \frac{u_c}{p^2}\, p_\theta + \left[\frac{u_{Lc}f_\theta}{w^2} - \frac{2u_L}{w^3}\, w_\theta\right] pf_L + \frac{u_L}{w^2}\, [p_\theta f_L + pf_{L\theta}]\, ,\ \frac{-2u_L}{w^3}\, w_x pf_L\right\}.$$

$$\tilde{Q}_1 = u_{Lc}f_k + \beta\, \frac{u_c}{p}\, \frac{w_k}{1 + x}$$

$$\tilde{Q}_2 = -u_{Lc} + \beta\left[\frac{u_{cc}f_k}{p} - \frac{u_c}{p^2}\, p_k\right] \frac{w}{1 + x} + \frac{\beta u_c}{p}\, \frac{w_{k'}}{1 + x}$$

$$\tilde{Q}_3 = -\beta\left[\frac{u_{cc}}{p} + \frac{u_c}{p^2}\, p_{k'}\right] \frac{w}{1 + x}$$

$$\tilde{Q}_4 = u_{Lc}f_L + u_{LL} + \beta\, \frac{u_c}{p}\, \frac{w_L}{1 + x}$$

$$\tilde{Q}_5 = \beta\left[\frac{u_{cc}f_L + u_{cL}}{p} - \frac{u_c}{p^2}\, p_L\right] \frac{w}{1 + x}$$

$$\tilde{Q}_6 = \beta\, \frac{u_c}{p}\, \frac{w_n}{1 + x}$$

$$\tilde{Q}_7 = \frac{-\beta u_c}{p^2}\, p_n\, \frac{w}{1 + x}$$

$$\tilde{Q}_8 = \left\{u_{Lc}f_\theta + \beta\, \frac{u_c}{p}\, \frac{w_\theta}{1 + x},\ \beta\, \frac{u_c}{p}\left[\frac{w_x}{1 + x} - \frac{w}{(1+x)^2}\right]\right\}$$

$$\tilde{Q}_9 = \left\{\beta\left[\frac{u_{cc}f_\theta}{p} - \frac{u_c}{p^2}\, p_\theta\right] \frac{w}{1 + x},\ 0\right\}.$$

$$\tilde{W}_1 = \left[\frac{u_{Lc}f_k}{w} - \frac{u_L}{w^2}\, w_k\right] \frac{p^2}{w}\, f_L + \frac{u_L}{w}\left\{\frac{2pp_k}{w}\, f_L - \left(\frac{p}{w}\right)^2 [w_k f_L - wf_{Lk}]\right\}$$

$$\hat{W}_2 = \left\{\frac{-u_{Lc}}{w} - \frac{u_L}{w^2} w_{k'}\right\} \frac{p^2}{w} f_L + \frac{u_L}{w}\left\{\frac{2pp_{k'}}{w} f_L - \frac{p^2}{w^2} w_{k'} f_L\right\}$$

$$- \beta\left\{\frac{u_{Lc}f_k}{w} - \frac{u_L}{w^2} w_k\right\} pf_k - \frac{\beta u_L}{w}\{p_k f_k + pf_{kk}\}$$

$$\hat{W}_3 = -\beta\left\{\frac{-u_{Lc}}{w} - \frac{u_L}{w^2} w_{k'}\right\} pf_k - \beta\,\frac{u_L}{w} p_{k'} f_k$$

$$\hat{W}_4 = \left[\frac{u_{Lc}f_L + u_{LL}}{w} - \frac{u_L}{w^2} w_L\right] \frac{p^2}{w} f_L + \frac{u_L}{w}\left\{\frac{2pp_L}{w} f_L - \left(\frac{p}{w}\right)^2 w_L f_L + \frac{p^2}{w} f_{LL}\right\}$$

$$\hat{W}_5 = -\beta\left\{\frac{u_{Lc}f_L + u_{LL}}{w} - \frac{u_L}{w^2} w_L\right\} pf_k - \frac{\beta u_L}{w}\,[p_L f_k + pf_{kL}]$$

$$\hat{W}_6 = \frac{-u_L}{w^2} w_n \frac{p^2}{w} f_L + \frac{u_L}{w}\left\{\frac{2pp_n}{w} f_L - \left(\frac{p}{w}\right)^2 w_n f_L\right\}$$

$$\hat{W}_7 = \beta\,\frac{u_L}{w^2} w_n p f_k - \beta\,\frac{u_L}{w} p_n f_k$$

$$\hat{W}_8 = \left\{\left[\frac{u_{Lc}f_\theta}{w} - \frac{u_L}{w^2} w_\theta\right] \frac{p^2}{w} f_L + \frac{u_L}{w}\,\frac{2pp_\theta}{w} f_L + \frac{u_L}{w}\,\frac{p^2}{w} f_{L\theta} - (p/w)^2 w_\theta f_L u_L/w,\ \frac{-2u_L}{w^3} w_x p^2 f_L\right\}$$

$$\hat{W}_9 = \left\{-\beta\left[\frac{u_{Lc}f_\theta}{w} - \frac{u_L}{w^2} w_\theta\right] pf_k - \beta\,\frac{u_L}{w} p_\theta f_k - \beta\,\frac{u_L}{w} pf_{k\theta},\ \beta\,\frac{u_L}{w^2} w_x p f_k\right\}.$$