

Econometric Exogeneity and Alternative
Estimators of Portfolio Balance Schedules
for Hyperinflations: A Note

by

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Note: Phillip Cagan drew my attention
to the issue discussed in this paper.

Rodney Jacobs has implemented an estimator of Cagan's portfolio balance schedule that is statistically consistent under the condition that the money supply is econometrically exogenous in the portfolio balance equation---a condition under which Cagan's estimator is not in general consistent. Jacobs' estimator gives very different results from Cagan's, and recovers estimates of the "stability parameter" close to and sometimes exceeding the critical value of unity.

In this note, I evaluate the statistical properties of Jacobs' estimator under circumstances where money fails to be econometrically exogenous in the portfolio balance schedule. For concreteness and simplicity, I assume that Cagan's adaptive expectations scheme is equivalent with rational expectations---which delivers a parsimonious model in which money fails to be exogenous with respect to inflation. The calculations imply that if this model is correct, Jacobs' (biased) estimator of the stability parameter will be distributed about its population value of unity---a finding which rationalizes Jacobs' estimates for five of the six hyperinflations he studied. In addition to this substantive result, the note aims to illustrate some computational techniques that are useful for calculating probability limits of distributed lag estimators. If the reader doesn't like the model of the money-price process that I assume, he can use these techniques with his own preferred model to evaluate Jacobs' estimator (and Cagan's estimator too). Related techniques were applied to Cagan's estimator in Sargent [].

Phillip Cagan's model of portfolio balance during hyperinflation consists of the two equations

$$(1) \quad m_t - p_t = \alpha \pi_t + u_t \quad \alpha < 0$$

$$(2) \quad \pi_t = \frac{1 - \lambda}{1 - \lambda L} (1 - L) p_t \quad |\lambda| < 1$$

where m_t is the natural log of the money supply, p_t is the log of the price level, π_t is the rate of inflation now expected by the public to prevail over some horizon, L is the lag operator ($L^n x_t = x_{t-n}$), and u_t is a random disturbance. Cagan implemented his model by substituting (2) into (1) to derive the estimable equation

$$m_t - p_t = \alpha \left(\frac{1 - \lambda}{1 - \lambda L} \right) (1 - L) p_t + u_t$$

or

$$(3) \quad m_t - p_t = \alpha(1 - \lambda) \sum_{i=0}^{\infty} \lambda^i (p_{t-i} - p_{t-i-1}) + u_t$$

Cagan estimated (3) by nonlinear least squares. That method leads to consistent parameter estimates in (3) provided that u_t is orthogonal to current and past $(p_t - p_{t-1})$'s, that is, provided that

$$(4) \quad E u_t \cdot (p_{t-j} - p_{t-j-1}) = 0 \quad \text{for all } t \text{ and for all } j \geq 0$$

Suppose we make the assumption that m_t is strictly econometrically exogenous in (3), which is the condition

$$(5) \quad E[m_t \cdot u_s] = 0 \quad \text{for all } t, s$$

Condition (5) states that the money supply at all leads and lags is orthogonal to the disturbance u in portfolio balance. Under the

condition (5) that m is exogenous in (3), the orthogonality condition (4) required for the consistency of Cagan's estimates will in general fail. For under (5), movements in the u 's on average leave m unaltered and lead to equilibrating changes in p . For example, a high value of u causes the demand for real balances to be high, which because m does not respond, causes a fall in the price level in order to increase real balances. This sets up a correlation between p_t and u_t , thus violating (4).

Rodney Jacobs has pointed out that where m is exogenous in (3), the appropriate thing to do is to "invert" (3) and solve for a relationship in which m is the regressor. Solving (3) for p as a function of m 's and u 's gives

$$p_t = \left[\frac{1}{1 + \alpha(1 - \lambda)} \right] \left[\frac{1 - \lambda L}{1 - \delta L} \right] m_t - \left[\frac{1}{1 + \alpha(1 - \lambda)} \right] \left[\frac{1 - \lambda L}{1 - \delta L} \right] u_t$$

where $\delta = \frac{\lambda + \alpha(1 - \lambda)}{1 + \alpha(1 - \lambda)}$. Solving for $m_t - p_t$ gives

$$m_t - p_t = \frac{\alpha(1 - \lambda)}{1 + \alpha(1 - \lambda)} \cdot \frac{1}{1 - \delta L} (1 - L)m_t + \frac{1}{1 + \alpha(1 - \lambda)} \left[\frac{1 - \lambda L}{1 - \delta L} \right] u_t$$

or

$$(6) \quad m_t - p_t = \frac{\alpha(1 - \lambda)}{1 + \alpha(1 - \lambda)} \sum_{i=0}^{\infty} \delta^i (m_{t-i} - m_{t-i-1}) + u_t'$$

where $u_t' = \frac{1}{1 + \alpha(1 - \lambda)} \left[\frac{1 - \lambda L}{1 - \delta L} \right] u_t$.

On the hypothesis that m is exogenous with respect to u , Equation (6) has a disturbance u_t' that is orthogonal to the regressors at all lags, and so is consistently estimated by nonlinear least squares. This is the procedure used by Jacobs.

Jacobs found that estimating (6) gave very different estimates of α , λ and the "stability parameter" δ than Cagan obtained. In particular, for five of the six hyperinflations studied by Jacobs, he found δ to be distributed around the borderline-unstable value of unity, three being somewhat above unity, two being somewhat below.¹ According to Cagan's analysis, δ had to be below unity in order for one to conclude that the hyperinflations were responses to excessive monetary growth and were not self-sustaining explosive processes. Cagan's estimates had on the whole been consistent with δ being below unity, so that Jacobs' procedure does lead to substantially different conclusions than Cagan's.

As emphasized above, the *raison d'être* of Jacobs' procedure is the hypothesis that money is exogenous with respect to disturbances to portfolio balance. Empirical work by Wallace and me raises strong doubts about that hypothesis. Using Sims's econometric exogeneity test, we found that in the hyperinflation data there is usually strong evidence for rejecting the hypothesis that m is exogenous with respect to p . This implies that the data are not compatible with there being a structural one-sided distributed lag of p on m with a disturbance orthogonal to past, present, and future m 's, which is the form of regression estimated by Jacobs.

On a more casual level, it has been widely remarked that the German monetary authority was operating under a "real bills" regime during the hyperinflation, a regime which is bound to set up feedback from inflation to money creation, thereby rendering invalid the assumption that m is exogenous in the portfolio balance equation.

Given this evidence, it seems useful to ask whether one can account for the pattern of Jacobs' estimates as resulting from a failure of his exogeneity assumption. Here I calculate the theoretical values that one would obtain estimating Jacobs equation (6) in a world described by the bivariate inflation-money creation model which must prevail for Cagan's adaptive expectations scheme to be compatible with rational expectations.² There are three reasons for using this model in the present exercise. First, the model is compatible with the broad outlines of the feedback structure between inflation and money creation indicated by Granger-Sims causality tests (inflation Granger-causes money creation with no reverse causality; see Sargent and Wallace). Second, the model is parsimonious in terms of its parameterization, which is useful for calculations like those to be performed here. Third, the assumption of rational expectations provides an economic rationale for using Cagan's model in the first place.

Wallace and I showed that Cagan's model is compatible with rational expectations where the inflation-money creation process is governed by

$$(7) \quad x_t = \phi(1 - \lambda L)(\varepsilon_t - \eta_t)$$

$$(8) \quad M_t = \phi(1 - \lambda)(\varepsilon_t - \eta_t) + (1 - L)\varepsilon_t$$

where $x_t \equiv (1 - L)^2 p_t$

$$M_t = (1 - L)^2 m_t$$

$$\phi = (\lambda + \alpha(1 - \lambda))^{-1},$$

$$\eta_t = u_t - u_{t-1}$$

and where η_t and ε_t are each serially independent random variables with mean zero and finite variances σ_η^2 and σ_ε^2 , respectively.

I assume that $E_{t-1}\varepsilon_t = E_{t-1}\eta_t = 0$ and that $E(\varepsilon_t \eta_t) = \tau_\varepsilon \eta$.

Under (7) and (8), the first difference of inflation, x_t , and the first difference of percentage money creation, M_t , are stationary, correlated first-order moving average processes. The covariogram of M is defined as

$$c(\tau) = E[M(t)M(t - \tau)].$$

It is straightforward to calculate

$$c(0) = [\phi(1 - \lambda) + 1]^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \phi^2(1 - \lambda)^2 \sigma_\eta^2 - 2[\phi(1 - \lambda) + 1]\phi(1 - \lambda)\sigma_\varepsilon\sigma_\eta$$

$$c(1) = c(-1) = -[\phi(1 - \lambda) + 1]\sigma_\varepsilon^2$$

$$c(\tau) = 0 \quad , \quad |\tau| > 1 .$$

The cross-covariogram of x and M is defined as

$$r(\tau) = E[x_t M_{t-\tau}] .$$

It is straightforward to calculate

$$r(0) = [\phi(\phi(1 - \lambda) + \phi 1) + \phi \lambda] \sigma_{\epsilon}^2 + \phi^2(1 - \lambda) \sigma_{\eta}^2 \\ - [\phi^2(1 - \lambda) + \phi(\phi(1 - \lambda) + 1)] \sigma_{\epsilon \eta}$$

$$r(1) = -\phi \lambda [\phi(1 - \lambda) + 1] \sigma_{\epsilon}^2 - \phi^2 \lambda (1 - \lambda) \sigma_{\eta}^2 \\ + \{\phi \lambda [\phi(1 - \lambda) + 1] + \phi^2 \lambda (1 - \lambda)\} \sigma_{\epsilon \eta}$$

$$r(-1) = -\phi(\sigma_{\epsilon}^2 + \sigma_{\epsilon \eta})$$

$$r(\tau) = 0 \quad |\tau| > 1 .$$

We will write the covariance generating function of M or z -transform of $c(\tau)$ as

$$c(z) = \sum_{\tau=-1}^{\infty} c(\tau) z^{\tau} ,$$

the coefficient on z^{τ} being the covariance $c(\tau)$. The cross-covariance generating function is defined as

$$r(z) = \sum_{\tau=-1}^{\infty} r(\tau) z^{\tau} .$$

It is useful to factor $c(z)$ as

$$c(z) = c(1) z^{-1} + c(0) + c(1) z \\ = (b_0 + b_1 z)(b_0 + b_1 z^{-1}) .$$

Expanding and equating powers of z , we find that b_0 and b_1 are determined from

$$b_0 b_1 = c(1)$$

$$b_0^2 + b_1^2 = c(0)$$

So we have the factorization

$$c(z) = (b_0 + b_1 z)(b_0 + b_1 z^{-1})$$

or

$$c(z) = b_0^2 \left(1 + \frac{b_1}{b_0} z\right) \left(1 + \frac{b_1}{b_0} z^{-1}\right)$$

$$c(z) = b_0^2 (1 + bz)(1 + bz^{-1})$$

where $b = b_1/b_0$.

Using the classic Wiener-Kolmogorov formula³, we can calculate the z-transform (or lag-generating function) of the population regression of x_t on current and past M 's as

$$x_t = \theta(L)M_t + \text{residual}_t$$

where

$$\theta(z) = \frac{b_0^{-2}}{1 + bz} \left[\frac{r(1)z^{-1} + r(0) + r(1)z}{1 + bz^{-1}} \right]_+$$

where $[]_+$ means "ignore all negative powers of z ". Expanding the geometric series $1/(1+bz^{-1})$, we have, assuming $|b| < 1$,

$$\begin{aligned} \theta(z) &= \frac{b_0^{-2}}{1 + bz} \left[(r(1)z^{-1} + r(0) + r(1)z) \sum_{j=0}^{\infty} (-b)^j z^{-j} \right]_+ \\ &= \frac{b_0^{-2}}{1 + bz} \left[(r(0) + r(1)z)(1 - bz^{-1} + \dots) \right]_+ \end{aligned}$$

$$(9) \quad \theta(z) = \frac{b_0^{-2}}{1 + bz} [(r(0) - r(1)b) + r(1)z]$$

Using (9) it is easy to calculate the regression of $M_t - x_t$ against current and past M_t 's as

$$\begin{aligned} M_t - x_t &= h(L)M_t + \text{residual}_t \\ &= (1 - \theta(L))M_t + \text{residual} \end{aligned}$$

which implies that the lag generating function $h(z)$ obeys

$$\begin{aligned} h(z) &= 1 - \theta(z) \\ &= \frac{1 + bz}{1 + bz} - \frac{b_0^{-2}}{1 + bz} [(r(0) - r(1)b) + r(1)z] \\ h(z) &= \frac{[1 - b_0^{-2}(r(0) - r(1)b)] + (b - b_0^{-2}r(1))z}{1 + bz} \end{aligned}$$

The lag generating function $h(z)$ can be written more compactly as

$$h(z) = \frac{h_0 + h_1 z}{1 - h_2 z}$$

where

$$\begin{aligned} h_0 &= 1 - b_0^{-2}(r(0) - r(1)b) \\ (10) \quad h_1 &= b - r(1)b_0^{-2} \\ h_2 &= -b \end{aligned}$$

In summary, where the model (7) - (8) is correct, the population regression of $M_t - x_t$ against current and past M 's will be

$$M_t - x_t = \frac{h_0 + h_1 L}{1 - h_2 L} M_t + \text{residual}_t$$

It follows that the regression of $m_t - p_t$ against current and lagged m 's is given by⁴

$$m_t - p_t = \left(\frac{h_0 + h_1 L}{1 - h_2 L} \right) m_t + \text{residual}_t .$$

It is easily deduced that the population regression against current and lagged values of the change in m is given by

$$(11) \quad m_t - p_t = \frac{h_0 + h_1 L}{(1 - h_2 L)(1 - L)} (1 - L)m_t + \text{residual}_t .$$

Now (11) is the population regression of $m - p$ against current and lagged m 's where the distributed lag coefficients are not constrained in any way. Recall from (6), however, that Jacobs estimated the constrained version of (11)

$$(12) \quad m_t - p_t = \frac{\gamma_0}{1 - \gamma_1 L} (1 - L)m_t + \text{residual}_t ,$$

where he interpreted γ_0 as estimating $\frac{\alpha(1 - \lambda)}{1 + \alpha(1 - \lambda)}$ and γ_1 as estimating the stability parameter δ .

If the rational expectations model (7) and (8) is correct, the constrained parameterization (12) is a binding restriction. Where the true regression on current and past $(1 - L)m$'s is given by (11), least squares estimation of Jacob's parameterization (12) will minimize the approximation criterion (see Sims)

$$\int_{-\pi}^{\pi} \left| \frac{\gamma_0}{1 - \gamma_1 e^{-iw}} - \frac{(h_0 - h_1 e^{-iw})}{(1 - h_2 e^{-iw})(1 - e^{-iw})} \right|^2 S_{\Delta m}(w) dw$$

where $S_{\Delta m}(w)$ is the spectral density of $(1 - L)m$. The integrand can be rearranged to be

$$\begin{aligned} & \left| \left(\frac{1}{1 - e^{-iw}} \right) \left(\frac{\gamma_0(1 - e^{-iw})}{1 - \gamma_1 e^{-iw}} - \frac{(h_0 - h_1 e^{-iw})}{1 - h_2 e^{-iw}} \right) \right|^2 S_{\Delta m}(w) \\ &= \left| \frac{\gamma_0(1 - e^{-iw})}{1 - \gamma_1 e^{-iw}} - \frac{h_0 - h_1 e^{-iw}}{1 - h_2 e^{-iw}} \right|^2 S_m(w) \end{aligned}$$

where $S_m(w) = \left| \frac{1}{1 - e^{-iw}} \right|^2 S_{\Delta m}$. Here $S_m(w)$ is the spectral density of the level of the (log of the) money supply.⁵ Thus, least squares applied to (12) minimizes the approximation criterion

$$(13) \quad \int_{-\pi}^{\pi} \left| \frac{\gamma_0(1 - e^{-iw})}{1 - \gamma_1 e^{-iw}} - \frac{(h_0 - h_1 e^{-iw})}{1 - h_2 e^{-iw}} \right|^2 S_m(w) dw.$$

Now it is an implication of the model (7) - (8) that the spectral density $S_m(w)$ of the logarithm of the money supply is unbounded at $w = 0$ (the process m_t is not stationary, so that strictly speaking the spectrum $S_m(w)$ is not defined, though it is effectively defined by a certain limiting process). Consequently, the approximation criterion (13) will be dominated by its behavior at (and very close to) $w = 0$. We therefore minimize (13) by minimizing the integrand at $w = 0$. By inspection, it is directly verified that at $w = 0$ the integrand is driven to zero by

$$(14) \quad \begin{aligned} \gamma_1 &= 1 \\ \gamma_0 &= \frac{h_0 - h_1}{1 - h_2} \end{aligned}$$

If the model summarized by (7) and (6) is correct, the population values of Jacobs' regression (12) will obey (14), where the h 's appearing in (14) are connected to the parameters of (7) and (8) by the relations given in (10). It is evident that the population regression parameters γ_1 and γ_0 do not recover the theoretical structural parameters that Jacobs took them to be estimating (remember that Jacobs' procedure interpreted γ_0 as estimating $\frac{\alpha(1-\lambda)}{1+\alpha(1-\lambda)}$ and interpreted γ_1 as estimating the "stability parameter"

$\delta = \frac{\lambda + \alpha(1-\lambda)}{1 + \alpha(1-\lambda)}$). If the model (7) - (8) is correct, the least squares estimates of γ_0 and γ_1 are thus not statistically consistent estimates of the parameters Jacobs was trying to estimate.

Least squares estimates of Jacobs' equation (12) will recover consistent estimates of the parameters in (14). Our calculations thus imply that in sufficiently large samples, Jacobs' estimate of γ_1 ($= \delta$) will be distributed about unity. That prediction is borne out in five of the six hyperinflations studied by Jacobs.

References

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Footnotes

¹In Jacobs' notation, my $\delta = e^{-k}$. Jacobs' estimates were as follows:

<u>Country</u>	<u>k</u>	<u>$\delta = e^{-k}$</u>
Austria	.143	.87
Germany	-.131	1.14
Greece	-.262	1.30
Hungary	-.199	1.22
Poland	.139	.87
Russia	.857	.43

²The model is analyzed in detail and tested in Sargent [].

³See Whittle [, p. 66-67, pp. 41-43].

⁴There is a delicate point here. In going from the preceding equation in the text to the following one, we are in effect operating twice on M_t and x_t with the summation operator $(1 - L)^{-1} = 1 + L + L^2 + \dots$. That is, we are using the fact that

$$(1 - L)^2 p_t = x_t$$

to work back from x_t to p_t . But, for example, the process

$$(1 + L + L^2 + \dots)x_t$$

is not a well defined stationary random process, so that the projection theory underlying our calculations is strictly speaking not applicable. However, if we regard p_t as being defined by

$$p_t = (1 - \rho L)^{-2} x_t$$

where $(1 - \rho L)^{-1} = (1 + \rho L + \rho^2 L^2 + \dots)$ and where ρ is arbitrarily close to unity from below, then p_t is a well defined stationary process. I will think of m_t and p_t as being defined as the limit points of such stationary processes.

⁵We have that the spectrum of m is given by

$$S_m(w) = \left(\frac{1}{1 - e^{-iw}}\right)^2 \left(\frac{1}{1 - e^{iw}}\right)^2 (c(1)e^{iw} + c(0) + c(1)^{-iw})$$

This is unbounded at $w = 0$, so that it is not really well defined.

However, we can regard $S_m(w)$ as the limit of spectral densities defined by

$$S_m(w)_\rho = \left(\frac{1}{1 - \rho e^{-iw}}\right)^2 \left(\frac{1}{1 - \rho e^{iw}}\right)^2 (c(1)e^{iw} + c(0) + c(1)^{-iw})$$

as ρ approaches unity from below. $S_m(w)_\rho$ is well defined and is the spectrum of a stationary process for all $|\rho| < 1$.