

Dynamic Analysis of a  
Keynesian Model

Thomas J. Sargent

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## Dynamic Analysis of a Keynesian Model

These notes describe the dynamics of the Keynesian macroeconomic model under two alternative assumptions about the formation of expectations of inflation. One assumption is that expectations of inflation are formed "adaptively" so that  $\pi$  is governed by the differential equation

$$D\pi = \beta \left( \frac{Dp}{p} - \pi \right), \beta > 0$$

where  $D$  is the right-hand time derivative operator. The solution of the above differential equation is

$$\pi(t) = \pi(t_0) e^{-\beta(t-t_0)} + \beta \int_{t_0}^t e^{-\beta(t-s)} \frac{\dot{p}(s)}{p} ds,$$

so that  $\pi(t)$  is formed as a geometric "distributed lag" of past actual rates of inflation. The other assumption under which the model will be analyzed is that of perfect foresight, so that  $\pi = Dp/p$ . Changing from the first to the second assumption about expectations will be seen to convert the particular model that we analyze from a Keynesian one to a classical one, even though the "structural" equations of the model remain the same.

The Model with Adaptive Expectations

The model is identical with the Keynesian model described above, except that it is augmented with a Phillips curve. We will take advantage of the linear homogeneity of the production function and write it in the intensive form

$$\frac{Y}{K} = F\left(1, \frac{N}{K}\right) = f\left(\frac{N}{K}\right)$$

or

$$(1) \quad y = f(\lambda), \quad f'(\lambda) > 0, \quad f''(\lambda) < 0,$$

where  $y = \frac{Y}{K}$  and  $\lambda = N/K$ .

The marginal product condition for employment can be written.

$$(2) \quad \frac{w}{p} = f'(\lambda) \left( = \frac{\partial}{\partial N} K f\left(\frac{N}{K}\right) = \frac{\partial}{\partial N} F(K, N) \right).$$

The Keynesian investment schedule will be written in the intensive form

$$\frac{\dot{K}}{K} = I\left(F_K - (r + \delta - \pi)\right), \quad I' > 0$$

or

$$(3) \quad i = \frac{I}{K} = I\left(f(\lambda) - \lambda f'(\lambda) - (r + \delta - \pi)\right).$$

Notice that

$$F_K(K, N) = \frac{\partial}{\partial K} \left( K f\left(\frac{N}{K}\right) \right) = f\left(\frac{N}{K}\right) + K f'\left(\frac{N}{K}\right) \left[ -\frac{N}{K^2} \right] = f(\lambda) - f'(\lambda).$$

The consumption function is assumed to be linear and "proportional," which permits us to write it in the capital intensive form

$$\frac{C}{K} = z \cdot \left[ \frac{Y}{K} - \frac{T}{K} - \frac{\delta K}{K} \right], \quad 0 < z < 1$$

or

$$(4) \quad c = z(y - t - \delta)$$

where  $t = T/K$ . The parameter  $z$  is the marginal propensity to consume.

We write the national income identity as

$$(5) \quad y = c + i + g + \delta,$$

where  $g = G/K$  and  $c = c/K$ .

The portfolio equilibrium condition is assumed to take the form

$$(6) \quad \frac{M}{pK} = m(r, y),$$

which can be rationalized by assuming the demand for money  $m(r, Y)$  is homogeneous of degree one in output.

We posit that the evolution of the money wage is governed by the Phillips curve

$$(7) \quad \frac{Dw}{w} = h\left(\frac{N}{N^S}\right) + \pi, \quad h' > 0, \quad h(1) = 0.$$

where  $N^S$  is the labor supply. Given  $\pi$ , equation (7) depicts a trade-off between the rate of employment relative to the labor supply and the rate of wage inflation. An increase in  $\pi$  shifts the Phillips curve upward by the amount of that increase.

The labor supply is exogenous and is governed by

$$(8) \quad N^S(t) = N^S(t_0)e^{n(t-t_0)}$$

where  $n$  is the proportionate rate of growth of the labor supply.

The model is completed by specifying that expectations of inflation obey the adaptive scheme

$$(9) \quad \pi(t) = \pi(t_0)e^{-\beta(t-t_0)} + \beta \int_{t_0}^t e^{-\beta(t-s)} \frac{Dp(s)}{p(s)} ds.$$

Collecting equations, the complete model is:

$$(1) \quad y = f(\lambda)$$

$$(2) \quad \frac{w}{p} = f'(\lambda)$$

$$(3) \quad i = I(F(\lambda) - f'(\lambda) - (r + \delta - \pi)) = \frac{\dot{K}}{K}$$

$$(4) \quad c = z(y - \bar{t} - \delta)$$

$$(5) \quad y = c + i + g$$

$$(6) \quad \frac{M}{pK} = m(r, y)$$

$$(7) \quad \frac{Dw}{w} = h\left(\frac{\lambda K}{N^s}\right) + \pi$$

$$(8) \quad N^s(t) = N^s(t_0)e^{n(t-t_0)}$$

$$(9) \quad \pi(t) = \pi(t_0)e^{-\beta(t-t_0)} + \beta \int_{t_0}^t e^{-\beta(t-s)} \frac{Dp(s)}{p(s)} ds.$$

$$K(t) = K(t_0) + \int_{t_0}^t i(s)K(s) ds$$

Given the initial conditions  $w(t_0)$ ,  $\pi(t_0)$ ,  $K(t_0)$ , and given time paths for the exogenous variables  $M$ ,  $g$ , and  $\bar{t}$  for  $t \geq t_0$ , the model will generate time paths of the endogenous variables  $y$ ,  $\lambda$ ,  $K$ ,  $c$ ,  $w$ ,  $p$ ,  $r$  and  $\pi$ . Notice that even though  $w$ ,  $\pi$ , and  $K$  are fixed or "exogenous" at a point in time, being inherited from the past according to (3), (7), and (9), they are endogenous variables from the point of view of our dynamic analysis. The analysis determines their evolution over time.

The momentary equilibrium of our system can be determined by solving equations (1)-(6) for IS and LM curves. The IS curve gives the

combinations of  $r$  and  $y$  that make the demand for output equal to the supply. It is derived by substituting (3) and (4) into (5):

$$(10) \quad y = z(y - \bar{t} - \delta) + I(f(\lambda) - \lambda f(\lambda) - (r + \delta - \pi)) + g + \delta.$$

Since  $f'(\lambda) > 0$ , we can invert (1) and obtain

$$\lambda = \lambda(y), \quad \lambda'(y) = \frac{1}{f'(\lambda)} > 0, \quad \lambda''(y) = \frac{-f''(\lambda)\lambda'(y)}{f'(\lambda)^2} > 0.$$

Substituting this into (10) yields the IS curve:

$$y = z(y - \bar{t} - \delta) + I\left(y - \frac{\lambda(y)}{\lambda'(y)} - (r + \delta - \pi)\right) + g + \delta.$$

The slope of the IS curve in the  $r$ - $y$  plane is given by

$$\frac{dy}{dr}_{IS} = \frac{-I'}{1 - z - I' \frac{\lambda(y)\lambda''(y)}{\lambda'(y)^2}},$$

which is of ambiguous sign since  $\lambda''(y) > 0$ . The denominator of the above expression is Hicks's "supermultiplier," the term  $I'\lambda\lambda''/\lambda'^2$  being the marginal propensity to invest out of income. We will assume that this term is less than the marginal propensity to save, so that the IS curve is downward sloping. The position of the IS curve depends on the parameters  $g$ ,  $\bar{t}$ , and  $\pi$  in the usual way. An increase in  $\pi$  shifts the IS curve upward by the amount of that increase.

We can write the marginal productivity condition for labor as

$$p = w\lambda'(y).$$

Substituting the expression for  $p$  into (6) yields the LM curve:

$$M = w\lambda'(y)K_m(r, y),$$

the slope of which is easily verified to be positive in the  $r$ - $y$  plane. The LM curve shows the combinations of  $r$  and  $y$  that guarantee portfolio balance. Its position depends on  $M$ ,  $w$ , and  $K$ , all of which are parameters at a point in time.

The momentary equilibrium of the system is determined at the intersection of the IS and LM curves. That equilibrium will in general be a nonstationary one, the interest rate, the real wage, and the capital-labor ratio possibly changing over time. However, given fixed values of  $g$ ,  $\tilde{t}$ , and  $\dot{M}/M$ , the system may over time approach a steady state in which the interest rate, real wage, and employment-capital ratio are fixed, while prices and wages change at a rate equal to  $\dot{M}/M$  minus  $n$ . We will use two curves to characterize the steady-state growth path in the  $r$ - $y$  plane. The first is simply a vertical line at the steady-state output-capital ratio, which we denote by  $y^*$ . From (5), the rate of growth of capital is

$$i = y - z(y - \tilde{t} - \delta) - g - \delta.$$

Since  $D(K/N)/(K/N) = DK/K - DN/N = i - n$ ,

we have

$$\frac{D(K/N)}{K/N} = y - z(y - \tilde{t} - \delta) - g - \delta - n.$$

Setting  $D(K/N)$  to zero and solving for  $y$  yields the value of  $y^*$ :

$$y^* = \frac{n + g + \delta(1 - z) - z\tilde{t}}{1 - z}.$$

This is the value of the output-capital ratio at which the capital-labor ratio is stationary, i.e., unchanging through time. We show  $y^*$  as a

vertical line in Figure 1. On our assumptions, the steady-state value of  $y$  is independent of the interest rate.

If firms are to be content to increase the capital stock at the steady-state rate  $n$ , so that  $i-n$  equals zero, we require

$$(11) \quad I(y - \frac{\lambda(y)}{\lambda'(y)} - (r + \delta - \pi)) - n = 0$$

which is implicitly an equation that tells us what  $r + \delta - \pi$  must be if the system is to be in a steady-state equilibrium at a given  $y$ . Taking the total differential of the above equation and rearranging gives

$$\frac{d(r + \delta - \pi)}{dy} = \frac{\lambda''\lambda}{\lambda'^2} > 0$$

as the slope of the locus of points in the  $r$ - $y$  plane along which (11) is satisfied. The slope is positive, reflecting the direct dependence of the marginal product of capital on the output-capital ratio. We call (11) the capital-market equilibrium curve and label it KE. Note that an increase in  $\pi$  causes the KE curve to shift upward by the full amount of the increase.

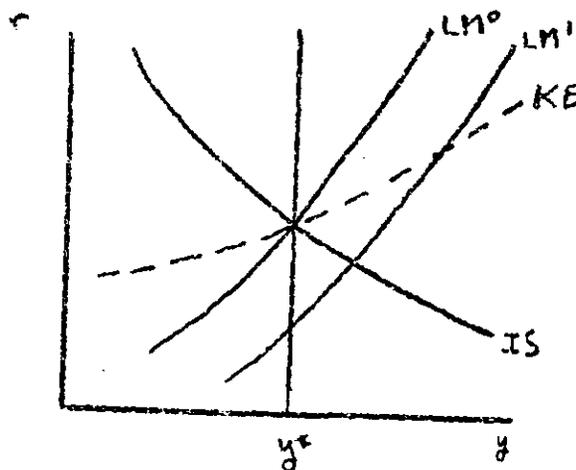


Figure 1

The determination of momentary and steady-state equilibrium can be illustrated with Figure 1. Notice that the IS curve has been drawn so that it intersects the KE curve at  $y^*$ , the steady-state output-capital ratio. That this must be so can be verified as follows. Along the KE curve, equation (11) is satisfied. Substituting for  $I(\quad)$  from (11) into the IS curve gives

$$y = z(y - t - \delta) + n + g + \delta$$

or

$$y = \frac{n + g + \delta(1 - z) - zt}{1 - z},$$

which is identical with our expression for  $y^*$ . A steady state is determined at the intersection of the KE and IS curves. Momentary equilibrium is determined at the intersection of the IS and LM curves. If the IS and LM curve intersect at an  $r$ - $y$  combination below the KE curve, capital is growing more rapidly than  $n$  at that moment. The model possesses mechanisms propelling over time the intersection of the IS and LM curve toward the intersection of the IS and KE curves. The dynamics of capital and the money wage are the key elements in the mechanism.

To illustrate how the model works, suppose that the system is initially in a full, steady-state equilibrium, the IS, LM, and KE curves all intersecting at  $y^*$ , as in Figure 1. Suppose that  $\dot{M}/M = n$ , so that the equilibrium rate of inflation is zero. We know this because in the steady-state  $r$  and  $y$  and therefore  $m(r, y)$  are constant through time. Therefore,  $\dot{M}/pK$  must be constant through time, so that  $\dot{M}/M - \dot{p}/p - \dot{K}/K = 0$  or  $\dot{M}/M - \dot{p}/p - n = 0$ . The system is in a steady state so that the LM, IS, and KE curves are unchanging through time. Since in that steady state  $\dot{w}/w = \dot{p}/p = \pi = \dot{M}/M - n = 0$ , we know from the Phillips curve that  $N/N^B$  must

equal unity. Now suppose that at some point in time there occurs a once-and-for-all jump in  $M$ , engineered via an open-market operation that leaves  $M/M$  unaltered. To simplify matters, we will suppose that  $\pi$  remains fixed at zero, its steady-state value, during the movement to a new steady-state. So we temporarily suspend (9) and substitute  $\pi(t) = 0$ . We also assume that  $g$  and  $t$  are constant over time. The immediate effect of the jump in the money supply is to shift the LM curve to the right, say to  $LM$ , in Figure 1. The result is an instantaneous jump in employment and the output-capital ratio. Employment now exceeds the labor supply, causing the money wage to adjust upward over time, as described by the Phillips curve (7). In addition, the nominal interest rate has fallen, creating a larger discrepancy between the marginal product of capital, which has risen, and the real rate of interest. Firms respond by adding to the capital stock at a rate exceeding  $n$ —this occurs at each moment the momentary solution is at an  $r$ - $y$  combination below the KE curve. Since capital is growing faster than the money supply, and since money wages are rising over time, the LM curve shifts upward over time, from  $LM$ , toward  $LM_0$  in Figure 1. To show this, notice that, for fixed values of  $y$ , logarithmic differentiation of the LM curve with respect to time yields

$$(12) \quad \dot{r}|_{LM} = \frac{m(r, y)}{m_r} \left[ \frac{\dot{M}}{M} - \frac{\dot{K}}{K} - \frac{\dot{w}}{w} \right].$$

If the expression in brackets is negative, then at each value of  $y$ , the  $r$  that maintains portfolio balance is increasing over time. Now when enough time has passed to move the LM curve back to  $LM^0$ , so that  $y$  and  $r$  are back at their initial values, the adjustment is not yet complete. When the LM curve has shifted back to  $LM^0$ , the value of  $y$  and  $\lambda$  are

at their initial values. But since  $\dot{K}/K$  exceeded  $n$  all during the intervening period, we know that  $N/N^S$ , which had the initial value of unity, since  $\dot{w}/w = 0$  initially, now exceeds unity. For  $N/N^S = \lambda K/N^S$ ;  $\lambda$  has returned to its initial value, but  $K$  has grown faster than  $n$  at each intervening moment, so that  $N/N^S$  must now exceed unity. The Phillips curve therefore implies that  $\dot{w}/w > 0$ , implying that the LM curve continues to shift up over time (see (12)) since  $\dot{M}/M - \dot{K}/K - \dot{w}/w$  is still less than zero even though  $\dot{K}/K$  has returned to its steady-state value once the LM curve has come back to its initial position. The system must therefore "overshoot," having  $y$  fall below  $y^*$  as wages rise and the LM curve moves to an intersection with the IS curve above the KE curve. Indeed, the system must spend some moments at which  $N/N^S$  is less than unity along the path returning to steady-state equilibrium. For at the moment at which the LM curve has just returned to  $LM^0$ , having been below it all during the intervening period, the price and wage level have risen by just the "right" amount in the following sense: If somehow  $\dot{w}/w$  could be zero, the LM curve would cease to move over time and the system would remain in steady-state equilibrium. But as we have seen,  $\dot{w}/w$  must be rising at the moment in question, which means that  $w$  overshoots its new steady-state value. Since  $w$  overshoots its steady-state value, we know from the Phillips curve

$$\frac{\dot{w}}{w} = h\left(\frac{N}{N^S}\right), \quad h(1) = 0, \quad h' > 0$$

that for  $w$  to fall back to its steady-state value, the system must spend some time during which  $N/N^S < 1$ , which can be characterized as a period of recession. So in this system, a once-and-for-all jump in the money supply sets off a boom followed by alternating periods of recession and

boom. The return to steady state is oscillatory. The model thus has implicit in it a theory of business cycles.

The adjustment process in response to the once-and-for-all jump in  $M$  would be even more complex if we were to permit  $\pi$  to respond to the occurrence of actual inflation, say by restoring our equation (9). For then the IS curve and the KE curve would shift upward during the early part of the transition as  $\pi$  increases in response to the emergence of inflation. It is easily verified that when  $\pi$  changes, the IS curve continues to intersect the KE curve at  $y^*$ . The result of allowing  $\pi$  to depend on past values of  $\dot{p}/p$  is to accentuate the "overshooting" or cyclical phenomenon. Following the original jump in  $M$ , the LM curve will now be shifting toward an intersection of  $y^*$  with an IS curve associated with a positive rate of expected inflation, since inflation has occurred during the transition up to that moment. Since  $\pi$  is positive and, as we verified earlier,  $N/N^e > 1$  at the moment the LM curve has shifted back enough to intersect an IS curve at  $y^*$ , wages are rising even faster at that moment than they would have been had  $\pi$  been zero throughout the transition. Assuming that the system is dynamically stable, the final resting place for all variables will be the same as if  $\pi$  had remained at its steady-state value of zero throughout the adjustment process; but the path to steady-state equilibrium may be much different.

The final effect of the once-and-for-all jump in  $M$ , once the system has returned to its steady state, is to leave all real variables unaltered and to increase the price level and money wage proportionately with the money supply. The variables  $r$  and  $y$  have steady-state values determined at the intersection of the IS and KE curves, which aren't

affected by the jump in  $M$ . In the new steady-state  $\dot{w}/w$  must be zero, which means that  $N/N^S$  must be unity, which means since  $\lambda$  is at its initial steady-state value, that  $K/N^S$  must be at its initial steady-state value. Since  $M/pK = m(r, y)$  is not moved from its steady-state value,  $p$  must have increased proportionately with the initial jump in  $M$  once the new steady state is finally achieved.

Though this model is clearly Keynesian in its momentary or point-in-time behavior, its steady-state or long-run properties are "classical" in the sense that real variables are unaffected by the money supply. The real variables are determined at the intersection of the KE and IS curves which are determined by the propensities to save and invest and the government's fiscal policies. In the steady-state, the price level must adjust so that the LM curve passes through the intersection of the IS and KE curves.

Perfect Foresight ( $\pi = Dp/p$ )

We now abandon equation (9) and for it substitute the assumption of perfect foresight or "rationality":

$$(9') \quad \pi(t) = Dp(t)/p(t),$$

where we continue to interpret  $D$  as the right-hand time derivative operator. Equation (9') asserts that people accurately perceive the right-hand time derivative of the log of the price level, the rate at which inflation is proceeding.

The dynamics of the model in response to shocks is very much different when (9') replaces (9). To solve the model, we begin by substituting (9') into (7) to obtain.

$$(13) \quad \frac{Dw}{w} = h\left(\frac{\lambda K}{N^S}\right) + \frac{Dp}{p}.$$

Differentiating (2) logarithmically with respect to time gives

$$(14) \quad \frac{Dw}{w} = \frac{f''(\lambda)}{f'(\lambda)} D\lambda + \frac{Dp}{p}.$$

Equating (13) with (14) gives

$$(15) \quad h\left(\frac{\lambda K}{N^S}\right) = \frac{f''(\lambda)}{f'(\lambda)} D\lambda, \text{ where } \frac{f''(\lambda)}{f'(\lambda)} < 0.$$

Now (15) is a differential equation in the employment capital ratio  $\lambda$ , which may be solved for  $\lambda$  in terms of past values of  $K$  and  $N^S$ . To illustrate, suppose that  $f(\lambda)$  is Cobb-Douglas, so that

$$y = f(\lambda) = A\lambda^{(1-\alpha)}.$$

Then we have

$$f'(\lambda) = A(1-\alpha)\lambda^{-\alpha}$$

$$f''(\lambda) = -\alpha(1-\alpha)A\lambda^{-\alpha-1}$$

$$\frac{f''(\lambda)}{f'(\lambda)} = \frac{-\alpha}{\lambda}$$

Also suppose that  $h(\lambda K/N^S)$  takes the form

$$\begin{aligned} h\left(\frac{\lambda K}{N^S}\right) &= \gamma \cdot \log \frac{N}{N^S} \\ &= \gamma \log N - \gamma \log N^S. \end{aligned}$$

where  $\log$  denotes the natural logarithm. Then (15) becomes

$$(16) \quad \gamma \log N - \gamma \log N^S = -\alpha \frac{\lambda}{\lambda} = -\alpha D \log N + \alpha D \log K.$$

Rearranging, we have

$$(\gamma + \alpha D) \log N = \gamma \log N^S + \alpha D \log K$$

or

$$\left(\frac{\gamma}{\alpha} + D\right) \log N = \frac{\gamma}{\alpha} \log N^S + D \log K.$$

This is a linear, first-order differential equation in  $\log N$ . To find its solution, divide through by  $\frac{\gamma}{\alpha} + D$  to get

$$\log N = \frac{1}{D + \frac{\gamma}{\alpha}} \left[ \frac{\gamma}{\alpha} \log N^S + D \log K \right].$$

Notice that

$$\frac{1}{D + \frac{\gamma}{\alpha}} = \frac{e^{(s-t)\left(\frac{\gamma}{\alpha} + D\right)}}{\frac{\gamma}{\alpha} + D} \Bigg|_{s=0}^t$$

$$= \int_{-\infty}^t e^{(s-t)\left(\frac{Y}{\alpha} + D\right)} ds. \quad -15-$$

$$= \int_{-\infty}^t e^{(s-t)\frac{Y}{\alpha}} (s-t)D ds.$$

Recalling that  $e^{(s-t)D} x(t) = x(s)$ , we have

$$\log N(t) = \int_{-\infty}^t e^{(s-t)\frac{Y}{\alpha}} (s-t)D \left[ \frac{Y}{\alpha} \log N^S(t) + D \log K \right] ds$$

$$(17) \quad \log N(t) = \frac{Y}{\alpha} \int_{-\infty}^t e^{(s-t)\frac{Y}{\alpha}} \log N^S(s) ds + \int_{-\infty}^t e^{(s-t)\frac{Y}{\alpha}} \frac{DK(s)}{K(s)} ds.$$

Equation (17) is the solution to equation (16), and expresses  $\log N$  at  $t$  as distributed lags of past values of the labor supply and capital stock. Since these are predetermined at time  $t$ , we immediately know that employment and hence output will not respond at  $t$  to the imposition of shocks to the system at  $t$ .

Given the value of  $N$  at  $t$  determined from some version of (17), and given the quantity of  $K$  inherited from the past, output is determined by equation (1), the real wage by (2), and  $c$  by (3). Given  $c$  and  $y$ , (5) then determines  $i$ . Given  $i$  and  $\lambda$ , equation (3) determines  $r - \pi$  at  $t$ . Equation (7) determines  $Dw/w - Dp/p$ .

Notice that given entire time paths from now until forever for the fiscal variables  $g$  and  $\tau$ , the model determines entire time paths of the real variables without using the portfolio balance equation. In effect, then, this model dichotomizes.

All real variables have now been determined and it remains only to determine the values of  $p$  and  $Dp/p$  at instant  $t$ . They are determined by the portfolio balance condition in the manner indicated in our notes on the classical model. To illustrate, suppose the portfolio balance equation (6) assumes the special form

$$(18) \quad \frac{M}{pK} = e^{\beta r} y, \quad \beta < 0$$

We know that in this system,  $r$  is determined by (5) which we express by inverting (5) and writing

$$r = f(\lambda) - \lambda f'(\lambda) - \delta + \pi + \xi(i), \quad \xi' < 0.$$

Substituting this into (18) gives

$$\frac{M}{pK} = y \exp \beta (f(\lambda) - \lambda f'(\lambda) - \delta + \pi + \xi(i)).$$

or

$$\log M - \log p - \log K = \log y + \beta [f(\lambda) - \lambda f'(\lambda) - \delta + \pi + \xi(i)].$$

Substituting  $Dp/p$  for  $\pi$  and rearranging gives

$$\left[ \frac{1}{\beta} + D \right] \log p = \frac{1}{\beta} [\log M - \log K - \beta \{f(\lambda) - \lambda f'(\lambda) - \delta + \xi(i)\}].$$

Pursuing a line of reasoning identical to that in our Notes on the Classical Model, it is verified that the solution of this differential equation is

$$(19) \quad \log_e p(t) = - \frac{1}{\beta} \int_t^{\infty} e^{(s-t)/\beta} [\log M(s) - \log K(s) - \beta \{f(\lambda(s)) - \lambda(s) f'(\lambda(s)) - \delta + \xi(i(s))\}] ds.$$

(We are again imposing a terminal condition that suppresses the term  $e^{-t/\beta}$  that should be added to the above solution.) Equation (19) expresses the current price level as a function of the entire future paths of the money supply, the capital stock, the employment-capital ratio  $\lambda$ , and the rate of investment. The value of  $\pi$  is also determined by (19), and can be obtained by differentiating the path given by (19)

with respect to time from the right. Notice that in this model, the entire time paths of the variables appearing on the right side of (19) can all be determined before the current price level is determined. That is, we showed how (17) determines the values of  $N$  and  $\lambda$  at  $t$ , and how this readily enables us to calculate the rate of growth of capital,  $i$ . This enables us to update the capital stock, and so to determine subsequent values of  $N$ . Proceeding in this way, given the exogenous fiscal policy variables, the entire time paths of all the real variables can be determined before determining the price level at any moment.

Since all of the structural equations remain the same, the model continues to be characterized by the IS, LM, and KE curves. The steady-state is determined at the intersection of the IS and KE curves. But now the instantaneous value of  $y$  is determined by substituting the value of  $\lambda$  determined by the solution to (15), e.g., (17), into the production function. The momentary equilibrium value of  $y$  so determined, call it  $\hat{y}$  depicted in Figure 2, when substituted into the IS curve gives the momentary value of  $r-\pi$ . The LM curve must pass through the intersection of the IS curve and the vertical line at  $\hat{y}$ ; jumps in the price level occur to assure this.

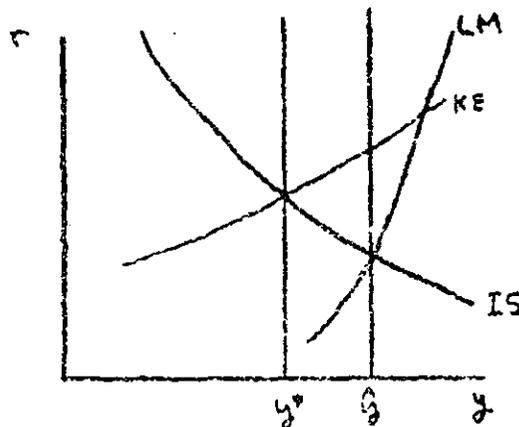


Figure 2

If we return to the experiment performed earlier under adaptive expectations, we shall see how radically the substitution of (9') for (9) alters the adjustment dynamics. We assume that the system is initially in a steady state and that  $\dot{M}/M = n$  so that  $\dot{w}/w = \dot{p}/p = \pi = 0$ . At the current moment there occurs an unexpected jump in  $M$  that leaves the right-hand derivative  $DM/M$  unchanged.\* The only result is an instantaneous jump in  $p$  and  $w$  proportional to the jump in  $M$  with all real variables unaltered. Since the solution to (15), e.g., (17), implies that  $y$  will not jump in response to the jump in  $M$ , the price level must jump to keep the LM curve passing through the intersection of the KE and IS curves. To show this explicitly for the sample portfolio balance equation that leads to (19) as the solution for the price level, suppose that the initial path of the log of the money supply was expected to be

$$\log M(s), s \geq t.$$

After the once-and-for-all jump in the money supply that leaves  $DM/M$  unaltered, the new path of the log of the money supply is expected to be

$$\emptyset + \log M(s), s \geq t, \emptyset > 0.$$

The new price level  $p'(t)$  differs from the initial price level given by (19) according to

$$\begin{aligned} \log p'(t) - \log p(t) &= -\frac{1}{\beta} \int_t^{\infty} e^{(s-t)/\beta} [\log M(s) + \emptyset] ds \\ &+ \frac{1}{\beta} \int_t^{\infty} e^{(s-t)/\beta} \log M(s) ds \\ &= -\frac{1}{\beta} \int_t^{\infty} \emptyset e^{(s-t)/\beta} ds \\ &= \emptyset. \end{aligned}$$

So the price level jumps by the same multiplicative factor  $\phi$  as does the money supply. Notice that in the above calculations we use the fact that the paths of the variables  $K(s)$ ,  $\lambda(s)$ , and  $i(s)$  will be unaltered when the money supply jumps.

Now consider the response of the system to a jump in  $M$  which had previously been anticipated. Suppose that the money supply follows the path

$$\log M(s) = \log M(t_0) + n(s - t_0), \quad t_0 + \theta \geq s,$$

$$\log M(s) = \log M(t_0) + \phi + n(s - t_0), \quad t_0 + \theta \leq s,$$

so that the money supply is expected to and does jump at  $t_0 + \theta$ . Using (19) and ignoring the terms in the real variables, we compute the price path as

$$\log p(t) = -\frac{1}{\beta} \int_t^{t_0 + \theta} e^{(s-t)/\beta} n(s - t_0) ds - \frac{1}{\beta} \int_t^{t_0 + \theta} e^{(s-t)/\beta} (n(s - t_0) + \phi) ds$$

$$+ \text{other terms, } t_0 \leq t \leq t_0 + \theta$$

$$= -\frac{1}{\beta} \int_t^{\infty} e^{(s-t)/\beta} n(s - t_0) ds - \frac{\phi}{\beta} \int_t^{t_0 + \theta} e^{(s-t)/\beta} ds + \text{other terms}$$

$$= -\frac{1}{\beta} \int_t^{\infty} e^{(s-t)/\beta} n(s - t_0) ds + \phi e^{(t_0 + \theta - t)/\beta} + \text{other terms.}$$

$$t_0 \leq t \leq t_0 + \theta$$

For  $t \geq t_0 + \theta$ , the solution for the price level is

$$\log p(t) = -\frac{1}{\beta} \int_t^{\infty} e^{(s-t)/\beta} n(s - t_0) ds + \phi + \text{other terms.}$$

The jump in the money supply at  $t_0 + \theta$  is reflected in earlier values of the price level. The above calculations show that the price level is continuous at  $t_0 + \theta$ , so that no jump occurs in  $p$  at the moment when  $M$  jumps. However, the expected rate of inflation  $\pi$  does jump at  $t_0 + \theta$ . For notice that

$$\frac{d}{dt} e^{\delta(t_0 + \theta - t)} = -\delta e^{\delta(t_0 + \theta - t)}$$

At  $t_0 + \theta$ , therefore, the left-hand derivative of the log of the price level exceeds the right-hand derivative by  $-\delta\beta > 0$ . So there is a sudden fall in  $\pi$  at  $t_0 + \theta$ . It is this downward jump in  $\pi$ , leading to a downward jump in  $r$ , that stimulates the demand for money enough at  $t_0 + \theta$  to guarantee that portfolio balance is maintained in the face of the jump in  $M$  that occurs at that moment.

The two experiments just performed show how essential it is to distinguish between jumps in policy variables that are anticipated and unanticipated in the model with perfect foresight.

The effect of substituting the perfect foresight assumption (9') for the adaptive expectations mechanism (9) has been to convert the model from one with Keynesian momentary behavior to one with classical momentary behavior. In the system with perfect foresight, (9'), money is a "veil," momentarily as well as in the long run. Jumps in the money supply don't cause any real movements of the sort that they do in the system with adaptive expectations.

The distinction between the models under (9) and (9') is mathematically subtle. Under the adaptive expectations scheme (9), the model must be manipulated under the "Keynesian" assumption that the money wage does not jump at a point in time, so that the Phillips curve

(7) gives the time derivative of the wage (= the right-hand time derivative = the left-hand time derivative). Essentially, this is because at any moment  $t$ , equations (8) and (9) make  $N^B(t)$  and  $\pi(t)$  predetermined from past variables. Of course  $K(t)$  is also inherited from the past. Equations (1)-(7) then form a system of seven equations in the seven endogenous variables  $y(t)$ ,  $\lambda(t)$ ,  $i(t)$ ,  $c(t)$ ,  $p(t)$ ,  $r(t)$ , and  $Dw(t)/w(t)$ . The model is incapable of restricting any additional variables, in particular  $w(t)$ , at the moment  $t$ . So  $w(t)$  must be regarded as fixed and inherited from the past at each point in time.

However, in the system with  $\pi = Dp/p$ , it is employment that is predetermined at any moment in time by the differential equation (15). Since employment is predetermined at  $t$ , say by (17),  $y$ ,  $\lambda$ , and  $w/p$  are also predetermined and constrained to change continuously as functions of time. They cannot jump at a point in time. But if  $w/p$  can't jump, and neither can  $K$  or  $y$ , then if  $M$  jumps at a point in time we know that  $p$  and  $w$  must jump in order to satisfy the portfolio balance equation at each moment.

Exercises

1. Under both adaptive expectations and perfect foresight, perform a dynamic analysis of Tobin's Dynamic Aggregative Model, formed by substituting for equation (3) the following equation

$$r + \delta - \pi = f(\lambda) - f'(\lambda).$$

Now instantaneous equilibrium occurs at the intersection of the KE and LM curves, while steady-state equilibrium occurs at the intersection of the KE curve and  $y^*$  line.

2. For both adaptive expectations and perfect foresight, analyze the dynamics of the model when the Phillips curve is modified to assume the form.

$$\frac{\dot{w}}{w} = h\left(\frac{N}{N^s}\right) + \alpha\pi, \quad h' > 0, \quad h(1) = 0, \quad 0 < \alpha < 1.$$

3. For both adaptive expectations and perfect foresight, analyze the dynamics of the model where the monetary authority pegs the nominal interest rate  $r$  at each instant, letting the money supply be whatever it must to guarantee portfolio balance.

Footnote

If the jump in  $M$  that occurs at some moment  $t$  had previously been expected to occur, it would not cause a jump in  $p$  at  $t$ . This follows from equation (19), which implies that  $p(s)$  will be a continuous function of time at  $t$ , even if there is a discontinuity in  $M(s)$  at  $t$  or anywhere else. However,  $\pi(t)$  will jump at  $t$  in response to a previously anticipated jump in  $M$ . By contrast, the experiment that we are performing here is one in which the monetary authority at  $t$  suddenly and unexpectedly moves from a previously planned money supply path of  $M(s)$ ,  $s \geq t$ , to a new planned path of  $e^{\phi} M(s)$  for  $s \geq t$ .