Seasonality and Portfolio Balance
Under Rational Expectations

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Seasonality and Portfolio Balance Under Rational Expectations

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1. Introduction

For a deterministic model, Sargent and Wallace [4] gave a formula describing the behavior of prices where expectations are formed rationally and where Cagan's portfolio balance schedule rules. To illustrate that formula they showed the response of prices to a known future increase in the money supply.

This paper gives a formula describing the covariation of prices and money in the more realistic case in which money is a purely indeterministic stochastic process, which means that future money creation (and therefore also inflation) can be predicted only imperfectly. We do assume that money is strictly econometrically exogenous in the portfolio balance schedule, which is restrictive, but parallels assumptions made by Lucas in his applications of rational expectations. Our interest in this exercise was sparked by our access to data from Argentina, which display a dramatic seasonal in the money supply, a consequence of the government's practice of giving workers a deficit-financed bonus every December. One aim of this paper is to provide a framework for studying how exogenous seasonal variation in the money supply gets translated into seasonal variation in prices; in particular, how the portfolio balance schedule makes seasonal movements in money cause seasonal movements in prices of modified amplitude and shifted phase (or timing).

From a technical point of view, this paper illustrates a useful, somewhat novel application of Lucas's undetermined coefficients method for calculating rational expectations equilibria. We formulate the existence problem in the frequency domain, and by exploiting the assumed exogeneity of the money supply process, reduce the number of
undetermined coefficients to one, regardless of the autoregressive order
of the money process. This affords a considerable simplification as
compared with comparable calculations in the time domain. The method is
applicable to a broad class of linear, rational expectations models with
exogenous driving variables.

Section 2 contains the detailed derivation of our results,
aside from a proof of uniqueness which we have placed in the appendix.
The nontechnical reader can skip most of Section 2 and proceed to
Section 3, where we illustrate our formula describing the covariation
between money creation and inflation.

2. Derivation of the Formula

The portfolio balance schedule assumes Cagan's form

\[(1) \quad m_t - p_t = \phi(E_t p_{t+1} - p_t) + u_t \quad \phi < 0\]

where \(m_t\) is the natural log of the money supply at time \(t\), \(p_t\) is the
natural log of the price level at \(t\), and \(E_{t} p_{t+1}\) is the linear least
squares estimate of \(p_{t+1}\) conditioned on current and lagged \(p\)'s and \(m\)'s.
We assume that \(m\) is strictly econometrically exogenous in (1), by which
is meant that \(u\) is orthogonal to past, present, and future \(m\).\(^{1/}\) Solving
(1) for \(p_t\) gives

\[p_t = \frac{1}{1-\phi} m_t - \frac{\phi}{1-\phi} E_t p_{t+1} - \frac{1}{1-\phi} u_t\]

or

\[(2) \quad p_t = \alpha m_t + \gamma E_t p_{t+1} + e_t\]

where \(\alpha = \frac{1}{1-\phi}\), \(\gamma = -\frac{\phi}{1-\phi}\), \(e_t = -\frac{1}{1-\phi} u_t\).
Notice that $\phi < 0$ implies $0 < \alpha < 1$, $0 < \gamma < 1$. First differencing (2) gives

$$p_t - p_{t-1} = \alpha(m_t - m_{t-1}) + \gamma[(E_t p_{t+1} - p_t) + (p_t - p_{t-1})] + e_t - e_{t-1}$$

or

$$x_t = \alpha u_t + \gamma E_t x_{t+1} + \gamma x_t - \gamma E_{t-1} x_t + \eta_t$$

where $x_t \equiv p_t - p_{t-1}$, $u_t = m_t - m_{t-1}$, $E_t x_{t+1} \equiv E_t (p_{t+1} - p_t)$, $\eta_t \equiv e_t - e_{t-1}$. Rearranging the preceding equation gives

$$x_t = \frac{\alpha}{1-\gamma} u_t + \frac{\gamma}{1-\gamma} (E_t x_{t+1} - E_{t-1} x_t) + \frac{1}{1-\gamma} \eta_t$$

or

$$x_t = u_t - \phi(1-L)E_t x_{t+1} + \eta'_t$$

where $\eta'_t = \eta_t/(1-\gamma)$, and where $L$ is the lag operator defined by

$$L^n x_t = x_{t-n}$$

Equation (3) can be written

$$x_t = u_t - \phi(1-L)E[x_{t+1} | u_t, u_{t-1}, \ldots] + \eta'_t$$

$$- \phi(1-L)\{E_t x_{t+1} - E[x_{t+1} | u_t, u_{t-1}, \ldots]\}.$$  

We claim that by virtue of the strict econometric exogeneity of $u$, the composite random disturbance $\eta'_t - \phi(1-L)\{E_t x_{t+1} - E[x_{t+1} | u_t, u_{t-1}, \ldots]\}$ is orthogonal to the entire $u$ process. This implies that the projection of $x_t$ on the entire $u$ process is given by the first two terms on the
right side of the above equation. For this reason, the preceding equation itself completely characterizes the covariance structure between the \( x \) and \( \mu \) processes. To verify the composite disturbance in the equation is orthogonal to the entire \( \mu \) process, consider the following argument.

The linear least squares forecast \( E_t x_{t+1} \) made on the basis of the information \( \{ \mu_t, \mu_{t-1}, \ldots, x_t, x_{t-1}, \ldots \} \) can be written

\[
E_t x_{t+1} = \xi(L)\mu_t + \psi(L)x_t
\]

where \( \xi(L) = \xi_0 L^0 + \xi_1 L^1 + \ldots \), \( \psi(L) = \psi_0 L^0 + \psi_1 L^1 + \ldots \). Calculating the linear least squares regression (projection) of \( E_t x_{t+1} \) against the set of current and past \( \mu \)'s \( \{ \mu_t, \mu_{t-1}, \ldots \} \), we have
\[ E[E_t x_{t+1} \mid \{\mu_t, \mu_{t-1}, \ldots\}] = \xi(L) \mu_t + \psi_0 E[x_t \mid \mu_t, \mu_{t-1}, \ldots] + \psi_1 E[x_{t-1} \mid \mu_t, \mu_{t-1}, \ldots] + \psi_2 E[x_{t-2} \mid \mu_t, \mu_{t-1}, \ldots] + \sum_{j=0}^{\infty} \psi_j E[x_{t-j} \mid \mu_t, \mu_{t-1}, \ldots]. \]

Now calculate \( r_t = E_t x_{t+1} - E\{E_t x_{t+1} \mid \mu_t, \mu_{t-1}, \ldots\} \), which is the error in taking \( E_t x_{t+1} \mid \mu_t, \mu_{t-1}, \ldots \) as an estimate of people's expectations of inflation. We have

\[
r_t = E_t x_{t+1} - E\{x_{t+1} \mid \mu_t, \mu_{t-1}, \ldots\} = \sum_{j=0}^{\infty} \psi_j (x_{t-j} - E[x_{t-j} \mid \mu_t, \mu_{t-1}, \ldots]).
\]

(4)

Now the assumption that \( \mu \) is strictly econometrically exogenous in the portfolio balance schedule implies that the regression of \( x_t \) on the entire \( \mu \) process—past, present, and future \( \mu \)'s—is one-sided on the past and present. Thus, we can write

\[
x_{t-j} = E[x_{t-j} \mid \mu_t, \mu_{t-1}, \ldots] + w_{t-j} = E[x_{t-j} \mid \mu_{t-j}, \mu_{t-j-1}, \ldots] + w_{t-j}
\]

where \( w_{t-j} \) is a second-order stationary random variable that is orthogonal to \( \mu \) at all leads and lags. Therefore, (4) becomes

\[
r_t = E_t x_{t+1} - E\{x_{t+1} \mid \mu_t, \mu_{t-1}, \ldots\} = \sum_{j=0}^{\infty} \psi_j w_{t-j}.
\]

(5)

Since \( w_t \) is orthogonal to \( \mu \) at all lags, so is \( r_t \), being a linear combination of current and past \( w \)'s.
Using these results, (3) can now be written as

\[ x_t = \mu_t - \phi(1-L) E[x_{t+1}|\mu_t, \mu_{t-1}, \ldots] + \eta_t' + r_t \]

or

\[ x_t = \mu_t - \phi(1-L) E_{\mu} x_{t+1} + a_t \]

where \( E_{\mu} x_{t+1} \equiv E x_{t+1}|\mu_t, \mu_{t-1}, \ldots \) and \( a_t = r_t + \eta_t' \). In (6), \( \mu_t \) is orthogonal to \( a_t \) at all lags by virtue of its orthogonality to \( r \) and \( \eta' \) at all lags. Thus, \( \mu_t \) is strictly econometrically exogenous in (6).

It follows that the covariances between \( x \) and \( \mu \) at all lags are determined by the single relationship (6). That is, (6) is a valid statistical "final form" with strictly exogenous processes on the right-hand side.

Let the covariance generating function of \( \mu \) be \( S_{\mu}(z) = \sum_{\tau=-\infty}^{\infty} R_{\mu}(\tau) z^\tau \), where the coefficient on \( z^\tau \) is the covariance \( R_{\mu}(\tau) = E((\mu_t - E\mu)(\mu_{t-\tau} - E\mu)) \). We assume that \( \mu \), in addition to being second-order stationary is linearly regular or indeterministic. Under this specification, \( S_{\mu}(z) \) has the representation

\[ S_{\mu}(z) = \sigma^2 B(z) B(z^{-1}) \]

where \( B(z) = \sum_{j=0}^{\infty} b_j z^j \). We assume that \( B(z) \) has a one-sided inverse, which means that \( \mu \) has an autoregressive representation. The cross-covariance generating function \( S_{x\mu}(z) \) is \( \sum_{\tau=-\infty}^{\infty} R_{x\mu}(\tau) z^\tau \) where

\[ R_{x\mu}(\tau) = E((x_t - Ex)(\mu_{t-\tau} - E\mu)) \]

Let us write \( E_{\mu} x_{t+1} \) as

\[ E_{\mu} x_{t+1} = h(L) \mu_t \]

where \( h(L) = \sum_{j=0}^{\infty} h_j L^j \). The classic Wiener-Kolmogorov formula for \( h(z) \) is
\[
    h(z) = \frac{1}{\sigma^2 B(z)} \left[ \frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} \right] + \left[ \right]_+ \\
    \text{where } [ \ ]_+ \text{ means ignore all negative powers of } z. \text{ Using the above equality, (6) implies}
\]

\[
    S_{x\mu}(z) = S_{\mu}(z) - \phi(1-z) \frac{1}{\sigma^2 B(z)} \left[ \frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} \right] + S_{\mu}(z)
\]
or

\[
    S_{x\mu}(z) = \sigma^2 B(z)B(z^{-1}) - \phi(1-z) \frac{1}{\sigma^2 B(z)} \left[ \frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} \right] + \sigma^2 B(z)B(z^{-1})
\]
or

\[
    S_{x\mu}(z) = B(z)B(z^{-1}) - \phi(1-z) \frac{1}{\sigma^2 B(z)} \left[ \frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} \right] + B(z^{-1})
\]
where we have normalized \( B(z) \) by setting \( \sigma^2 = 1 \), as we are free to do.

The above equality can be written

\[
    (7) \quad S_{x\mu}(z) = B(z^{-1}) \{ B(z) - \phi(1-z) \frac{1}{\sigma^2 B(z^{-1})} \}.
\]

A rational expectations equilibrium is a pair \( (B(z), S_{x\mu}(z)) \) that satisfies the functional equation (7). It is proper to consider \( S_{x\mu}(z) \) as a function of the generating function \( B(z) \) that characterizes the second-order properties of \( \mu \). Consider the operator

\[
    \mathcal{K} \left[ \frac{S_{x\mu}(z)}{B(z^{-1})} \right] = \{ B(z) - \phi(1-z) \frac{1}{\sigma^2 B(z^{-1})} \} ,
\]
defined on the space of functions \( \frac{S_{x\mu}(z)}{B(z^{-1})} \) defined on the unit circle and for which

\[
    \frac{1}{2\pi i} \int_C \left| \frac{S_{x\mu}(z)}{B(z^{-1})} \right|^2 \frac{dz}{z} < \infty ,
\]
where \( C \) is the unit circle. This space, which is complete, is the space of all cross-covariance generating functions between pairs of strictly indeterministic covariance stationary stochastic processes. Under suitable restrictions on \( \phi \), it is possible to show that \( \ell[ \ ] \) is a contraction mapping. Then by the contraction mapping theorem, there exists a unique \( S_{x\mu}(z) \) that satisfies equation (7). Details are relegated to the appendix.

To find a formula for \( S_{x\mu}(z) \), we use the method of undetermined coefficients. From (7) we have that

\[
S_{x\mu}(z) = \left\{ \frac{B(z) - \phi(1-z)}{B(z)} \left[ \frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} \right] \right\},
\]

which is one-sided in \( z \) by virtue of the exogeneity of \( \mu \). (Remember that \( S_{x\mu}(z)/S_{\mu}(z) \) is the \( z \)-transform of the coefficients of the projection of \( x \) on current, past, and future \( \mu \)'s, which is one-sided, as Sims's theorem 2 assures us.) The preceding equation can be written

\[
\frac{S_{x\mu}(z)}{B(z^{-1})} - B(z) = -\phi(1-z) \left[ \frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} \right] \quad .
\]

Now the exogeneity of \( \mu \) implies that the projection of \( x \) on past and present \( \mu \)'s equals the projection on past, present, and future \( \mu \)'s, which is to say

\[
\left[ \frac{S_{x\mu}(z)}{B(z^{-1})} \right] + \frac{1}{B(z)} = \frac{S_{x\mu}(z)}{B(z)B(z^{-1})}
\]

which is one-sided by virtue of the exogeneity of \( \mu \). Since \( B(z) \) is one-sided, it follows that \( S_{x\mu}(z)/B(z^{-1}) \) is one-sided. Therefore

\[
\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} \]

has at most one term in negative powers of \( z \). We can write
\[
\frac{S_{x_H}(z)z^{-1}}{B(z^{-1})} = g z^{-1} + \left[ \frac{S_{x_H}(z)z^{-1}}{B(z^{-1})} \right] + \]

where \( g \) is a coefficient to be determined.

Now substituting (9) into (8) gives

\[
\frac{S_{x_H}(z)}{B(z^{-1})} - B(z) = \phi (1 - z) \left( \frac{S_{x_H}(z)z^{-1}}{B(z^{-1})} - g z^{-1} \right)
\]

which can be written, after a little algebra,

\[
(1 - \phi + \phi z^{-1}) S_{x_H}(z) = S_{x_H}(z) + \phi g z^{-1} B(z^{-1}) - \phi g B(z^{-1})
\]

Noting that \( 1 - \phi = 1/(1 - \gamma) \) and multiplying by \( (1 - \gamma) \) gives

\[
(1 - \gamma z^{-1}) S_{x_H}(z) = a S_{x_H}(z) - \gamma g z^{-1} B(z^{-1}) + \gamma g B(z^{-1})
\]

or

\[
(10) \quad \frac{S_{x_H}(z)}{S_{\mu}(z)} = \frac{a}{1 - \gamma z^{-1}} - \frac{\gamma g z^{-1}}{(1 - \gamma z^{-1}) B(z)} + \frac{\gamma g}{(1 - \gamma z^{-1}) B(z)}
\]

Now each side of (10) must be one-sided, by the exogeneity of \( \mu \),

which gives us the condition needed to determine \( g \). Assuming that \( \mu \) is
governed by an \( n \)th-order autoregression gives

\[
(11) \quad B(z) = \frac{1}{1 - \delta_1 z^{-1} - \ldots - \delta_n z^n}
\]

Under (11), (10) becomes

\[
\frac{S_{x_H}(z)}{S_{\mu}(z)} = \left[ \frac{a + \alpha \gamma z^{-1}}{1 - \gamma z^{-1}} - \frac{\gamma (1 - \gamma) g [1 - \delta_1 \gamma - \ldots - \delta_n \gamma^n] z^{-1}}{1 - \gamma z^{-1}} + f(z) \right]
\]

where \( f(z) \) is one-sided. For \( S_{x_H}(z)/S_{\mu}(z) \) to be one-sided we require

\[
\alpha \gamma - \gamma (1 - \gamma) g [1 - \delta_1 \gamma - \ldots - \delta_n \gamma^n] = 0
\]
or

\[(12) \quad g = \frac{\alpha}{(1-\gamma)(1-\delta_1 \gamma - \ldots - \delta_n \gamma^n)}.
\]

Equations (10) and (12) allow us to compute \(S_{x\mu}(z)/S_{\mu}(z)\) as a function of \(S_{\mu}(z)\), i.e., the \(\delta\)'s, and \(\alpha\) and \(\gamma\). Defining \(k(z) \equiv S_{x\mu}(z)/S_{\mu}(z)\), and \(S_{a}(z)\) as the spectrum of the \(a\)'s in (6), we have that the spectrum (autocovariance generating function) of \(x_t\) obeys

\[S_{x}(z) = |k(z)|^2 S_{\mu}(z) + S_{a}(z)\]

where

\[k(z) = \frac{\alpha}{1-\gamma z^{-1}} - \frac{\gamma g z^{-1}}{(1-\gamma z^{-1})B(z)} + \frac{\gamma g}{(1-\gamma z^{-1})B(z)}\]

\[B(z) = \frac{1}{1-\delta_1 z^{-1} - \ldots - \delta_n z^n}\]

\[g = \frac{\alpha}{(1-\gamma)(1-\delta_1 \gamma - \ldots - \delta_n \gamma^n)} = \frac{1}{1-\delta_1 \gamma - \ldots - \delta_n \gamma^n},\]

since \(\alpha/(1-\gamma) = 1\). The \(z\)-transform of the coefficients in the regression of \(x_t\) on past, present, and future \(\mu\)'s is given by \(k(z)\).

3. Illustrations

Summarizing our results, we have assumed the structure

\[(1) \quad m_t - p_t = \phi(E_t p_{t+1} - p_t) + u_t, \quad \phi < 0\]

which upon first differencing gives

\[(3) \quad x_t = \mu_t - \phi (1-L)E_t x_{t+1} + \eta_t',\]

where

\[x_t \equiv p_t - p_{t-1}, \quad \mu_t \equiv m_t - m_{t-1}, \quad E_t x_{t+1} \equiv E_t p_{t+1} - p_t, \text{ and } \eta_t' \text{ satisfies}\]
\( E[\eta_t, \mu_s] = 0 \) for all \( t \) and \( s \), which means that \( \{\mu_t\} \) is assumed to be strictly econometrically exogenous in (3). We also assume that \( \mu_t \) follows the stationary \( n^{th} \)-order autoregressive process

\[
\mu_t = \delta_1 \mu_{t-1} + \delta_2 \mu_{t-2} + \ldots + \delta_n \mu_{t-n} + \psi_t
\]

or

\[
(1-\delta_1 L-\ldots-\delta_n L^n) \mu_t = \psi_t
\]

where \( E[\mu_t | \mu_{t-\tau} = \psi_{t-\tau}] = 0 \) for all \( \tau \geq 1 \). Under these conditions, \( x_t \) can be expressed as a one-sided distributed lag of \( \mu_t \),

\[
(13) \quad x_t = k(L) \mu_t + a_t
\]

where \( E[a_s | \mu_t] = 0 \) for all \( t \) and \( s \), and where \( k(L) \) is given by

\[
(14) \quad k(L) = \frac{\alpha}{1-\gamma L^{-1}} + \frac{\gamma g(1-L^{-1})(1-\delta_1 L-\ldots-\delta_n L^n)}{(1-\gamma L^{-1})}
\]

\[
g = \frac{1}{(1-\delta_1 \gamma-\delta_2 \gamma^2-\ldots-\delta_n \gamma^n)}
\]

\[
\alpha = \frac{1}{1-\phi}
\]

\[
\gamma = -\frac{\phi}{1-\phi}
\]

We can write \( k(L) = \sum_{j=0}^{\infty} k_j L^j \), where \( k_j \) is given by

\[
(14a) \quad k_j = \begin{cases} 
\gamma g \left[ \sum_{i=1}^{n} \delta_i \gamma^{i-1} - \sum_{i=0}^{n} \delta_i \gamma^i \right] + \alpha & j = 0 \\
\gamma g \left[ \sum_{i=1}^{n} \delta_i \gamma^{i-j-1} - \sum_{i=j+1}^{n} \delta_i \gamma^{i-j} \right] & 1 \leq j < n \\
-\gamma g \delta_n & j = n \\
0 & \text{Otherwise}
\end{cases}
\]
with $\delta_0 = -1$.

It is straightforward to verify that $k(L)$ is one-sided on nonnegative powers of $L$, despite the presence of terms in $L^{-1}$ on the right-hand side of (14). \[5/\]

Using (14), we can evaluate the sum of the lag weights $k_j$ as

\[
k(L) = \sum_{j=0}^{\infty} k_j = \frac{\alpha}{1-\gamma} = 1.
\]

Notice that the sum of lag weights is unity regardless of the values of the parameters $\lambda$ and $\phi$.

It is interesting to calculate the frequency response function of $k(L)$, which is given by

\[
k(e^{+i\omega}) = \sum_{j=0}^{\infty} k_j e^{i\omega j}.
\]

The frequency response function $k(e^{+i\omega})$ gives the response of $x_t$ in (13) to an input $\mu_t = e^{-i\omega t}$:

\[
x_t = \sum_{j=0}^{\infty} k_j e^{-i\omega(t-j)} = e^{-i\omega t} \sum_{j=0}^{\infty} k_j e^{i\omega j} = e^{-i\omega t} k(e^{+i\omega}).
\]

Similarly, the response of $x_t$ in (13) to an input $\mu_t = e^{+i\omega t}$ is $e^{+i\omega t} k(e^{-i\omega})$. Write $k(e^{+i\omega})$ in the polar form

\[
k(e^{i\omega}) = g(\omega) e^{i\theta(\omega)}.
\]
The quantity $g(w)$ is termed the "gain" while $\theta(w)$ is called the "phase shift." Then the response of $x_t$ in (13) to a sinusoidal input

$$\mu_t = 2 \cos tw = e^{iwt} + e^{-iwt}$$

is given by

$$x_t = e^{iwt}k(e^{-iw}) + e^{-iwt}k(e^{+iw})$$

$$= g(w)(e^{i(wt-\theta(w))} + e^{-i(wt-\theta(w))})$$

(16) \[ x_t = g(w) 2 \cos[wt-\theta(w)] \]

Thus, the response of $x_t$ in (15) to a $\mu$ input of $\cos wt$ is a cosine wave with phase shifted by $-\theta(w)$ and amplitude multiplied by $g(w)$. It must be emphasized, however, that our device of inserting a cosine wave input into (13) is only a heuristic one, since to derive (13) we have assumed that $\mu_t$ is governed by an autoregressive process. Such an indeterministic $\mu$ process has an arbitrarily small proportion of its variance occurring in an arbitrarily small band about any frequency $w$. If the money supply input $\mu_t$ were really of the form $\cos wt$, the relationship (13) would break down because the formula for the linear least squares forecast used to get (13) does not represent the best forecast for such a deterministic $\mu$ process. In effect, under the assumption that $\mu$ is governed by an $n^{th}$-order autoregression, $\mu_t$ is composed of a continuum of components $\cos wt$ with $w$ between $-\pi$ and $+\pi$. The importance of the various bands of components is controlled by the $S_j$'s. Equation (16) gives a picture of how the component of $\mu$ at a given frequency is shifted in phase and multiplied in amplitude in contributing to the composition of inflation $x$.

For purposes of comparison, it is useful to derive the analogue of (16) when $\mu_t$ is governed by a deterministic process (i.e., a process
perfectly predictable from its own past), such as

$$u_t = \cos wt.$$  

For convenience, assume that (3) holds with $\eta_t = 0$, which, because then

$$E_t x_{t+1} = x_{t+1},$$  

implies

$$x_t = u_t - \phi x_{t+1} + \phi x_t$$

or

$$x_t = u_t + \gamma x_{t+1}.$$

This difference equation can be solved for $x_t,

(17) \quad x_t = \alpha \sum_{j=0}^{\infty} \gamma^j u_{t+j} = \frac{\alpha}{1 - \gamma L - \gamma^2} u_t.

The response in (17) of $x_t$ to $u_t = e^{iwt}$ is

$$x_t = e^{iwt} \alpha \sum_{j=0}^{\infty} \gamma^j e^{iwj} = e^{iwt} \frac{\alpha}{1 - \gamma e^{iw}}.$$

The response of $x_t$ to $u_t = e^{-iwt}$ is

$$x_t = e^{-iwt} \frac{\alpha}{1 - \gamma e^{-iw}}.$$

We can express $\alpha/(1 - \gamma e^{iw})$ in the polar form

$$\frac{\alpha}{1 - \gamma e^{iw}} = r(w)e^{i\theta(w)}$$

$$r(w) = \frac{\gamma}{\sqrt{1 + \gamma^2 - 2\gamma \cos w}}$$

$$\theta(w) = \arctan \left[ \frac{-\gamma \sin w}{1 - \gamma \cos w} \right].$$
The response of $x_t$ to an input $2 \cos wt = e^{iwt} + e^{-iwt}$ is easily deduced to be

$$x_t = r(w) \cos[wt - \theta(w)].$$

Equation (18) gives the response of $x$ to a deterministic component of $\mu$, i.e., a component corresponding to a spike in the spectral density of $\mu$ at frequency $w$.

To illustrate our formula 14, we assume that the money creation process $\mu_t$ is strictly exogenous and is governed by the Markov process

$$\begin{align*}
(1-\lambda L^{12})\mu_t &= \varepsilon_t \\
\lambda < 1
\end{align*}$$

where $\varepsilon_t$ is a white noise process that is fundamental for $\mu_t$. Equation (19) can be inverted to yield the moving average representation for $\mu$:

$$\begin{align*}
\mu_t &= (1-\lambda L^{12})^{-1}\varepsilon_t \\
\mu_t &= (1 + \lambda L^{12} + \lambda^2 L^{24} + \ldots)\varepsilon_t
\end{align*}$$

The random variable $\varepsilon_t$ is the "innovation" in $\mu_t$; that is, it is the one-step-ahead prediction error for $\mu$, which is the unexpected part of money creation. From (20) we know that the spectrum for $\mu$ is given by

$$S_{\mu}(\omega) = \frac{\sigma^2}{(1-\lambda^2)(1-\lambda^{12})},$$

which for $\lambda > 0$ is characterized by peaks at the "seasonal" frequencies. We think of time having the units of months.
For two values of \( \lambda \) and two values of \( \phi \), Tables 1, 2, and 3 record the gain \( g(\omega) \), phase \( \theta(\omega) \), and distributed lag weights \( k_j \) obtained by using formulas (14a) and (15). Graphs of the phase and gain are also recorded in Graphs 1, 2, 3, and 4. For each value of \( \phi \), Table 4 records the gain and phase that obtain in the deterministic case.

The gain and phase reveal the following patterns. At the seasonal periodicities of 12, 6, 4, 3, 2.4, and 2 months, the gain is less than one indicating that the seasonal power in \( \mu \) is being attenuated or transmitted to \( x \) with diminished power. This attenuation becomes more pronounced, i.e., the gain becomes smaller, as \( \phi \) becomes larger, given \( \lambda \), and as \( \lambda \) becomes larger, given \( \phi \). Notice that for each \( \phi \) the gain at each seasonal frequency gets closer to the gain for the deterministic case as \( \lambda \) goes from .1 to .9. This is consistent with the fact that at \( \lambda = .9 \) a much larger proportion of the seasonal variation in \( \mu \) is predictable. Notice that the gain for \( \lambda = .9 \) is still slightly above the gain at the seasons for the deterministic case.

The phase shift at the seasonal frequencies is negative, which indicates a lead of inflation over money creation at those frequencies. (The phase is recorded in radians. To convert to units of time, divide the phase by the frequency, which is also recorded in radians.) The magnitude of the lead grows as \( \lambda \) increases, given \( \phi \). The lead is greatest in the deterministic case. The lead grows with \( \lambda \) because as \( \lambda \) increases, the extent to which seasonal movements are predictable increases. For large enough \( \lambda \), the magnitude of the lead grows with increases in the absolute value of \( \phi \).

The tendency for the gain to fall and the phase shift to increase as \( \phi \) increases is reasonable, since \( \phi \) governs the extent to
which expectations of future money creation influence current inflation. This influence grows with the absolute value of $\phi$. The greater attenuation and phase shift that accompany increases in $\phi$ is the counterpart of the bigger initial jump in price that occurs in response to a known future increase in money in Sargent and Wallace's diagram.

Table 3 presents the distributed lag coefficients and indicates how they depend on $\phi$ and $\lambda$. Notice that $k_{12}$ is negative in each instance and that the lag distribution is not "smooth" in the vicinity of $k_{11}$, $k_{12}$, $k_{13}$. It is, of course, well known that where lag distributions reflect optimal forecasting procedures, they often will be very choppy. Notice also that as $\lambda$ and $\phi$ each increase in absolute value, the absolute values of coefficients at lags of 10, 11, and 12 increase. Table 3 should serve as a warning that rational expectations models do not imply lag distributions of inflation on money creation that are concentrated at zero lag.6/

The lag distribution $k(L)$ characterizes the projection of $x_t$ on $\mu_t$,

\begin{equation}
(21) \quad x_t = k(L)\mu_t + a_t
\end{equation}

where $E a_s \mu_t = 0$ for all $t,s$. Combining (21) with (20), we can derive the lag distribution linking $x_t$ to $\varepsilon_t$, which is the unexpected change in $\mu_t$ (that is, $\varepsilon_t$ is the one-step-ahead prediction error in predicting $\mu_t$ from its own past):

\begin{equation}
(22) \quad x_t = \frac{k(L)}{1-\lambda L^{12}} \varepsilon_t + a_t
= h(L)\varepsilon_t + a_t,
\end{equation}

where $h(L) = k(L)(1+\lambda L^{12}+\lambda^2 L^{24}+\ldots)$. 
It is straightforward to verify that the coefficients of \( h(L) \) obey

\[
h_{1} = k_{1} \quad i = 0, \ldots, 11
\]

\[
h_{12} = k_{12} + \lambda k_{0}
\]

\[
h_{1i} = \lambda k_{1i-12} \quad i = 13, \ldots, 23
\]

\[
h_{24} = \lambda k_{12} + \lambda^2 k_{0}
\]

\[
h_{25} = \lambda^2 k_{1i-24} \quad i = 25, \ldots, 35
\]

\[
\vdots
\]

Indeed, writing out \( h(L) = k(L)(1+\lambda L^{12}+\lambda^2 L^{24} + \ldots) \), we have

\[
h(L) = k_0 + k_1 L + \ldots + k_{11} L^{11} + k_{12} L^{12} + \lambda k_0 L^{12} + \lambda k_{1} L^{13} + \ldots
\]

\[
+ \lambda k_{11} L^{23} + \lambda k_{12} L^{24} + \lambda^2 k_0 L^{24} + \ldots
\]

The response of (the rate of change of) real balances to an unexpected change in money creation can be deduced by subtracting (22) from (21) to get

\[
\mu_t - x_t = \left[ \frac{1}{1-\lambda L^{12}} - h(L) \right] \varepsilon_t - a_t
\]

\[
= \left[ \frac{1-k(L)}{1-\lambda L^{12}} \right] \varepsilon_t - a_t
\]

(23) \quad \mu_t - x_t = v(L) \varepsilon_t - a_t

where \( v(L) = (1-k(L))/(1-\lambda L^{12}) \). It is straightforward to determine that
\[ v_0 = 1 - h_0 \]
\[ v_j = -h_j \quad j > 1, j \neq 12, 24, 36, \ldots \]
(24)
\[ v_{12} = \lambda - h_{12} \]
\[ v_{12} = \lambda^n - h_{12n} \quad n = 1, 2, 3 \ldots . \]

or in terms of \( k_j \)'s we can write

\[ v_0 = 1 - k_0 \]
\[ v_j = -k_j \quad j = 1, \ldots, 11 \]
\[ v_{12} = \lambda v_0 - k_{12} \]
\[ v_j = \lambda v_{j-12} \quad j > 12 \]

Using (14), it is straightforward to show that

\[ h_0 = k_0 = \frac{1}{1 - \lambda \gamma_{12}} \]

which exceeds 1 so long as \( \lambda > 0 \). Therefore, so long as \( \lambda > 0 \), we have that

\[ v_0 = 1 - h_0 < 0. \]

Since for \( j = 1, \ldots, 11, k_j > 0 \) for \( \phi < 0 \) and \( \lambda > 0 \), we have that

\[ v_j = -h_j = -k_j < 0 \quad \text{for } j = 1, \ldots, 11. \]

Using (14a) we have that

\[ k_{12} = \frac{-\gamma \lambda}{1 - \lambda \gamma_{12}} \]

so that \( h_{12} = k_{12} + \lambda k_0 \)

\[ = \frac{-\gamma \lambda}{1 - \lambda \gamma_{12}} + \frac{\lambda}{1 - \lambda \gamma_{12}} \]
\[ h_{12} = \frac{\lambda (1-\gamma)}{1-\lambda \gamma^{12}}. \]

Hence, from (24), \( v_{12} \) becomes

\[
v_{12} = \lambda - \frac{\lambda (1-\gamma)}{1-\lambda \gamma^{12}} \]
\[
= \frac{\lambda (1-\lambda \gamma^{12})}{1-\lambda \gamma^{12}} > 0
\]
because \( 0 < \lambda < 1 \) and \( 0 < \gamma = \frac{-\phi}{1-\phi} < 1 \). Thus, we have

\[
(25) \quad \begin{cases} 
v_j < 0 & \text{for } j = 0, \ldots, 11 \\
v_j > 0 & \text{for } j = 12 
\end{cases}
\]

This pattern of \( v_j \)'s implies that an unexpected increase in money creation of \( \epsilon_t \) can be expected to cause (the rate of change of) real balances to fall at time \( t \), and to continue falling from time \( t+1 \) to time \( t+11 \), and then to increase at time \( t+12 \). This pattern is more pronounced the larger is \( \lambda \), for a given \( \phi \), and the larger is \( \phi \) for a given \( \lambda \). The pattern mimicks the graph presented by Sargent and Wallace for the response to a perfectly foreseen future increase in the money supply. In that graph, at the time that the previously unannounced increase is announced, real balances fall and continue falling until the previously announced change in the money supply actually occurs, at which moment the real money supply rises and expected inflation falls. That the pattern of \( v_j \)'s is mimicking that graph can be seen by noting that for \( \lambda > 0 \), an unexpected money creation of \( \epsilon_t \) causes previous expectations of money creation at time \( t+12 \) to be revised upward by \( \lambda \epsilon_t^{12} \). The expected response of real balances to this newly forecast increase in \( h_{t+12} \) is exactly like that predicted by the deterministic Sargent-Wallace diagram. As in the Sargent-Wallace diagram, the extent
to which real balances fall prior to the date of the anticipated increase in \( \mu \) is governed by the parameter \( \phi \). The parameter \( \lambda \), which was not a factor in Sargent and Wallace's deterministic analysis, enters here because it is the factor by which \( \varepsilon_t \) must be multiplied to help determine the future \( \mu \) at time \( t+12 \) expected as of time \( t \).
### Table 1
**GAIN**

<table>
<thead>
<tr>
<th>Frequency Radians</th>
<th>Period Months</th>
<th>$\lambda = 0.1$</th>
<th>$\phi = -0.5$</th>
<th>$\phi = -2.0$</th>
<th>$\lambda = 0.9$</th>
<th>$\phi = -0.5$</th>
<th>$\phi = -2.0$</th>
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<tbody>
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<td>0.</td>
<td>$\infty$</td>
<td>1.0000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.0000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
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<td>96</td>
<td>1.002</td>
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<td>1.020</td>
<td>1.073</td>
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<td></td>
<td></td>
<td></td>
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<th>Frequency Radians</th>
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<th>$\phi = -0.5$</th>
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Table 3

Distributed Lag Coefficients

\[ k(L) = \sum_{j=0}^{\infty} k_j L^j \]

<table>
<thead>
<tr>
<th>Lag</th>
<th>( \lambda = 0.1 )</th>
<th>( \lambda = 0.9 )</th>
</tr>
</thead>
<tbody>
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<td>( \phi = -0.5 )</td>
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<td>0.000</td>
</tr>
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Table 4

Deterministic Case

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<th>Frequency Radians</th>
<th>Period Months</th>
<th>GAIN $\phi = -0.5$</th>
<th>GAIN $\phi = -2.0$</th>
<th>PHASE (Radians) $\phi = -0.5$</th>
<th>PHASE (Radians) $\phi = -2.0$</th>
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</thead>
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Table 5
Distributed Lag Coefficients of
\( \mu_t - x_t \) on \( \varepsilon_t \)

<table>
<thead>
<tr>
<th>Lag</th>
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<th></th>
<th>( \lambda = 0.9 )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \phi = -0.5 )</td>
<td>( \phi = -2.0 )</td>
<td>( \phi = -0.5 )</td>
<td>( \phi = -2.0 )</td>
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<td>-0.000</td>
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<tr>
<td>( v_j ) ( j &gt; 12 )</td>
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<td>( \lambda v_{j-12} )</td>
<td>( \lambda v_{j-12} )</td>
<td>( \lambda v_{j-12} )</td>
</tr>
</tbody>
</table>
Graph 1
Gain and Phase for $\lambda = 0.1, \phi = -0.5$
Graph 3
Gain and Phase for $\lambda = 0.9$, $\phi = -0.5$
Graph 4
Gain and Phase for $\lambda = 0.9$, $\phi = -2.0$
Appendix

Proof of Uniqueness

Consider the operator

\[(Al) \quad \ell \left( \frac{S_{x \mu}(z)}{B(z^{-1})} \right) = B(z) - \phi(1-z) \left[ \frac{S_{x \mu}(z)z^{-1}}{B(z^{-1})} \right] \]

defined on the space \( F \) consisting of function \( f(z) = \frac{S_{x \mu}(z)}{B(z^{-1})} \)
defined on the unit circle \( |z| = 1 \) and for which

\[ \frac{1}{2\pi i} \int_C \left| \frac{S_{x \mu}(z)}{B(z^{-1})} \right|^2 \frac{dz}{z} < \infty \]

where \( C \) is the unit circle. This space of functions is complete and unitarily equivalent to \( \ell_2(-\infty, \infty) \), the space of two-sided sequences \( \{x_j\} \) with \( \sum_{j=-\infty}^{\infty} |x_j|^2 < \infty \). Indeed, each \( f(z) \) in \( F \) has Fourier series representation

\[ f(z) = \sum_{j=-\infty}^{\infty} f_j z^j \]

with the sequence \( \{f_j\} \) belonging to \( \ell_2(-\infty, \infty) \), and where the \( f_j \)'s are given by

\[ f_j = \frac{1}{2\pi} \int_C f(z) z^{-j} \frac{dz}{z} . \]

The spaces \( F \) and \( \ell_2(-\infty, \infty) \) are unitarily equivalent, the mapping defined by the preceding equation being a unitary mapping, i.e., one that preserves inner products. We can thus express the norm of \( f(z) \) in terms of the norm of \( \{f_j\} \), so that

\[ \|f(z)\| = \left( \sum_{j=-\infty}^{\infty} |f_j|^2 \right)^{1/2} \]

Now consider the two functions \( x(z) \) and \( y(z) \) belonging to \( F \). From (Al) we have
(A2) \[ \ell(x(z)) - \ell(y(z)) = -\phi(1-z)[(x(z)-y(z))z^{-1}]_+ . \]

The linear operator \((1-z)\) which maps \(\ell_2(-\infty, \infty)\) into itself has operator norm 2. That is,

\[
\|\| (1-z) \| \| = \sup_{x \neq 0} \frac{\| (1-z)x(z) \|}{\| x(z) \|} \\
= \sup_{x \neq 0} \left( \sum_{j=-\infty}^{\infty} (x_j - x_{j-1})^2 \right)^{1/2} \\
= \sup_{x \neq 0} \left( \frac{\sum_{j=-\infty}^{\infty} x_j^2}{\sum_{j=-\infty}^{\infty} x_j^2} \right)^{1/2} \\
= 2 ,
\]

where the last step follows from the following special case of the Schwartz inequality,

\[
\left| \sum_{j=-\infty}^{\infty} x_j x_{j-1} \right| \leq \sum_{j=-\infty}^{\infty} x_j^2 .
\]

Now since

\[
\| (x(z)-y(z))z^{-1} \|_+ \leq \| x(z) - y(z) \| ,
\]

we have in (A2) that

\[
(A3) \quad \| \ell(x(z)) - \ell(y(z)) \| = -\phi \| (1-z)[(x(z)-y(z))z^{-1}]_+ \| \\
\leq -2\phi \| x(z) - y(z) \| .
\]

Inequality (A3) establishes that if \(-2\phi < 1\), \(\ell(x(z))\) is a contraction mapping on a complete metric space. Therefore, the contraction mapping
Theorem guarantees that the functional equation (7) has a unique solution. The preceding proves the existence of a unique solution to (7) under the condition that $-\phi < 1/2$. This is a sufficient, though not a necessary, condition for a unique equilibrium to exist.

A much weaker condition on $\phi$ can be obtained if it can be assumed that $m$ and $p$ are covariance stationary (rather than only assuming $\mu$ and $x$ covariance stationary, as in the preceding). For then we can use equation (2) to get

$$S_{pm}(z) = \alpha c(z^{-1}) + \gamma \left[ \frac{S_{pm}(z)z^{-1}}{c(z^{-1})} \right] c(z^{-1})$$

or

\[(A4) \quad \frac{S_{pm}(z)}{c(z^{-1})} = \alpha c(z) + \gamma \left[ \frac{S_{pm}(z)z^{-1}}{c(z^{-1})} \right] + \]

where $S_m(z) = c(z)c(z^{-1})$, $c(z) = \sum_{j=0}^{\infty} c_j z^j$. Define the linear operator $h(x(z))$

\[(A5) \quad h(x(z)) = \alpha c(z) + \gamma [x(z)z^{-1}]_+ \]

on the same space $F$ defined above. We have

$$||h(x(z)) - h(y(z))|| = ||\gamma [x(z) - y(z)]z^{-1}]_+||$$

$$\leq \gamma ||x(z) - y(z)||,$$

so that (A5) is a contraction so long as $0 < \gamma < 1$. But $\gamma \equiv \frac{-\phi}{1-\phi} < 1$ so long as $\phi < 0$, i.e., so long as the demand schedule for money is sloped downward. Therefore, by the contraction mapping theorem, (A4) possesses a unique solution $S_{pm}(z)/c(z^{-1})$ so long as $\phi < 0$. 
Footnotes

1/ See the appendix of Sims [5] for a useful characterization of strict econometric exogeneity by way of its relationship to the (negation of) Granger causality.

2/ This is a consequence of the material in Sims's appendix [5].

3/ A good treatment of the aspects of stationary stochastic processes exploited here is Whittle [6].

4/ See Whittle [6, p. 66-67, 41-43].

5/ To illustrate Lucas's time domain method, consider our "level" model, suppressing the disturbance, to get

\( p_t = \alpha E_t p_{t+1} + \beta m_t \).

The money process is assumed to obey the Markov law

\( E_t m_{t+1} = \sum_{i=1}^{n} w_i m_{t-i+1} \).

Lucas's method is to guess a solution of the form

\( p_t = \pi_0 m_t + \pi_1 m_{t-1} + \pi_2 m_{t-2} + \ldots \).

Substituting (b) and (c) into (a) then gives

\[ p_t = \alpha(\pi_0 \sum_{i=1}^{n} w_i m_{t-i+1} + \pi_1 m_t + \ldots) + \beta m_t. \]

or

\[ p_t = (\alpha \pi_0 w_1 + \pi_1 + \beta)m_t + (\alpha \pi_0 w_2 + \pi_2)m_{t-1} + (\alpha \pi_0 w_3 + \pi_3)m_{t-2} + \ldots. \]

Equations (c) and (d) involve identical variables and must therefore have coefficients that match. This condition gives us

\[ \pi_0 = \alpha \pi_0 w_1 + \pi_1 + \beta, \]
\[ \pi_1 = (\alpha \pi_0 w_2 + \pi_2), \]
\[ \pi_2 = (\alpha \pi_0 w_3 + \pi_3), \]
\[ \ldots, \]
\[ \pi_{n-1} = \alpha \pi_0 w_n. \]

which is a set of n nonlinear equations that determine the \( \pi \)'s as functions of \( \alpha, \beta \), and the \( w_i \)'s.
Gordon [2, p. 647] has interpreted rational expectations models as implying that inflation on money creation lag distributions are concentrated at zero lag.

Since $\mu_{t+2} = \varepsilon_{t+12} + \lambda \varepsilon_t + \lambda^2 \varepsilon_{t-12} + \ldots$, we have that $E[\mu_{t+12}|\varepsilon_t, \varepsilon_{t-1}, \ldots] = \lambda \varepsilon_t + \lambda^2 \varepsilon_{t-12} + \ldots$. As before, we use $E[x|y]$ to denote the linear least squares projection of $x$ on $y$. 
References


