Seasonality and Portfolio Balance
Under Rational Expectations

Rusdu Saracoglu

and

Thomas J. Sargent

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Rusdu Saracoglu

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Thomas J. Sargent

University of Minnesota

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1. Introduction

For a deterministic model, Sargent and Wallace [4] gave a formula describing the behavior of prices where expectations are formed rationally and where Cagan's portfolio balance schedule rules. To illustrate that formula they showed the response of prices to a known future increase in the money supply.

This paper gives a formula describing the covariation of prices and money in the more realistic case in which money is a purely indeterministic stochastic process, which means that future money creation (and therefore also inflation) can be predicted only imperfectly. We do assume that money is strictly econometrically exogenous in the portfolio balance schedule, which is restrictive, but parallels assumptions made by Lucas in his applications of rational expectations. Our interest in this exercise was sparked by our access to data from Argentina, which display a dramatic seasonal in the money supply, a consequence of the government's practice of giving workers a deficit-financed bonus every December. One aim of this paper is to provide a framework for studying how exogenous seasonal variation in the money supply gets translated into seasonal variation in prices; in particular, how the portfolio balance schedule makes seasonal movements in money cause seasonal movements in prices of modified amplitude and shifted phase (or timing).

From a technical point of view, this paper illustrates a useful, somewhat novel application of Lucas's undetermined coefficients method for calculating rational expectations equilibria. We formulate the existence problem in the frequency domain, and by exploiting the assumed exogeneity of the money supply process, reduce the number of undetermined coefficients to one, regardless of the autoregressive order of the money process. This affords a considerable simplification as compared with comparable calculations in the time domain. The method is applicable to a broad class of linear, rational expectations models with exogenous driving variables.

Section 2 contains the detailed derivation of our results, aside from a proof of uniqueness which we have placed in the appendix. The nontechnical reader can skip most of Section 2 and proceed to Section 3, where we illustrate our formula describing the covariation between money creation and inflation.

2. Derivation of the Formula

The portfolio balance schedule assumes Cagan's form

$$(1) \qquad m_t - p_t = \phi(E_t p_{t+1} - p_t) + u_t \qquad \phi < 0$$

where m_t is the natural log of the money supply at time t, p_t is the natural log of the price level at t, and $E_t p_{t+1}$ is the linear least squares estimate of p_{t+1} conditioned on current and lagged p's and m's. We assume that m is strictly econometrically exogenous in (1), by which is meant that u is orthogonal to past, present, and future m.[1/] Solving (1) for p_t gives

$$p_t = \frac{1}{1-\phi} m_t - \frac{\phi}{1-\phi} E_t p_{t+1} - \frac{1}{1-\phi} u_t$$

or

$$(2) \qquad p_t = \alpha m_t + \gamma E_t p_{t+1} + e_t$$

where $\alpha \equiv \frac{1}{1-\phi}$, $\gamma \equiv -\frac{\phi}{1-\phi}$, $e_t \equiv -\frac{1}{1-\phi} u_t$.

Notice that $\phi < 0$ implies $0 < \alpha < 1$, $0 < \gamma < 1$. First differencing (2) gives

$$p_t - p_{t-1} = \alpha(m_t - m_{t-1}) + \gamma[(E_t p_{t+1} - p_t) + (p_t - p_{t-1})$$

$$- (E_{t-1} p_t - p_{t-1})] + e_t - e_{t-1}$$

or

$$x_t = \alpha\mu_t + \gamma E_t x_{t+1} + \gamma x_t - \gamma E_{t-1} x_t + \eta_t$$

where $x_t \equiv p_t - p_{t-1}$, $\mu_t = m_t - m_{t-1}$, $E_t x_{t+1} \equiv E_t(p_{t+1} - p_t)$, $\eta_t \equiv \varepsilon_t - \varepsilon_{t-1}$. Rearranging the preceding equation gives

$$x_t = \frac{\alpha}{1-\gamma}\mu_t + \frac{\gamma}{1-\gamma}(E_t x_{t+1} - E_{t-1} x_t) + \frac{1}{1-\gamma}\eta_t$$

or

$$(3) \qquad x_t = \mu_t - \phi(1-L)E_t x_{t+1} + \eta_t'$$

where $\eta_t' = \eta_t/(1-\gamma)$, and where L is the lag operator defined by $L^n x_t = x_{t-n}$.

Equation (3) can be written

$$x_t = \mu_t - \phi(1-L)E[x_{t+1}|\mu_t, \mu_{t-1}, \ldots] + \eta_t'$$

$$- \phi(1-L)\{E_t x_{t+1} - E[x_{t+1}|\mu_t, \mu_{t-1}, \ldots]\} .$$

We claim that by virtue of the strict econometric exogeneity of μ, the composite random disturbance $\eta_t' - \phi(1-L)\{E_t x_{t+1} - E[x_{t+1}|\mu_t, \mu_{t-1}, \ldots]\}$ is orthogonal to the entire μ process. This implies that the projection of x_t on the entire μ process is given by the first two terms on the right side of the above equation. For this reason, the preceding equation itself completely characterizes the covariance structure between the x and μ processes. To verify the composite disturbance in the equation is orthogonal to the entire μ process, consider the following argument.

The linear least squares forecast $E_t x_{t+1}$ made on the basis of the information $\{\mu_t, \mu_{t-1}, \ldots, x_t, x_{t-1}, \ldots\}$ can be written

$$E_t x_{t+1} = \xi(L)\mu_t + \psi(L)x_t$$

where $\xi(L) = \xi_0 L^0 + \xi_1 L^1 + \ldots$, $\psi(L) = \psi_0 L^0 + \psi_1 L^1 + \ldots$. Calculating the linear least squares regression (projection) of $E_t x_{t+1}$ against the set of current and past μ's $\{\mu_t, \mu_{t-1}, \ldots\}$, we have

$$E[E_t x_{t+1} | \{\mu_t, \mu_{t-1}, \ldots\}] = \xi(L)\mu_t$$

$$+ \psi_0 E[x_t | \mu_t, \mu_{t-1}, \ldots]$$

$$+ \psi_1 E[x_{t-1} | \mu_t, \mu_{t-1}, \ldots]$$

$$+ \psi_2 E[x_{t-2} | \mu_t, \mu_{t-1}, \ldots]$$

$$= \xi(L)\mu_t + \sum_{j=0}^{\infty} \psi_j E[x_{t-j} | \mu_t, \mu_{t-1}, \ldots] .$$

Now calculate $r_t = E_t x_{t+1} - E\{E_t x_{t+1} | \mu_t, \mu_{t-1}, \ldots\}$, which is the error in taking $Ex_{t+1} | \mu_t, \mu_{t-1}, \ldots$ as an estimate of people's expectations of inflation. We have

$$r_t = E_t x_{t+1} - E\{x_{t+1} | \mu_t, \mu_{t-1}, \ldots\}$$

$$(4) \qquad = \sum_{j=0}^{\infty} \psi_j (x_{t-j} - E[x_{t-j} | \mu_t, \mu_{t-1}, \ldots]).$$

Now the assumption that μ is strictly econometrically exogenous in the portfolio balance schedule implies that the regression of x_t on the entire μ process--past, present, and future μ's--is one-sided on the past and present.[2/] Thus, we can write

$$x_{t-j} = E[x_{t-j} | \mu_t, \mu_{t-1}, \ldots] + w_{t-j}$$

$$= E[x_{t-j} | \mu_{t-j}, \mu_{t-j-1}, \ldots] + w_{t-j}$$

where w_{t-j} is a second-order stationary random variable that is orthogonal to μ at all leads and lags. Therefore, (4) becomes

$$(5) \qquad r_t = E_t x_{t+1} - E\{x_{t+1} | \mu_t, \mu_{t-1}, \ldots\} = \sum_{j=0}^{\infty} \psi_j w_{t-j} .$$

Since w_t is orthogonal to μ at all lags, so is r_t, being a linear combination of current and past w's.

Using these results, (3) can now be written as

$$x_t = \mu_t - \phi(1-L)\ E[x_{t+1}|\mu_t,\mu_{t-1},\ \ldots] + \eta_t' + r_t$$

or

$$(6)\qquad x_t = \mu_t - \phi(1-L)\ E_\mu x_{t+1} + a_t$$

where $E_\mu x_{t+1} \equiv Ex_{t+1}|\mu_t,\mu_{t-1},\ \ldots$ and $a_t = r_t + \eta_t'$. In (6), μ_t is orthogonal to a_t at all lags by virtue of its orthogonality to r and η' at all lags. Thus, μ_t is strictly econometrically exogenous in (6). It follows that the covariances between x and μ at all lags are determined by the single relationship (6). That is, (6) is a valid statistical "final form" with strictly exogenous processes on the right-hand side.

Let the covariance generating function of μ be $S_\mu(z) = \sum_{\tau=-\infty}^{\infty} R_\mu(\tau)z^\tau$, where the coefficient on z^τ is the covariance $R_\mu(\tau) = E\{(\mu_t-E\mu)(\mu_{t-\tau}-E\mu)$. We assume that μ, in addition to being second-order stationary is linearly regular or indeterministic.[3/] Under this specification, $S_\mu(z)$ has the representation

$$S_\mu(z) = \sigma^2B(z)B(z^{-1})$$

where $B(z) = \sum_{j=0}^{\infty} b_jz^j$. We assume that B(z) has a one-sided inverse, which means that μ has an autoregressive representation. The cross-covariance generating function $S_{x\mu}(z)$ is $\sum_{\tau=-\infty}^{\infty} R_{x\mu}(\tau)z^\tau$ where

$$R_{x\mu}(\tau) \equiv E\{(x_t-Ex)(\mu_{t-\tau}-E\mu)\}\ .$$

Let us write $E_\mu x_{t+1}$ as

$$E_\mu x_{t+1} = h(L)\mu_t\ ,$$

where $h(L) = \sum_{j=0}^{\infty} h_jL^j$. The classic Wiener-Kolmogorov formula for h(z) is[4]

$$h(z) = \frac{1}{\sigma^2 B(z)} \left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_+$$

where $[\ \ ]_+$ means ignore all negative powers of z. Using the above equality, (6) implies

$$S_{x\mu}(z) = S_\mu(z) - \phi(1-z)\frac{1}{\sigma^2 B(z)} \left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_+ S_\mu(z)$$

or

$$S_{x\mu}(z) = \sigma^2 B(z)B(z^{-1}) - \phi(1-z)\frac{1}{\sigma^2 B(z)} \left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_+ \sigma^2 B(z)B(z^{-1})$$

or

$$S_{x\mu}(z) = B(z)B(z^{-1}) - \phi(1-z) \left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_+ B(z^{-1})$$

where we have normalized B(z) by setting $\sigma^2 = 1$, as we are free to do. The above equality can be written

(7) $$S_{x\mu}(z) = B(z^{-1})\ \{B(z) - \phi(1-z) \left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_+ \}.$$

A rational expectations equilibrium is a pair $(B(z), S_{x\mu}(z))$ that satisfies the functional equation (7). It is proper to consider $S_{x\mu}(z)$ as a function of the generating function B(z) that characterizes the second-order properties of μ. Consider the operator

$$\ell\left[\frac{S_{x\mu}(z)}{B(z^{-1})}\right] = \{B(z) - \phi(1-z) \left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_+ \},$$

defined on the space of functions $\frac{S_{x\mu}(z)}{B(z^{-1})}$ defined on the unit circle and for which

$$\frac{1}{2\pi i}\int_C \left|\frac{S_{x\mu}(z)}{B(z^{-1})}\right|^2 \frac{dz}{z} < \infty ,$$

where C is the unit circle. This space, which is complete, is the space of all cross-covariance generating functions between pairs of strictly indeterministic covariance stationary stochastic processes.[6/] Under suitable restrictions on ϕ, it is possible to show that $\ell[\quad]$ is a contraction mapping. Then by the contraction mapping theorem, there exists a unique $S_{x\mu}(z)$ that satisfies equation (7). Details are relegated to the appendix.

To find a formula for $S_{x\mu}(z)$, we use the method of undetermined coefficients. From (7) we have that

$$\frac{S_{x\mu}(z)}{S_{\mu}(z)} = \frac{\left\{ B(z)-\phi(1-z)\left[\dfrac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_{+} \right\}}{B(z)},$$

which is one-sided in z by virtue of the exogenity of μ. (Remember that $S_{x\mu}(z)/S_{\mu}(z)$ is the z-transform of the coefficients of the projection of x on current, past, and future μ's, which is one-sided, as Sims's theorem 2 assures us.) The preceding equation can be written

$$\text{(8)} \qquad \frac{S_{x\mu}(z)}{B(z^{-1})} - B(z) = -\phi(1-z)\left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_{+}.$$

Now the exogeneity of μ implies that the projection of x on past and present μ's equals the projection on past, present, and future μ's, which is to say

$$\left[\frac{S_{x\mu}(z)}{B(z^{-1})}\right]_{+}\frac{1}{B(z)} = \frac{S_{x\mu}(z)}{B(z)B(z^{-1})}$$

which is one-sided by virtue of the exogeneity of μ. Since B(z) is one-sided, it follows that $S_{x\mu}(z)/B(z^{-1})$ is one-sided. Therefore $\dfrac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}$ has at most one term in negative powers of z. We can write

$$\text{(9)} \qquad \frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} = gz^{-1} + \left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_+$$

where g is a coefficient to be determined.

Now substituting (9) into (8) gives

$$\frac{S_{x\mu}(z)}{B(z^{-1})} - B(z) = -\phi(1-z)\left(\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})} - gz^{-1}\right)$$

which can be written, after a little algebra,

$$(1-\phi+\phi z^{-1})S_{x\mu}(z) = S_\mu(z) + \phi g\ z^{-1}B(z^{-1}) - \phi g\ B(z^{-1}) \ .$$

Noting that $1 - \phi = 1/(1-\gamma)$ and multiplying by $(1-\gamma)$ gives

$$(1-\gamma z^{-1})S_{x\mu}(z) = \alpha S_\mu(z) - \gamma g z^{-1}B(z^{-1}) + \gamma g B(z^{-1})$$

or

$$\text{(10)} \qquad \frac{S_{x\mu}(z)}{S_\mu(z)} = \frac{\alpha}{1-\gamma z^{-1}} - \frac{\gamma g z^{-1}}{(1-\gamma z^{-1})B(z)} + \frac{\gamma g}{(1-\gamma z^{-1})B(z)}$$

Now each side of (10) must be one-sided, by the exogeneity of μ, which gives us the condition needed to determine g. Assuming that μ is governed by an n^{th}-order autoregression gives

$$\text{(11)} \qquad B(z) = \frac{1}{1-\delta_1 z-\ldots-\delta_n z^n} \ .$$

Under (11), (10) becomes

$$\frac{S_{x\mu}(z)}{S_\mu(z)} = \left[\alpha + \frac{\alpha\gamma z^{-1}}{1-\gamma z^{-1}} - \frac{\gamma(1-\gamma)g[1-\delta_1\gamma-\ldots-\delta_n\gamma^n]z^{-1}}{1-\gamma z^{-1}} + f(z)\right]$$

where $f(z)$ is one-sided. For $S_{x\mu}(z)/S_\mu(z)$ to be one-sided we require

$$\alpha\gamma - \gamma(1-\gamma)g[1-\delta_1\gamma-\ldots-\delta_n\gamma^n] = 0$$

or

$$(12) \qquad g = \frac{\alpha}{(1-\gamma)[1-\delta_1\gamma-\ldots-\delta_n\gamma^n]} .$$

Equations (10) and (12) allow us to compute $S_{x\mu}(z)/S_{\mu}(z)$ as a function of $S_{\mu}(z)$, i.e., the δ's, and α and γ. Defining $k(z) \equiv S_{x\mu}(z)/S_{\mu}(z)$, and $S_a(z)$ as the spectrum of the a's in (6), we have that the spectrum (autocovariance generating function) of x_t obeys

$$S_x(z) = |k(z)|^2\, S_{\mu}(z) + S_a(z)$$

where

$$k(z) = \frac{\alpha}{1-\gamma z^{-1}} - \frac{\gamma g z^{-1}}{(1-\gamma z^{-1})B(z)} + \frac{\gamma g}{(1-\gamma z^{-1})B(z)}$$

$$B(z) = \frac{1}{1-\delta_1 z-\ldots-\delta_n z^n}$$

$$g = \frac{\alpha}{(1-\gamma)[1-\delta_1\gamma-\ldots-\delta_n\gamma^n]} = \frac{1}{1-\delta_1\gamma-\ldots-\delta_n\gamma^n} ,$$

since $\alpha/(1-\gamma) = 1$. The z-transform of the coefficients in the regression of x_t on past, present, and future μ's is given by $k(z)$.

3. Illustrations

Summarizing our results, we have assumed the structure

$$(1) \qquad m_t - p_t = \phi(E_t p_{t+1} - p_t) + u_t, \quad \phi < 0$$

which upon first differencing gives

$$(3) \qquad x_t = \mu_t - \phi(1-L)E_t x_{t+1} + \eta_t'$$

where

$x_t \equiv p_t - p_{t-1}$ $\mu_t \equiv m_t - m_{t-1}$, $E_t x_{t+1} \equiv E_t p_{t+1} - p_t$, and η_t' satisfies

$E\{\eta_t'\mu_s\} = 0$ for all t and s, which means that $\{\mu_t\}$ is assumed to be strictly econometrically exogenous in (3). We also assume that μ_t follows the stationary n^{th}-order autoregressive process

$$\mu_t = \delta_1\mu_{t-1} + \delta_2\mu_{t-2} + \dots + \delta_n\mu_{t-n} + \psi_t$$

or

$$(1-\delta_1 L-\dots-\delta_n L^n)\mu_t = \psi_t$$

where $E\psi_t \ \mu_{t-\tau} = 0$ for all $\tau \geq 1$. Under these conditions, x_t can be expressed as a one-sided distributed lag of μ_t,

$$x_t = k(L)\mu_t + a_t \tag{13}$$

where $E \ a_s \ \mu_t = 0$ for all t and s, and where k(L) is given by

$$k(L) = \frac{\alpha}{1-\gamma L^{-1}} + \frac{\gamma g(1-L^{-1})(1-\delta_1 L-\dots-\delta_n L^n)}{(1-\gamma L^{-1})} \tag{14}$$

$$g = \frac{1}{(1-\delta_1\gamma-\delta_2\gamma^2-\dots-\delta_n\gamma^n)}$$

$$\alpha = \frac{1}{1-\phi}$$

$$\gamma = -\frac{\phi}{1-\phi} .$$

We can write $k(L) = \sum_{j=0}^{\infty} k_j L^j$, where k_j is given by

$$k_j = \begin{cases} \gamma g \ [\sum_{i=1}^{n} \delta_i\gamma^{i-1} - \sum_{i=0}^{n} \delta_i\gamma^i] + \alpha & j=0 \\ \gamma g \ [\sum_{i=j+1}^{n} \delta_i\gamma^{i-j-1} - \sum_{i=j}^{n} \delta_i\gamma^{i-j}] & 1 \leq j < n \\ -\gamma g \ \delta_n & j = n \\ 0 & \text{Otherwise} \end{cases} \tag{14a}$$

with $\delta_0 = -1$.

It is straightforward to verify that k(L) is one-sided on nonnegative powers of L, despite the presence of terms in L^{-1} on the right-hand side of (14).[5/]

Using (14), we can evaluate the sum of the lag weights k_j as

$$\begin{aligned} k(1) &= \sum_{j=0}^{\infty} k_j \\ &= \frac{\alpha}{1-\gamma} = 1. \end{aligned}$$

Notice that the sum of lag weights is unity regardless of the values of the parameters λ and ϕ.

It is interesting to calculate the frequency response function of k(L), which is given by

$$k(e^{+iw}) = \sum_{j=0}^{\infty} k_j e^{iwj} .$$

The frequency response function $k(e^{+iw})$ gives the response of x_t in (13) to an input $\mu_t = e^{-iwt}$:

$$\begin{aligned} x_t &= \sum_{j=0}^{\infty} k_j e^{-iw(t-j)} \\ &= e^{-iwt} \sum_{j=0}^{\infty} k_j \ e^{iwj} \\ &= e^{-iwt} k(e^{+iw}). \end{aligned}$$

Similarly, the response of x_t in (13) to an input $\mu_t = e^{+iwt}$ is $e^{+iwt}k(e^{-iw})$. Write $k(e^{+iw})$ in the polar form

$$k(e^{iw}) = g(w)e^{i\theta(w)} \quad (15)$$

The quantity $g(w)$ is termed the "gain" while $\theta(w)$ is called the "phase shift." Then the response of x_t in (13) to a sinusoidal input $\mu_t = 2 \cos wt = e^{iwt} + e^{-iwt}$ is given by

$$x_t = e^{iwt}k(e^{-iw}) + e^{-iwt}k(e^{+iw})$$

$$= g(w)(e^{i(wt-\theta(w))} + e^{-i(wt-\theta(w))})$$

$$x_t = g(w)\ 2\cos[wt-\theta(w)] \tag{16}$$

Thus, the response of x_t in (15) to a μ input of cos wt is a cosine wave with phase shifted by $-\theta(w)$ and amplitude multiplied by $g(w)$. It must be emphasized, however, that our device of inserting a cosine wave input into (13) is only a heuristic one, since to derive (13) we have assumed that μ_t is governed by an autoregressive process. Such an indeterministic μ process has an arbitrarily small proportion of its variance occurring in an arbitrarily small band about any frequency w. If the money supply input μ_t were really of the form cos wt, the relationship (13) would break down because the formula for the linear least squares forecast used to get (13) does not represent the best forecast for such a deterministic μ process. In effect, under the assumption that μ is governed by an n^{th}-order autoregression, μ_t is composed of a continuum of components cos wt with w between $-\pi$ and $+\pi$. The importance of the various bands of components is controlled by the δ_j's. Equation (16) gives a picture of how the component of μ at a given frequency is shifted in phase and multiplied in amplitude in contributing to the composition of inflation x.

For purposes of comparison, it is useful to derive the analogue of (16) when μ_t is governed by a deterministic process (i.e., a process perfectly predictable from its own past), such as

$$\mu_t = \cos wt.$$

For convenience, assume that (3) holds with $\eta_t \equiv 0$, which, because then $E_t x_{t+1} = x_{t+1}$, implies

$$x_t = \mu_t - \phi x_{t+1} + \phi x_t$$

or

$$x_t = \alpha\mu_t + \gamma x_{t+1} .$$

This difference equation can be solved for x_t,

$$x_t = \alpha \sum_{j=0}^{\infty} \gamma^j \mu_{t+j} = \frac{\alpha}{1-\gamma L^{-1}} \mu_t . \tag{17}$$

The response in (17) of x_t to $\mu_t = e^{iwt}$ is

$$\begin{aligned} x_t &= e^{iwt} \alpha \sum_{j=0}^{\infty} \gamma^j e^{iwj} \\ &= e^{iwt} \frac{\alpha}{1-\gamma e^{iw}} . \end{aligned}$$

The response of x_t to $\mu_t = e^{-iwt}$ is

$$x_t = e^{-iwt} \frac{\alpha}{1-\gamma e^{-iw}} .$$

We can express $\alpha/(1-\gamma e^{+iw})$ in the polar form

$$\frac{\alpha}{1-\gamma e^{iw}} = r(w)e^{i\theta(w)}$$

$$r(w) = \frac{\gamma}{\sqrt{1+\gamma^2-2\gamma\cos w}}$$

$$\theta(w) = \arctan\left[\frac{-\gamma \sin w}{1-\gamma\cos w}\right] .$$

The response of x_t to an input $2 \cos wt = e^{iwt} + e^{-iwt}$ is easily deduced to be

$$(18) \qquad x_t = r(w) \cos[wt-\theta(w)].$$

Equation (18) gives the response of x to a deterministic component of μ, i.e., a component corresponding to a spike in the spectral density of μ at frequency w.

To illustrate our formula 14, we assume that the money creation process μ_t is strictly exogenous and is governed by the Markov process

$$(19) \qquad (1-\lambda L^{12})\mu_t = \varepsilon_t \qquad\qquad (\lambda < 1)$$

where ε_t is a white noise process that is fundamental for μ_t. Equation (19) can be inverted to yield the moving average representation for μ:

$$(20) \qquad \mu_t = (1-\lambda L^{12})^{-1}\varepsilon_t$$

or

$$\mu_t = (1+\lambda L^{12}+\lambda^2 L^{24}+ \ldots)\varepsilon_t \ .$$

The random variable ε_t is the "innovation" in μ_t; that is, it is the one-step-ahead prediction error for μ, which is the unexpected part of money creation. From (20) we know that the spectrum for μ is given by

$$S_\mu(z) = \frac{\sigma_\varepsilon^2}{(1-\lambda z^{12})(1-\lambda z^{-12})} \ ,$$

which for $\lambda > 0$ is characterized by peaks at the "seasonal" frequencies. We think of time having the units of months.

For two values of λ and two values of ϕ, Tables 1, 2, and 3 record the gain $g(w)$, phase $\theta(w)$, and distributed lag weights k_j obtained by using formulas (14a) and (15). Graphs of the phase and gain are also recorded in Graphs 1, 2, 3, and 4. For each value of ϕ, Table 4 records the gain and phase that obtain in the deterministic case.

The gain and phase reveal the following patterns. At the seasonal periodicities of 12, 6, 4, 3, 2.4, and 2 months, the gain is less than one indicating that the seasonal power in μ is being attenuated or transmitted to x with diminished power. This attenuation becomes more pronounced, i.e., the gain becomes smaller, as ϕ becomes larger, given λ, and as λ becomes larger, given ϕ. Notice that for each ϕ the gain at each seasonal frequency gets closer to the gain for the deterministic case as λ goes from .1 to .9. This is consistent with the fact that at $\lambda = .9$ a much larger proportion of the seasonal variation in μ_t is predictable. Notice that the gain for $\lambda = .9$ is still slightly above the gain at the seasonals for the deterministic case.

The phase shift at the seasonal frequencies is negative, which indicates a lead of inflation over money creation at those frequencies. (The phase is recorded in radians. To convert to units of time, divide the phase by the frequency, which is also recorded in radians.) The magnitude of the lead grows as λ increases, given ϕ. The lead is greatest in the deterministic case. The lead grows with λ because as λ increases, the extent to which seasonal movements are predictable increases. For large enough λ, the magnitude of the lead grows with increases in the absolute value of ϕ.

The tendency for the gain to fall and the phase shift to increase as ϕ increases is reasonable, since ϕ governs the extent to which expectations of future money creation influence current inflation. This influence grows with the absolute value of ϕ. The greater attenuation and phase shift that accompany increases in ϕ is the counterpart of the bigger initial jump in price that occurs in response to a known future increase in money in Sargent and Wallace's diagram.

Table 3 presents the distributed lag coefficients and indicates how they depend on ϕ and λ. Notice that k_{12} is negative in each instance and that the lag distribution is not "smooth" in the vicinity of k_{11}, k_{12}, k_{13}. It is, of course, well known that where lag distributions reflect optimal forecasting procedures, they often will be very choppy. Notice also that as λ and ϕ each increase in absolute value, the absolute values of coefficients at lags of 10, 11, and 12 increase. Table 3 should serve as a warning that rational expectations models do not imply lag distributions of inflation on money creation that are concentrated at zero lag.[6/]

The lag distribution k(L) characterizes the projection of x_t on μ_t,

$$x_t = k(L)\mu_t + a_t \tag{21}$$

where $E\, a_s \mu_t = 0$ for all t,s. Combining (21) with (20), we can derive the lag distribution linking x_t to ε_t, which is the unexpected change in μ_t (that is, ε_t is the one-step-ahead prediction error in predicting μ from its own past):

$$\begin{aligned} x_t &= \frac{k(L)}{1-\lambda L^{12}} \varepsilon_t + a_t \\ &= h(L)\varepsilon_t + a_t \ , \end{aligned} \tag{22}$$

where $h(L) = k(L)(1+\lambda L^{12}+\lambda^2 L^{24}+ \ldots)$.

It is straightforward to verify that the coefficients of h(L) obey

$$h_i = k_i \qquad i = 0, \ldots, 11$$

$$h_{12} = k_{12} + \lambda k_0$$

$$h_i = \lambda k_{i-12} \qquad i = 13, \ldots, 23$$

$$h_{24} = \lambda k_{12} + \lambda^2 k_0$$

$$h_i = \lambda^2 k_{i-24} \qquad i = 25, \ldots, 35$$

$$\vdots$$

Indeed, writing out $h(L) = k(L)(1+\lambda L^{12}+\lambda^2 L^{24} + \ldots)$, we have

$$h(L) = k_0 + k_1 L + \ldots + k_{11}L^{11} + k_{12}L^{12} + \lambda k_0 L^{12} + \lambda k_1 L^{13} + \ldots$$
$$+ \lambda k_{11}L^{23} + \lambda k_{12}L^{24} + \lambda^2 k_0 L^{24} + \ldots \; .$$

The response of (the rate of change of) real balances to an unexpected change in money creation can be deduced by subtracting (22) from (21) to get

$$\mu_t - x_t = \left[\frac{1}{1-\lambda L^{12}} - h(L)\right] \varepsilon_t - a_t$$
$$= \left[\frac{1-k(L)}{1-\lambda L^{12}}\right] \varepsilon_t - a_t$$

(23) $$\mu_t - x_t = v(L)\varepsilon_t - a_t$$

where $v(L) = (1-k(L))/(1-\lambda L^{12})$. It is straightforward to determine that

$$
(24) \quad \begin{aligned}
v_0 &= 1 - h_0 \\
v_j &= -h_j \qquad j \geq 1,\ j \neq 12, 24, 36, \ldots \\
v_{12} &= \lambda - h_{12} \\
v_{12} &= \lambda^n - h_{12n} \quad n = 1, 2, 3 \ldots \ .
\end{aligned}
$$

or in terms of k_j's we can write

$$
\begin{aligned}
v_0 &= 1 - k_0 \\
v_j &= -k_j \qquad j = 1, \ldots, 11 \\
v_{12} &= \lambda v_0 - k_{12} \\
v_j &= \lambda v_{j-12} \quad j > 12
\end{aligned}
$$

Using (14), it is straightforward to show that

$$h_0 = k_0 = \frac{1}{1-\lambda\gamma^{12}}$$

which exceeds 1 so long as $\lambda > 0$. Therefore, so long as $\lambda > 0$, we have that

$$v_0 = 1 - h_0 < 0.$$

Since for $j = 1, \ldots, 11$, $k_j > 0$ for $\phi < 0$ and $\lambda > 0$, we have that

$$v_j = -h_j = -k_j < 0 \qquad \text{for } j = 1, \ldots, 11.$$

Using (14a) we have that

$$k_{12} = \frac{-\gamma\lambda}{1-\lambda\gamma^{12}} \ ,$$

so that $h_{12} = k_{12} + \lambda k_0$

$$= \frac{-\gamma\lambda}{1-\lambda\gamma^{12}} + \frac{\lambda}{1-\lambda\gamma^{12}}$$

$$h_{12} = \frac{\lambda(1-\gamma)}{1-\lambda\gamma^{12}} \quad .$$

Hence, from (24), v_{12} becomes

$$v_{12} = \lambda - \frac{\lambda(1-\gamma)}{1-\lambda\gamma^{12}}$$
$$= \frac{\lambda\gamma(1-\lambda\gamma^{11})}{1-\lambda\gamma^{12}} > 0$$

because $0 < \lambda < 1$ and $0 < \gamma = \frac{-\phi}{1-\phi} < 1$. Thus, we have

$$(25) \qquad \begin{array}{ll} v_j < 0 & \text{for } j = 0, \ldots, 11 \\ v_j > 0 & \text{for } j = 12 \end{array} \quad .$$

This pattern of v_j's implies that an unexpected increase in money creation of ε_t can be expected to cause (the rate of change of) real balances to <u>fall</u> at time t, and to continue <u>falling</u> from time t+1 to time t+11, and then to <u>increase</u> at time t+12. This pattern is more pronounced the larger is λ, for a given ϕ, and the larger is ϕ for a given λ. The pattern mimicks the graph presented by Sargent and Wallace for the response to a perfectly foreseen future increase in the money supply. In that graph, at the time that the previously unannounced increase is announced, real balances fall and continue falling until the previously announced change in the money supply actually occurs, at which moment the real money supply rises and expected inflation falls. That the pattern of v_j's is mimicking that graph can be seen by noting that for $\lambda > 0$, an unexpected money creation of ε_t causes previous expectations of money creation at time t+12 to be revised upward by $\lambda\varepsilon_t$.[7/] The expected response of real balances to this newly forecast increase in μ_{t+12} is exactly like that predicted by the deterministic Sargent-Wallace diagram. As in the Sargent-Wallace diagram, the extent to which real balances fall prior to the date of the anticipated increase in μ is governed by the parameter ϕ. The parameter λ, which was not a factor in Sargent and Wallace's deterministic analysis, enters here because it is the factor by which ε_t must be multiplied to help determine the future μ at time t+12 expected as of time t.

Table 1

GAIN

Frequency Radians	Period Months	λ = 0.1, φ = -0.5	λ = 0.1, φ = -2.0	λ = 0.9, φ = -0.5	λ = 0.9, φ = -2.0
0.	∞	1.0000	1.000	1.000	1.000
0.06545	96	1.002	1.008	1.020	1.073
0.1309	48	1.006	1.024	1.058	1.217
0.2618	24	1.003	1.025	1.035	1.266
0.3927	16	0.9829	0.9611	0.8477	0.7362
0.5236	12	0.9891	0.9497	0.9189	0.6390
1.047	6	0.9717	0.9294	0.7755	0.4212
1.571	4	0.9602	0.9237	0.6648	0.3379
2.094	3	0.9539	0.9216	0.5968	0.3014
2.618	2.4	0.9509	0.9208	0.5611	0.2852
3.142	2	0.9500	0.9206	0.5500	0.2806

Table 2

PHASE

Frequency Radians	Period Months	λ = 0.1, φ = -0.5	λ = 0.1, φ = -2.0	λ = 0.9, φ = -0.5	λ = 0.9, φ = -2.0
0.	∞	0.0	0.0	0.0	0.0
0.06545	96	-0.00245	-0.01033	-0.02168	-0.08795
0.1309	48	-0.00084	-0.00741	-0.00719	-0.05645
0.2618	24	0.01227	0.03615	0.1073	0.2682
0.3927	16	0.00695	0.04176	0.0726	0.5162
0.5236	12	-0.02105	-0.04011	-0.2053	-0.5701
1.047	6	-0.02547	-0.02644	-0.2913	-0.5566
1.571	4	-0.02083	-0.01654	-0.2742	-0.4219
2.094	3	-0.01397	-0.00982	-0.2023	-0.2755
2.618	2.4	-0.00692	-0.00461	-0.1058	-0.1352
3.142	2	0.0	0.0	0.0	0.0

Table 3

Distributed Lag Coefficients

$$k(L) = \sum_{j=0}^{\infty} k_i L^i$$

	$\lambda = 0.1$		$\lambda = 0.9$	
Lag	$\phi = -0.5$	$\phi = -2.0$	$\phi = -0.5$	$\phi = -2.0$
0	1.000	1.001	1.000	1.007
1	0.000	0.000	0.000	0.003
2	0.000	0.001	0.000	0.005
3	0.000	0.001	0.000	0.008
4	0.000	0.001	0.000	0.012
5	0.000	0.002	0.000	0.018
6	0.000	0.003	0.001	0.027
7	0.000	0.004	0.002	0.040
8	0.001	0.007	0.007	0.060
9	0.002	0.010	0.022	0.090
10	0.007	0.015	0.067	0.134
11	0.022	0.022	0.200	0.201
12	-0.033	-0.067	-0.300	-0.604
Sum	1.000	1.000	1.000	1.000

Table 4

Deterministic Case

Frequency Radians	Period Months	GAIN		PHASE (Radians)	
		$\phi = -0.5$	$\phi = -2.0$	$\phi = -0.5$	$\phi = -2.0$
0.0	∞	1.000	1.000	0.0	0.0
0.06545	96	0.9984	0.9874	-0.0327	-0.1295
0.1309	48	0.9936	0.9523	-0.0649	-0.2512
0.2618	24	0.9754	0.8425	-0.1266	-0.4513
0.3927	16	0.9474	0.7229	-0.1823	-0.5863
0.5236	12	0.9125	0.6193	-0.2302	-0.6678
1.047	6	0.7559	0.3780	-0.3335	-0.7137
1.571	4	0.6325	0.2774	-0.3218	-0.5880
2.094	3	0.5547	0.2294	-0.2426	-0.4086
2.618	2.4	0.5131	0.2068	-0.1286	-0.2083
3.142	2	0.5000	0.2000	0.0	0.0

Table 5

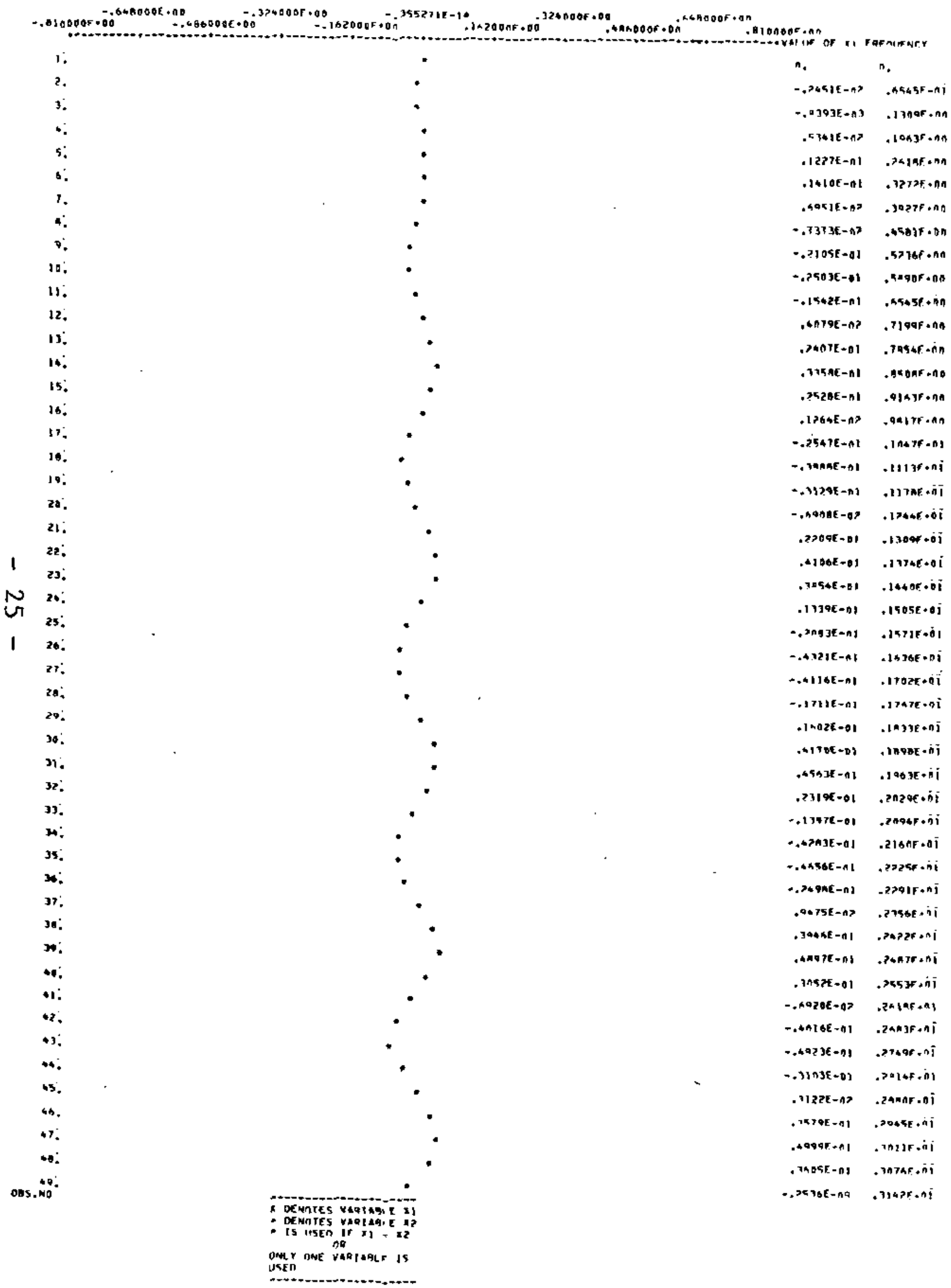
Distributed Lag Coefficients of
$\mu_t - x_t$ on ε_t

	$\lambda = 0.1$		$\lambda = 0.9$	
Lag	$\phi = -0.5$	$\phi = -2.0$	$\phi = -0.5$	$\phi = -2.0$
0	-0.000	-0.001	-0.000	-0.007
1	-0.000	-0.000	-0.000	-0.003
2	-0.000	-0.001	-0.000	-0.005
3	-0.000	-0.001	-0.000	-0.008
4	-0.000	-0.001	-0.000	-0.012
5	-0.000	-0.002	-0.000	-0.018
6	-0.000	-0.003	-0.001	-0.027
7	-0.000	-0.004	-0.002	-0.040
8	-0.001	-0.007	-0.007	-0.060
9	-0.002	-0.010	-0.022	-0.090
10	-0.007	-0.015	-0.067	-0.134
11	-0.022	-0.022	-0.200	-0.201
12	0.033	0.0669	0.300	0.5977
v_j $j>12$	λv_{j-12}	λv_{j-12}	λv_{j-12}	λv_{j-12}

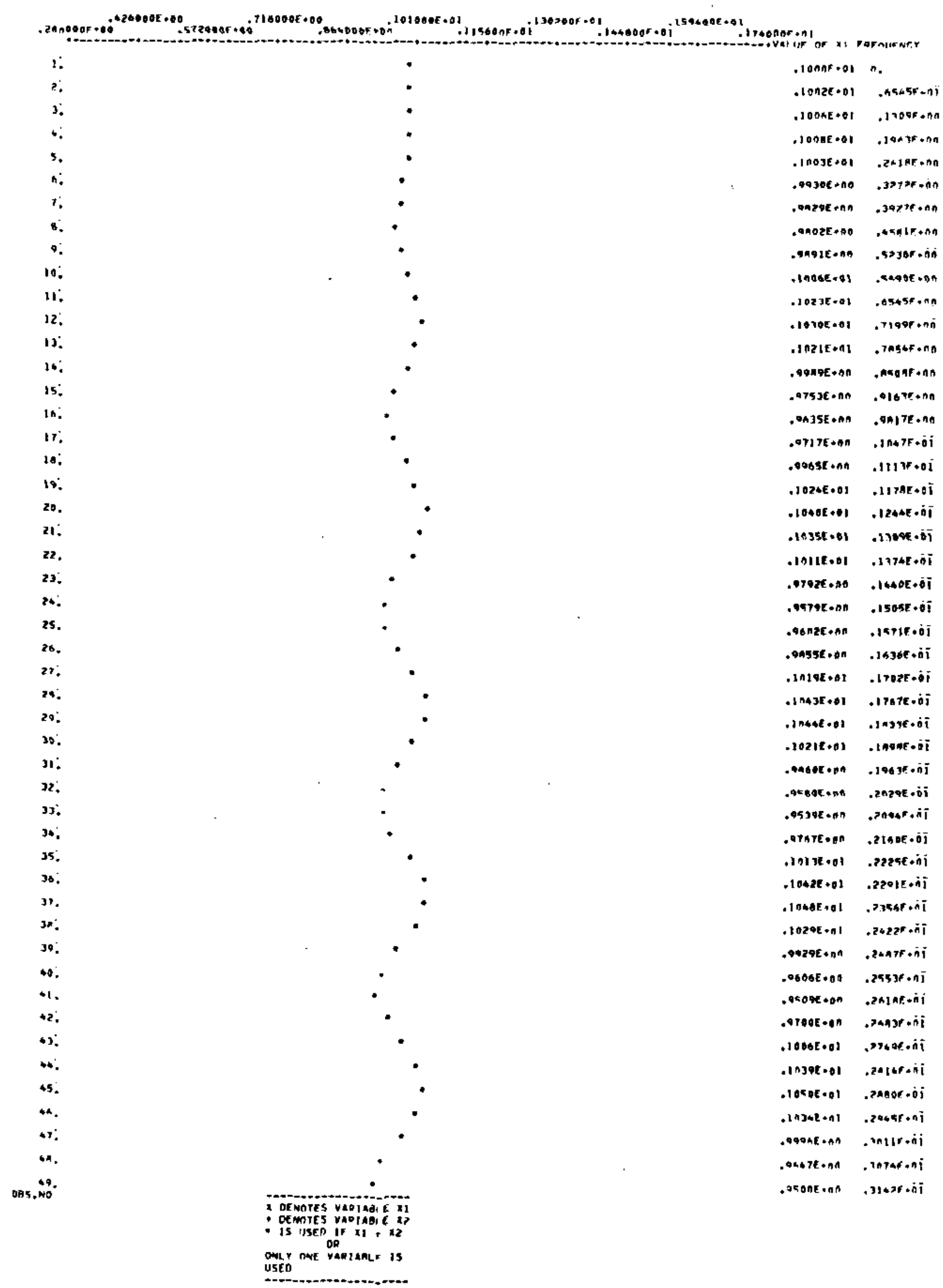
Graph 1

Gain and Phase for $\lambda = 0.1$, $\phi = -0.5$

***** CALCULATED PHASE OF FILTER *****



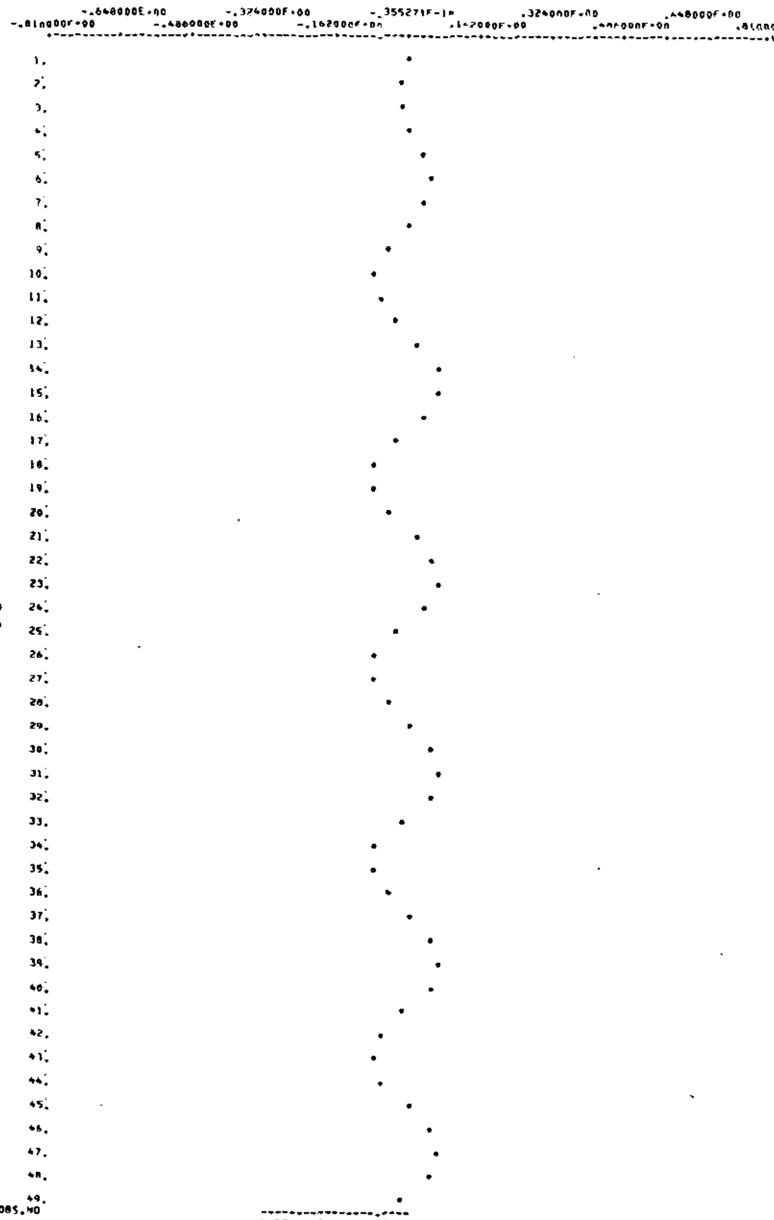
***** CALCULATED GAIN OF FILTER *****



Graph 2

Gain and Phase for $\lambda = 0.1$, $\phi = -2.0$

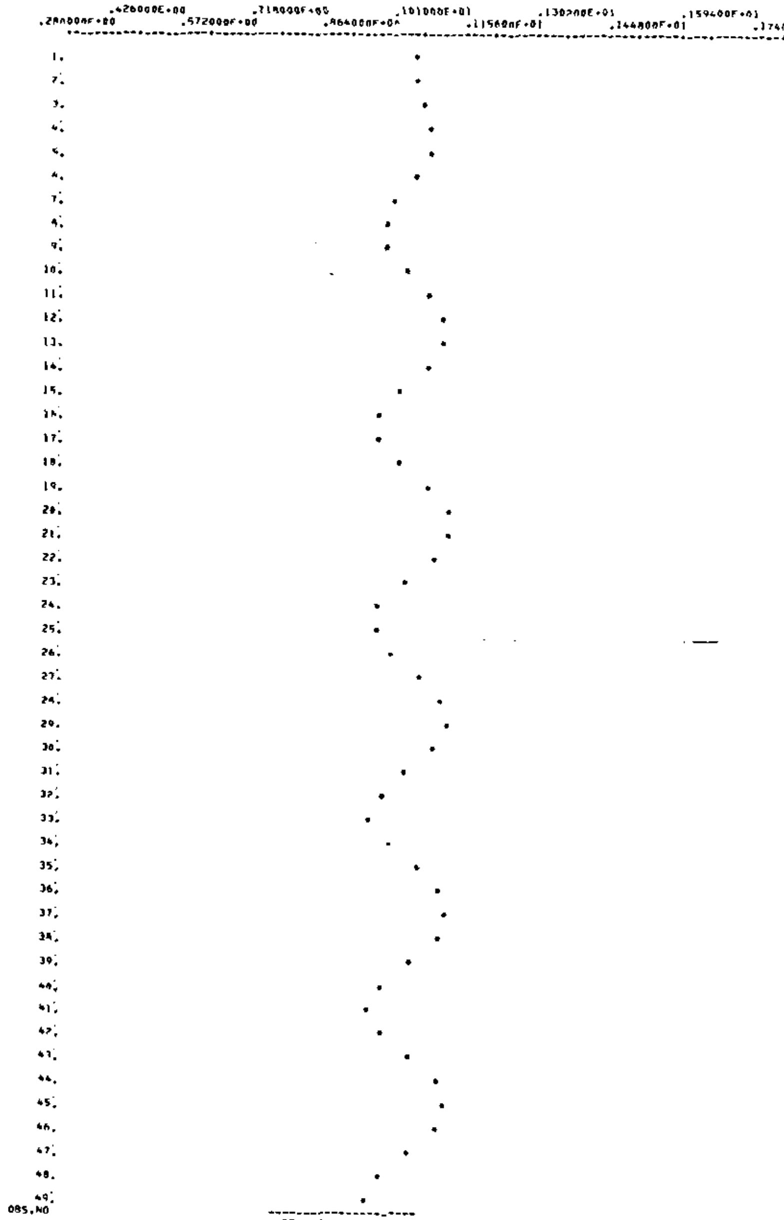
***** CALCULATED PHASE OF FILTER *****



OBS.NO	VALUE OF X1	FREQUENCY
1.	0.	0.
2.	-.1933E-01	.6545E-01
3.	-.7406E-02	.1309E+00
4.	.1125E-01	.1963E+00
5.	.3615E-01	.2618E+00
6.	.5123E-01	.3272E+00
7.	.4176E-01	.3927E+00
8.	.5261E-02	.4581E+00
9.	-.4811E-01	.5236E+00
10.	-.6596E-01	.5890E+00
11.	-.5707E-01	.6545E+00
12.	-.1913E-01	.7199E+00
13.	.2964E-01	.7854E+00
14.	.6682E-01	.8508E+00
15.	.7079E-01	.9163E+00
16.	.3754E-01	.9817E+00
17.	-.2644E-01	.1047E+01
18.	-.6991E-01	.1113E+01
19.	-.7157E-01	.1178E+01
20.	-.3530E-01	.1244E+01
21.	.1931E-01	.1309E+01
22.	.6389E-01	.1374E+01
23.	.7738E-01	.1440E+01
24.	.4565E-01	.1505E+01
25.	-.1654E-01	.1571E+01
26.	-.6770E-01	.1636E+01
27.	-.7433E-01	.1702E+01
28.	-.4316E-01	.1767E+01
29.	.1121E-01	.1833E+01
30.	.6052E-01	.1898E+01
31.	.7936E-01	.1963E+01
32.	.5208E-01	.2029E+01
33.	-.9923E-02	.2094E+01
34.	-.6501E-01	.2160E+01
35.	-.7841E-01	.2225E+01
36.	-.4900E-01	.2291E+01
37.	.4146E-02	.2356E+01
38.	.5748E-01	.2422E+01
39.	.7995E-01	.2487E+01
40.	.5638E-01	.2553E+01
41.	-.4610E-02	.2618E+01
42.	-.6234E-01	.2683E+01
43.	-.7944E-01	.2749E+01
44.	-.5157E-01	.2814E+01
45.	.1964E-02	.2880E+01
46.	.5440E-01	.2945E+01
47.	.7992E-01	.3011E+01
48.	.5953E-01	.3076E+01
49.	-.4244E-09	.3142E+01

X DENOTES VARIABLE X1
+ DENOTES VARIABLE X2
* IS USED IF X1 = X2
OR
ONLY ONE VARIABLE IS
USED

***** CALCULATED GAIN OF FILTER *****



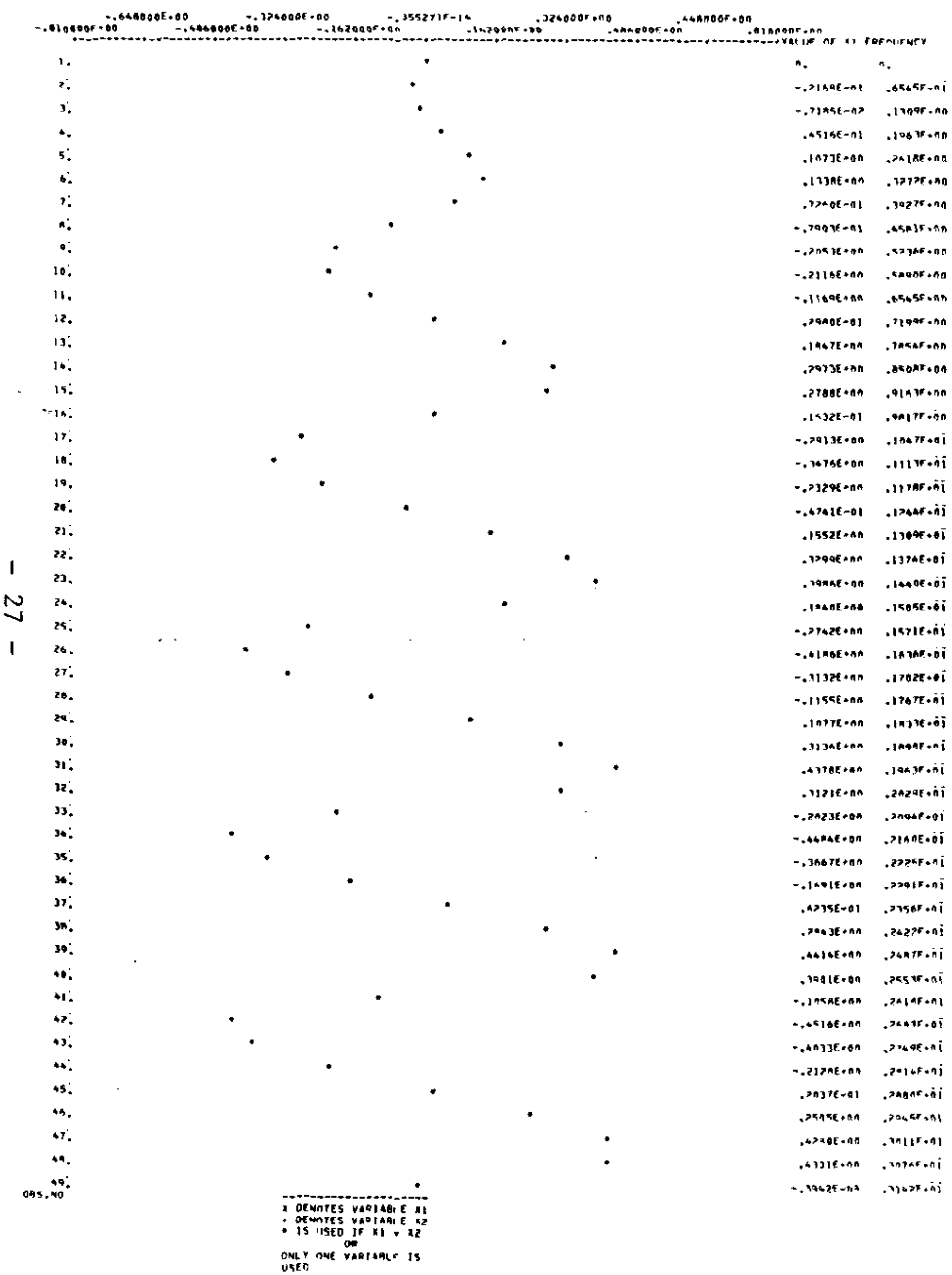
OBS.NO	VALUE OF X1	FREQUENCY
1.	.1000E+01	0.
2.	.1008E+01	.6545E-01
3.	.1024E+01	.1309E+00
4.	.1034E+01	.1963E+00
5.	.1025E+01	.2618E+00
6.	.9947E+00	.3272E+00
7.	.9411E+00	.3927E+00
8.	.9398E+00	.4581E+00
9.	.9497E+00	.5236E+00
10.	.9891E+00	.5890E+00
11.	.1037E+01	.6545E+00
12.	.1068E+01	.7199E+00
13.	.1066E+01	.7854E+00
14.	.1029E+01	.8508E+00
15.	.9751E+00	.9163E+00
16.	.9320E+00	.9817E+00
17.	.9294E+00	.1047E+01
18.	.9681E+00	.1113E+01
19.	.1025E+01	.1178E+01
20.	.1068E+01	.1244E+01
21.	.1076E+01	.1309E+01
22.	.1043E+01	.1374E+01
23.	.9862E+00	.1440E+01
24.	.9359E+00	.1505E+01
25.	.9237E+00	.1571E+01
26.	.9583E+00	.1636E+01
27.	.1017E+01	.1702E+01
28.	.1064E+01	.1767E+01
29.	.1079E+01	.1833E+01
30.	.1050E+01	.1898E+01
31.	.9932E+00	.1963E+01
32.	.9392E+00	.2029E+01
33.	.9216E+00	.2094E+01
34.	.9526E+00	.2160E+01
35.	.1012E+01	.2225E+01
36.	.1063E+01	.2291E+01
37.	.1080E+01	.2356E+01
38.	.1054E+01	.2422E+01
39.	.9984E+00	.2487E+01
40.	.9423E+00	.2553E+01
41.	.9208E+00	.2618E+01
42.	.9486E+00	.2683E+01
43.	.1007E+01	.2749E+01
44.	.1060E+01	.2814E+01
45.	.1081E+01	.2880E+01
46.	.1058E+01	.2945E+01
47.	.1003E+01	.3011E+01
48.	.9453E+00	.3076E+01
49.	.9206E+00	.3142E+01

X DENOTES VARIABLE X1
+ DENOTES VARIABLE X2
* IS USED IF X1 = X2
OR
ONLY ONE VARIABLE IS
USED

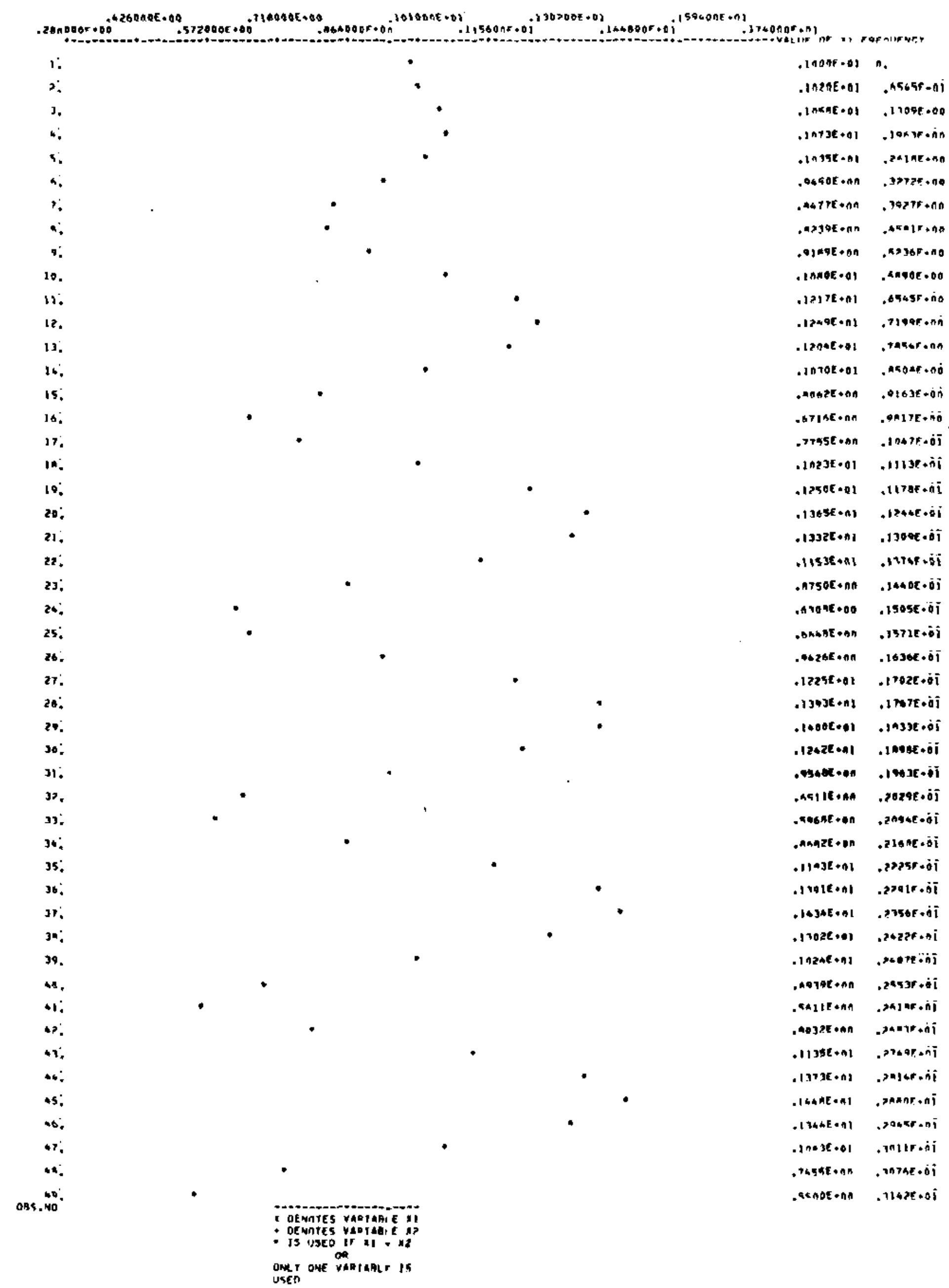
Graph 3

Gain and Phase for $\lambda = 0.9$, $\phi = -0.5$

***** CALCULATED PHASE OF FILTER *****

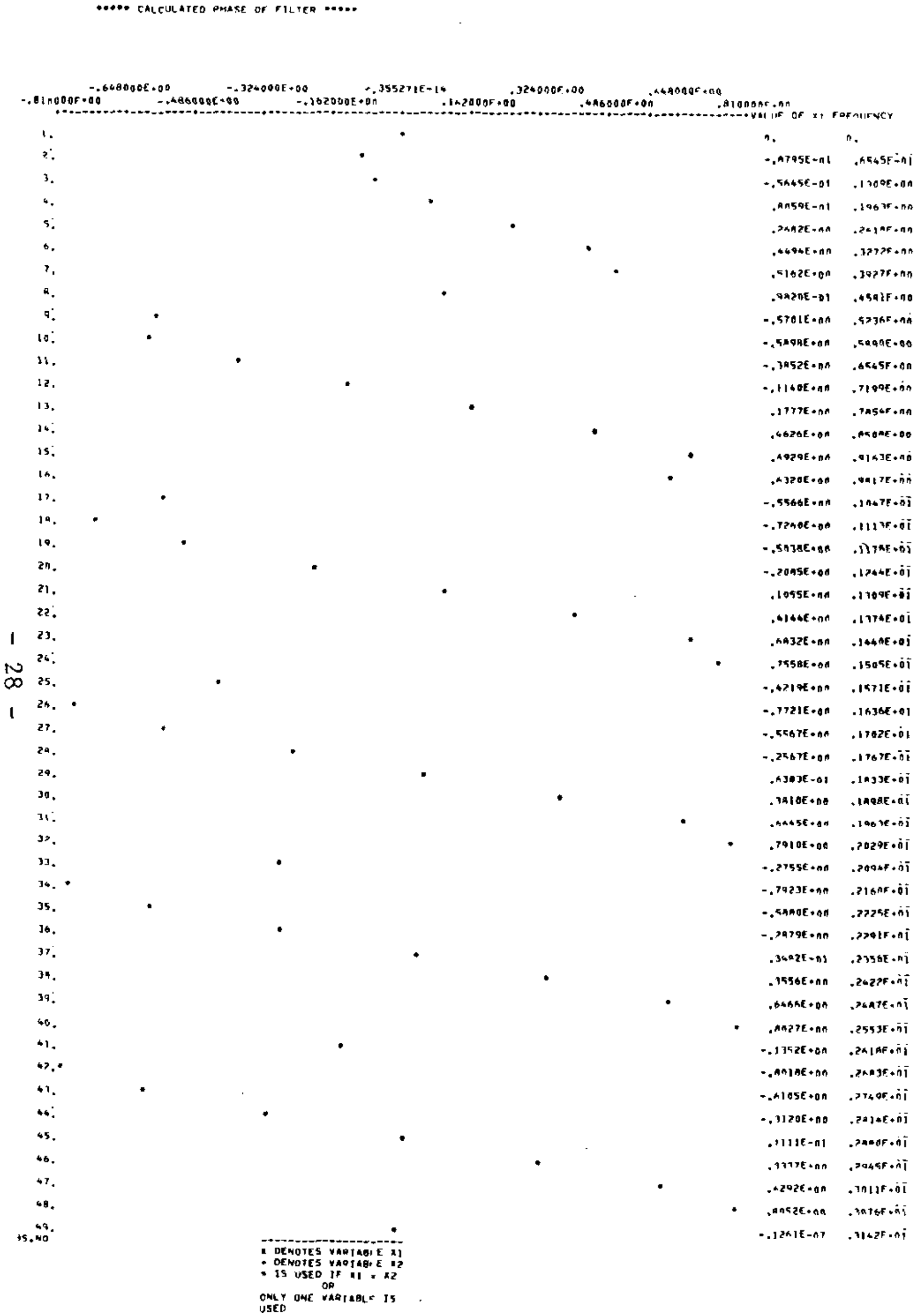


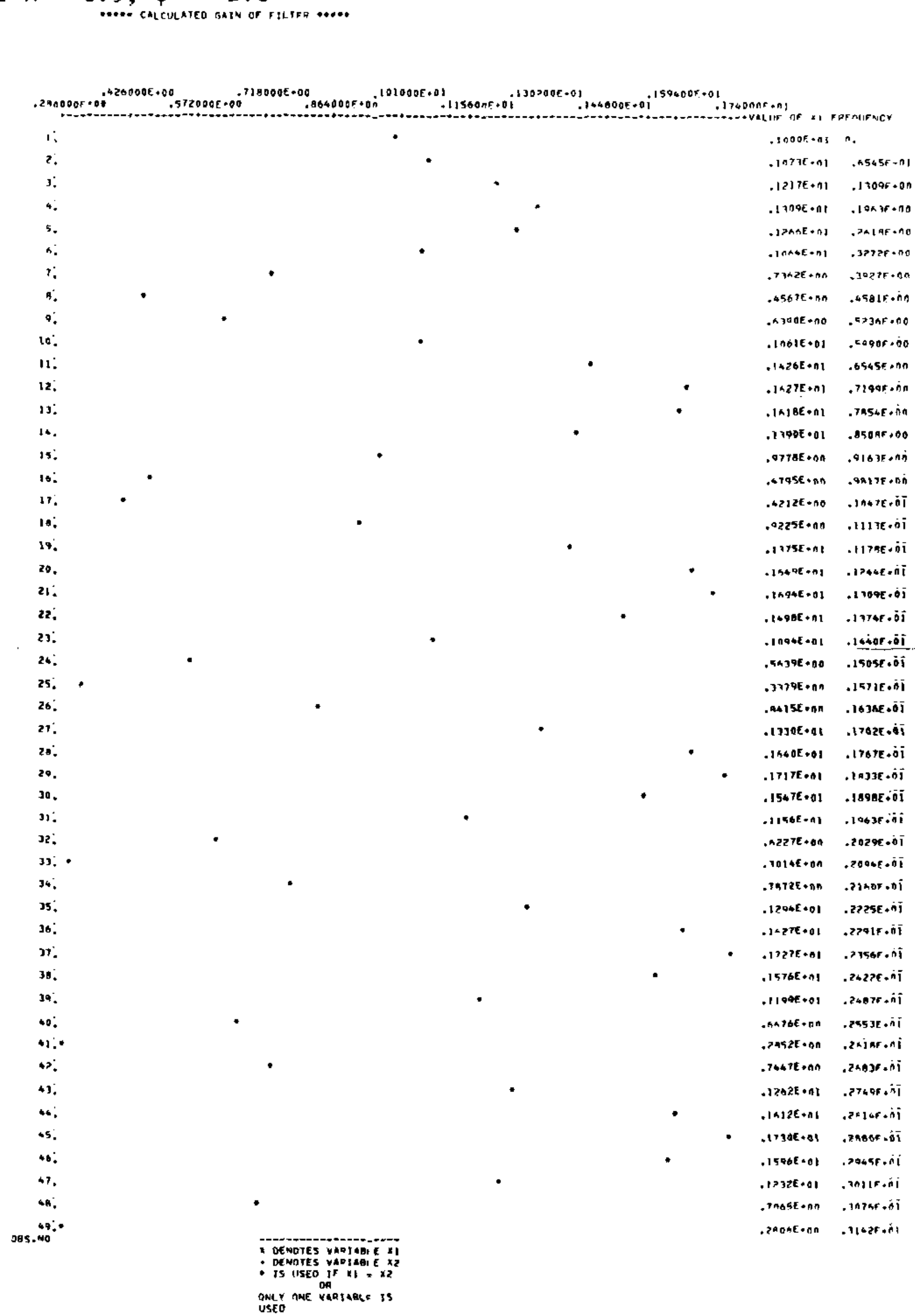
***** CALCULATED GAIN OF FILTER *****



Graph 4

Gain and Phase for $\lambda = 0.9$, $\phi = -2.0$

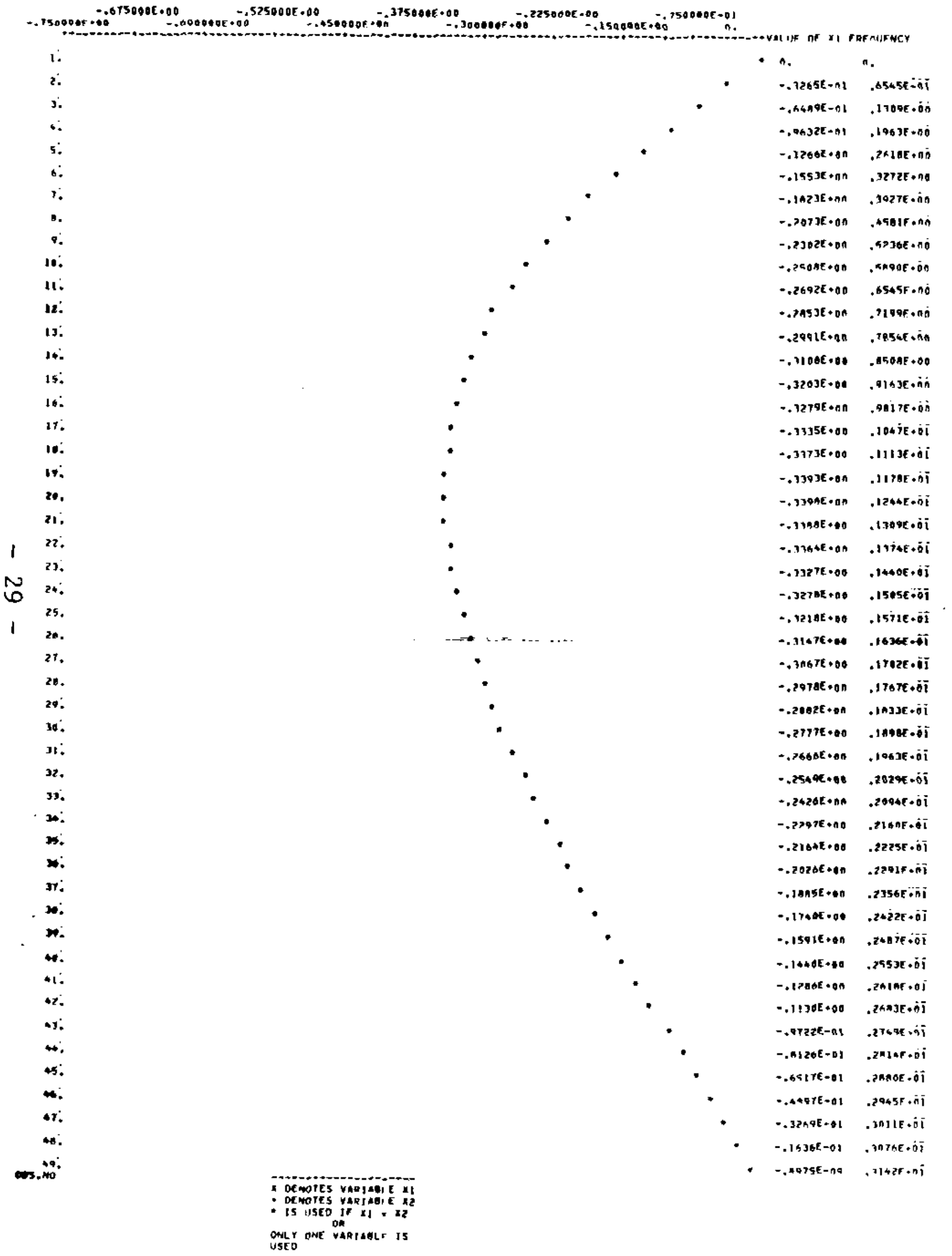




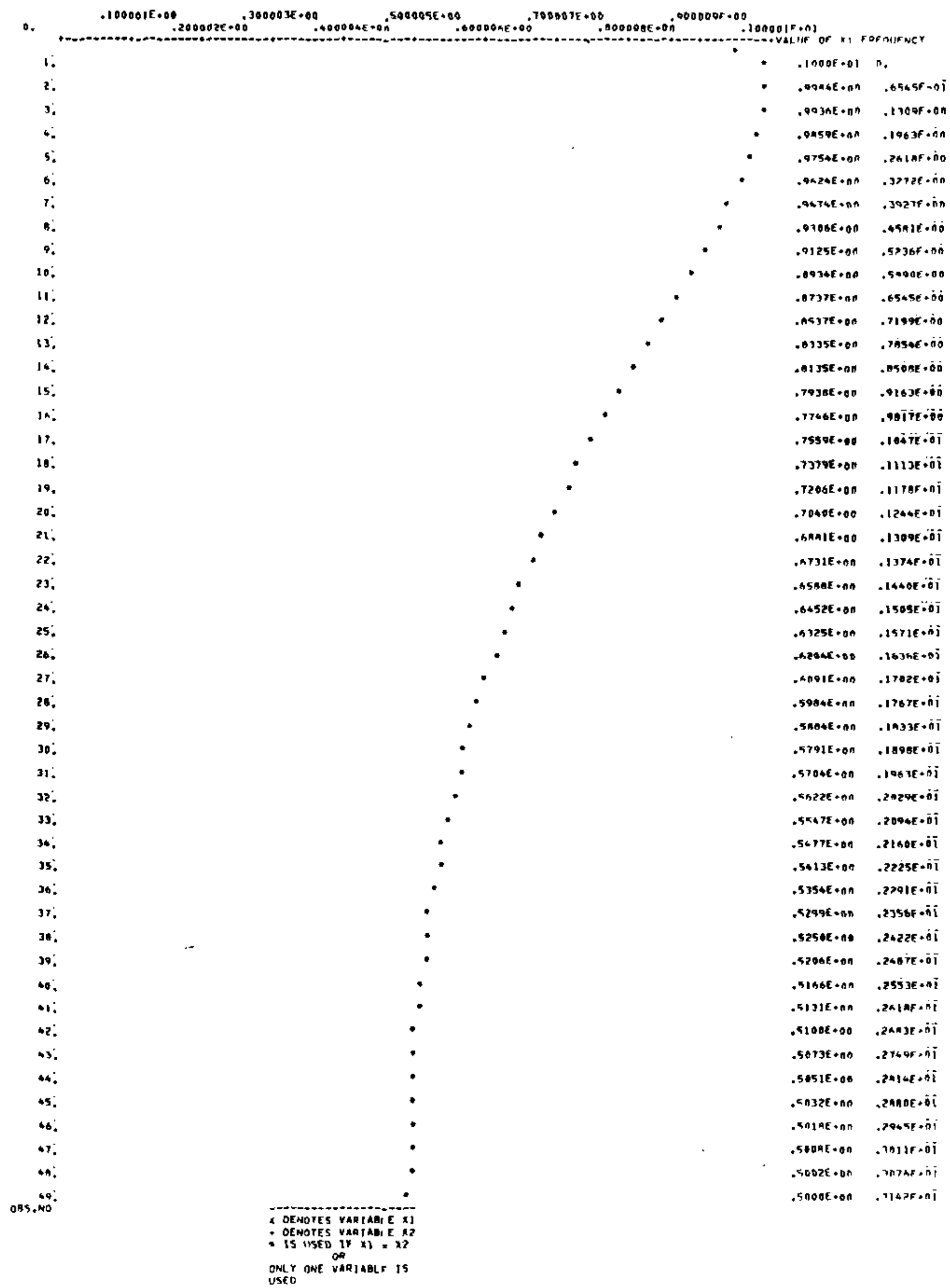
Graph 5

Deterministic Case for $\phi = -0.5$

***** PHASE OF DETERMINISTIC CASE *****

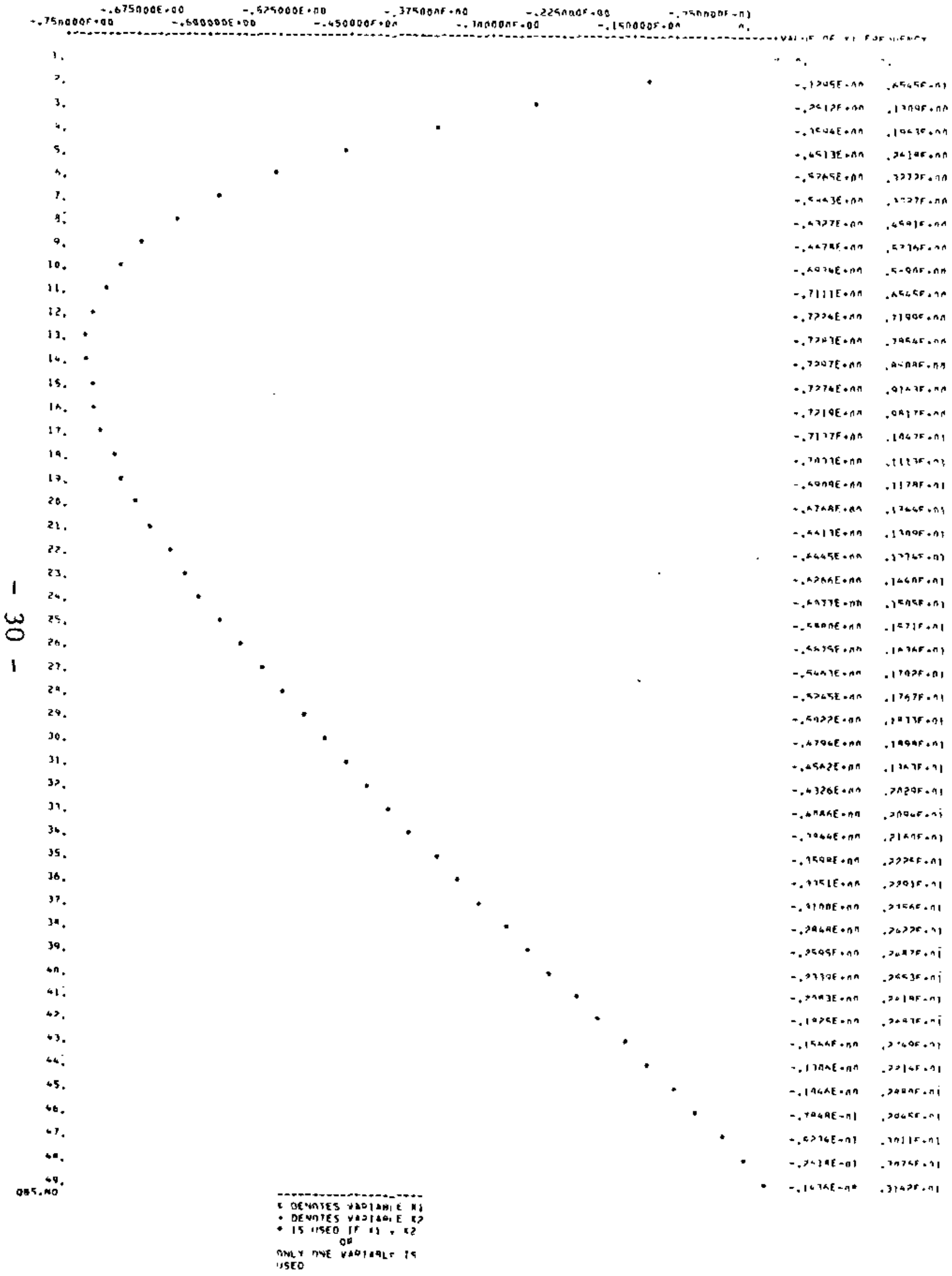


***** AMPLITUDE OF DETERMINISTIC CASE *****

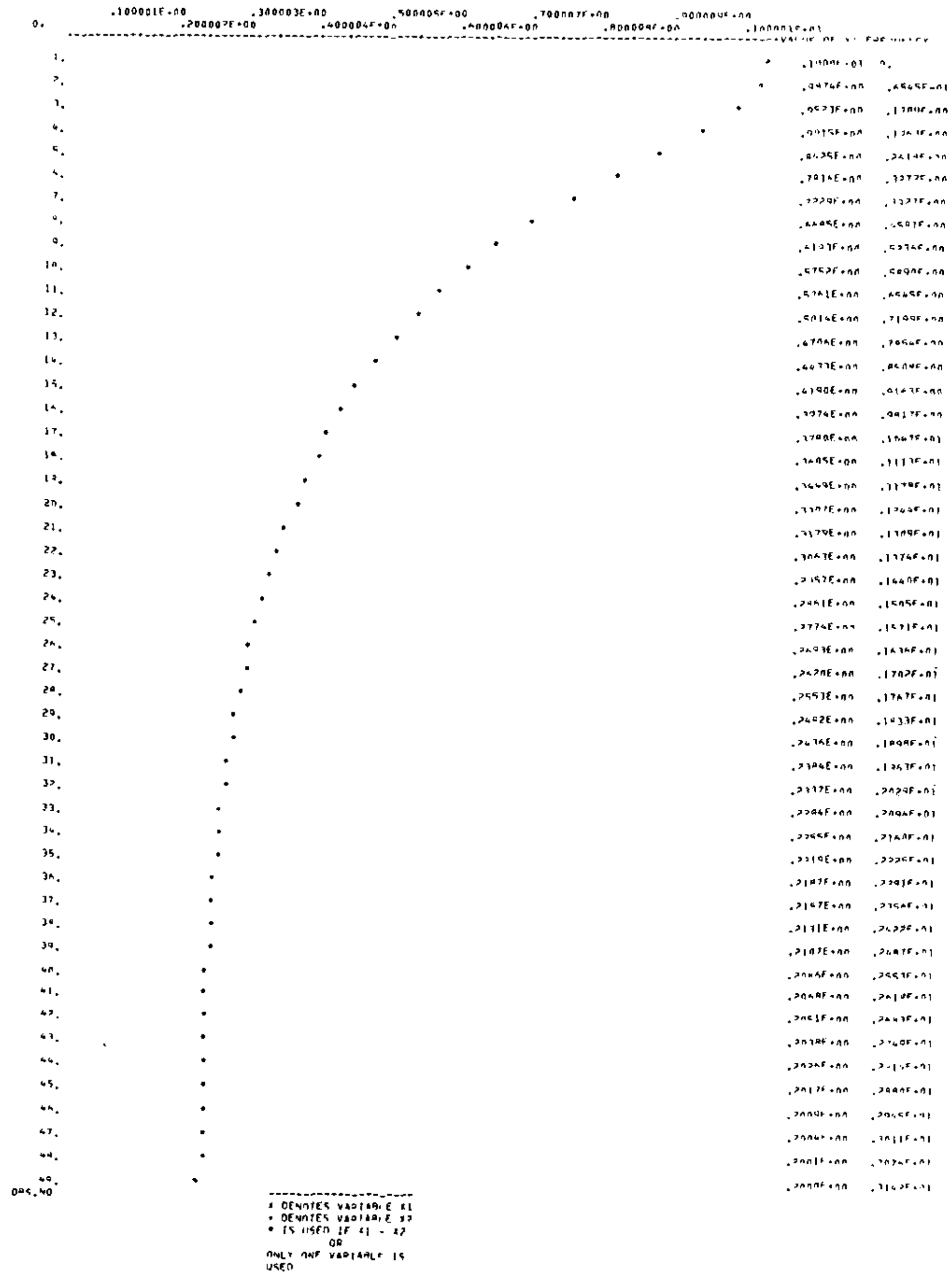


Graph 6
Deterministic Case for $\phi = -2.0$

***** PHASE OF DETERMINISTIC CASE *****



***** AMPLITUDE OF DETERMINISTIC CASE *****



Appendix
Proof of Uniqueness

Consider the operator

(A1) $$\ell\left(\frac{S_{x\mu}(z)}{B(z^{-1})}\right) = B(z) - \phi(1-z)\left[\frac{S_{x\mu}(z)z^{-1}}{B(z^{-1})}\right]_+$$

defined on the space F consisting of function $f(z) = S_{x\mu}(z)/B(z^{-1})$ defined on the unit circle $|z| = 1$ and for which

$$\frac{1}{2\pi i}\int_C \left|\frac{S_{x\mu}(z)}{B(z^{-1})}\right|^2 \frac{dz}{z} < \infty$$

where C is the unit circle. This space of functions is complete and unitarily equivalent to $\ell_2(-\infty, \infty)$, the space of two-sided sequences $\{x_j\}$ with $\sum_{j=-\infty}^{\infty} |x_j|^2 < \infty$. Indeed, each f(z) in F has Fourier series representation

$$f(z) = \sum_{j=-\infty}^{\infty} f_j z^j$$

with the sequence $\{f_j\}$ belonging to $\ell_2(-\infty, \infty)$, and where the f_j's are given by

$$f_j = \frac{1}{2\pi}\int_C f(z)z^{-j}\frac{dz}{z}.$$

The spaces F and $\ell_2(-\infty, \infty)$ are unitarily equivalent, the mapping defined by the preceding equation being a unitary mapping, i.e., one that preserves inner products. We can thus express the norm of f(z) in terms of the norm of $\{f_j\}$, so that

$$||f(z)|| = \left(\sum_{j=-\infty}^{\infty} |f_j|^2\right)^{1/2}$$

Now consider the two functions x(z) and y(z) belonging to F. From (A1) we have

(A2) $\qquad \ell(x(z)) - \ell(y(z)) = -\phi(1-z)[(x(z)-y(z))z^{-1}]_+ .$

The linear operator (1-z) which maps $\ell_2(-\infty, \infty)$ into itself has operator norm 2. That is,

$$||(1-z)|| \equiv \sup_{x\neq 0} \frac{||(1-z)x(z)||}{||x(z)||}$$

$$= \sup_{x\neq 0} \frac{\left(\sum_{j=-\infty}^{\infty} (x_j - x_{j-1})^2\right)^{1/2}}{\left(\sum_{j=-\infty}^{\infty} x_j^2\right)^{1/2}}$$

$$= \sup_{x\neq 0} \frac{\left(2\sum_{-\infty}^{\infty} x_j^2 - 2\sum_{-\infty}^{\infty} x_j x_{j-1}\right)^{1/2}}{\left(\sum_{j=-\infty}^{\infty} x_j^2\right)^{1/2}}$$

$$= 2 ,$$

where the last step follows from the following special case of the Schwartz inequality,

$$\left|\sum_{j=-\infty}^{\infty} x_j x_{j-1}\right| \leq \sum_{j=-\infty}^{\infty} x_j^2 .$$

Now since

$$||[(x(z)-y(z))z^{-1}]_+|| \leq ||x(z) - y(z)|| ,$$

we have in (A2) that

(A3) $\qquad ||\ell(x(z)) - \ell(y(z))|| = \mp\phi||(1-z)[(x(z)-y(z))z^{-1}]_+||$

$$\leq -2\phi||x(z) - y(z)|| .$$

Inequality (A3) establishes that if $-2\phi < 1$, $\ell(x(z))$ is a contraction mapping on a complete metric space. Therefore, the contraction mapping theorem guarantees that the functional equation (7) has a unique solution. The preceding proves the existence of a unique solution to (7) under the condition that $-\phi < 1/2$. This is a sufficient, though not a necessary, condition for a unique equilibrium to exist.

A much weaker condition on ϕ can be obtained if it can be assumed that m and p are covariance stationary (rather than only assuming μ and x covariance stationary, as in the preceding). For then we can use equation (2) to get

$$S_{pm}(z) = \alpha c(z)c(z^{-1}) + \gamma\left[\frac{S_{pm}(z)z^{-1}}{c(z^{-1})}\right]_+ c(z^{-1})$$

or

$$\text{(A4)} \qquad \frac{S_{pm}(z)}{c(z^{-1})} = \alpha c(z) + \gamma\left[\frac{S_{pm}(z)z^{-1}}{c(z^{-1})}\right]_+$$

where $S_m(z) = c(z)c(z^{-1})$, $c(z) = \sum_{j=0}^{\infty} c_j z^j$. Define the linear operator $h(x(z))$

$$\text{(A5)} \qquad h(x(z)) = \alpha c(z) + \gamma[x(z)z^{-1}]_+$$

on the same space F defined above. We have

$$||h(x(z)) - h(y(z))|| = ||\gamma[(x(z)-y(z))z^{-1}]_+||$$

$$\leq \gamma||x(z)-y(z)|| \ ,$$

so that (A5) is a contraction so long as $0 < \gamma < 1$. But $\gamma \equiv \frac{-\phi}{1-\phi} < 1$ so long as $\phi < 0$, i.e., so long as the demand schedule for money is sloped downward. Therefore, by the contraction mapping theorem, (A4) possesses a unique solution $S_{pm}(z)/c(z^{-1})$ so long as $\phi < 0$.

Footnotes

1/ See the appendix of Sims [5] for a useful characterization of strict econometric exogeneity by way of its relationship to the (negation of) Granger causality.

2/ This is a consequence of the material in Sims's appendix [5].

3/ A good treatment of the aspects of stationary stochastic processes exploited here is Whittle [6].

4/ See Whittle [6, p. 66-67, 41-43].

5/ To illustrate Lucas's time domain method, consider our "level" model, suppressing the disturbance, to get

(a) $$p_t = \alpha E_t p_{t+1} + \beta m_t \ .$$

The money process is assumed to obey the Markov law

(b) $$E_t m_{t+1} = \sum_{i=1}^{n} w_i m_{t-i+1} \ .$$

Lucas's method is to guess a solution of the form

(c) $$p_t = \pi_0 m_t + \pi_1 m_{t-1} + \pi_2 m_{t-2} + \cdots \ .$$

Substituting (b) and (c) into (a) then gives

$$p_t = \alpha(\pi_0 \sum_{i=1}^{n} w_i m_{t-i+1} + \pi_1 m_t + \ldots) + \beta m_t$$

or

(d) $$p_t = (\alpha\pi_0 w_1 + \pi_1 + \beta) m_t + (\alpha\pi_0 w_2 + \pi_2) m_{t-1} + (\alpha\pi_0 w_3 + \pi_3) m_{t-2} + \cdots \ .$$

Equations (c) and (d) involve identical variables and must therefore have coefficients that match. This condition gives us

$$\begin{aligned}
\pi_0 &= \alpha\pi_0 w_1 + \pi_1 + \beta \\
\pi_1 &= (\alpha\pi_0 w_2 + \pi_2) \\
\pi_2 &= (\alpha\pi_0 w_3 + \pi_3) \\
&\ \vdots \\
\pi_{n-1} &= \alpha\pi_0 w_n
\end{aligned}$$

which is a set of n nonlinear equations that determine the π's as functions of α, β, and the w_i's.

6/ Gordon [2, p. 647] has interpreted rational expectations models as implying that inflation on money creation lag distributions are concentrated at zero lag.

7/ Since $\mu_{t+2} = \varepsilon_{t+12} + \lambda\varepsilon_t + \lambda^2\varepsilon_{t-12} + \dots$, we have that $E[\mu_{t+12}|\varepsilon_t, \varepsilon_{t-1}, \dots] = \lambda\varepsilon_t + \lambda^2\varepsilon_{t-12} + \dots$. As before, we use $E[x|y]$ to denote the linear least squares projection of x on y.

References

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[4] Sargent, Thomas, and Neil Wallace, "The Stability of Models of Money and Growth Under Perfect Foresight," Econometrica, 1973.

[5] Sims, C. A., "Money, Income, and Causality," American Economic Review, 62, 540-552, 1972.

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