Federal Reserve Bank of Minneapolis
Research Department

Consumer Search and Firm Growth*

Erzo G.J. Luttmer

Working Paper 645

October 2006

ABSTRACT

This paper presents a simple model of search and matching between consumers and firms. The firm size distribution has a Pareto-like right tail if the population of consumers grows at a positive rate and the mean rate at which incumbent firms gain customers is also positive. This happens in equilibrium when entry is sufficiently costly. As entry costs grow without bound, the size distribution approaches Zipf's law. The slow rate at which the right tail of the size distribution decays and the 10% annual gross entry rate of new firms observed in the data suggest that more than a third of all consumers must switch from one firm to another during a given year. A substantially lower consumer switching rate can be inferred only if part of the observed firm entry rate is attributed to factors outside the model. The realized growth rates of large firms in the model are too smooth.

*Luttmer, University of Minnesota and Federal Reserve Bank of Minneapolis. This is a report on ongoing research. Comments welcome.

1. Introduction

The dispersion in firm size is large. Among employer firms in the U.S. economy, employment ranges from 1 to about 1.8 million employees. To a large extent, this is simply because some firms have many more customers than other firms. One alternative interpretation would be that firms produce differentiated products and that some products account for a much larger share of every consumer's budget than do other products. But the annual expenditures of a typical consumer would then consist of pennies on most products and tens of thousands of dollars on a few products supplied by very large firms. This paper presents a very simple model of the firm size distribution that avoids this troublesome implication.

In the model, firms are identified with specific goods and consumers are matched with firms whose goods match their taste. New firms gain their first customers through a process that can be interpreted as costly advertising. Matches between customers and existing firms are broken off randomly and new matches with existing firms are formed by word-of-mouth advertising. Searching consumers randomly sample customers of incumbent firms and then switch to the firms with which those customers are trading. Although all possible contacts between consumers are equally likely, a randomly sampled consumer is more likely to be a customer of a large firm than of a small firm. As a result, a firm gains customers at an average rate that is proportional to the size of its existing customer base.

Trade results in a surplus for firms that is taken as an exogenous parameter on a per-customer basis. The value of a firm is a constant multiple of the size of its customer base, because mean firm growth rates are independent of firm size. This multiple is an increasing function of the mean growth rate of a firm. In equilibrium, this mean growth rate adjusts so that the per-customer value of a firm equals the cost of finding a first customer. Specifically, high entry costs must go together with high mean firm growth rates in equilibrium.

The population of consumers is assumed to grow over time at a non-negative rate, taken to be exogenous. Incumbent firms can gain customers because of this, but they also lose customers to new entrants. If entry costs are high, then entry rates must be low, or else incumbent firms cannot grow at a high enough rate.

In the absence of population growth, the stochastic process of firm size that arises is exactly the one obtained by Klette and Kortum [2004]. The resulting mean growth rate of incumbent firms is negative, and the implied stationary size distribution is the logarithmic series distribution. This distribution has tail probabilities that decline geometrically with firm size. In contrast, the number of firms of size at least n behaves more like $1/n$ in U.S. data (Simon and Bonini [1958], Axtell [2001]). This type of stationary distribution —Zipf's law— arises here when there is population growth and the mean growth rate of incumbent firms is positive and close to the population growth rate. For this the cost of entry must be sufficiently high.

The mechanism generating Zipf's law described here is essentially that proposed by Simon [1955].[1] A version of Simon [1955] adapted to the consumer search economy of this paper would have consumer-firm matches last forever. This would imply a firm entry rate equal to the population growth rate. The variance of the growth rate of the smallest firm would be less than the population growth rate. Both predictions are far from the data. Allowing for the break-up of consumer-firm matches is essential for an interpretation of firm entry and size data.

In a labor market context, a model analogous to the one described here would have workers find jobs through referrals, contacting other workers to find jobs. In Burdett and Vishwanath [1988] the probability of workers sampling a firm is proportional to firm size, and this can be interpreted in this way.[2] A far more commonly used search mechanism is to assume that workers sample firms at random. This mechanism leads to a size distribution that is Poisson if there is no population growth. The right tail of the Poisson distribution declines even faster than the logarithmic series distribution, and the firm size predictions of such a search mechanism would therefore be in sharp contrast to what is found in the data. If there is population growth and matches are not destroyed, the size distribution is geometric, and this is again contradicted by the data.

2. Consumers and Firms

Time is continuous. There is a continuum $N(t)$ of infinitely-lived agents. These agents all act as consumers, and they can choose to be entrepreneurs at the same time. The number of agents grows according to

$$\mathrm{D}N(t) = \gamma N(t), \tag{1}$$

[1] More recent research on network formation is also in this vein. For examples, see Jackson and Rogers [2006] and the review of Jackson [2006].

[2] For a recent example of consumer search driven by word-of-mouth advertising, see Rob and Fishman [2005]. Ellison and Fudenberg [1995] study the role of word-of-mouth communication in learning by a population of agents.

where γ is a non-negative parameter. Each agent is endowed with a flow of $\omega \in \mathbb{N}$ indivisible and non-storable outside goods that provide no utility. There is a continuum of firms owned by the agents who are also entrepreneurs. Each of these firms has the ability to convert one unit of the outside good into one unit of another indivisible good that does provide utility to the consumer, for each of the consumers with whom the firm trades. Every firm can convert only one unit per customer, and so consumers must trade with ω different firms. When trading with n consumers, the owner of a firm receives πn units of utility per unit of time. The per-customer surplus π is positive and treated as an exogenous parameter. All agents discount utility flows at a positive rate ρ.

2.1 Consumer Search

A newly born consumer samples another consumer in the economy and learns about a firm with which this other consumer trades. The newly born consumer develops a taste for the good supplied by this firm and decides to trade with the same firm. This happens for each of the ω goods the consumer has available to trade, independently across goods. At any point in time, every consumer is a customer of ω firms, and so there are $\omega N(t)$ customers.

At a rate σ, a consumer loses his or her taste for the good supplied by one of the firms with which the consumer trades. When this happens, the consumer samples another consumer at random from the economy-wide population of consumers and learns about the firm with which this other consumer trades. Again, the consumer develops a taste for the good supplied by this firm and starts trading with the firm. At a rate $\nu\omega$, the consumer can also be contacted by a new firm and develop a taste for the good it supplies. As a result, the consumer loses his or her taste for one of the ω firms, randomly selected, with which the consumer trades.

The firm switching rate σ is taken to be exogenous, but the new-firm contact rate ν will be determined by equilibrium conditions. Its equilibrium value will be a key parameter characterizing the firm size distribution.

2.2 Entrepreneurs

2.2.1 Entry

Agents can choose to be entrepreneurs and attempt to set up new firms. An entrepreneur can set up a firm by finding a consumer who is willing to trade. The arrival of a first customer is random and comes at a cost. As long as the entrepreneur pays a flow utility cost κ, a first customer will arrive after an exponentially distributed waiting time with unit mean. Once the entrepreneur has found the first customer, the number of customers of the firm changes as consumers switch, randomly, to and from the firm.

At a potentially very small rate $\epsilon\omega N(t) > 0$, agents in the economy succeed in setting up a firm and attracting a first customer at no cost. By a stroke of luck, some agents end up with a firm that produces a positive per-customer surplus π, without even trying. This assumption ensures that some entry will always take place, even if the surplus flow π is not large enough to justify the strictly positive flow expense κ required to set up a firm. This assumption is enough to guarantee the existence of a stationary size distribution.

Recall that $\nu\omega$ is the rate at which individual consumers are contacted by new firms. The aggregate rate at which new firms are set up is therefore $\nu\omega N(t)$. Since entrepreneurs can set up new firms at a unit rate, this means that the fraction of agents who are also entrepreneurs is $\nu\omega$. If $\nu\omega > 1$ then some agents must be setting up more than one firm per unit of time.

2.2.2 Firm Growth

Given that consumers trade with ω firms at the same time, an existing firm can lose any particular customer due to consumer search or new-firm advertising at a rate

$$\lambda = \sigma + \nu. \tag{2}$$

A firm with n customers therefore loses customers at a rate λn. When the last customer switches to another firm, the firm is assumed to exit—finding a new first customer is now again a matter of costly advertising. A firm with $n \in \mathbb{N}$ customers gains customers because its existing customers are contacted by searching consumers who switch. Since each consumer trades with ω firms, the aggregate flow of such contacts coming from the population of consumers at time t is $\sigma\omega N(t)$. Since the population grows at a rate γ and new consumers establish ω contacts, there is also a flow $\gamma\omega N(t)$ of contacts established by new consumers with older consumers. The rate at which any particular consumer is contacted is thus $\omega(\sigma + \gamma)$. Since this consumer trades with ω different firms, the rate at which one of the firms trading with this consumer gains a customer is

$$\mu = \sigma + \gamma. \tag{3}$$

The rate at which an existing firm with n customers gains a new customer is therefore μn.

The average growth rate of a firm in this environment is independent of its size $n \in \mathbb{N}$, and equal to $\mu - \lambda = \gamma - \nu$. Firms must lose customers on average if the common rate ν at which consumers are contacted by new firms for each of the ω goods exceeds the growth rate of the population.

2.2.3 The Value of a Firm

Let $V(n)$ be the discounted utility gain, relative to being a regular consumer, of an entrepreneur who owns a firm with n customers. Thus $V(0) = 0$. For any $n \in \mathbb{N}$, the value function V must satisfy the Bellman equation

$$\rho V(n) = \pi n + \mu n \left[V(n+1) - V(n)\right] + \lambda n \left[V(n-1) - V(n)\right], \tag{4}$$

The solution is simply $V(n) = sn$, where

$$s = \frac{\pi}{\rho - (\mu - \lambda)}. \tag{5}$$

The value of a firm is well defined and finite as long as $\rho > \mu - \lambda$. Because the value function is linear in the number of firm customers, s is both the value of the first customer and of each additional customer.

2.2.4 Entry Decisions

The value of being an entrepreneur —paying a flow cost κ to receive a present-value gain s at a unit rate— is $(s - \kappa)/\rho$. There is no bound on the number of new-firm projects entrepreneurs can undertake. The value of being an entrepreneur can therefore not be positive in equilibrium. Hence

$$\kappa \geq s, \text{ with equality if } \nu > \epsilon. \tag{6}$$

Everyone can become an entrepreneur, and so the value of being an entrepreneur must be zero if some agents choose to be entrepreneurs in equilibrium.

2.3 Equilibrium Entry and Mean Firm Growth Rates

In a steady state, the equilibrium entry rate is determined by (2), (3), (5) and (6). Note from (2), (3), (5) that $s = \pi/(\rho - \gamma + \nu)$.

Suppose first that $\rho + \epsilon > \gamma$. If $\kappa > \pi/(\rho - \gamma + \epsilon)$, then it is not profitable to become an entrepreneur and the only form of entry is random entry. Hence $\nu = \epsilon$. Alternatively, if $\kappa < \pi/(\rho - \gamma + \epsilon)$, then there would be positive profits associated with becoming an entrepreneur if the only form of entry was random entry. In equilibrium, sufficiently many agents will have to choose to become entrepreneurs, raising ν above ϵ, in order to drive these profits down to zero. The equilibrium entry rate is therefore

$$\nu = \max\left\{\epsilon, \gamma + \frac{\pi}{\kappa} - \rho\right\}. \tag{7}$$

Next, suppose that $\rho + \epsilon < \gamma$. Then the gains from being an entrepreneur would be unbounded if nobody chose to be an entrepreneur. In a steady state equilibrium, enough agents will choose to be entrepreneurs to drive these profits down to zero. The equilibrium entry rate is again determined by (7). In turn, ν determines the rate λ at which firms lose customers via (2), and from this the mean firm growth rate $\mu - \lambda$ follows.

2.4 Competition

In this economy, consumers receive the same fixed surplus at all times. The surplus that accrues to firms is also fixed on a per-customer basis. Entry of more firms is just a form of rent seeking and does not reduce this surplus. A higher entry rate reduces the scope for the customer base of incumbent firms to grow. The result is a lower firm growth rate, and hence a lower firm value. In equilibrium, the value of an entering firm should equal to the cost of entry. If entry costs are low, then the value of a firm must be low. Given the fixed per-customer surplus, this can only happen if firms grow slowly. This in turn requires entry at a high rate, or a lot of rent seeking.

In a richer model, both the surplus π and the contact rates μ and λ would depend on the price posted by the firm, relative to prices posted by other firms. The equilibrium surplus would then depend on how sensitive these contact rates are to price. In a dynamic version of Burdett and Vishwanath [1988] adapted to the consumer search problem studied here, there is an equilibrium in which all firms charge the same price and extract all the surplus from consumers. In such a model, competition is limited by the fact that firms can only affect the rate λ at which they lose customers, and then only if price is below the reservation utility of the consumer.

3. Market Share Dynamics[3]

The measure of firms with n customers at time t is denoted by $M_n(t)$. Since each consumer is a customer of exactly ω firms, it must be that $\{M_n(t)\}_{n=1}^{\infty}$ satisfies

$$\omega N(t) = \sum_{n=1}^{\infty} n M_n(t). \tag{8}$$

If this holds at the initial date, then the mechanism by which customers are matched to firms will ensure that this continues to hold over time. The change in the number of firms with one customer is

$$\mathrm{D}M_1(t) = \lambda 2 M_2(t) + \nu\omega N(t) - (\lambda + \mu) M_1(t). \tag{9}$$

The number of firms with one customer increases because firms with two customers lose one, or because of entry. The number declines because firms with one customer gain or lose a customer. Similarly, the number of firms with $n \in \mathbb{N} \setminus \{1\}$ customers evolves according to

$$\mathrm{D}M_n(t) = \lambda(n+1)M_{n+1}(t) + \mu(n-1)M_{n-1}(t) - (\lambda + \mu) n M_n(t). \tag{10}$$

The joint dynamics of $N(t)$ and $\{M_n(t)\}_{n=1}^{\infty}$ is fully described by (1) and (9)-(10). The initial conditions for $N(t)$ and $\{M_n(t)\}_{n=1}^{\infty}$ are assumed to satisfy (8).

3.1 The Stationary Distribution

A stationary distribution exists if (9)-(10) has a solution for which $M_n(t)/N(t)$ is constant. Since $N(t)$ grows at a rate γ, this means that $\mathrm{D}M_n(t) = \gamma M_n(t)$ for all $n \in \mathbb{N}$. Given that $N(t)$ and $M_n(t)$ grow at the common rate γ, one can define

$$P_n = \frac{M_n(t)}{\sum_{n=1}^{\infty} M_n(t)}$$

for all $n \in \mathbb{N}$. This is the fraction of firms that serve n customers. Analytically more convenient is the fraction of customers who trade with firms of size n, defined by

$$Q_n = \frac{n M_n(t)}{\sum_{n=1}^{\infty} n M_n(t)}$$

[3] The definitions (2)-(3) imply that $\mu \geq \gamma$ and $\lambda \geq \nu$. These restrictions play no role in the following. They would not necessarily hold if searching consumers can be matched with new firms.

for all $n \in \mathbb{N}$. The mean number of customers per firm can be written in terms of the two stationary distributions $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ as

$$\frac{\sum_{n=1}^{\infty} nM_n(t)}{\sum_{n=1}^{\infty} M_n(t)} = \sum_{n=1}^{\infty} nP_n = \left(\sum_{n=1}^{\infty} \frac{1}{n} Q_n \right)^{-1}.$$

The numerator of the left-hand side adds up to the total measure of customers in the economy. This is finite by construction. Hence the mean firm size is well defined and finite by construction. The right-hand side is the reciprocal of the mean number of firms per customer, provided that this mean is calculated against the distribution of customers by size of firm with which they trade.

Recall that $\mu - \lambda = \gamma - \nu$. Using this, (9) can now be written as

$$\gamma Q_1 = \lambda Q_2 + \gamma - (\mu - \lambda) - (\lambda + \mu) Q_1, \tag{11}$$

and (10) implies that

$$\frac{1}{n} \gamma Q_n = \lambda Q_{n+1} + \mu Q_{n-1} - (\lambda + \mu) Q_n, \tag{12}$$

for $n \in \mathbb{N} \backslash \{1\}$. Condition (8) corresponds to the requirement that the fractions Q_n add up to one,

$$\sum_{n=1}^{\infty} Q_n = 1. \tag{13}$$

Any sequence $\{Q_n\}_{n=1}^{\infty} \subset [0, 1]$ that satisfies (11)-(13) defines a stationary size distribution $\{P_n\}_{n=1}^{\infty}$ via $Q_n \propto P_n/n$. Note that (11)-(13) only depend on the parameters μ/λ and γ/λ.

The difference equation (12) is a second-order equation in $\{Q_n\}_{n=1}^{\infty}$. It comes with two boundary conditions, (11) and (13). To solve (11)-(13), it is convenient to reduce (12) to a first-order equation in the variables

$$Z_{n+1} = \frac{1}{\beta_{n+1}} \left[Q_n - \beta_{n+1} Q_{n+1} \right], \tag{14}$$

for all $n \in \mathbb{N}$ and some sequence $\{\beta_n\}_{n=1}^{\infty}$. Set $\beta_2 = 1/(1 + (\gamma + \mu)/\lambda)$. Then the initial condition (11) translates into

$$Z_2 = \frac{1}{\lambda} \left[\gamma - (\mu - \lambda) \right] \tag{15}$$

Also, (12) can be written as

$$Z_{n+1} = \left(\frac{\mu \beta_n}{\lambda} \right) Z_n \tag{16}$$

for all $n \in \mathbb{N} \setminus \{1\}$ if

$$\beta_{n+1} = \left(1 + \frac{\gamma + \mu n}{\lambda n} - \frac{\mu \beta_n}{\lambda}\right)^{-1}. \tag{17}$$

From the definition of β_2, observe that the recursion (17) holds for all $n \in \mathbb{N}$ if initialized by $\beta_1 = 0$. The recursion (17) is depicted in Figure I for the case $\mu > \lambda$. Note in particular that the curve defined by (17) shifts upwards as n increases. Using this observation and the diagram, as well as an analogous diagram for $\lambda > \mu$, one can verify that $\{\beta_n\}_{n=1}^{\infty}$ converges monotonically from $\beta_1 = 0$ to $\min\{1, \lambda/\mu\}$.

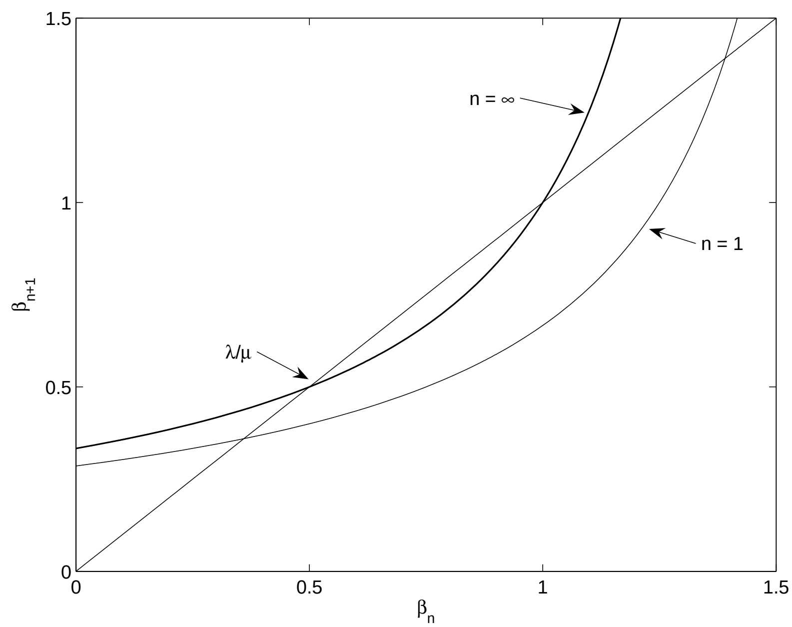

Figure I. The Dynamics of $\{\beta_n\}_{n=1}^{\infty}$.

The sequence $\{Z_n\}_{n=2}^{\infty}$ is completely determined by (15)-(16). Observe from (14) that $Q_n = \beta_{n+1}(Q_{n+1} + Z_{n+1})$. The boundary condition (13) together with the fact that $\beta_n \leq 1$ implies that $Q_N \prod_{n=1}^{N} \beta_n$ must converges to zero as N becomes large. Thus one can iterate forward to obtain the solution for $\{Q_n\}_{n=1}^{\infty}$. The following proposition presents this solution and provides upper and lower bounds Q_n when n is large.

Proposition 1 *Suppose that μ, λ, γ and $\nu = \gamma - (\mu - \lambda)$ are positive. Define the sequence $\{\beta_n\}_{n=1}^{\infty}$ by the recursion* (17) *and the initial condition $\beta_1 = 0$. This sequence is monotone and converges to* $\min\{1, \lambda/\mu\}$. *The solution to* (11)-(13) *is given by*

$$Q_n = \frac{\nu}{\lambda} \sum_{k=n+1}^{\infty} \left(\prod_{m=n+1}^{k} \beta_m\right) \left(\frac{\mu \beta_k}{\lambda}\right)^{-1} \prod_{m=2}^{k} \frac{\mu \beta_m}{\lambda}. \tag{18}$$

Take any $\varepsilon > 0$*. If* $\mu > \lambda$ *then*

$$\frac{\nu}{(1+\varepsilon)\mu - \frac{\lambda}{1+\varepsilon}} \le \left(\prod_{m=2}^{n} \frac{\mu\beta_m}{\lambda}\right)^{-1} Q_n \le \frac{\nu}{\mu - \lambda} \tag{19}$$

for all large enough n*. If* $\mu < \lambda$ *then*

$$\frac{\nu}{(1+\varepsilon)\lambda - \frac{\mu}{1+\varepsilon}} \le \left(\prod_{m=2}^{n} \frac{\mu\beta_m}{\lambda}\right)^{-1} Q_n \le \frac{\nu}{\lambda - \mu} \tag{20}$$

for all large enough n*.*

The proof in Appendix A shows that the solution (18) satisfies (13). The distribution $\{P_n\}_{n=1}^{\infty}$ follows immediately from $P_n \propto Q_n/n$. The assumption in Proposition 1 that the entry rate ν is positive is implied by the equilibrium condition (7) and the assumption that $\epsilon > 0$. The consumer switching rate σ is non-negative by definition, the population growth rate γ is assumed to be non-negative, and $\nu \ge \epsilon > 0$. Thus μ is non-negative and strictly positive if the consumer switching rate σ is positive or if the population growth rate is strictly positive. The rate λ at which firms lose customers is automatically positive.

3.2 The Right Tail

As shown in (19)-(20),

$$Q_n \sim \prod_{k=2}^{n} \frac{\mu\beta_k}{\lambda} \tag{21}$$

for all large enough n. When $\lambda > \mu$, the properties of this product are quite different from what they are when $\mu < \lambda$. If $\lambda > \mu$, then Q_n is bounded above by a multiple of the geometrically declining sequence $(\mu/\lambda)^n$. On the other hand, if $\mu > \lambda$ then $\mu\beta_n/\lambda \uparrow 1$, and hence the right-hand side of (21) declines at a rate that is slower than any given geometric rate. The proof of Proposition 1 shows that the right-hand side of (21) is nevertheless summable. The following proposition gives a further characterization of the right tail of the distribution.

PROPOSITION 2 *Suppose that* $\gamma > 0$*,* $\mu > \lambda$ *and* $\gamma > \mu - \lambda$*. Then the right tail probabilities of the stationary firm size distribution satisfy*

$$\limsup_{N\to\infty} N^z \sum_{n=N}^{\infty} P_n = 0$$

for any z *smaller than the tail index* $\zeta = \gamma/(\mu - \lambda)$.

The proof is in Appendix B. This proposition implies that

$$\ln\left(\sum_{n=N}^{\infty} P_n\right) \sim c - \zeta \ln(N) \tag{22}$$

for some constant c. The limiting tail index $\zeta = 1$ associated with Zipf's law arises when the entry rate $\nu = \gamma - (\mu - \lambda)$ converges to zero.

3.3 Entry and Exit Rates

The entry rate $\nu\omega N(t)$ represents the rate per unit of time at which consumers are contacted by new firms. Each new firm starts with one customer, and so $\nu\omega N(t)$ is also the number of new firms that enters per unit of time. The firm entry rate η as a fraction of the number of incumbent firms is therefore equal to $\nu\omega N(t)$ divided by the number of firms in the economy,

$$\sum_{n=1}^{\infty} M_n(t) = \omega N(t) \times \sum_{n=1}^{\infty} \frac{1}{n} Q_n.$$

This implies a firm entry rate equal to

$$\eta = \frac{\nu}{\sum_{n=1}^{\infty} \frac{1}{n} Q_n}. \tag{23}$$

That is, the firm entry rate is equal to customer contact rate ν times the average number of customers per firm. The only firms that exit in this economy are one-customer firms. There are $\omega N(t) Q_1$ such firms, and they each exit at a rate λ. The balance $[\nu - \lambda Q_1]\omega N(t)$ of firms entering and exiting per unit of time must equal γ times the total number of firms. On a per-customer basis, this gives

$$\underbrace{\nu}_{\text{gross entry per customer}} = \underbrace{\gamma \sum_{n=1}^{\infty} \frac{1}{n} Q_n}_{\text{net entry per customer}} + \underbrace{\lambda Q_1}_{\text{exit per customer}}. \tag{24}$$

This can be verified mechanically by adding up (11) and (12) over all n. In terms of $\{P_n\}_{n=1}^{\infty}$ one can summarize (23)-(24) more concisely as

$$\eta = \nu \sum_{n=1}^{\infty} n P_n = \gamma + \lambda P_1.$$

Eliminating the mean number of customers per firm from (23) and (24) yields $\eta/\gamma = 1/(1-(\lambda/\nu)Q_1)$ and then the mean firm size is $\eta/\nu = (\gamma/\nu)(1-(\lambda/\nu)Q_1)$. Together with the expression for Q_1 implied by (18) this gives explicit solutions for the firm entry rate and the mean firm size.

3.4 Special Cases

As will be shown in Section 4, the empirically relevant firm size process is one with large μ and λ, where $\mu > \lambda$ and where $\nu = \gamma - (\mu - \lambda)$ is positive but very close to zero. These conditions are violated in some well-known special cases of the firm size process (8)-(10). Taking the limit $\nu \downarrow 0$ gives a tractable special case that is very close to what is observed in U.S. data.

3.4.1 The Logarithmic Series Distribution

Suppose there is no growth in the aggregate number of consumers, so that $\gamma = 0$. Then $\mu < \lambda$ and hence the size distribution must have a geometrically declining right tail. The sequence $\{\beta_n\}_{n=1}^{\infty}$ is simply $\beta_1 = 0$ and $\beta_n = 1$ for all larger n. The transition (12) simplifies to $[Q_{n+1} - Q_n] = (\mu/\lambda)[Q_n - Q_{n-1}]$. Clearly, $Q_{n+1} - Q_n \propto (\mu/\lambda)^n$, and then $Q_n \propto (\mu/\lambda)^n$ as well. The resulting size distribution $\{P_n\}_{n=1}^{\infty}$ is R.A. Fischer's logarithmic series distribution

$$P_n = \frac{\frac{1}{n}(\mu/\lambda)^n}{\ln\left(\frac{1}{1-\mu/\lambda}\right)}.$$

This is the distribution that arises in Klette and Kortum [2004]. The mean of this distribution is easy to compute, and $\nu = \lambda - \mu$. The resulting firm entry rate (23) equals the exit rate λP_1. This can be written as

$$\frac{\eta}{\lambda} = \frac{\mu/\lambda}{\ln\left(\frac{1}{1-\mu/\lambda}\right)}.$$

This ratio ranges from 1 to 0 as μ/λ ranges from 0 to 1. To obtain a size distribution with a right tail that decays at a slow geometric rate one needs μ close to $\lambda > \mu$. This implies η/λ close to 0. High observed entry rates then imply high values of λ.

Randomly Sampling Firms Although the right tail of the size distribution declines geometrically when $\gamma = 0$, the distribution is still more skewed than would be the case if searching consumers sampled firms directly, instead of by randomly sampling customers.

Continue to suppose that new firms sample from the whole population of customers. The stationary size distribution $\{P_n\}_{n=1}^{\infty}$ would then have to satisfy the initial condition $0 = \lambda 2P_2 + \nu \sum_{n=1}^{\infty} nP_n - (\lambda+\mu)P_1$ and the recursion $0 = \lambda(n+1)P_{n+1} + \mu P_{n-1} - (\lambda n + \mu)\, P_n$, where $\lambda = \sigma + \nu$ and $\mu = \sigma \sum_{n=1}^{\infty} nP_n$. This implies $P_n \propto (\mu/\lambda)^n/n!$, which is the Poisson distribution restricted to $\mathbb{N}$. The factor $1/n!$ instead of $1/n$ makes the tail of the Poisson distribution decline much more rapidly than the tail of the logarithmic series distribution. In this environment, all firms gain customers at the same rate, equal to the customer switching rate times the average number of customers per firm. Larger firms no longer have an advantage over smaller firms in attracting customers, and this makes it much harder for large firms to arise the population.

3.4.2 The Yule Process

Although not consistent with (2)-(3), consider the case $\lambda = 0$ and $\mu \in (0, \gamma)$. In this scenario, firms can only grow. A stationary size distribution arises because not all firms have had the same time to grow, and the population of firms itself grows. The entry rate is $\nu = \gamma - \mu$, and the resulting stochastic process is known as the Yule process. It was used by Simon [1955] as a model for various skewed empirical distributions, including the city size distribution. The difference equations (11)-(12) simplify to $Q_1 = (\gamma - \mu)/(\gamma + \mu)$ and

$$Q_n = \left(\frac{n}{n + \frac{\gamma}{\mu}} \right) Q_{n-1},$$

for all $n \in \mathbb{N} \setminus \{1\}$. Working out this recursion and using $P_n \propto Q_n/n$ gives

$$P_n = \frac{\gamma}{\mu} \frac{\Gamma(n)\Gamma\left(1 + \frac{\gamma}{\mu}\right)}{\Gamma\left(n + 1 + \frac{\gamma}{\mu}\right)}$$

where Γ is the gamma function. The right tail probabilities are

$$\sum_{n=N}^{\infty} P_n = \frac{\Gamma(N)\,\Gamma\left(1 + \frac{\gamma}{\mu}\right)}{\Gamma\left(N + \frac{\gamma}{\mu}\right)}$$

Since $N^{1-\alpha}\Gamma(N+\alpha)/\Gamma(N+1)$ converges to 1 as N becomes large, these tail probabilities behave like $N^{-\gamma/\mu}$ for large N, as predicted by Proposition 2.

The size distribution has a mean $1/(1 - \mu/\gamma)$, and so the formula for the firm entry rate (23) reduces to $\eta = \gamma$, as expected. The limiting distribution as $\mu \uparrow \gamma$, or $\nu \downarrow 0$, is $P_n = 1/[n(n+1)]$. This is the discrete analog of the Pareto distribution that corresponds to Zipf's law. The right tail probabilities of this limiting distribution are exactly $1/n$, and so the distribution has no well defined mean.

One can verify that the stationary distribution would be $P_n = (\gamma/\mu)(1+\gamma/\mu)^{-n}$ if individual firms gain customers at a rate μ instead of μn. This corresponds to an environment in which a fraction $(\mu/\gamma)/\sum_{n=1}^{\infty} nP_n$ of the flow $\gamma\omega N(t)$ of new customers is matched with incumbent firms and the remaining fraction $1/\sum_{n=1}^{\infty} nP_n$ with new firms. By itself, population growth is not enough to generate the heavy right tail. It needs to be combined with geometric growth of the individual firms.

3.4.3 The Limiting Case $\nu \downarrow 0$

As before, assume that $\gamma > 0$ and $\lambda > 0$. Letting ν approach zero from above implies that $\gamma = \mu - \lambda$ in the limit. In this limit, the recursion (12) for $Q_n \propto nP_n$ becomes

$$(\mu-\lambda)P_n = \lambda(n+1)P_{n+1} + \mu(n-1)P_{n-1} - (\lambda+\mu)nP_n,$$

for all $n \in \mathbb{N}\setminus\{1\}$. This can also be written as

$$P_n = \frac{\lambda}{\mu}\left(P_{n+1} + X_{n+1}\right) \tag{25}$$

together with

$$X_{n+1} = \left(\frac{n-1}{n+1}\right) X_n \tag{26}$$

for all $n \in \mathbb{N}\setminus\{1\}$. Iterating on (26) gives $X_{n+1} = (2/[n(n+1)])X_2$. Since the P_n have to add up to 1, it must be that $P_n \to 0$. The fact that $\lambda/\mu < 1$ then implies that we can solve (25) forward. The result is

$$P_n = \frac{1}{\ln(\mu/\gamma)} \sum_{k=n}^{\infty} \frac{(\lambda/\mu)^{k+1-n}}{k(k+1)}.$$

where we have used $\gamma/\mu = 1 - \lambda/\mu$. The right tail probabilities are

$$\sum_{n=N}^{\infty} P_n = \frac{1}{\ln(\mu/\gamma)} \sum_{n=N}^{\infty} \frac{1}{n}\left(\frac{\lambda}{\mu}\right)^{n+1-N}, \tag{27}$$

and these satisfy

$$\lim_{N\to\infty} N \sum_{n=N}^{\infty} P_n = \lim_{N\to\infty} \frac{1}{\ln(\mu/\gamma)} \sum_{m=0}^{\infty} \frac{N}{N+m}\left(\frac{\lambda}{\mu}\right)^{m+1} = \frac{1}{\ln(\mu/\gamma)}\frac{1}{\mu/\lambda - 1}$$

by the dominated convergence theorem. Thus the right tail probabilities behave like $1/N$, and the log right tail probabilities expressed as a function of $\ln(N)$ must asymptote to a straight line with slope -1. The distribution does not have a finite mean.

Although the rate ν at which customers are contacted by new firms converges to zero, the number of firms per customer also goes to zero. But the entry rate of new firms satisfies $\eta = \gamma + \lambda P_1$, and this converges to a positive value. A calculation yields $\eta = \lambda / \ln(\mu/\gamma)$. Together with $\gamma = \mu - \lambda$ this gives

$$\frac{\eta}{\gamma} = \frac{\mu/\gamma - 1}{\ln(\mu/\gamma)}. \tag{28}$$

This allows one to infer μ and $\lambda = \mu - \gamma$ simply from the ratio of the firm entry rate η and the population growth rate γ.

4. U.S. Employer Firms

It is not difficult to modify the economy described so far by letting firms produce the goods they deliver to n consumers using labor inputs $l = \tau n$, so that τ is the labor required to trade with one customer. Agents supply labor in a competitive market and are paid with transferrable utility. In such an economy, one can measure firm size by employment.

U.S. Internal Revenue Service statistics contain more than 26 million corporations, partnerships and non-farm proprietorships. Business statistics collected by the U.S. Census consist of both non-employer firms and employer firms. In 2002 there were more than 17 million non-employer firms, many with very small receipts, and close to 6 million employer firms. In the following, Census data on employer firms assembled by the U.S. Small Business Administration (SBA) will be considered. Economic activity involving non-employer firms is interpreted as arising in an independent parallel economy. For employer firms, part-time employees are included in employee counts, as are executives. But proprietors and partners of unincorporated business are not (Armington [1998, p.9]). This is likely to create significant biases in measured employment for small firms.

Figure IV below shows the 2002 number of firms in the right tail of the size distribution of U.S. employer firms. Employer firms reported to have 0 to 4 employees during the observation period in March 2002 are interpreted to have τ to 4 employees, where $\tau = 1/5$. The calibration of τ will be discussed below. The tail index for this data is $\zeta \approx 1.06$—note from (22) that ζ does not depend on the units in which firm size is measured. U.S. population growth is around 1% per annum. These two numbers imply that incumbent firms gain customers at an average net annual rate of $\mu - \lambda = \gamma/\zeta \approx .94\%$. Since ζ is so close to 1, this is only slightly below the population growth rate. Observe that $\nu = \gamma - (\mu - \lambda) = \gamma(1 - 1/\zeta) \approx .06\%$. The rate ν at which consumers are contacted by new firms, for trade in each of the ω goods, is only about .06% per annum.

4.1 A First Estimate of μ and λ

As an approximation, consider the limiting case of $\nu = 0$, and thus $\zeta = 1$. The firm entry rate then satisfies (28). For 2002, the U.S. Small Business Administration reports that new employer firms were set up at a rate of a little over 10% per annum. Figure II shows the functions $\ln(\mu/\gamma)$ and $(\mu/\gamma - 1)/(\eta/\gamma)$ for $\eta/\gamma = 10$. The two curves intersect at $\mu/\gamma \approx 37$, and $\gamma = .01$ then gives $\mu = .37$ and $\lambda = .36$.The implied switching rate is $\sigma = \lambda = .36$. Thus the 1% per annum average growth rate of firms masks large gross flows of customers switching firms. Note however that the estimate of μ/γ implied by (28) and Figure II is quite sensitive to the value of η/γ.

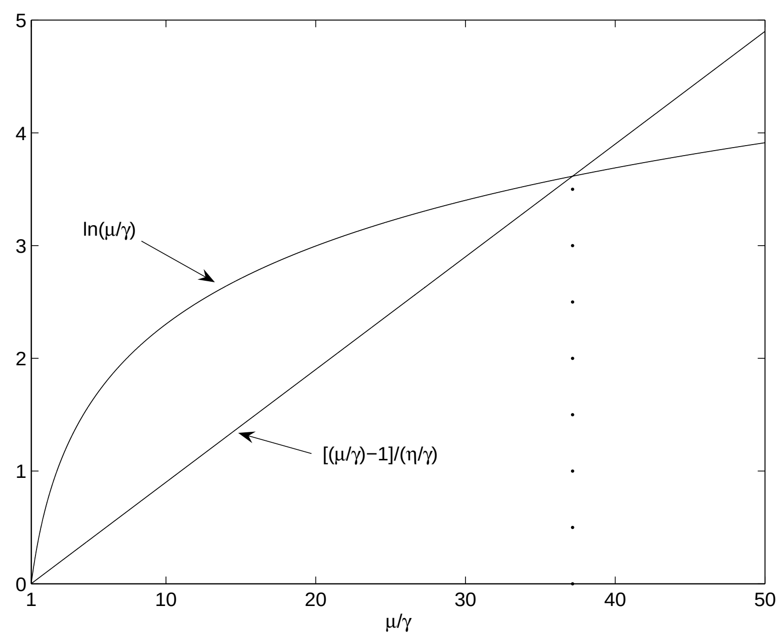

Figure II. Inferring μ/γ When $\zeta = 1$.

4.2 A Second Estimate of μ and λ

At the estimate $\zeta = 1.06$, the implied ν must be positive and the formula (28) for η/γ can only be an approximation. The entry rate of new firms and the right tail index of the stationary distribution can still be used to infer μ and λ.

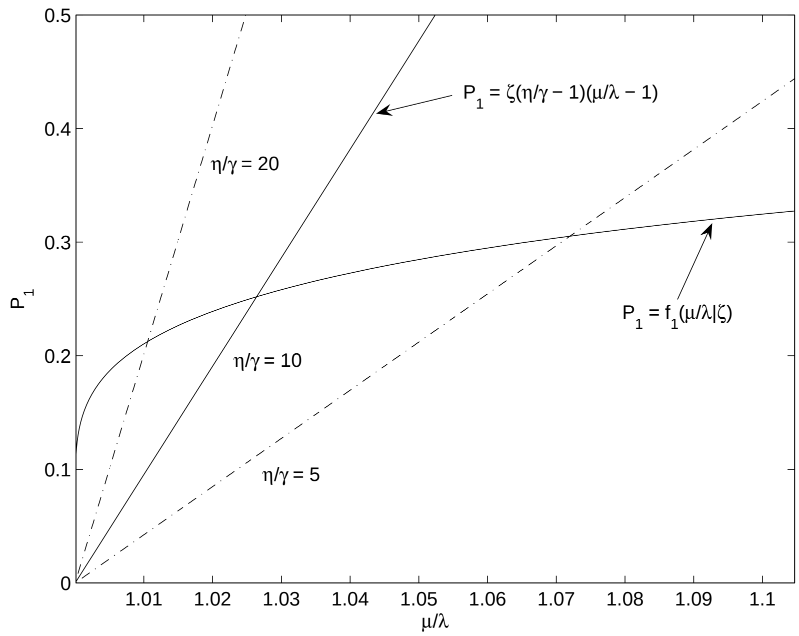

Figure III. Inferring μ/λ When $\zeta = 1.06$.

The stationary probabilities P_n implied by Proposition 1 are functions only of μ/λ and γ/λ. In turn, μ/λ and γ/λ are related via $\gamma/\lambda = \zeta(\mu/\lambda - 1)$, from the definition of the tail index ζ. Hence $P_n = f_n(\mu/\lambda|\zeta)$ for known functions f_n. Once the tail index is inferred from the observed tail probabilities for large firms, only one parameter, μ/λ, completely describes the rest of the size distribution. In principle, any partial sum of P_n for n not too large can be used to infer μ/λ. The quasi-maximum likelihood estimator would make use of all observed sums of P_n. Given that firm size is actually measured by employment τn rather than by the number of customers n, one would have to estimate the scale parameter τ along with μ/λ.

A convenient alternative way to estimate μ/λ that does not rely on the scale parameter τ is to go beyond using only the stationary size distribution and exploit the fact that $\eta = \gamma + \lambda P_1$ is the entry rate of firms. Together with the tail index $\zeta = \gamma/(\mu - \lambda)$, this implies that

$$P_1 = \zeta \left(\frac{\eta}{\gamma} - 1 \right) \left(\frac{\mu}{\lambda} - 1 \right). \tag{29}$$

This adds to $P_1 = f_1(\mu/\lambda|\zeta)$ a second restriction on P_1 and μ/λ. The two curves are shown in Figure III for the estimated $\zeta = 1.06$ and $\eta/\gamma = 10$. The unique intersection occurs at $\mu/\lambda = 1.0264$. Together with the tail index $\zeta = \gamma/(\mu - \lambda) = 1.06$, this gives

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \frac{\gamma}{\zeta} \frac{1}{\mu/\lambda - 1} \begin{bmatrix} 1 \\ \mu/\lambda \end{bmatrix} \approx \begin{bmatrix} .3574 \\ .3668 \end{bmatrix}.$$

Observe that these estimates are very close to the approximate values obtained for $\zeta = 1$. Setting $\zeta = 1.10$ gives $[\lambda, \mu] \approx [.3537, .3628]$ and at $\zeta = 1.20$ one obtains $[\lambda, \mu] \approx [.3487, .3570]$. These estimates hardly budge as one varies ζ in a reasonable range. Neither the line (29) nor the function f_1 are very sensitive to ζ. To examine the dependence of λ and μ on estimates of the entry rate η, note that the function f_1 does not depend on η since it only relies on properties of the stationary distribution. The restriction (29) does depend on η and the estimates of λ and μ are extremely sensitive to the observed entry rate. Between $\eta = .05$ and $\eta = .20$ the estimates of λ and μ range from $[\lambda, \mu] = [.1310, .1405]$ to $[\lambda, \mu] = [.8900, .8994]$.

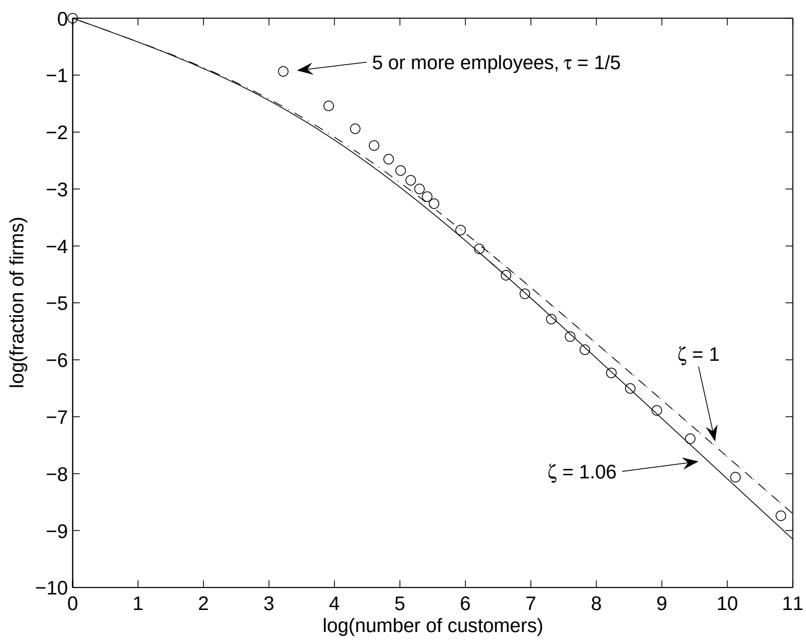

Figure IV. The Right Tail of the Size Distribution.

The parameters γ, μ and λ completely determine the stationary distribution of firm size, when firm size is measured by the number of customers. The estimated distributions are shown in Figure IV for the approximation $\zeta = 1$ and for the more precise estimate $\zeta = 1.06$. In the data, firm size is measured by employment. Comparing the fitted and empirical distributions requires an estimate of τ, the number of employees per customer. In Figure IV, this parameter is set at $\tau = 1/5$ to ensure that the number of relatively large firms matches the data. Alternative choices for τ give rise to horizontal shifts by $\ln(1/\tau) - \ln(5)$ of the curve representing the data in Figure IV.

4.3 An Entry Puzzle

Figure V shows the empirical and fitted cumulative distribution functions. Included is also the distribution function of $P_n = 1/[n(n+1)]$ that corresponds to Zipf's law. Very much apparent in Figures IV and V is the fact that the fitted distributions predict fewer firms in the 5 to 20 employee range than are observed in the data. Instead, the model predicts more very small firms. The observed number of firms with 5 or more employees can be matched by setting $\tau = 1/2$, but then the estimated stationary distributions predict too many large firms. A very close fit to the data displayed in Figures IV and V can be obtained by assuming that the entry rate is only $\eta = .025$ and setting $\tau = 1$. This implies $[\lambda, \mu] = [.03969, .04912]$, or much lower rates at which firms gain or lose customers. This fits the stationary distribution, but then leads to an entry puzzle: given the fairly low rates at which firms move up and down the size distribution, why is there so much entry? Jovanovic (1982) is a classic answer that is abstracted from in the model described here.

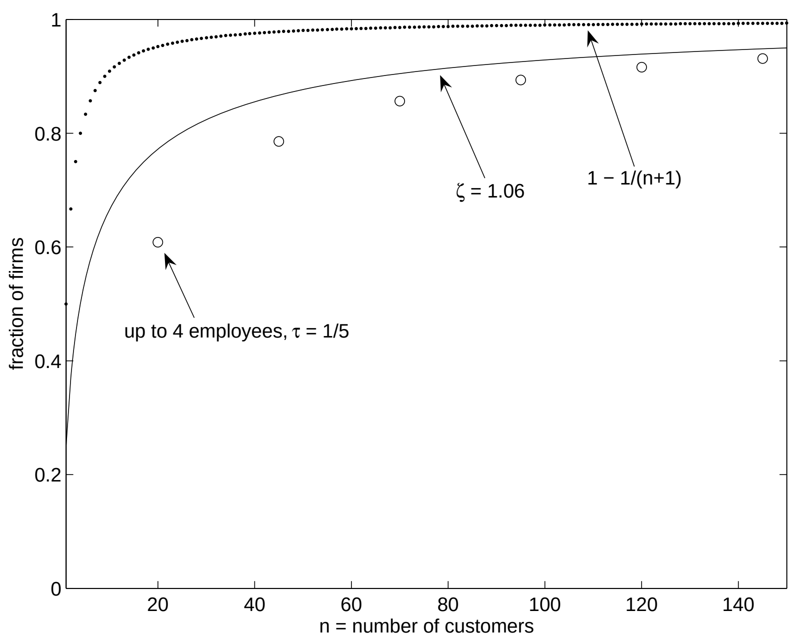

Figure V. The Left Tail of the Size Distribution.

4.4 Employee and Consumer Counts

By (23), the ratio η/ν must equal the average number of customers per firm,

$$\left[\sum_{n=1}^{\infty} \frac{1}{n} Q_n\right]^{-1} = \frac{\eta}{\nu} = \frac{\eta/\gamma}{1 - \frac{1}{\zeta}} \approx 176.67. \tag{30}$$

In 2002, the number of employees of U.S. firms with at least one employee was about 110 million. If this is taken to be an estimate of aggregate employment then the number of customers per firm estimated in (30) suggests another estimate of τ,

$$\tau = \frac{\text{firms}}{\text{customers}} \times \frac{\text{employees}}{\text{firms}} = \frac{110}{176.67 \times 6} = .10, \tag{31}$$

which is half the $\tau = 1/5$ that is shown in Figure IV. In other words, the fitted model in Figure IV implies that there should be 220 million employees instead of the 110 million observed in U.S. employer firms. To some extent this can be attributed to the fact that the average firm size computed in (30) is extremely sensitive to small changes in ζ. At $\zeta = 1.10$ the average number of customers per firm predicted by (30) is only 110 and then (31) implies $\tau = 1/6$. A related source of discrepancy between (31) and the $\tau = 1/5$ estimated in Figure IV is the fact that the empirical size distribution necessarily has a finite support. Because the estimated tail index ζ is so close to 1, truncating the estimated distribution can lead to large reductions in its mean.[4]

The parameter that remains to be identified is the number ω of "customers per consumer." The U.S. population measures close to 300 million individuals and there are somewhat fewer than 6 million employer firms. Ignoring the fact that many firms do not produce final goods, this makes for an average of around $50 \times \omega$ customers per employer firm. Using (30),

$$\omega = \frac{\text{customers}}{\text{firms}} \times \frac{\text{firms}}{\text{consumers}} = \frac{176.67 \times 6}{300} = 3.53 \tag{32}$$

One interpretation is that 53% of consumers trade with 4 firms and the remainder with 3 firms. Of course, ω will rise if one uses households instead of individuals to measure the number of consumers.

4.5 The Variance of Firm Growth Rates

Over short intervals of time, the variance of firm growth is $(\lambda + \mu)/n$ for a firm with n customers. The standard deviation is shown in Figure VI for the estimated parameters $\lambda = .3574$ and $\mu = .3668$, and n ranging from 1 to 500 customers. If $\tau = 1/5$, then this corresponds to firms with up to 100 employees.

Since $\lambda + \mu \approx .72$, the growth rates of the smallest firms are extremely volatile. But the standard deviation of firm growth is down to 3.8% per annum for firms with 100

[4] Gabaix [2005] and Atkeson and Burstein [2006] emphasize the sensitivity of predictions of models of this type for aggregates and employment levels in the right tail of the distribution.

employees, given $\tau = 1/5$. For firms with 1000 employees the number would be only 1.2%. In the model, the return on the value of a firm equals $w + \pi$ times the growth rate of a firm, and hence the standard deviation of the value of a firm should correspond to the standard deviation of the firm growth rate. The standard deviation of NYSE traded stock returns is about 30% per annum. No doubt, leverage and other factors make these returns more volatile than the standard deviation of firm growth. But a 1.2% standard deviation seems low.

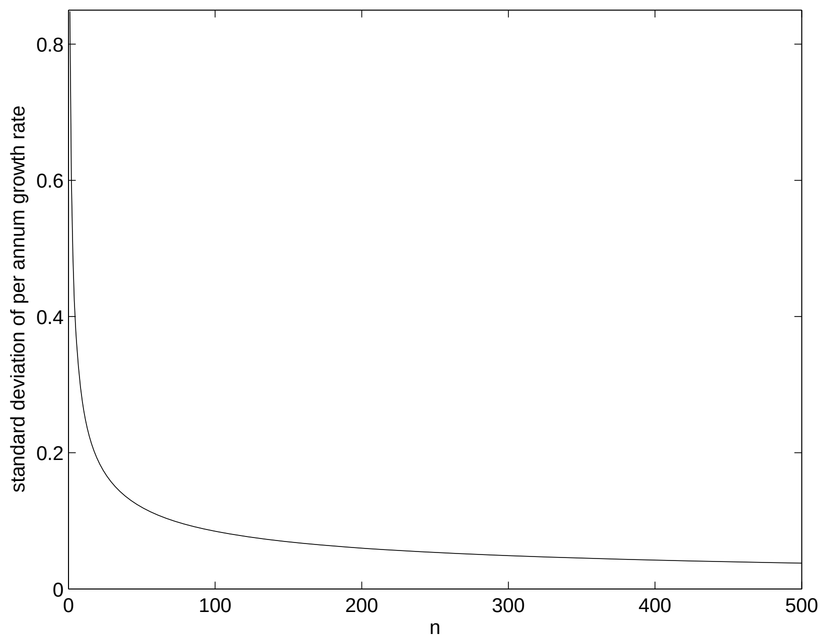

Figure VI. Standard Deviation of Firm Growth.

As emphasized by Klette and Kortum [2004], the empirical evidence suggests that the variance of firm growth rates declines more slowly than $1/n$. Hymer and Pashigian (1962) compared standard deviations of firm growth rates across size quartiles and found that firms in the largest quartile were significantly more volatile than predicted by the $1/n$ rule. More recently, Stanley et al [1996] and Sutton [2002] find that the variance of the growth rate of Compustat firms behaves like $1/n^{1/3}$. Tentative interpretations are given in Stanley et al [1996] and Sutton [2002, 2006].

5. Concluding Remarks

U.S. firm data exhibit (i) high entry rates, (ii) growth rate standard deviations that decline with firm size at a rate that is slower than one over the square root of firm size, and (iii) a size distribution with many small firms and a very long right tail. Luttmer [2006] and the current paper provide alternative interpretations of (iii). Both papers require large amounts of randomness to deal with (i). The two papers seem to be on opposite sides of the data when it comes to (ii). Finding tractable equilibrium models of this phenomenon is task for further research.

Skewed firm size distributions are interpreted as reflecting skewed productivity distributions in Hopenhayn [1992], Klette and Kortum [2004], Lentz and Mortensen [2006] and Luttmer [2006]. Furthermore, the continuous reallocation of resources across firms plays a crucial role in generating aggregate growth. In Lucas [1978] and Gabaix and Lanier [2006] an assignment problem relates the firm size and talent distributions. By contrast, in Simon and Bonini [1958] and Ijiri and Simon [1964] firm size does not affect aggregate outcomes because all firms operate under constant returns to scale. Similarly, firms in the search environment described here are all equally productive. The relative importance of each of these interpretations of the firm size distribution remains to be sorted out.

A Proof of Proposition 1

Write the candidate solution (18) as

$$\left(\prod_{m=2}^{n}\frac{\mu\beta_m}{\lambda}\right)^{-1} Q_n = \frac{\nu}{\lambda}\sum_{k=n+1}^{\infty}\left(\prod_{m=2}^{k}\beta_m\right)\left(\frac{\mu\beta_k}{\lambda}\right)^{-1}\prod_{m=2}^{k}\frac{\mu\beta_m}{\lambda}$$

Since $\beta_n \leq \min\{1, \lambda/\mu\}$ this implies the upper bounds in (19) and (20). Take some $\varepsilon > 0$. The lower bounds rely on $\beta_n \uparrow \min\{1, \lambda/\mu\}$. If $\mu > \lambda$, then eventually $\beta_n \geq (\lambda/\mu)(1+\varepsilon)$, and this gives the lower bound in (19). If $\mu < \lambda$, then $1/(1+\varepsilon) \leq \beta_n$ for all large enough n, and this implies (20). Thus the sums defining $\{Q_n\}_{n=1}^{\infty}$ converge and (19) and (20) hold. By construction, the candidate solution satisfies (11)-(12). It remains to prove the adding-up condition (13), which ensures that $\{Q_n\}_{n=1}^{\infty}$ is in fact a probability distribution.

Define $F_1 = 1$ and

$$F_n = n\prod_{k=2}^{n}\frac{\mu\beta_k}{\lambda},$$

for all $n \in \mathbb{N}\setminus\{1\}$. Note from the bounds (19)-(20) that the sequence $\{Q_n\}_{n=1}^{\infty}$ is summable if and only if $\{F_n/n\}_{n=1}^{\infty}$ is summable. Define

$$X_n = n\left(\frac{\mu\beta_n}{\lambda} - 1\right)$$

for all $n \in \mathbb{N}$. The recursion (17) is equivalent to

$$X_{n+1} = \left(1 + \frac{1}{n}\right)\left(\frac{X_n - \frac{\gamma}{\lambda}}{\frac{\gamma+n\mu}{n\lambda} - \frac{1}{n}X_n}\right) \tag{33}$$

Observe from this that $X_{n+1} < -1$ if and only if X_n satisfies

$$X_n < \frac{\gamma - (\mu - \lambda)}{\lambda} - 1 \tag{34}$$

Since $\gamma > \mu - \lambda$, this is true if $X_n \leq -1$. But this follows by induction starting from $X_1 = -1$. The fixed point for the $n = \infty$ version of (33) is $-\gamma/(\mu - \lambda) < -1$. One can verify that X_n converges to this fixed point starting from $X_1 = -1$. The fact that $\lim_{n\to\infty} X_n < -1$ implies that $\{F_n/n\}_{n=1}^{\infty}$ is summable, by Raabe's test. The inequality $X_{n+1} < 1$ is equivalent to $F_{n+1} < F_n$, and so $F_n \downarrow F_\infty$ for some $F_\infty \geq 0$. Therefore

$$\sum_{n=2}^{N} \frac{1}{n} F_n \geq F_\infty \sum_{n=1}^{N} \frac{1}{n}$$

for all N. Since the left-hand side is summable, it must be that $F_\infty = 0$.

Write (11) as $\gamma Q_1 = \lambda\left[Q_2 - Q_1\right] + \gamma - (\mu - \lambda) - \mu Q_1$ and (12) as

$$\gamma Q_n = \lambda n\left[Q_{n+1} - Q_n\right] - \mu n\left[Q_n - Q_{n-1}\right],$$

for $n \in \mathbb{N} \setminus \{1\}$. Adding up gives over all n gives

$$\gamma \sum_{n=1}^{\infty} Q_n = \gamma - (\mu - \lambda) + \lambda \sum_{n=1}^{\infty} n\left[Q_{n+1} - Q_n\right] - \mu\left(Q_1 + \sum_{n=2}^{\infty} n\left[Q_n - Q_{n-1}\right]\right). \tag{35}$$

Note that $n[Q_{n+1} - Q_n] = (n+1)Q_{n+1} - nQ_n - Q_{n+1}$ and $n[Q_n - Q_{n-1}] = nQ_n - (n-1)Q_{n-1} - Q_{n-1}$, and observe that the candidate solution (18) satisfies $\lim_{n\to\infty} nQ_n = 0$, since $F_\infty = 0$. Using summation-by-parts for the two sums on the right-hand side of (35) one obtains

$$\sum_{n=1}^{\infty} n\left[Q_{n+1} - Q_n\right] = Q_1 + \sum_{n=2}^{\infty} n\left[Q_n - Q_{n-1}\right] = -\sum_{n=1}^{\infty} Q_n. \tag{36}$$

Together with $\gamma > \mu - \lambda$, (35) and (36) imply that the sequence $\{Q_n\}_{n=1}^{\infty}$ adds up to 1.

B Proof of Proposition 2

Recall that $P_n \sim F_n/n^2$ and define $R_N = \sum_{n=N}^{\infty} F_n/n^2$. Observe

$$N^z R_N = \sum_{n=N}^{\infty} \left(\frac{N}{n}\right)^z n^{z-1} \prod_{k=2}^{n} \frac{\mu\beta_k}{\lambda} \leq \sum_{n=N}^{\infty} n^{z-1} \prod_{k=2}^{n} \frac{\mu\beta_k}{\lambda}.$$

If the sum on the right-hand side is finite, then $\lim_{N\to\infty} N^z R_N = 0$. A sufficient condition for the sum to converge is a version of Raabe's test, $\lim_{n\to\infty} X_n > 1$, where now

$$X_n = n\left(\left[1 - \frac{1}{n}\right]^{z-1} \frac{\lambda}{\mu\beta_n} - 1\right).$$

The recursion (17) for β_n is equivalent to

$$X_{n+1} = \left(1 + \frac{1}{n}\right)\left(A_n + \left(1 - \frac{1}{n}\right)^{z-1}\left[\frac{\gamma}{\mu} + \frac{\lambda}{\mu}\frac{X_n - A_n}{1 + \frac{1}{n}X_n}\right]\right)$$

where

$$A_n = n\left(\left[1 - \frac{1}{n}\right]^{z-1} - 1\right).$$

Observe that $\lim_{n\to\infty} A_n = 1 - z$. The limiting recursion for X_n is therefore

$$X_{n+1} - [1 - z] \approx \frac{\gamma}{\mu} + \frac{\lambda}{\mu}\left(X_n - [1 - z]\right),$$

and this has the unique fixed point $1 - z + \zeta$. One can verify that X_n converges to this fixed point starting from $X_1 = -1$. Thus $z < \zeta$ guarantees convergence.

C The Limiting Case $\nu \downarrow 0$

Write

$$P_n = \xi \sum_{k=n}^{\infty} \frac{(\lambda/\mu)^{k+1-n}}{k(k+1)},$$

and note that

$$\begin{aligned} \frac{1}{\xi}\sum_{n=N}^{\infty} P_n &= \sum_{n=N}^{\infty}\sum_{k=n}^{\infty} \frac{(\lambda/\mu)^{k+1-n}}{k(k+1)} \\ &= \sum_{m=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^{m+1} \sum_{n=N}^{\infty} \frac{1}{(m+n)(m+n+1)} = \sum_{m=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^{m+1} \frac{1}{m+N}. \end{aligned}$$

For $N = 1$ this gives:

$$\frac{1}{\xi}\sum_{n=1}^{\infty} P_n = \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\lambda}{\mu}\right)^m = -\ln\left(1 - \frac{\lambda}{\mu}\right),$$

and hence $\xi = -\ln(1 - \lambda/\mu) = \ln(\gamma/\mu)$. Using the fact that $\gamma = \mu - \lambda$, one can write the entry rate as

$$\eta = \gamma + \lambda P_1 = \gamma + \frac{\lambda}{-\ln\left(1 - \frac{\lambda}{\mu}\right)} \sum_{k=1}^{\infty} \frac{(\lambda/\mu)^k}{k(k+1)} = \lambda\xi,$$

which is the result reported in (28).

References

[1] Atkeson, Andrew and Ariel Burstein, personal communication, 2006.

[2] Armington, Catherine, "Statistics on U.S. Businesses—Microdata and Tables," Small Business Administration (1998).

[3] Axtell, Robert L., "Zipf Distribution of U.S. Firm Sizes," *Science*, CCXCIII (2001), 1818-1820.

[4] Burdett, Kenneth and Tara Vishmanath, "Balanced Matching and Labor Market Equilibrium," *Journal of Political Economy*, vol. 96, no. 5 (1988), 1048-1065.

[5] Ellison, Glenn and Drew Fudenberg, "Word-of-Mouth Communication and Social Learning," *Quarterly Journal of Economics*, vol. 110 , no. 1 (1995), 93-125.

[6] Gabaix, Xavier, "The Granular Origins of Aggregate Fluctuations," MIT (2005).

[7] Gabaix, Xavier and Augustin Landier, "Why Has CEO Pay Increased so Much," MIT and New York University (2006).

[8] Hopenhayn, Hugo, "Entry, Exit and Firm Dynamics in Long Run Equilibrium," *Econometrica*, vol. 60, no. 5 (1992), 1127-1150.

[9] Hymer, Stephen and Peter Pashigian, "Firm Size and Rate of Growth," *Journal of Political Economy*, vol. 70, no. 6 (1962), 556-569.

[10] Jackson, Matthew O., "The Economics of Social Networks," in *Proceedings of the 9th World Congress of the Econometric Society*, edited by Richard Blundell, Whitney Newey, and Torsten Persson, Cambridge University Press (2006).

[11] Jackson, Matthew O. and Brian W. Rogers, "Meeting Strangers and Friends of Friends: How Random are Social Networks," Caltech (2006).

[12] Ijiri, Yuji, and Herbert A. Simon, "Business Firm Growth and Size," *American Economic Review*, LIV (1964), 77-89.

[13] Jovanovic, Boyan, "Selection and the Evolution of Industry," *Econometrica*, C (1982), 649-670.

[14] Klette, Tor Jakob, and Samuel S. Kortum, "Innovating Firms and Aggregate Innovation," *Journal of Political Economy*, vol. 112, no. 5 (2004), 986-1018.

[15] Lentz, Rasmus. and Dale T. Mortensen, "An Empirical Model of Growth Through Product Innovation" University of Wisonsin and Northwestern University (2006).

[16] Lucas, Robert E., Jr. "On the Size Distribution of Business Firms," *Bell Journal of Economics*, vol. 9, no. 2 (1978), 508-523.

[17] Luttmer, Erzo G.J., "Selection, Growth, and the Size Distribution of Firms, *Quarterly Journal of Economics*, forthcoming (2006).

[18] Rob, Rafael and Arthur Fishman, "Is Bigger Better? Customer Base Expansion through Word-of-Mouth Reputation," *Journal of Political Economy*, vol. 113, no. 5 (2005), 1146-1162.

[19] Stanley, R. Michael, L.A.N. Amaral, S.V. Buldyrev, S. Harlin, H. Leschorn, P. Maass, M.A. Salinger, H.E. Stanley, "Scaling Behavior in the Growth of Companies," *Nature* 319 (1996), 804-806.

[20] Simon, Herbert A., "On a Class of Skew Distribution Functions," *Biometrika*, vol. 42, no. 3/4 (1955), 425-440.

[21] Simon, Herbert A., and Charles P. Bonini, "The Size Distribution of Business Firms," *American Economic Review*, XCVIII (1958), 607-617.

[22] Sutton, John, "The Variance of Firm Growth Rates: The 'Scaling' Puzzle," *Physica A*, CCCXII (2002), 577-590.

[23] Sutton, John, "Market Share Dynamics and the 'Persistence of Leadership' Debate," *American Economic Review*, forthcoming (2006).