AN INQUIRY INTO THE CONDITIONS
UNDER WHICH A SINGLE ENDOWMENT
TRUST FUND WILL PROVIDE PERPETUAL
FUNDING FOR AN EXPENSE
STREAM GROWING AT A COMPOUND
ANNUAL RATE

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Ron Kaatz

Research Department
Federal Reserve Bank of Minneapolis
January 26, 1970
THE PROBLEM

A single contribution trust fund is to be established. Determine the minimum combination of endowment and rate of return required to provide a specified annual income stream. The annual income must be sufficient to cover the fund managers fee and a given annual expense stream which grows at a constant annual rate. The fund and expense stream last forever (in perpetuity).

NOTATION

\( t \) = time, measured in years

\( X \) = endowment made at \( t = 0 \)

\( r \) = annual rate of return earned by the fund

\( f \) = percentage rate used to determine fund managers fee

\( F_t \) = management fee due at the end of the \( t \)th year

\( E_1 \) = expense amount due at the end of the first year

\( E_t \) = expense amount due at the end of the \( t \)th year

\( i \) = annual rate of growth of the expense stream

\( B_t \) = balance in the trust fund at the end of the \( t \)th year, after paying the expense amount (\( E_t \)) and management fee (\( F_t \))
FURTHER DESCRIPTION OF THE PROBLEM

Graphically we're dealing with a world that looks like this . . .

At $t = 0$ (the beginning of the first year) an initial endowment of $X$ dollars is provided. At the end of the first year the fund has earned income and capital appreciation at the rate of $r\%$. The first year expense amount ($E_1$ dollars) and management fee ($F_1$ dollars) are paid leaving a fund balance of $B_1$ dollars. The second year begins (with balance $B_1$). Income and appreciation accrue at $r\%$ during the year and at $t = 2$ the annual expense ($E_2$) and the management fee ($F_2$) are paid leaving a balance of $B_2$. The process is repeated year after year. One of two outcomes will occur:

Outcome A: Income and appreciation of the fund are adequate to cover the expense stream and management fee in perpetuity.

Outcome B: The rate of return and/or the endowment is not adequate and eventually the fund balance will deplete to zero.

The expense fund grows at an annual rate of $i\%$ . . .

$$E_t = E_1 (1 + i)^{t-1}$$
The management fee is \( f \% \) of the sum of the balance at the beginning of the year plus income and appreciation earned during the year.

\[
F_t = f \left( B_{t-1} + r \cdot B_{t-1} \right)
\]

**SOLUTION**

Given particular values for the management fee rate \( f \), the annual growth rate of the expense stream \( i \) and the beginning expense amount \( E_1 \), two conditions must be satisfied in order to bring about outcome A (i.e. provide perpetual funding for the expense stream).

The two conditions are:

1. \( r > \frac{f + i}{1 - f} \)

2. \( x \geq \frac{E_1}{(1 + r) \left( (1 - f) - (1 + i) \right)} \)

The tables on the following pages provide minimum combinations of rate of return and endowment required to provide perpetual funding.

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1/ Proof of these conditions is given in Appendix A.

2/ The computer program used to derive the tables is given in Appendix B.
TABLES OF MINIMUM ENDOWMENT PER ONE DOLLAR OF EXPENSE IN YEAR ONE

FUND MANAGERS FEE RATE = 0.2%

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TABLES OF MINIMUM ENDOWMENT PER ONE DOLLAR OF EXPENSE IN YEAR ONE

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TABLES OF MINIMUM ENDOWMENT PER ONE DOLLAR OF EXPENSE IN YEAR ONE

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APPENDIX A

THE MATHEMATICS

The Process

(P-1) \( B_o = X \)
(P-2) \( E_t = E_1 (1+i) \) for \( t = 1, 2, ... \)
(P-3) \( B_t = B_{t-1} (1+r)(1-f) - E_t \) for \( t = 1, 2, ... \)

Conditions on the Variables and Parameters

(C-1) \( t = 0, 1, 2, 3, ... \)
(C-2) \( r \geq 0 \)
(C-3) \( 0 \leq f < 1 \)
(C-4) \( 0 \leq E_1 < \infty \)
(C-5) \( i \geq 0 \)
(C-6) \( 0 < X < \infty \)

Theorem 1: Given (P-1), ..., (P-3) and (C-1), ..., (C-6) the fund balance at the end of the \( t \)th year is...

\[
B_t = X[(1+r)(1-f)]^t - E_1 \sum_{k=1}^{t} [(1+r)(1-f)]^{(t-k)}(1+i)^{(k-1)}
\]

for \( t = 1, 2, ... \)

Proof (by induction):

A. By (P-1), (P-3), and inspection it is true for \( t = 1 \)

B. Assume it is true for \( t = M \).

Then by assumption...

\[
B_M = X[(1+r)(1-f)]^M - E_1 \sum_{k=1}^{M} [(1+r)(1-f)]^{(M-k)}(1+i)^{(k-1)}
\]

by (P-3)
by (P-3)

\[ B_{M+1} = B_M (1+r)(1-f) - E_{M+1} \]

by (P-2)

\[ B_{M+1} = B_M (1+r)(1-f) - E_1 (1+i)^M \]

by the induction assumption...

\[ B_{M+1} = X [(1+r)(1-f)]^{M+1} (1+r)(1-f) - E_1 \sum_{k=1}^{M} [(1+r)(1-f)]^{M-k}(1+i)^{(k-1)} \]

\[ - E_1 (1+i)^M \]

\[ B_{M+1} = X [(1+r)(1-f)]^{M+1} - E_1 \sum_{k=1}^{M} [(1+r)(1-f)]^{M-k}(1+i)^{(k-1)} \]

\[ + (1+i)^M \sum_{k=1}^{M} [(1+r)(1-f)]^{M-k}(1+i)^{(k-1)} \]

\[ = X [(1+r)(1-f)]^{M+1} - E_1 \sum_{k=1}^{M} [(1+r)(1-f)]^{M-k}(1+i)^{(k-1)} \]

\[ = X [(1+r)(1-f)]^{M+1} - E_1 \sum_{k=1}^{M} [(1+r)(1-f)]^{M-k}(1+i)^{(k-1)} \quad q.e.d. \]

\[ B_{M+1} = X [(1+r)(1-f)]^{M+1} - E_1 \sum_{k=1}^{M} [(1+r)(1-f)]^{M-k}(1+i)^{(k-1)} \]

\[ \text{Theorem 2: Outcome A will occur if and only if } B_t > 0 \quad \forall t = 1, 2, \ldots \]

\[ \text{Proof:} \]

1. \textbf{Necessary.} Outcome A asserts \( B_t (1+r)(1-f) \geq E_{t+1} \quad \forall t = 1, 2, \ldots \)

   Where \( 0 \leq E_{t+1} \quad \text{by (C-4) and (P-2)} \)

   \( 0 \leq f < 1 \quad \text{by (C-3)} \)

   \( 0 \leq r \quad \text{by (C-2)} \)

   So \( B_t \geq 0 \).

   if \( B_t = 0 \) then \( E_{t+1} = -E_t < 0 \)

   So Outcome A = \( B_t > 0 \quad t = 1, 2, \ldots \)

2. \textbf{Sufficient.} to show that

   \( B_p > 0 \quad \forall p = B_q (1+r)(1-f) \geq E_{q+1} \quad \forall q \)

   is equivalent to showing

   \( B_q (1+r)(1-f) < E_{q+1} \) for some \( q \Rightarrow B_p < 0 \) for some \( p \).
from (P-3)
\[ B_{q+1} = B_q (1+r)(1-f) - E_{q+1} \]

therefore \[ B_q (1+r)(1-f) < E_{q+1} 
\rightarrow B_{q+1} < 0 \]
q.e.d.

By rearranging the equation of Theorem 1
\[ B_t = [(1+r)(1-f)]^t X - \frac{E_1 t}{1+i} \sum_{k=1}^{\infty} \left[ \frac{1+i}{(l+r)(1-f)} \right]^k \]

Theorem 3: Let the vector \( \langle f, E_1, i \rangle \) have Domain \( \mathcal{D} \) given by (C-3), (C-4), (C-5).

For any \( D = \langle f, E_1, i \rangle \in \mathcal{D} \)

\[ B_t > 0 \quad \forall \quad t=1,2,3,... \]

\[ \begin{cases} \text{r >} & \text{Condition 1} \\ \frac{f+i}{1-f} & \text{and} \\ \frac{E_1}{(1+r)(1-f)-(1+i)} & \text{Condition 2} \end{cases} \]

Proof:

Necessary
\[ A) \quad \frac{1+i}{(1+r)(1-f)} > 0 \text{ by (C-2), (C-3), (C-5).} \]

Given \( D \in \mathcal{D} \), \( B_t > 0 \quad \forall \quad t=1,2,... \)

\[ B) \quad X > \frac{E_1}{1+i} \sum_{k=1}^{\infty} \left[ \frac{1+i}{(l+r)(1-f)} \right]^k \quad \forall \quad t=1,2,... \]

A, B, and (C-6) \( \Rightarrow \)

\[ C) \lim_{t \to \infty} \sum_{k=1}^{t} \left[ \frac{1+i}{(l+r)(1-f)} \right]^k < \infty \]

\[ C = -1 < \frac{1+i}{(1+r)(1-f)} < 1 \]

\[ \therefore \text{since A holds by assumption,} \]

\[ C = \frac{1+i}{(1+r)(1-f)} < 1 \]

or equivalently \( r > \frac{f+i}{1-f} \) Condition 1

Condition 1 \( \Rightarrow \lim_{t \to \infty} \sum_{k=1}^{t} \left[ \frac{1+i}{(l+r)(1-f)} \right]^k = \frac{1+i}{(l+r)(1-f)-(1+i)} \)
Since this series is strictly monotonically increasing, it follows that the limit of the series is its least upper bound:

\[ \mathcal{V}_{t=1,2,3} \sum_{k=1}^{t} \left[ \frac{1+i}{(1+r)(1-f)} \right]^k < \frac{1+i}{(1+r)(1-f)-(1+i)} \]

and \( \forall y; y < \frac{1+i}{(1+r)(1-f)-(1+i)} \Rightarrow \exists t^* \geq 1 \exists \)

\[ \mathcal{V}_t \geq t^* \sum_{k=1}^{t} \left[ \frac{1+i}{(1+r)(1-f)} \right]^k > y \]

Note: \( \{ B_t > 0 \mathcal{V}_{t=1,2,\ldots} \Rightarrow \text{Condition 1} \} \) has been shown for any \( X \)

\[ \therefore B_t > 0 \mathcal{V}_{t=1,2,\ldots} \text{ and Condition 1} \Rightarrow \]

\[ X \geq \frac{E_1}{1+i} \lim_{t \to \infty} \sum_{k=1}^{t} \left[ \frac{1+i}{(1+r)(1-f)} \right]^k \]

or equivalently

\[ X \geq \frac{E_1}{(1+r)(1-f)-(1+i)} \text{ Condition 2} \]

**Sufficiency**

Suppose \( r > \frac{f+i}{1-f} \) and \( X = \frac{E_1}{(1+r)(1-f)-(1+i)} \)

then \( (1+r)(1-f) > 1 \),

\[ 0 < \frac{1+i}{(1+r)(1-f)} < 1, \]

\[ \lim_{t \to \infty} \sum_{k=1}^{t} \left[ \frac{1+i}{(1+r)(1-f)} \right]^k = \frac{1+i}{(1+r)(1-f)-(1+i)} \]

\[ \therefore B_t = [(1+r)(1-f)]^t \left[ X - \frac{E_1}{1+i} \sum_{k=1}^{t} \left[ \frac{1+i}{(1+r)(1-f)} \right]^k \right] > 0 \quad \forall t=1,2,\ldots \quad \text{q.e.d.} \]

**Corollary**

\[ B_t > 0 \mathcal{V}_{t=1,2,3,\ldots} \lim_{t \to \infty} B_t = 0 \]
Proof:

\[ B_t > 0 \quad \forall t \geq 1,2,3,\ldots \quad \Rightarrow r > \frac{f+i}{1-f} \quad \text{and} \quad \exists \text{by Theorem } 3 \quad x \geq \frac{E_1}{(1+r)(1-f)-(1+i)} \]

Case 1

Suppose \( r > \frac{f+i}{1-f} \), \( x > \frac{E_1}{(1+r)(1-f)-(1+i)} \), and let \( \Delta = \frac{E_1}{(1+r)(1-f)-(1+i)} > 0 \),

then \( \lim_{t \to \infty} B_t = \lim_{t \to \infty} \left[ (1+r)(1-f) \right]^t \lim_{t \to \infty} \left[ \frac{E_1}{1+i} \sum_{k=1}^{t} \left( \frac{1+i}{(1+r)(1-f)} \right)^k \right] = \infty \cdot \Delta = \infty \)

Case 2

Suppose \( r > \frac{f+i}{1-f} \) and \( x \geq \frac{E_1}{(1+r)(1-f)-(1+i)} \), then

\[ B_t = (1+r)(1-f)^t \left[ \frac{E_1}{(1+r)(1-f)-(1+i)} - \frac{E_1}{1+i} \sum_{k=1}^{t} \left( \frac{1+i}{(1+r)(1-f)} \right)^k \right] \]

\[ \frac{B_t}{E_1} = \left[ \frac{(1+r)(1-f)}{(1+r)(1-f)-(1+i)} \right]^t - \sum_{k=1}^{t} \frac{(1+i)^{k-1} [(1+r)(1-f)]^{t-k}}{k} \]

let \( b = (1+r)(1-f) \)

\( \quad d = (1+i) \)

then \( b > d > 1 \) by (C-5) and Condition 1

\[ \frac{B_t}{E_1} = \left( \frac{b^t}{b-d} \right) - \sum_{k=1}^{t} d^{k-1} b^{-k} \]

\[ \frac{B_t}{E_1} = \left( \frac{b^t}{b-d} \right) - \sum_{k=1}^{t} d^{k-1} b^{-k} \]

let \( C = \frac{b-d}{E_1} > 0 \), then \( C B_t = b^t \left[ 1 - \sum_{k=1}^{t} \left( \frac{d}{b} \right)^{k-1} + \sum_{k=1}^{t} \left( \frac{d}{b} \right)^k \right] \)

\[ C B_t = b^t \left[ 1 + \left( \frac{d}{b} \right)^t - 1 \right] \]

\[ C B_t = d^t \]
or \( B_t = \frac{d^t}{C} \)

\[ \therefore \lim_{t \to \infty} B_t = \lim_{t \to \infty} \frac{d^t}{C} = \infty. \] \( \theta \in \mathbb{R} \)
LIST
3 E1=1
5 FOR H = 1 TO 5
10 F = (H+1)*(.001)
12 PRINT "FEE % = " F
13 WRITE (1,14)
14 FORMAT(11X,".06",".07",".08",".09",".10",".11")
15 FOR Y = 1 TO 11
20 R = .04 + Y*(.01)
25 FOR Z = 1 TO 6
30 T = .05 + Z*(.01)
35 IF R > (I+F)/(1-F) THEN 45
40 X(Z) = 0
42 GO TO 50
45 X(Z) = E1/((1+R)*(1-F)-(1+I))
46 X(Z) = INT(X(Z)*100+.5)/100
50 NEXT Z
55 WRITE (1,60) R,X(1),X(2),X(3),X(4),X(5),X(6)
60 FORMAT(F5.2,F6(4X,F7.2))
65 NEXT Y
70 PRINT
71 PRINT
72 PRINT
75 NEXT H
80 END