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Abstract

In this paper, we show that a substantial fraction of the volatility of real exchange rates between developed economies such as Germany, Japan, and the United Kingdom against the US dollar can be accounted for by shocks that affect the prices of primary commodities such as oil, aluminum, maize, or copper. Our analysis implies that existing models used to analyze real exchange rates between large economies that mostly focus on trade between differentiated final goods could benefit, in terms of matching the behavior of real exchange rates, by also considering trade in primary commodities.

Keywords: primary commodity prices, real exchange rate disconnect puzzle.
JEL classification: F31, F41.

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1 Introduction

This is a data paper: it shows that shocks that generate fluctuations in a small number of primary commodity prices can explain a substantial fraction of the movements in real exchange rates (RER) among industrialized countries. Specifically, we study the behavior of the bilateral RER of Germany, Japan, and the United Kingdom with the United States for the 1960-2014 period. A rough summary of the results is that with just four primary commodity prices (PCP), we can account for between one-third and one-half of the volatility of the RER between the United States and those three countries.

The relevance of these results is highlighted by the so-called *exchange rate disconnect puzzle*: the fact that real exchange rates across developed economies are very volatile, very persistent, and very hard to relate to fundamentals.[1] This difficulty opened the door for theoretical explorations of models with nominal rigidities as the source of RER movements, as in, for example Chari, Kehoe, and McGrattan (2002). We will ignore nominal rigidities in our analysis and explore how far one can go with shocks that affect relative prices of the main primary commodities.

The disconnect puzzle is not present in small open economies where exports of a few primary commodities are a sizable share of total exports.[2] For countries such as Australia, Chile, or Norway, changes in the international prices of the commodities each country exports are highly correlated with their real exchange rates.

As we show, a very similar idea can go a long way in explaining movements in RER among developed economies. The idea that we exploit in the paper is very simple: fluctuations in the prices of commodities affect manufacturing costs and therefore manufacturing prices, which in turn induce changes in final good costs. These cost fluctuations translate into price fluctuations at the country level. If changes in commodity prices have differential effects on the domestic cost of any two countries, primary commodity price changes will affect the real exchange rate between those two countries.

Relating PCP changes to RER changes is a promising avenue to explore for several reasons. First, PCP are very volatile (even more volatile than real exchange rates, as we show below) and very persistent, a feature that, as we mentioned, real exchange rates also exhibit. Second, the share of trade in primary commodities in total world trade is far from trivial: total trade in a few commodities (10) accounts for between 12% to 18% of total world trade in goods, depending on the year chosen.[3] This number clearly underestimates the true

[1]See, for example, in Meese and Rogoff (1983), Engel (1999), Obstfeld and Rogoff (2001), and Betts and Kehoe (2004).

[2]See Chen and Rogoff (2003) and Hevia and Nicolini (2013).

[3]It is close to 12% in 1990 and 18% in 2012. The main difference is that the first is a year of particularly

share of commodities, since trade shares are not value-added measures. Thus, when steel is exported, it is fully counted as a manufactured good, even though an important component of its cost depends on iron. The same happens when a car is exported. Third, primary commodities are at the bottom of the production chain, so they directly affect final good prices.[4] In addition, they may directly affect the prices of other domestic inputs – such as some types of labor and services in general – that are used jointly with primary commodities in the production of intermediate goods, and thus they may indirectly affect the costs of final goods. Because just a few commodities make up a high share of total trade, we only need to focus on a handful of prices. Finally, it is well known that the law of one price on those primary commodities holds, so no ambiguity with respect to their tradability exists. The literature on RER has struggled to separate the set of final goods into two categories: the ones that are traded, for which the law of one price is assumed to hold, and the ones that are not. We only need to assume that for the few commodities we analyze, the law of one price holds, and it is precisely for these prices that independent evidence that the law of one price holds is the strongest.

The exchange rate disconnect puzzle has been widely studied in the literature. Two recent attempts at quantitatively explaining several facts related to the puzzle are Itskhoki and Mukhin (2017), and Eaton, Kortum, and Neiman (2016). They provide very good descriptions of the state of the literature. The connection between RER and PCP has largely been ignored for the countries we focus on, and not only on the empirical side. Ever since the seminal contribution of Obstfeld and Rogoff (1995), the theoretical literature developed to study RER between the countries we consider in this paper has focused exclusively on the production and trade of final goods. Our evidence suggests that theoretical models of RER among developed economies that ignore primary commodity markets may fall short of providing a comprehensive explanation of RER movements.

In Section 2 we describe the data and motivate the analysis by showing some descriptive statistics. We also discuss several issues related to the empirical methodology used in the paper. In Section 3 we present the main results. In Section 4 we perform a series of Monte Carlo exercises to address issues related to small sample properties of the moments we estimate in Section 3. In Section 5 we partially spell out the production side of a totally standard model of an open economy that makes explicit the production of commodities and the use of commodities in the production of manufactured goods. We derive an equilibrium condition relating the bilateral RER between two countries to PCP that, in the case of

low primary commodity prices, while the second is a period of particularly high prices.

[4] This direct effect is substantial enough for monetary authorities in developed countries to focus attention on measures of "core" inflation, which abstract from the "volatile" effect of primary commodity prices (food and energy).

Cobb-Douglas production functions, is linear in the logarithms of the variables. This log-linear relationship rationalizes the one used in the empirical analysis presented in Section 3. Finally, in Section 6, we present an additional exercise, motivated by the model of Section 5. A brief discussion of the implications of the results is presented in a final concluding section.

2 The data and the methodology

We collected monthly data on consumer price levels and nominal exchange rates for the United States, Germany, the United Kingdom, and Japan. The bilateral RER are defined as the nominal exchange rates between Germany (DEU), Japan (JPN), and the United Kingdom (UK) against the US dollar multiplied by the ratio of CPIs. In the case of Germany, we use the mark until 2000 and the euro thereafter. In addition, we collected price data for the 10 primary commodities with the largest shares in world trade in 1990 and for which monthly price data are available from January 1960 to December 2014. Appendix A includes a detailed description of the data. Table 1 shows the selected primary commodities and their respective definitions and shares in world trade.[5]

Table 1: Primary Commodities List

Commodity	(%) share of world trade in 1990	SITC (rev.3)	Commodity	(%) share of world trade in 1990	SITC (rev.3)
(1) Petroleum	7.22	33	(6) Gold	0.42	971.01
(2) Fish	1.05	03	(7) Wheat	0.35	041
(3) Meat	0.89	011/012	(8) Maize	0.28	044
(4) Aluminium	0.49	285.1/684.1	(9) Timber	0.26	24
(5) Copper	0.45	283.1/682.1	(10) Cotton	0.22	263

Note: SITC (rev3) stands for Standard International Trade Classification (revision 3).
Source: Comtrade.

As already mentioned, the bilateral real exchange rates between the countries we analyze are known to be very persistent and volatile. This is one of the properties that make PCP attractive to relate to RER, since they are also known to be persistent and volatile: if shocks to PCP are to account for a large share of the volatility of real exchange rates, it must also share these same features. We now show that this is indeed the case.

Table 2 reports the results of unit root tests for the data in levels and in three-, four-, and five-year differences. In all cases, we took the log of the original data. As can be seen, there is evidence of unit roots for the raw data, whereas the evidence vanishes for the data

[5]We repeated the analysis using trade data in 2000, and the results remain the same. In this case, maize and cotton are replaced by platinum and coffee.

Table 2: Unit root tests (p-values)

	Level	three-year differences	four-year differences	five-year differences
Real Exchange Rates				
US-UK	0.018	0.001	0.003	0.003
US-DEU	0.117	0.005	0.028	0.027
US-JPN	0.809	0.001	0.018	0.027
Commodities				
Oil	0.485	0.079	0.128	0.356
Fish	0.352	0.001	0.027	0.009
Meat	0.523	0.019	0.047	0.304
Aluminium	0.145	0.001	0.001	0.003
Copper	0.319	0.009	0.025	0.103
Gold	0.508	0.001	0.016	0.025
Wheat	0.226	0.001	0.005	0.009
Maize	0.269	0.001	0.010	0.013
Timber	0.047	0.003	0.018	0.047
Cotton	0.592	0.005	0.016	0.015

Notes: variables are in logs and commodity prices are normalized by US CPI. We use the Dickey-Fuller test, in which the p-values are under the null hypothesis that the series follows a unit root process. The lag length is selected according to the Ng-Perron test. We assume a trend in the case of Japan.

in four-year differences. In Table 3, we report the first-order autocorrelation for all the series in four-year differences, which will be our benchmark case. As can be clearly seen, the high persistence of real exchange rates is also present in the commodity prices.

Table 3: First order autocorrelation of four-year differences

US-UK	US-DEU	US-JPN	Oil	Fish	Meat	Aluminium
0.98	0.98	0.98	0.97	0.98	0.97	0.98
Copper	Gold	Wheat	Maize	Timber	Cotton	
0.98	0.99	0.98	0.97	0.97	0.98	

Note: Variables are in logs and commodity prices are normalized by US CPI.

Next, we turn to analyze the volatility of the series. Tables 4.a and 4.b show the volatility (standard deviation) of the monthly data on the US bilateral real exchange rates against the United Kingdom, Germany, and Japan between 1960 and 2014, as well as for four subperiods.[6] The volatility is computed on the log of the series, so it can be interpreted as percentage variations. We also report the average volatility (simple and trade-weighted) of

[6] When specifying the subperiods, we opted for isolating 1960–1972, the period during which the Bretton Woods system was active. Then we chose the next subperiods so that they would have similar lengths.

the prices of the commodities listed in Table 1. Table 4.a presents the volatilities for the raw data, and Table 4.b presents the volatilities for the data in four-year differences.

As can be seen, the volatility of PCP is substantially higher than that of RER.[7] In addition, it is apparent that in the subperiods in which the volatility of PCP is high, so is the volatility of the RER. We find this issue particularly interesting, since the substantial increase in the volatility of real exchange rates after the breakdown of the Bretton-Woods system of fixed exchange rates is accompanied by an equally substantial increase in the volatility of commodity prices. The conventional interpretation has been that the increase in volatility after 1972 was the result of the regime change from fixed to flexible exchange rates.[8] An alternative interpretation is that the fundamentals that cause real exchange rates and commodity prices to move together were much more volatile after 1973 than before. To the extent that PCP are independent of the exchange rate regimes, our evidence points toward an alternative explanation of the increase in volatility post Bretton-Woods.

Note also that the reduction in real exchange rate volatility that ensued after the mid-1980s is also accompanied by a reduction in the volatility of commodity prices. As a complementary piece of evidence, Figure 1 shows rolling volatilities computed using windows of 10 years of data for the three real exchange rates and for the average volatility of the 10 primary commodity prices. The figure clearly points toward a positive association between the volatilities of the real exchange rate and commodity prices.[9] This positive association reinforces, in our view, the interest in associating RER with PCP.

Finally, in Table 5 we show the simple correlations of each of the bilateral RER and all the commodity prices we use. As can be seen, all simple correlations between the prices and the RER are sizable. In addition, the correlations across the PCP are also sizable in many cases.

2.1 Methodology

Our goal is to assess how much of the variability of the US bilateral real exchange rates with the United Kingdom, Germany, and Japan can be accounted for by a set of primary commodity prices. In following the small open economy literature, and pairing the United States and United Kingdom as an example, we analyze the following regression equation:

$$\ln P_t^{USA} - \ln P_t^{UK} + \ln S_t = \beta \mathbf{p}_t^{X,USA} + v_t. \tag{1}$$

[7] This is also the case for small open economies, where the ratio of the volatility of the relevant PCP is between 2.5 and 3.5 the volatility of the RER. These values are similar to the ones that can be obtained from Table 4.

[8] See Mussa (1986).

[9] The correlations are 0.40, 0.54, and 0.39 for the United Kingdom, Germany, and Japan, respectively.

Table 4: Volatilities of real exchange rates and primary commodity prices

	1960–2014	1960–1972	1973–1985	1986–1998	1999–2014
			(a) Level		
Real Exchange Rates:					
US-UK	0.12	0.06	0.15	0.08	0.08
US-DEU	0.18	0.07	0.21	0.09	0.13
US-JPN	0.37	0.13	0.13	0.12	0.11
Average across commodities:					
Simple	0.44	0.13	0.31	0.22	0.36
Trade weighted	0.57	0.13	0.37	0.23	0.46
			(b) Four-year differences		
Real Exchange Rates:					
US-UK	0.18	0.11	0.26	0.17	0.14
US-DEU	0.22	0.07	0.31	0.16	0.21
US-JPN	0.23	0.07	0.27	0.26	0.17
Average across commodities:					
Simple	0.38	0.17	0.44	0.30	0.36
Trade weighted	0.46	0.17	0.55	0.35	0.36

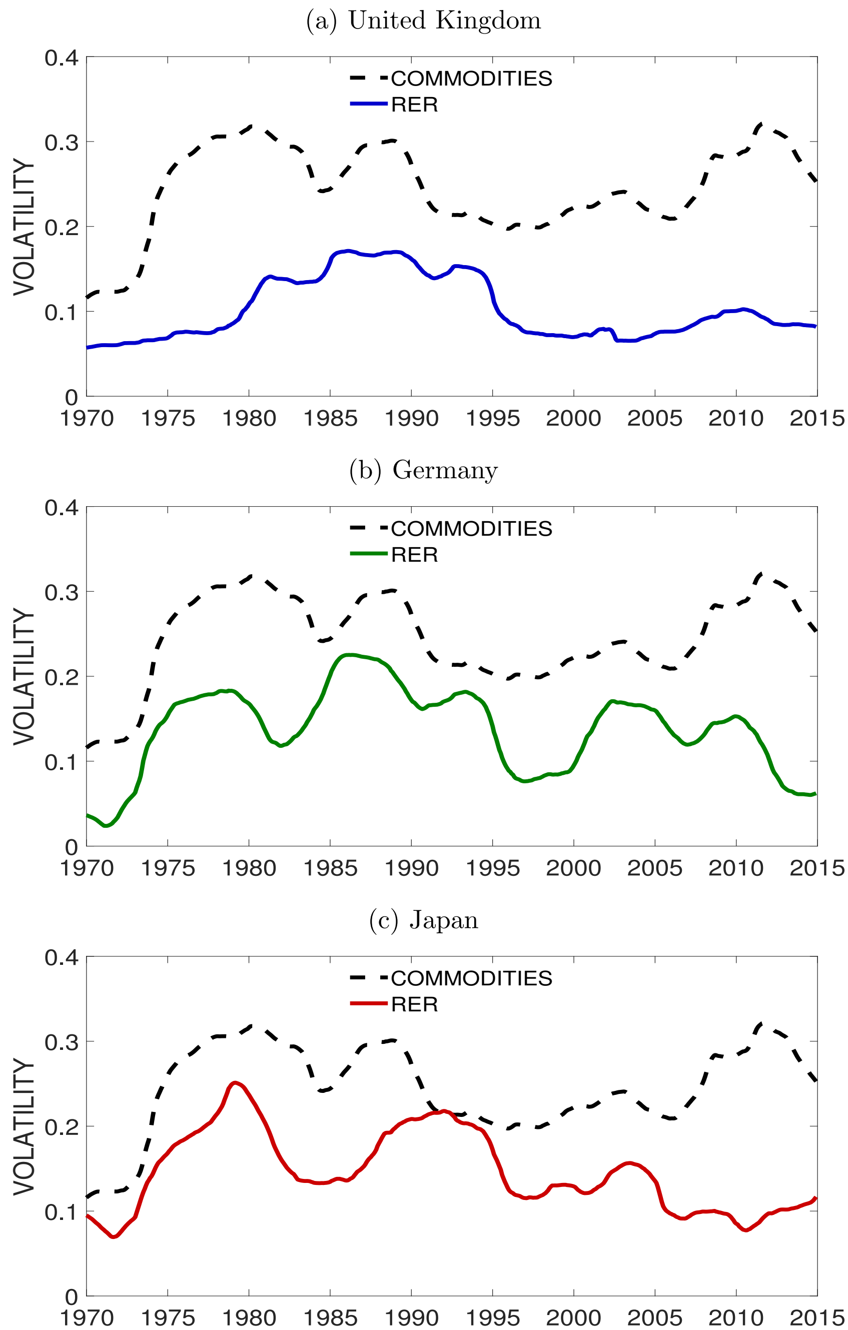
Notes: Variables are in logs and commodity prices are normalized by US CPI. Weights are based on the share of total trade in 1990. The set of primary commodities is oil, fish, meat, aluminum, copper, gold, wheat, maize, timber, and cotton.

The left-hand side of equation (1) is the bilateral real exchange rate: P_t^{USA} denotes the price level in the United States, P_t^{UK} denotes the price level in the United Kingdom, and S_t denotes the nominal exchange rate between US dollars and British pounds. On the right-hand side of equation (1), we have the vector $\mathbf{p}_t^{X,USA}$, which contains the log of the PCP normalized by the US price level, the vector of coefficients β, and the error term v_t.[10] The subscript t denotes the time period.

Following the small open economy literature at this stage entails a major difficulty. The assumption that PCP are exogenous for economies such as Norway or New Zealand is reasonable, but it is clearly unacceptable for the pairs of economies we analyze. Indeed, we have no hope of obtaining consistent estimators of the vector β. But the R^2 on that regression contains valuable information, precisely the information we need to answer the question of the paper. We now discuss why this is so.

[10] Since we use PCP in constant dollars, one might be worried that the price level of the United States enters both sides of the equation. In Section 3, however, we show that the results do not depend on that.

Figure 1: Rolling volatilities of real exchange rates and commodity prices (10-year windows)



The RER and the PCP in equation (1) are determined simultaneously. Let

$$r_t^{USA,UK} = \ln P_t^{USA} - \ln P_t^{UK} + \ln S_t$$

Table 5: Correlations (1960–2014)

	Oil	Fish	Meat	Alum.	Copper	Gold	Wheat	Maize	Timber	Cotton
RER										
US-UK	-0.47	0.00	0.30	0.11	0.09	-0.53	0.26	0.36	-0.40	0.30
US-DEU	-0.51	-0.24	0.16	0.08	-0.08	-0.62	0.06	0.14	-0.58	0.11
US-JPN	-0.49	0.25	0.59	0.52	0.41	-0.63	0.59	0.63	-0.44	0.55
Commodities										
Oil	1.00									
Fish	0.28	1.00								
Meat	-0.17	0.45	1.00							
Alum.	-0.20	0.36	0.73	1.00						
Copper	0.07	0.72	0.57	0.52	1.00					
Gold	0.88	0.25	-0.16	-0.22	0.03	1.00				
Wheat	-0.05	0.57	0.78	0.70	0.60	-0.07	1.00			
Maize	-0.11	0.58	0.81	0.70	0.62	-0.13	0.94	1.00		
Timber	0.39	0.10	0.18	0.17	0.10	0.56	0.21	0.15	1.00	
Cotton	-0.23	0.39	0.83	0.78	0.43	-0.21	0.84	0.86	0.24	1.00

Note: Variables are in logs and primary commodity prices are normalized by US CPI.

denote the (log) of the bilateral RER of interest and

$$\mathbf{p}_t^{X,USA} \in R^m, \text{ and } \xi_t \in R^n$$

be a vector of (log) primary commodity prices and a vector representing the state of the economy, also referred to as the fundamentals, respectively. The state vector may include endogenous state variables, such as stocks of capital, and exogenous state variables, such as productivity, preference, and policy shocks. Given any model, the equilibrium values for the RER and the PCP are functions of the fundamental shocks ξ_t:[11]

$$\begin{aligned} r_t^{USA,UK} &= f(\xi_t), \\ \mathbf{p}_t^{X,USA} &= g(\xi_t). \end{aligned} \tag{2}$$

Using a linear approximation to the previous equations, we obtain the following log-linear system of equations

$$\begin{aligned} r_t^{USA,UK} &= \theta' \xi_t, \\ \mathbf{p}_t^{X,USA} &= \Omega \xi_t, \end{aligned} \tag{3}$$

[11] In Section 5, we describe a model with a solution as in (2). That model implies an equilibrium relationship between the RER and the PCP that rationalizes equation (1).

where θ is an $n \times 1$ vector, Ω is an $m \times n$ matrix, and variables are measured as deviations from their long-run means. We treat the fundamental shocks of the economy as unobserved, so we can interpret the state variables ξ_t as orthogonal with an identity covariance matrix without loss of generality.[12]

Consider projecting the real exchange rate $r_t^{USA,UK}$ onto the commodity prices $\mathbf{p}_t^{X,USA}$,

$$Proj(r_t^{USA,UK}|\mathbf{p}_t^{X,USA}) = \beta' \mathbf{p}_t^{X,USA}.$$

By the orthogonality principle, β solves $E[r_t^{USA,UK}(\mathbf{p}_t^{X,USA})'] = \beta' E[\mathbf{p}_t^{X,USA}(\mathbf{p}_t^{X,USA})']$. Using that in equilibrium the RER and the PCP are related to the fundamental shocks through equation (3), $E[r_t^{USA,UK}(\mathbf{p}_t^{X,USA})'] = \theta'\Omega'$ and $E[\mathbf{p}_t^{X,USA}(\mathbf{p}_t^{X,USA})'] = \Omega\Omega'$, which gives $\beta' = (\theta'\Omega')(\Omega\Omega')^{-1}$. Finally, using $\mathbf{p}_t^{X,USA} = \Omega\xi_t$, the projection of the real exchange rate onto the commodity prices is equivalent to decomposing the real exchange rate into two orthogonal components:

$$r_t^{USA,UK} = \beta'\Omega\xi_t + (\theta' - \beta'\Omega)\xi_t. \tag{4}$$

The first term on the right side of equation (4) is the component of the real exchange rate that is correlated with primary commodity prices. It measures how much of the variability of the real exchange rate can be accounted for by fundamental shocks that affect primary commodity prices. The second component of the projection is orthogonal to the first and measures how much of the variability of the real exchange rate is accounted for by fundamental shocks that do not manifest themselves as fluctuations in commodity prices correlated with the real exchange rate. In terms of this decomposition, the R^2 of the regression (1) can be written as

$$R^2 = \frac{E[\beta'\Omega\xi_\mathbf{t}\xi_\mathbf{t}'\Omega'\beta]}{E[(\theta'\xi_t\xi_\mathbf{t}'\theta)]} = \frac{\beta'\Omega\Omega'\beta}{\theta'\theta}. \tag{5}$$

The underlying (implicit) assumption in much of the literature on bilateral real exchange rates between developed countries is that the component associated with commodity prices, $\beta'\Omega\xi_t$, can be safely ignored. We can express this no-relevance-of-commodities assumption as the requirement that the R^2 of the regression of real exchange rates on commodity prices is zero, which is true whenever $\beta'\Omega = 0$.

Let state variables be divided into two sets as $\xi_t = [\xi_{1t}' \; \xi_{2t}']'$, so that $r_t^{USA,UK} = \theta_1'\xi_{1t} + \theta_2'\xi_{2t}$ and $\mathbf{p}_t^{X,USA} = \Omega_1\xi_{1t} + \Omega_2\xi_{2t}$. It then follows that $\beta'\Omega = \theta_1'\Omega_1 + \theta_2'\Omega_2$. A necessary and sufficient condition for the R^2 of the regression to be zero is thus

$$\theta_1\Omega_1 = -\theta_2\Omega_2.$$

[12]If the shocks ξ_t have a nondiagonal covariance matrix $E(\xi_t\xi_t') = \Sigma$, we can create an observationally equivalent system with orthogonal state variables by letting $\tilde{\xi}_t = \Sigma^{-1/2}\xi_t$, $\tilde{\theta}' = \theta'\Sigma^{1/2}$, and $\tilde{\Omega} = \Omega\Sigma^{1/2}$.

A sufficient condition for this equality to hold is that $\theta_1 = 0$ and $\Omega_2 = 0$. This implies an equilibrium with a block-recursive structure in which the set of state variables that determine the real exchange rate are different from (and orthogonal to) those that determine primary commodity prices. If these conditions do not hold, then commodity prices will be (generically) correlated with the real exchange rate.

Measuring how much of the variability of the RER can be explained by this common component – the R^2 of equation (1) – is the objective of the following sections.

2.1.1 Higher-order terms

Linear approximations work well following shocks that imply small deviations from the steady state. The large and persistent movements in both RER and PCP imply that the approximation error may be large, so that second-order effects may be important to understand the comovement between the RER and the PCP. One way out of this would be to incorporate nonlinear terms in the regression equation (1). We view this paper as a first step toward understanding the role that PCP may play in helping us to understand the large and persistent movements in RER. We opted for simplicity, so we will only consider the linear terms in the analysis that follows. Accordingly, the interpretation of the R^2s we present below can be seen as a lower bound on the fraction of the volatility of the RER that can be accounted for by shocks that also move the PCP.

2.1.2 Time-varying coefficients

The coefficients in the linearized version (3) are evaluated at the equilibrium around which the linearization is made. The economies we are about to study have experienced major transformations during the more than five decades that cover the period we study. These transformations have changed not only the production structures but also the trade patterns. As such, it is reasonable to imagine that the equilibrium around which the linearization is made in the 1960s is different from the one in the 1980s.[13] Thus, there is no reason to believe that those coefficients would remain constant over time. In order to capture this possibility, we present our results for the whole period, but also for several subperiods. We also use this idea to motivate the out-of-sample fit exercises we do in the next section.

[13]The model in Section 5 makes these statements precise.

3 Results

We start our analysis by reporting the R^2 of the OLS regression of equation (1) using the price series of the primary commodities listed in Table 1. We run the regression for the whole period and also for four subperiods. The results are reported in Table 6.a.

Table 6: R^2

	1960–2014	1960–1972	1973–1985	1986–1998	1999–2014
(a) 10 commodities, level					
United Kingdom	0.50	0.76	0.73	0.67	0.54
Germany	0.59	0.87	0.86	0.57	0.67
Japan	0.81	0.87	0.60	0.75	0.75
(b) 10 commodities, 4-year differences					
United Kingdom	0.48	0.90	0.90	0.81	0.60
Germany	0.63	0.95	0.87	0.83	0.75
Japan	0.57	0.92	0.84	0.92	0.82
(c) 4 commodities (best fit), level					
United Kingdom	0.39	0.66	0.70	0.51	0.51
Germany	0.56	0.77	0.83	0.38	0.66
Japan	0.79	0.85	0.57	0.67	0.66
(d) 4 commodities (best fit), 4-year differences					
United Kingdom	0.33	0.72	0.82	0.63	0.58
Germany	0.56	0.84	0.87	0.81	0.74
Japan	0.48	0.88	0.76	0.86	0.80

Table 6.a shows R^2 are 0.50, 0.59, and 0.81 for the United Kingdom, Germany, and Japan, respectively. The R^2s are larger when we consider the subperiods, although as we will show below, they are largely the effect of smaller samples.

As we showed in Table 2, there is strong evidence that many of the PCP have unit roots. At the same time, there is clear evidence of a unit root for the case of Japan, while there is very weak evidence for Germany and no evidence for the case of the United Kingdom.[14] To check that the results do not hinge on that, in Table 6.b we report the R^2 of the OLS regression of equation (1) in four-year differences, since Table 2 shows that the hypothesis

[14] The evidence for Japan heavily depends on the first 15 years of the sample, where the RER was appreciating fast, due most likely to a Balassa-Samuelson effect, as can be seen in Figure 8.c. There is no evidence of a unit root if one considers the sample from 1975 to 2015.

of unit root processes can be easily rejected for all three RER in this case.[15] One could use higher-frequency data by taking differences over a shorter horizon. However, we are interested in the relatively long swings exhibited by the RER, those that last over a few years, and that is why we focus on the four-year differences. For the interested reader, we show in Appendix C the relationship between the R^2 depicted in Table 6.d in the period 1960–2014 and the number of months for which we take the differences.

The results in Table 6.b show that PCP still account for between 48% and 57% of the real exchange rate variation when we use the data in four-year differences. The R^2 of the regression for the whole period is smaller in the case of Japan (0.57 versus 0.81), which is the country for which the evidence of a unit root in the RER is very strong. For the other two countries, the differences are minimal.

As we showed in Table 5, the prices of the commodities we are using are well known to be highly correlated. One could then guess that it is possible to account for a large fraction of the real exchange rate volatility even if we considerably reduce the number of PCP. To show this, we start by running the regression with 10 PCP, and then we pick the 4 with highest t-statistics.[16] The results with the data in levels are reported in Table 6.c, and the results with the data in four-year differences are reported in Table 6.d. Tables 12–14 in Appendix D report the coefficients of the regressions in Tables 6.c and 6.d.[17]

Tables 6.c and 6.d show that by selecting only four commodity price series, we can still account for between 39% and 79% of the volatility of real exchange rates in levels, and for between 33% and 56% of the volatility of real exchange rates in four-year differences. As before, the problem of the unit root seems to be relevant only for the case of Japan. It is also important to emphasize that PCP can account for a large share of real exchange rate fluctuations in all subperiods we consider, and, in particular, there are no systematic differences in the relationship between PCP and real exchange rates before and after the Bretton Woods system. This goes in line with the alternative hypothesis about the increase in real exchange rate volatility following 1972: that it coincided with an increase in the volatility of fundamentals.

Finally, Figure 2 plots the data versus the respective fitted values for the regressions in four-year differences for the cases of both 10 and 4 PCPs, and also reports the respective correlation between the data and fitted values (equivalent to the square root of the R^2).[18]

[15] Cointegration tests such as Johansen (1991) or Stock and Watson (1993) do not provide evidence of cointegration between real exchange rates and primary commodity prices.

[16] Throughout the paper, we compute t-statistics using the Newey-West heteroskedasticity-and-autocorrelation-consistent standard errors.

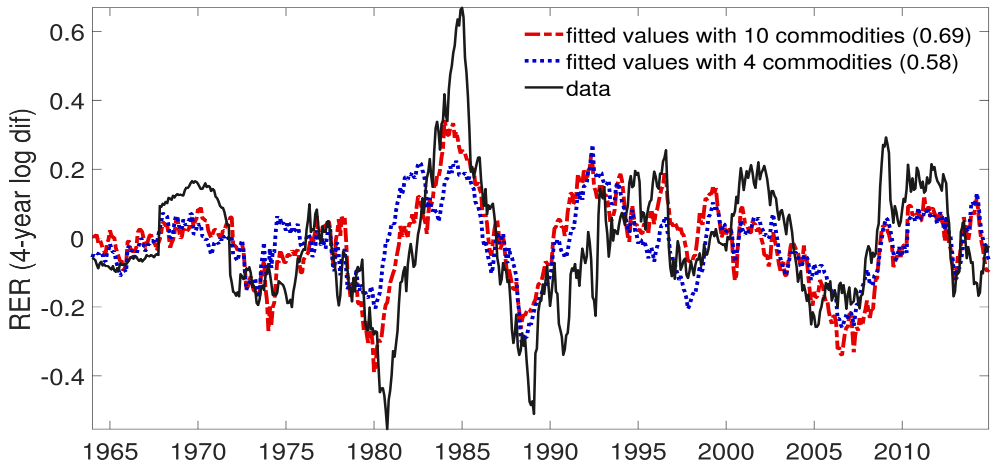
[17] In Appendix E we also show the results for the case in which we choose three commodities.

[18] For the case of the data in levels, see Figure 8 in Appendix B. The results are very similar, except for Japan, which is the country for which the unit root evidence is very high.

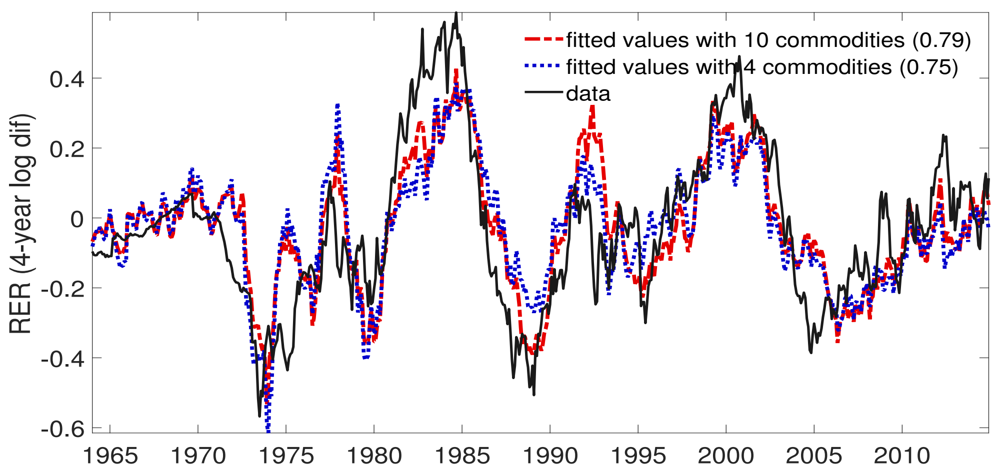
As can be seen, the match is very good in all cases.

Figure 2: Real exchange rates and fitted values, four-year differences.

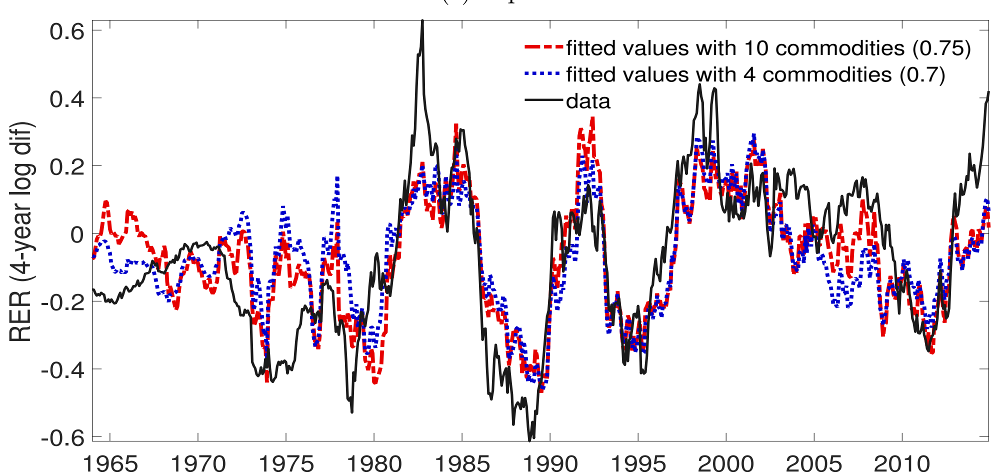
(a) United Kingdom



(b) Germany



(c) Japan



One concern about regression (1) is that the variables are expressed in constant US dollars, so the US CPI appears on both sides of the equation. If its volatility is sufficiently large relative to the volatility of the nominal exchange rate and foreign CPI, that would imply large R^2s. In Table 7.a we show that this is not the case. The table shows the same results as in Table 6.d, but for the case in which we use variables expressed in current US dollars; that is, we do not subtract the log of US CPI from either side of equation (1).[19] Table 7.a shows that the results are invariant to whether we use variables in current or in constant US dollars.

Table 7: R^2 using data in four-year differences with four commodities (best fit): Robustness

	1960–2014	1960–1972	1973–1985	1986–1998	1999–2014
(a) Nominal values					
United Kingdom	0.41	0.60	0.82	0.67	0.63
Germany	0.59	0.86	0.87	0.81	0.75
Japan	0.57	0.89	0.76	0.77	0.79
(c) Non-US pairs					
UK-DEU	0.37	0.73	0.53	0.68	0.72
UK-JPN	0.42	0.64	0.56	0.80	0.68
DEU-JPN	0.35	0.79	0.46	0.69	0.68

Another concern regarding regression (1) is that commodity prices are expressed in US dollars, so they might contain the nominal exchange rate, which in turn would imply that the nominal exchange rate appears on both sides of equation (1). Again, Table 7.b shows that this is not the case. Table 7.b shows the results for the case in which we run the regressions for the bilateral real exchange rates without including the United States. That is, we run the regression in (1) for the bilateral real exchange rates of the United Kingdom versus Germany and Japan, and for Germany versus Japan. The results show that four primary commodities still account for a large fraction of these bilateral real exchange rate fluctuations. And this result holds true for the whole period as well as the subperiods.

3.1 Out-of-sample fit

In the previous regressions, we chose the four primary commodities that obtain a good fit with the real exchange rate, so one could argue that the regressors have been chosen precisely

[19]This procedure is correct to the extent that the sum of the coefficients that multiply the price of the US CPI on the right-hand side is 1 in all cases. In Section 5, we provide a model that rationalizes that restriction.

to match the data. Even in this case, we find it remarkable that a linear combination of such a small number of variables comoves so well with the real exchange rate. To check the robustness of our results to the in-sample selection, we adopt the following procedure. We start by running a regression using data in four-year differences over the period 1960-1972. We drop the six commodities with the lowest t-statistics using the Newey-West standard errors and rerun the regression. Based on the four commodities selected by this procedure and their estimated coefficients, we use observed commodity prices over the following R periods to fit the real exchange rate and store the R fitted values. We next add one observation to the sample and repeat previous regressions to fit the real exchange rates over the following R periods. Repeating this procedure until the end of the sample, we construct time series of out-of-sample fitted real exchange rates over the following $r = 1, 2, ..., R$ periods.

The logic behind this exercise is related to the discussion in subsection 2.1. We interpret the linear regression as a linear approximation of the solution of a model in which the RER and the PCP are jointly determined, as described in equation (2). The constants on that linearization are evaluated at the equilibrium around which the linearization is made. The maintained assumption in this exercise is that those values will not change much in a relatively short period of time, so that the reduced-form estimates could work reasonably well for an interval of time that is not too long, particularly if no major changes occurred.

Figure 3 shows the actual and fitted real exchange rates for the case $r = 6$ months ahead. The out-of-sample fit is remarkable, with a correlation between the fitted and actual values of 0.45 for the United Kingdom, 0.73 for Germany, and 0.64 for Japan.

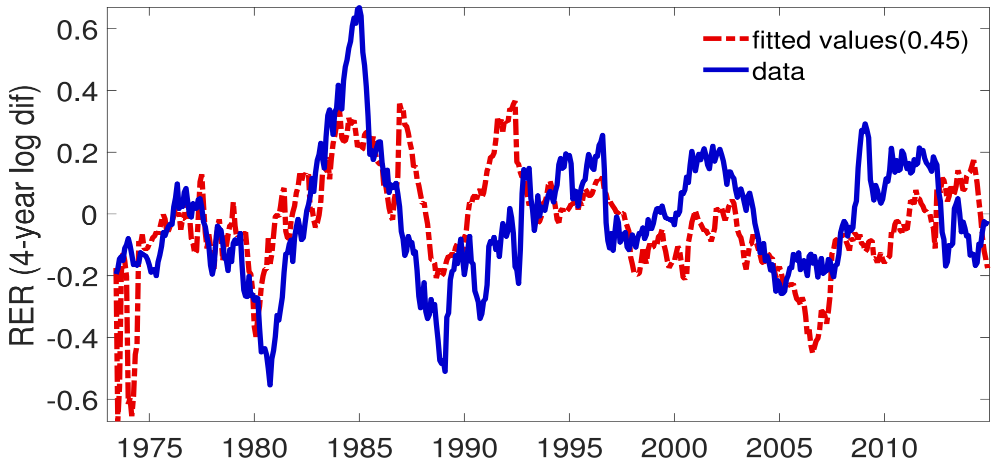
We summarize the results in Figure 4, in which we show the correlation between fitted and actual real exchange rates as we vary the forward window from $r = 1$ to $r = 60$ months ahead. Although the correlations decrease as the fitting horizon increases, they decrease slowly. There is a good out-of-sample fit even using data that are several years old to select the commodities and coefficients to fit real exchange rates today. Overall, we interpret these results as supporting our initial findings that shocks that affect just four commodity prices account for a substantial fraction of real exchange rate movements.

4 Are the results spurious?

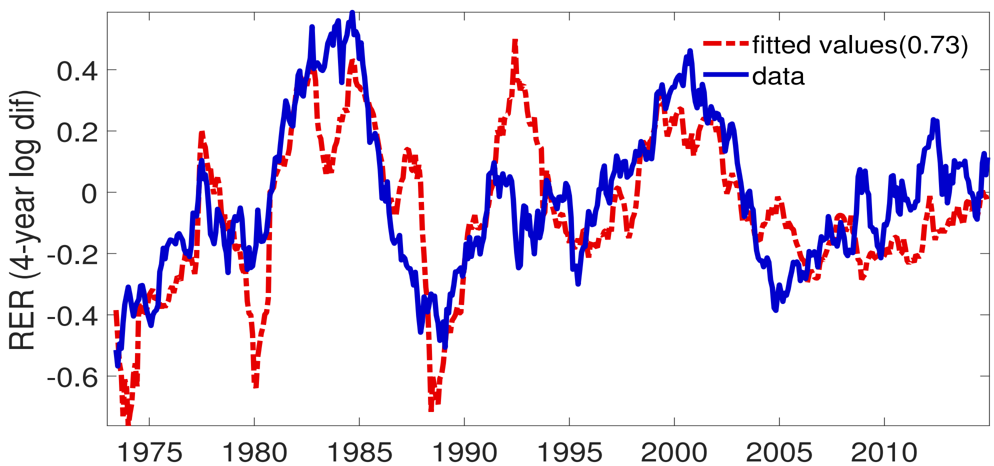
A concern with the previous regressions is to what extent the results could be due to a problem of small sample size. It is well known that, even with stationary series, regressing two orthogonal but highly persistent series could lead to a spurious correlation for moderate sample sizes. To explore this issue, we perform small sample inference by using a parametric bootstrap procedure that generates real exchange rate and commodity price data under the

Figure 3: Out-of-sample fit six months ahead with four commodities (best fit)

(a) United Kingdom



(b) Germany



(c) Japan

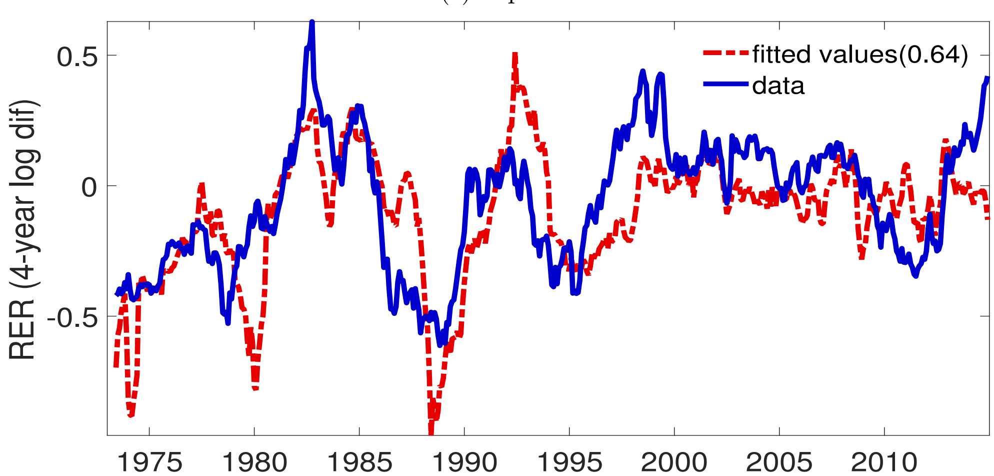
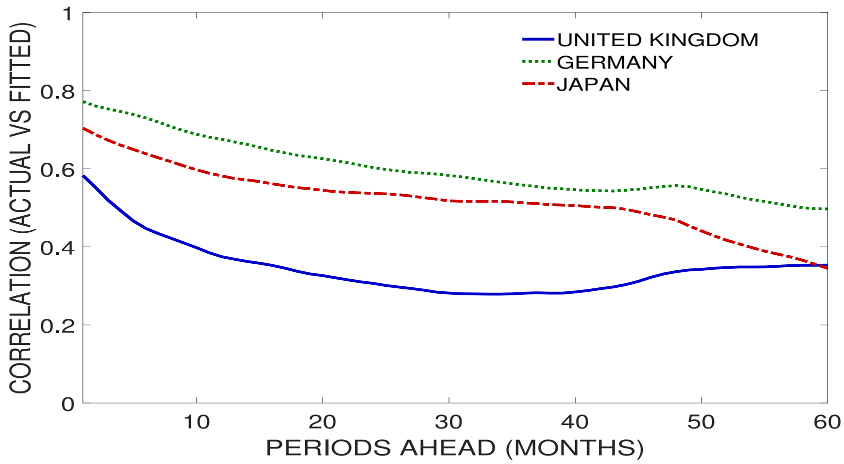


Figure 4: Out-of-sample fit, four commodities, correlations as a function of r (months ahead)



null hypothesis that commodity prices are orthogonal to real exchange rates.

Consider the regressions using data in four-year differences displayed in Table 6.d. Take, for example, the R^2 of 0.56 of Germany-US real exchange rate regression on the four commodity prices with the highest t-statistics. In this case, the bootstrap procedure under the null hypothesis that real exchange rates are orthogonal to commodity prices is as follows.

We first estimate an autoregressive (AR) process for the Germany-US real exchange rate and an independent vector autoregression (VAR) with the four commodity prices used in the regression. In both cases, we use data in four-year differences and choose the lag length of the estimated processes according to the Schwarz information criterion. We then simulate artificial series by feeding into those estimated processes shocks that are orthogonal. By construction, commodity prices and real exchange rates are orthogonal. Hence, a regression of the simulated real exchange rate on the simulated commodity prices from the artificial time series delivers an R^2 that converges to zero as the artificial sample size grows toward infinity. But for a sample size such as ours, the R^2 of the regression is positive. The question is, how common is it to obtain an R^2 of 0.56 under the null hypothesis of orthogonality given an artificial sample of data with the same number of observations that we have?

To compute the small sample distribution of the R^2, we draw 10,000 samples of length 660 (the number of months between January 1960 and December 2014) by resampling from the residuals of the estimated AR and VAR processes and computing artificial real exchange rate and commodity price data. Then, for each artificial sample, we run a regression of the real exchange rate on the four commodity prices and store the associated R^2. Finally, we compare the estimated R^2 of 0.56 with the small sample distribution of R^2s computed with the bootstrap procedure. We use the same procedure to compute small sample distributions of the R^2 for each regression and subsample in Table 6.d.

Figure 5: Small sample distribution of the R^2 over the period 1960–2014

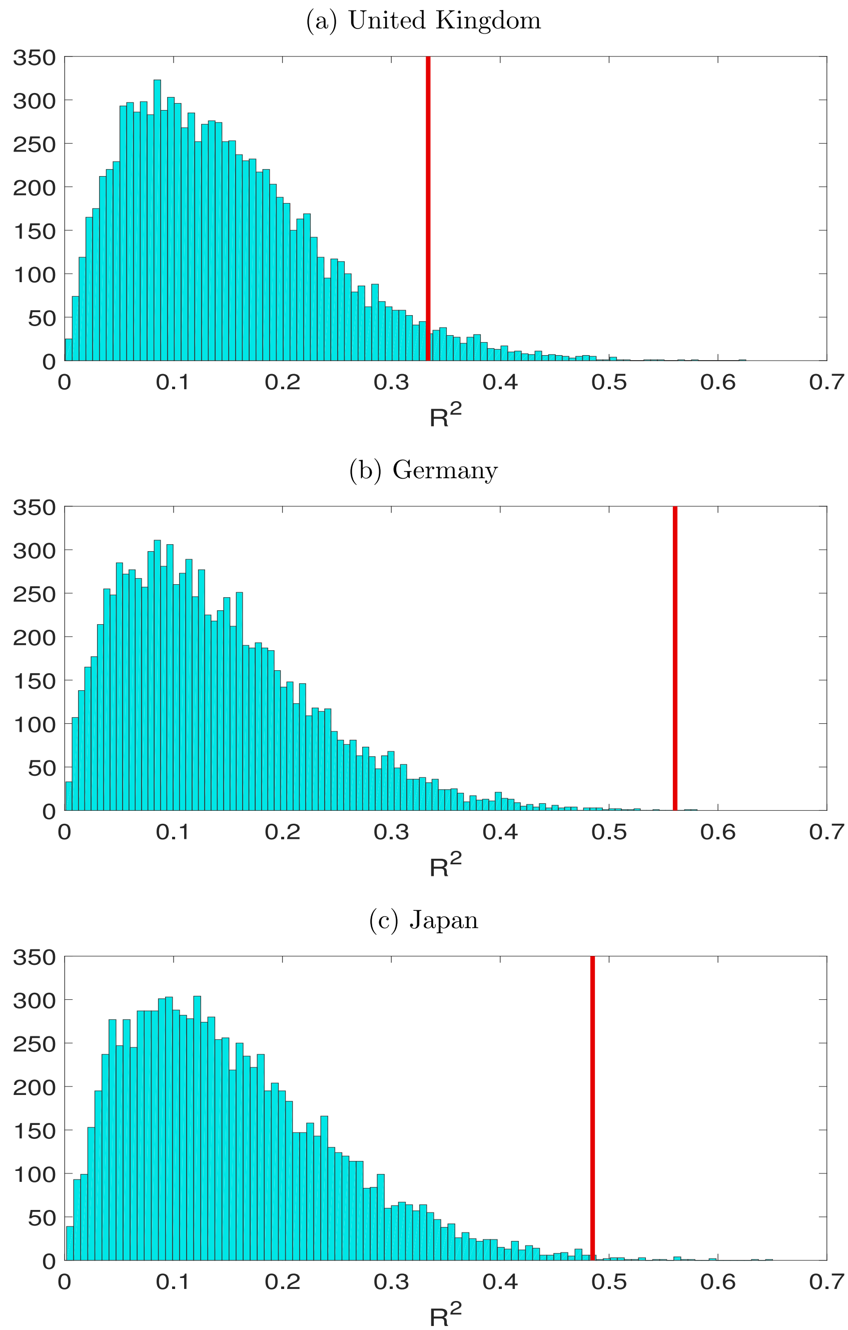


Figure 5 displays the small sample distributions of the R^2 under the null hypothesis of a spurious correlation over the entire sample period (1960-2014) for the three real exchange rates. The vertical lines are the estimated R^2 using the actual data. In all cases, the probability of obtaining an R^2 as large as that estimated in Table 6.d is smaller than 5% and as low as 0% for the case of Germany. The three distributions under the null hypothesis are positively skewed with a mode of about 0.1, which is much smaller than the estimated R^2 in the table.

Table 8: Bootstrap distributions of R^2 under the null hypothesis of orthogonality, with four commodities (best fit) in four-year differences

		Percentiles distribution of R^2				
	$\hat{R}^2$	Median	75	90	95	$\Pr(R^2 \geq \hat{R}^2)$
United Kingdom						
1960-2014	0.33	0.13	0.20	0.27	0.31	0.037
1960-1972	0.72	0.52	0.66	0.75	0.80	0.143
1973-1985	0.82	0.37	0.52	0.64	0.70	0.004
1986-1998	0.63	0.37	0.50	0.61	0.67	0.077
1999-2014	0.58	0.29	0.41	0.53	0.59	0.059
Germany						
1960-2014	0.56	0.13	0.19	0.26	0.31	0.000
1960-1972	0.84	0.56	0.69	0.79	0.83	0.032
1973-1985	0.87	0.49	0.63	0.73	0.78	0.005
1986-1998	0.81	0.40	0.54	0.65	0.71	0.007
1999-2014	0.74	0.30	0.43	0.55	0.61	0.007
Japan						
1960-2014	0.48	0.14	0.21	0.29	0.34	0.003
1960-1972	0.88	0.59	0.72	0.81	0.85	0.022
1973-1985	0.76	0.46	0.60	0.70	0.75	0.045
1986-1998	0.86	0.41	0.55	0.66	0.71	0.001
1999-2014	0.80	0.33	0.46	0.57	0.63	0.002

Table 8 shows statistics of the small sample distributions under the null of orthogonality for the three bilateral real exchange rates and for the five subperiods, together with the probability of observing an R^2 as large as that estimated under the null of orthogonality. For comparison, the table also includes the estimated R^2s from Table 6.d for each country and subperiod. Overall, these results suggest that the estimated correlations are robust for every subperiod and bilateral real exchange rate. Of course, for some subperiods and countries, the small sample distributions are more dispersed and it is not uncommon to observe a relatively large R^2 under the null of orthogonality, especially for smaller sample sizes. For example, although the estimated R^2 for Germany over the period 1964-1972 is 0.84, the median R^2 under the null of orthogonality is 0.56.

We also computed the small sample distributions of the out-of-sample fit exercise displayed in Figure 4. For each country, we created 2,000 artificial correlations as a function of $r = 1, 2, ..., R$ replicating the procedure in Figure 4 but under the null hypothesis that real exchange rates are orthogonal to commodity prices, as we did before. In particular, for each expanding subsample (beginning with the period 1960-1972), we estimate an AR process for the real exchange rate and a VAR process for the 10 commodities independent of the autoregressive process for the real exchange rate. We simulate a history of commodity prices and real exchange rates of the appropriate size and run a regression of the real exchange rate on the commodity prices in four-year differences. We keep the four commodities with the highest t-statistics and rerun the regression. With these commodities and estimated coefficients, we perform the same out-of-sample fit that we did above over the following $r = 1, 2, ..., R$ periods and store the fitted values. We next expand the sample by adding one observation and redo the entire estimation and out-of-sample fit procedure until we use all the available data. Then we compute the correlation of the real exchange rates with their out-of-sample fitted counterparts using the artificial time series. We repeat this procedure 2,000 times and compute a small sample distribution of out-of-sample correlations under the null hypothesis of orthogonality.

Figure 6 displays the median correlation for each country under the null hypothesis (solid line), and the shaded areas represent the 5th and 95th percentiles of the small sample distribution of the correlation as a function of the horizon $r = 1, 2, ..., R$. We also include in the plot the correlation that we estimated above in Figure 4 (dashed lines). In all the cases, we comfortably reject the null hypothesis of orthogonality.

5 The model

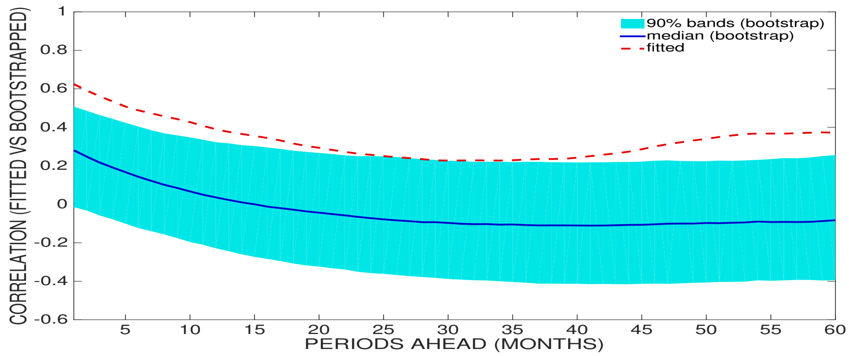
In this section, we present a model that, in equilibrium, delivers a system of equations as in (2). We go into detail, since we want to consider an economy with an input-output matrix that is slightly more complicated than the ones typically used in macro-trade models, so as to explicitly discuss the role of prices of primary commodities, such as oil and wheat, on final goods price indexes.

The discussion is made in the context of a simple Ricardian model, where trade is the result of differences in endowments and productivities. We will not characterize all the equilibrium conditions; rather, we emphasize how final good prices (and therefore real exchange rates) are related to prices of these primary commodities in a competitive equilibrium. Since we want to allow for heterogeneity in labor types and in differentiated intermediate goods, the notation is rather heavy, but the ideas are very simple and well known. The analysis also highlights under which conditions one should expect the PCP to be related to the real exchange rate. We would like to emphasize at the outset that we will derive a relationship between prices, all of them endogenous variables.

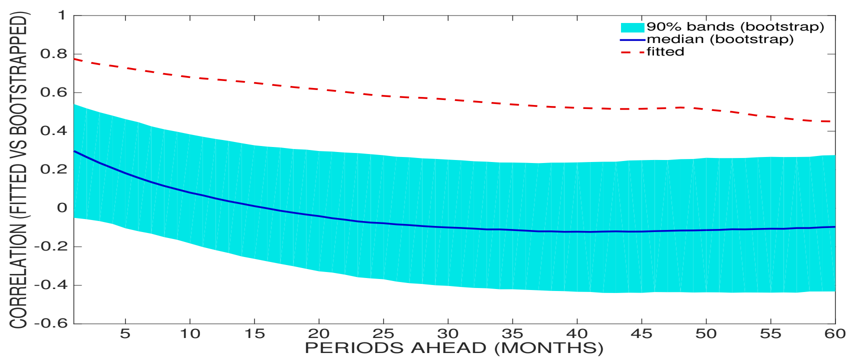
Specifically, we consider a world with a finite number of countries, each one inhabited

Figure 6: Fitted correlations and bootstrap bands under the null hypothesis of orthogonality (with four commodities, best fit)

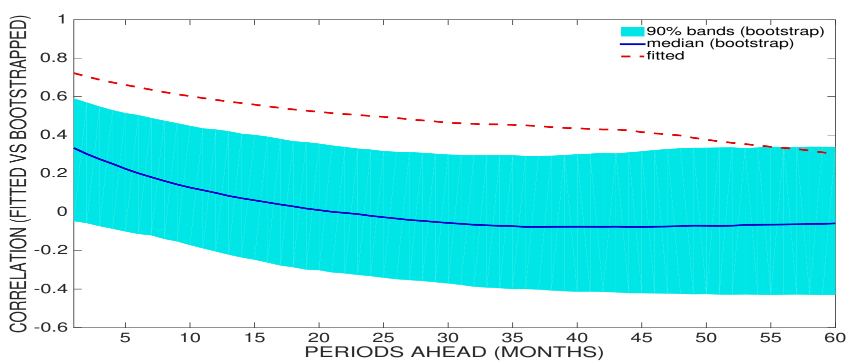
(a) United Kingdom



(b) Germany



(c) Japan



by a representative consumer with standard preferences over a final consumption good C_t. This final consumption good should be seen as an aggregate of a very large number of different varieties but, to save on notation, we maintain the assumption of a single final good.[20] We assume the final good to be nontraded in order to make the model consistent with the overwhelming evidence of the lack of a law of one price in final goods (Engel, 1999). In this sense, the model below adopts the view of Burstein et. al (2003), who argue that an important share of final good prices have a nontraded component. To motivate nominal magnitudes in each country, we assume that a cash-in-advance constraint of the form $P_t C_t \leq M_t$ is imposed on the representative consumer, where P_t is the price of the final good and M_t is the quantity of money.

We will not characterize equilibrium conditions for the household, since all we will exploit is the production structure of the economy. The preferences and the cash-in-advance constraint should be kept in the background for completeness in terms of thinking about how quantities and nominal prices are determined in an equilibrium.

In each country, there are different varieties of labor, intermediate goods, and commodities. In particular, we assume that there are

$$
\begin{aligned}
j &= 1, 2, ..., J \text{ types of labor} \\
i &= 1, 2, ..., N \text{ types of intermediate goods} \\
h &= 1, 2, ..., H \text{ types of commodities.}
\end{aligned}
$$

For simplicity, we assume that all varieties will be produced in each country. If not, the discussion below should consider the possibility that some varieties may not be produced in some countries, making the notation (yet) more cumbersome. This assumption does not affect the result, as will become apparent. We also assume that for each commodity, there is a nontradable fixed factor $E_t(h)$ for all h, which is used in the production of the primary commodities.[21] We imagine that the number of labor varieties and intermediate goods is very large, in the order of thousands. In contrast, in the empirical section, we focused attention on a handful (four) of primary commodities.

All technologies are assumed Cobb-Douglas, even though this assumption implies the unrealistic restriction that sector shares are constant over time.[22] Still, it makes the algebra very simple and the expressions easy to interpret. The theoretical equation implied by this assumption is linear in logs with parameters that are time invariant, so it naturally leads to an equation that can be used in the empirical analysis using the simplest techniques. In Appendix G we show that the relationship between the RER and commodity prices that we

[20] In Appendix H, we show how the model naturally extends to a continuum of nontraded final goods.

[21] For instance, in the case of oil, the oil fields are non-tradable; the oil extracted using the oil fields and other inputs is. In the case of wheat, the grain is tradable, the land used to produce it is not.

[22] For instance, in the United States between 1960 and 2000, the service sector grew from roughly 50% of GDP to 65%, while manufacturing dropped from 25% to 15% of GDP.

derived also holds for general constant returns to scale production functions, but it will not be log-linear.

In what follows, we describe in detail the production structure of one of the economies. To fix ideas, consider the economy whose currency is used for international transactions: the United States in our empirical application. We now describe the environment in this economy without any country-specific index. Those indexes will be introduced when we consider two countries and construct a measure of their bilateral real exchange rates.

Countries may differ in their endowments of labor, $n_t(j)$ for all j, endowments of primary commodities fixed factors, $E_t(h)$ for all h, and in the parameters of their production function, including the total factor productivity associated with each production function. The fixed factors used in the production of commodities and the different varieties of labor are non-traded, while commodities are internationally traded in perfectly competitive markets. For the expressions that we derive below, we do not need to take a stand on how tradable the intermediate goods are.

Production of all goods (final, intermediate, and commodities) requires, in general, inputs of all types of labor. Labor for the production of each of them is aggregated from all varieties using Cobb-Douglas production functions. The total labor endowment of each variety is equal to L_j, which can be country specific.

The final good is produced according to the technology

$$C_t = Z_t^C \left(\prod_{j=1}^{J} [n_t^C(j)]^{\psi^C(j)} \right)^{\alpha} \left(\prod_{i=1}^{N} q_t\left(i\right)^{\varphi(i)} \right)^{1-\alpha},$$

where Z_t^C is productivity, $n_t^C(j)$ is labor of type j, used in the production of final consumption, $q_t\left(i\right)$ is the quantity of intermediate good i used in the production of final consumption, $0 < \alpha < 1$, $\psi^C(j) \geq 0$ for all j, $\varphi(i) \geq 0$ for all i, $\sum_{j=1}^{J} \psi^C(j) = 1$, and $\sum_{i=1}^{N} \varphi(i) = 1$.[23]

Each variety of intermediate good i is produced using labor and primary commodities. The country-specific production function is

$$Q_t\left(i\right) = Z_t^Q\left(i\right) \left(\prod_{j=1}^{J} [n_t^{Q(i)}\left(j\right)]^{\psi^Q(i,j)} \right)^{\beta(i)} \left(\prod_{h=1}^{N} \left[x_t\left(i,h\right)\right]^{\phi(i,h)} \right)^{1-\beta(i)}, \text{ for all } i,$$

where $Q_t\left(i\right)$ is total output of intermediate i, $Z_t^Q\left(i\right)$ is productivity, $n_t^{Q(i)}\left(j\right)$ is the quantity of labor of type j used in the production of intermediate i, $x_t\left(i,h\right)$ is the quantity of primary commodity h used in the production of intermediate i, $\phi\left(i,h\right) \geq 0$ for all i and h, $\psi^Q(i,j) \geq 0$

[23]We write the production functions allowing for all possible inputs to be relevant for production in all cases. But we allow for some of the coefficients to be zero.

for all i and j, $\sum_{h=1}^{N} \phi(i,h) = 1$, $\sum_{j=1}^{J} \psi^{Q}(i,j) = 1$, and $0 < \beta(i) < 1$ for all i.

Finally, in each country there is a technology to produce the commodities given by

$$X_t(h) = Z_t^X(h) \left(\prod_{j=1}^{J} [n_t^{X(h)}(j)]^{\psi^X(h,j)} \right)^{\gamma(h)} E_t(h)^{1-\gamma(h)}, \text{for all } h,$$

where $X_t(h)$ is total output of commodity h, $Z_t^X(h)$ is productivity, $n_t^{X(h)}(j)$ is labor of type j used in the production of commodity h, $E_t(h)$ is the endowment of the fixed factor used in primary commodity h, $\psi^X(i,j) \geq 0$ for all i and j, $\sum_{j=1}^{J} \psi^X(i,j) = 1$ for all i, and $0 < \gamma(h) < 1$ for all h. As the endowment is not traded, as long as $E_t(h) > 0$, a positive amount of the commodity will be produced. Naturally, if $E_t(h) = 0$ for a particular country, production of that commodity will be zero and, as long as some is used in the production of intermediate goods, it will be imported.

5.1 Prices

With perfect competition, prices are equal to marginal costs. With Cobb-Douglas production functions, marginal costs are Cobb-Douglas functions of factor prices. Thus, the logarithm of the price level in the numeraire country will be

$$\ln P_t = \ln\left(\frac{\kappa^C}{Z_t^C}\right) + \alpha \sum_{j=1}^{J} \psi^C(j) \ln W_t(j) + (1-\alpha) \sum_{i=1}^{N} \varphi(i) \ln P_t^Q(i), \tag{6}$$

where P_t is the price of the final good, $W_t(j)$ is the nominal wage of type-j labor, $P_t^Q(i)$ is the price of intermediate good i, and κ^C is a constant that depends on the exponents in the Cobb-Douglas production function.

Similarly, the price of intermediate good i is

$$\ln P_t^Q(i) = \ln\left(\frac{\kappa^{Q(i)}}{Z_t^Q(i)}\right) + \beta(i) \sum_{j=1}^{J} \psi^Q(i,j) \ln W_t(j) + (1-\beta(i)) \sum_{h=1}^{H} \phi(i,h) \ln P_t^X(h), \tag{7}$$

where $P_t^X(h)$ is the price in domestic currency of primary commodity h, and $\kappa^{Q(i)}$ is a constant that depends on parameters of the production functions.

Combining (7) with (6) gives

$$
\begin{aligned}
\ln P_t &= \ln\left(\frac{\kappa^C}{Z_t^C}\right) + (1-\alpha)\sum_{i=1}^{N}\varphi(i)\ln\left(\frac{\kappa^{Q(i)}}{Z_t^Q(i)}\right) \\
&+ \sum_{j=1}^{J}\left[\alpha\psi^C(j) + (1-\alpha)\sum_{i=1}^{N}\varphi(i)\,\beta(i)\psi^Q(i,j)\right]\ln W_t(j) \\
&+ (1-\alpha)\sum_{h=1}^{H}\left[\sum_{i=1}^{N}\varphi(i)\,(1-\beta(i))\phi(i,h)\right]\ln P_t^X(h)
\end{aligned} \tag{8}
$$

Note that weights on all prices and wages are nonnegative, since they are products of exponents in the production functions. They also add up to one because of the Cobb-Douglas assumption on all production functions.[24]

Summarizing, the log of the aggregate price level is a log-linear function of some constants, productivity shocks in final and intermediate goods, $\ln Z_t^C$ and $\ln Z_t^Q(i)$ for all i, wages for the different types of labor, $\ln W_t(j)$ for all j, and prices of primary commodities, $\ln P_t^X(h)$ for all h.

If we let

$$
\begin{aligned}
\mathbf{w}_t &= \left[\ln W_t(1), \ln W_t(2), ..., \ln W_t(J)\right]', \\
\mathbf{p}_t^X &= \left[\ln P_t^X(1), \ln P_t^X(2), ..., \ln P_t^X(H)\right]', \\
\mathbf{z}_t^Q &= \left[\ln Z_t^{Q(1)}, \ln Z_t^{Q(2)}, ..., \ln Z_t^{Q(N)}\right]', \text{and} \\
z_t^C &= \ln Z_t^C,
\end{aligned}
$$

we can write (8) in vector notation as

$$
\ln P_t = a - z_t^C - \Psi_Q \mathbf{z}_t^Q + \Psi_w \mathbf{w}_t + \Psi_X \mathbf{p}_t^X, \tag{9}
$$

in which Ψ_Q, Ψ_w, Ψ_X are row vectors of coefficients which are functions of the exponents in the Cobb-Douglas production functions. As we argued above, the sum of the components of the vector Ψ_w plus the sum of the components of the vector Ψ_X are equal to 1.

Notice that the dimensions of the vectors $\mathbf{z}_t^Q$ and $\mathbf{w}_t$ are likely to be very large, since they involve all the different types of labor and intermediate goods that are used to produce the final good. On the contrary, as we argue below, with a very low dimension vector $\mathbf{p}_t^X$,

[24] The Cobb-Douglas assumption is not required for the property that the final good price is a constant returns to scale function of all factor prices: that property holds as long as technologies are all constant returns to scale. See Appendix G.

one can go a long way in accounting for real exchange rate variability.

Now, we use the fact that labor is used to produce commodities to relate the wages to primary commodity prices, and use those relations to replace the wages in equation (9). As long as the economy produces some commodity h, cost minimization in that industry implies that the type-j nominal wage is given by

$$\begin{aligned} \ln W_t(j) &= \ln P_t^X(h) + \gamma(h)\psi^X(h,j)\ln Z_t^X(h) + (1-\gamma(h))\ln\left(\frac{E_t(h)}{n_t^h(j)}\right) \\ &\quad + \gamma(h)\sum_{\widetilde{j}=1, j\neq\widetilde{j}}^{J} \psi^X\left(h,\widetilde{j}\right)\ln\left[\frac{n_t^{X(h)}(\widetilde{j})}{n_t^{X(h)}(j)}\right], \end{aligned} \tag{10}$$

for all j, as long as $\gamma(h)\psi^X(h,j) > 0$.

Notice that for this equation to hold, it is necessary that the country produces primary commodity h. Thus, if two different countries produce different commodities, the wages will be related to different commodity prices. This heterogeneity is important in order to identify a channel through which commodity prices affect real exchange rates.

Now let $\mathbf{n}_t(h,j)$ be a vector that contains the ratio of inputs in the production of commodity h, all normalized by labor of type j. That is,

$$\mathbf{n}_t(h,j) = \left[\frac{E_t(h)}{n_t^h(j)}, \frac{n_t^{X(h)}(\tilde{j})}{n_t^{X(h)}(j)} \text{ for all } \widetilde{j} = 1, ..., J \text{ and } \widetilde{j} \neq j.\right]'.$$

Then, we can express equation (10) as

$$\ln W_t(j) = \ln P_t^X(h) + \gamma(h)\psi^X(h,j)\ln Z_t^X(h) + \Psi_{\mathbf{n}(h,j)}\mathbf{n}_t(h,j), \tag{11}$$

where $\Psi_{\mathbf{n}(h,j)}$ is a vector of constants and also a function of the share parameters in the Cobb-Douglas production functions, whose elements also add up to 1.

Using (11) to substitute for all wages in equation (9), we can write the price level as a log-linear function of constants; productivity shocks in all sectors, $\mathbf{z}_t = [z_t^C, \mathbf{z}_t^{Q\prime}]'$; ratio of input allocations in some primary commodity industry, denoted by $\mathbf{n}_t$; and primary commodity prices, $\mathbf{p}_t^X$,

$$\ln P_t = a + \Gamma_z\mathbf{z}_t + \Gamma_n\mathbf{n}_t + \Gamma_X\mathbf{p}_t^X,$$

where the sum of the coefficients in the row vector Γ_X (the sum of the coefficients on all primary commodity prices) is equal to 1.[25] Note, also, that the vector of PCP, $\mathbf{p}_t^X$, is country specific despite of being traded goods, since prices are denominated in domestic currency.

[25]See details in Appendix I.

Using the United States as the benchmark economy, we now make explicit, through a supra-index, that the price level in the United States is given by

$$\ln P_t^{USA} = a^{USA} + \Gamma_z^{USA} \mathbf{z}_t^{USA} + \Gamma_n^{USA} \mathbf{n}_t^{USA} + \Gamma_X^{USA} \mathbf{p}_t^{X,USA}. \tag{12}$$

Likewise, we can write the price level in a different country, say the United Kingdom, as

$$\ln P_t^{UK} = a^{UK} + \Gamma_z^{UK} \mathbf{z}_t^{UK} + \Gamma_n^{UK} \mathbf{n}_t^{UK} + \Gamma_X^{UK} \mathbf{p}_t^{X,UK}.$$

Notice that, while the log-linear structure is similar, the coefficients in the equation are country specific, since they depend on each country's production functions. In addition, the shocks are also country specific (they are productivity shocks in the different sectors of each economy), and the way labor is allocated relative to the endowment input into the production of primary commodities, $\mathbf{n}_t^{UK}$, is also country specific. The vector of commodity prices $\mathbf{p}_t^{X,UK}$ is also different from that in the United States, $\mathbf{p}_t^{X,USA}$, but only because the prices are denominated in different currencies.

The law of one price for these primary commodities implies that the prices measured in US dollars, $p_t^{X,USA}(h) = \log P_t^{X,USA}(h)$, are related to the corresponding ones measured in British pounds, $p_t^{X,UK}(h) = \log P_t^{X,UK}(h)$, through

$$p_t^{X,UK}(h) = p_t^{X,USA}(h) + s_t \text{ for all } h, \tag{13}$$

where $s_t = \log S_t$ is the logarithm of the nominal exchange rate. Then, if we let $\iota = [1, 1, ..., 1]'$ denote a vector of ones with H elements, we can use (13) in the solution for the final good price in the United Kingdom to obtain

$$\ln P_t^{UK} = a^{UK} + \Gamma_z^{UK} \mathbf{z}_t^{UK} + \Gamma_n^{UK} \mathbf{n}_t^{UK} + \Gamma_X^{UK} \mathbf{p}_t^{X,USA} + \Gamma_X^{UK} \iota s_t.$$

But since the sum of the coefficients in the vector Γ_X^{UK} is equal to 1, we can write this expression as

$$\ln P_t^{UK} = a^{UK} + \Gamma_z^{UK} \mathbf{z}_t^{UK} + \Gamma_n^{UK} \mathbf{n}_t^{UK} + \Gamma_X^{UK} \mathbf{p}_t^{X,USA} + s_t \tag{14}$$

Subtracting (14) from (12), we obtain an equation relating the bilateral real exchange rate to the primary commodity prices, productivity shocks, and ratios of labor allocations,

$$\begin{aligned} \ln P_t^{USA} - \ln P_t^{UK} + \log S_t &= \left(a^{USA} - a^{UK}\right) + \left(\Gamma_z^{USA} \mathbf{z}_t^{USA} - \Gamma_z^{UK} \mathbf{z}_t^{UK}\right) \\ &\quad + \left(\Gamma_n^{USA} \mathbf{n}_t^{USA} - \Gamma_n^{UK} \mathbf{n}_t^{UK}\right) + \left(\Gamma_X^{USA} - \Gamma_X^{UK}\right) \mathbf{p}_t^{X,USA}. \end{aligned} \tag{15}$$

As long as $\left(\Gamma_X^{USA} - \Gamma_X^{UK}\right) \neq 0$, variations in commodity prices will affect the RER. This will generally be the case if, for instance, the parameters of the production functions or the endowment of commodities $E_t(h)$ differ across countries. Note also that the sum of the coefficients in both Γ_X^{USA} and Γ_X^{UK} is equal to 1, so the sum of the coefficients in $\left(\Gamma_X^{USA} - \Gamma_X^{UK}\right)$ is equal to zero. Thus, we can normalize the commodity prices in $\mathbf{p}_t^{X,USA}$. We can choose to do it with the price level in the United States. This equation provides one particular rationalization for the linear regression postulated in (1) in Section 2.1.

The first term on the right-hand side of (15) is a constant, and the second is a vector of total factor productivity shocks that we treat as unobservable. Available estimates of these shocks are much less volatile than the real exchange rates, so they have been disregarded as the main source of their fluctuations. This is one particular example of the exchange rate disconnect puzzle.

This discussion suggests that the volatility of the real exchange rate (left-hand side) ought to come from the last two terms on the right-hand side. The first of the two terms is differences in the allocation of labor types in the production of commodities, but we have no information on those. The second term is the one we considered: the primary commodity prices. As we already mentioned, they are an attractive candidate, since they are both very volatile and very persistent, a property that will be inherited by real exchange rates as long as $\left(\Gamma_X^{USA} - \Gamma_X^{UK}\right) \neq 0$. As we argued above, that means that the economies involved must have different production structures.

6 Selecting commodities based on US trade data

In Section 3 we showed the results with the four PCP that make the best fit with the real exchange rates. This set (possibly) varies by country pair and subperiod, and whether we use data in levels or in four-year differences. In this section, we explore an alternative approach based on the theory presented above: we choose the set of commodities based on US trade data and keep it fixed for all subperiods and country pairs.

Equation (15) shows that in order for the primary commodity price to explain a large fraction of real exchange rate fluctuations, a necessary condition is that the commodity price must be an important input (has a large share) in the production structure of one of the economies in the country pair. As we mentioned before, the difference in shares is what is crucial, but it can only be observed if the primary commodity is an important input in at least one of the economies. Based on that information, in this section we show the same set of results as in Section 3, but we choose as regressors the four commodities with the largest trade share for the United States. This is, admittedly, a very crude approximation to the data using the model of the previous section, but it has the advantage that the four commodities have not been chosen to fit the data. We see this exercise as a first approximation to using the model to discipline our choices in the empirical analysis.

Table 11 in Appendix A.1 shows the trade data for each country and commodity that we analyze in this paper. We choose the four commodities that are the most traded in the United States according to Table 11: petroleum, fish, timber, and gold. We report the results in Tables 9 and 10, and in Figure 7.

As can be seen, the results when we choose the set of four PCP based on US trade data are still very good. However, it is important to discuss some of the differences. First, the R^2s in Table 10 are lower than in the case for the best-fit commodities (Table 8). This should be expected, since the four commodities were chosen to maximize R^2. But, with the exception of the United Kingdom, the differences are not very large. Second, the bootstrap exercise based on the out-of-sample fit exercise shows very similar results for Germany and Japan and somewhat worse for the United Kingdom. Indeed, the main difference with the previous analysis is the out-of-sample fit for the UK, in which case the curve is within the 90% confidence interval, very marginally so for the first months but getting worse as the period length grows longer.

Table 9: R^2 with four commodities (largest US-trade share)

	1960–2014	1960–1972	1973–1985	1986–1998	1999–2014
			(a) Level		
United Kingdom	0.32	0.52	0.63	0.38	0.34
Germany	0.48	0.50	0.55	0.19	0.56
Japan	0.59	0.21	0.18	0.55	0.58
			(b) Four-year differences		
United Kingdom	0.25	0.63	0.73	0.25	0.46
Germany	0.53	0.89	0.71	0.50	0.69
Japan	0.44	0.71	0.52	0.82	0.68

7 Conclusion

In this paper, we provide empirical evidence that points toward a common factor that moves a handful of primary commodity prices on the one hand and real exchange rates between the United States and the United Kingdom, Germany, and Japan on the other. Both sets of variables are volatile and very persistent. Moreover, during decades in which commodity prices are particularly volatile, so are commodity prices. More specifically, with

Table 10: Bootstrap distributions of R^2 under the null hypothesis of orthogonality, with four commodities (largest US-trade share) in four-year differences

		Percentiles distribution of R^2				
	$\hat{R}^2$	Median	75	90	95	$\Pr(R^2 \geq \hat{R}^2)$
United Kingdom						
1960-2014	0.25	0.13	0.20	0.27	0.31	0.134
1960-1972	0.63	0.51	0.64	0.74	0.79	0.274
1973-1985	0.73	0.44	0.59	0.70	0.76	0.070
1986-1998	0.25	0.37	0.50	0.62	0.68	0.734
1999-2014	0.46	0.30	0.43	0.54	0.59	0.188
Germany						
1960-2014	0.53	0.13	0.20	0.27	0.33	0.000
1960-1972	0.89	0.53	0.68	0.77	0.81	0.005
1973-1985	0.71	0.45	0.60	0.72	0.77	0.106
1986-1998	0.50	0.40	0.54	0.66	0.71	0.313
1999-2014	0.69	0.33	0.47	0.58	0.64	0.024
Japan						
1960-2014	0.44	0.13	0.20	0.27	0.32	0.007
1960-1972	0.71	0.54	0.67	0.77	0.82	0.190
1973-1985	0.52	0.47	0.62	0.74	0.79	0.421
1986-1998	0.82	0.42	0.56	0.67	0.73	0.007
1999-2014	0.68	0.35	0.48	0.59	0.65	0.033

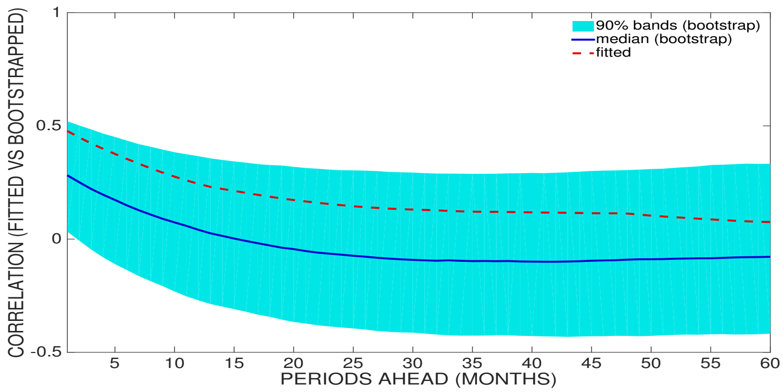
only four of these prices, one can account for between one-third to one-half of the volatility of the real exchange rates for a period that lasts more than half a century.

This relationship has been known to hold for small open economies, as has been long recognized in the literature. The results of this paper imply that similar relationships have the potential to go a long way in rationalizing the behavior of the three real exchange rates between these large developed economies.

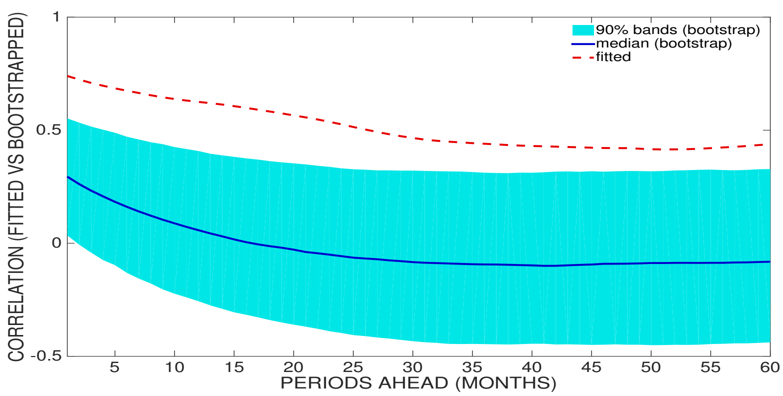
Theoretical models of real exchange rates for large developed economies have almost exclusively focused on trade in final differentiated goods. A challenge for the literature has been to deliver large fluctuations in prices (the real exchange rates) without large movements in quantities, since quantities do not seem to move nearly as much in the data. A natural way out of the puzzle is to impose small enough elasticities in the models, since these elasticities deliver large changes in prices without large changes in quantities. However, independent estimates of the substitutability between final goods deliver numbers that are much higher than what would be required to match the data. On the contrary, the elasticities of demand and supply for the primary commodities that we consider in the paper are known to be very small, so that could be a way out of the puzzle. Incorporating the commodities into a model

Figure 7: Fitted correlations and bootstrap bands under the null hypothesis of orthogonality (with four commodities, largest US-trade share)

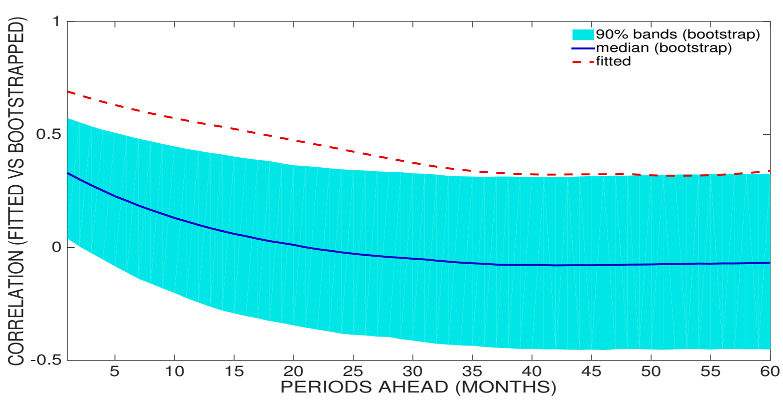
(a) United Kingdom



(b) Germany



(c) Japan



with multiple large economies places a long research path ahead of us. The results of this paper suggest that the path is worth pursuing.

References

[1] Betts, C. M. and T. J. Kehoe. “U.S. Real Exchange Rate Fluctuations and Relative Price Fluctuations,” Staff Report 334, Federal Reserve Bank of Minneapolis (2004).

[2] Burstein, A. T., J. C. Neves, and S. Rebelo. “Distribution Costs and Real Exchange Rate Dynamics During Exchange-Rate-Based Stabilizations,” *Journal of Monetary Economics*, 50(6), 1189-1214 (2003).

[3] Chari, V.V., P. J. Kehoe and E. R. McGrattan. “Can Sticky Price Models Generate Volatile and Persistent Real Exchange Rates?,” *Review of Economic Studies*, 69(3), 533-563 (2002).

[4] Chen, Y.-C. and K. Rogoff. “Commodity Currencies,” *Journal of International Economics*, 60(1), 133-160 (2003).

[5] Eaton, J., B. Neiman, and S. Kortum. “Obstfeld and Rogoff’s International Macro Puzzles: A Quantitative Assessment,” *Journal of Economic Dynamics and Control*, 72(C), 5-23 (2016).

[6] Engel, C. “Accounting for U.S. Real Exchange Rate Changes,” *Journal of Political Economy*, 107(3), 507-538 (1999).

[7] Hevia, C. and J. P. Nicolini. “Optimal Devaluations,” *IMF Economic Review*, 61(1), 22-51 (2013).

[8] Itskhoki, O. and D. Mukhin. “Exchange Rate Disconnect in General Equilibrium,” Working Paper, Princeton University (2017).

[9] Johansen, S. “Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models,” *Econometrica*, 59(6), 1551-1580 (1991).

[10] Meese, R. A. and K. Rogoff. “Empirical Exchange Rate Models of the Seventies: Do They Fit Out of Sample?,” *Journal of International Economics*, 14(1-2), 3-24 (1983).

[11] Mussa, M. “Nominal Exchange Rate Regimes and the Behavior of Real Exchange Rates: Evidence and Implications,” *Carnegie-Rochester Conference Series on Public Policy*, 25(1), 117-214 (1986).

[12] Obstfeld, M. and K. Rogoff. “Exchange Rate Dynamics Redux,” *Journal of Political Economy*, 103(3), 624-660 (1995).

[13] Obstfeld, M. and K. Rogoff. "The Six Mayor Puzzels in International Macroeconomics: Is There a Common Cause?," in B. S. Bernanke and K. Rogoff (eds.), *NBER Macroeconomics Annual 2000*, vol. 15, pp. 339-390. Cambridge, MA: MIT Press (2001).

[14] Stock, J. H. and M. W. Watson. "A Simple Estimator of Cointegrating Vectors in Higher Order Integrated Systems," *Econometrica*, 61(4), 783-820 (1993).

A Data

We used the official monthly series (end of period) for the nominal exchange rates, and the monthly Consumer Price Index (CPI) of each country for the price indexes. These series were downloaded from Global Financial Data. Regarding the commodity price series, they are from the World Bank *Commodity Price Data* (Pink Sheet) and the United Nations (UNCTAD*stat*). We excluded natural gas, coal, and iron because of data availability. We performed the experiments in the paper using sugar instead of gold (which is also a store of value), and the results are virtually the same. The data sources for the price series of each commodity are as follows:

(1) Petroleum - Brent crude oil. Source: Global Financial Data, Ticker: BRT_D.
(2) Fish - price of fish meal. Source: UNCTAD*stat*.
(3) Meat - price of beef. Source: World Bank *Commodity Price Data*.
(4) Aluminum - Source: World Bank *Commodity Price Data*.
(5) Copper: - Source: World Bank *Commodity Price Data*.
(6) Gold - Source: World Bank *Commodity Price Data*.
(7) Wheat - US, $n°$1, hard red winter. Source: World Bank *Commodity Price Data*.
(8) Maize - Source: World Bank *Commodity Price Data*.
(9) Timber - Logs, Malaysia. Source: World Bank *Commodity Price Data*.
(10) Cotton - Cotton Outlook A index. Source: World Bank *Commodity Price Data*.

A.1 Trade data

Trade data were obtained from the United Nations *Comtrade* Database.[26] World trade (exports+imports) for each commodity and its total were computed as the sum of trade over all the countries in the dataset.

[26] Available online at https://comtrade.un.org/data.

Table 11: Share of imports and exports in each country (% average in 1990–1999)

	United States		United Kingdom		Germany		Japan	
	Imports	Exports	Imports	Exports	Imports	Exports	Imports	Exports
Petroleum	8.5	1.2	3.4	6.0	5.2	0.7	13.1	0.4
Fish	1.0	0.6	0.7	0.5	0.5	0.2	5.0	0.2
Meat	0.3	0.9	0.7	0.6	1.1	0.4	2.4	0.0
Aluminium	0.4	0.2	0.3	0.2	0.5	0.1	1.5	0.0
Copper	0.2	0.2	0.3	0.0	0.4	0.1	1.2	0.1
Gold	0.3	0.9	0.0	0.1	0.3	0.2	0.8	0.1
Wheat	0.0	0.8	0.1	0.3	0.1	0.2	0.4	0.0
Maize	0.0	1.1	0.1	0.0	0.1	0.0	0.8	0.0
Timber	0.9	1.0	0.8	0.0	0.5	0.2	3.5	0.0
Cotton	0.0	0.5	0.0	0.0	0.1	0.0	0.3	0.0
SUM	**11.8**	**7.2**	**6.5**	**7.7**	**8.7**	**2.1**	**29.0**	**0.8**

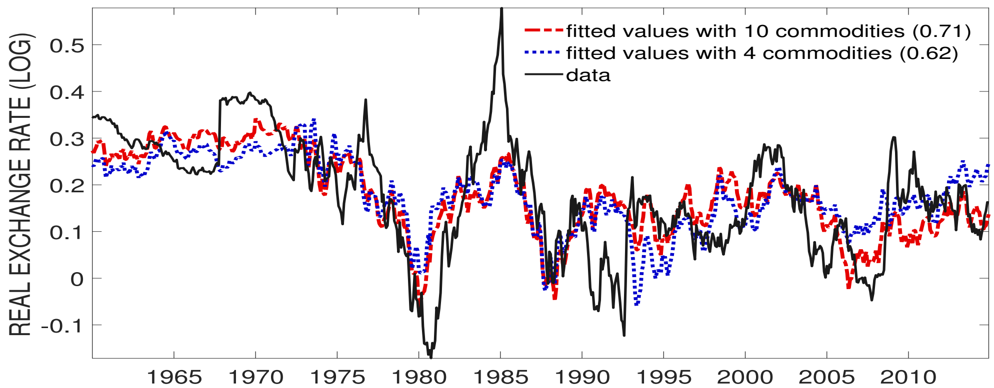
Source: Comtrade.

ONLINE APPENDIX
(not for publication)

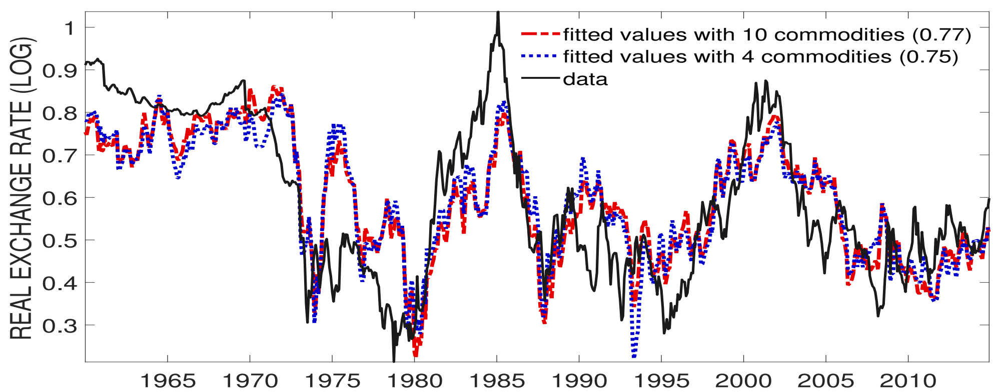
B Real exchange rates and fitted values (level)

Figure 8: Real exchange rates and fitted values, level.

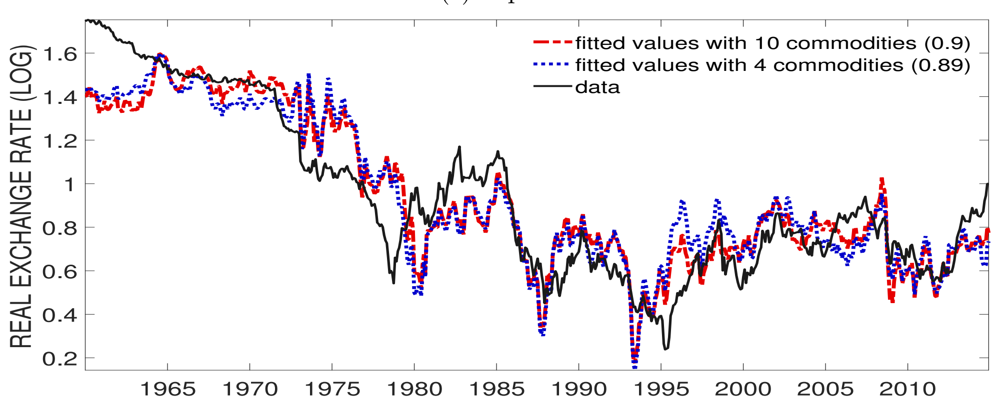
(a) United Kingdom



(b) Germany

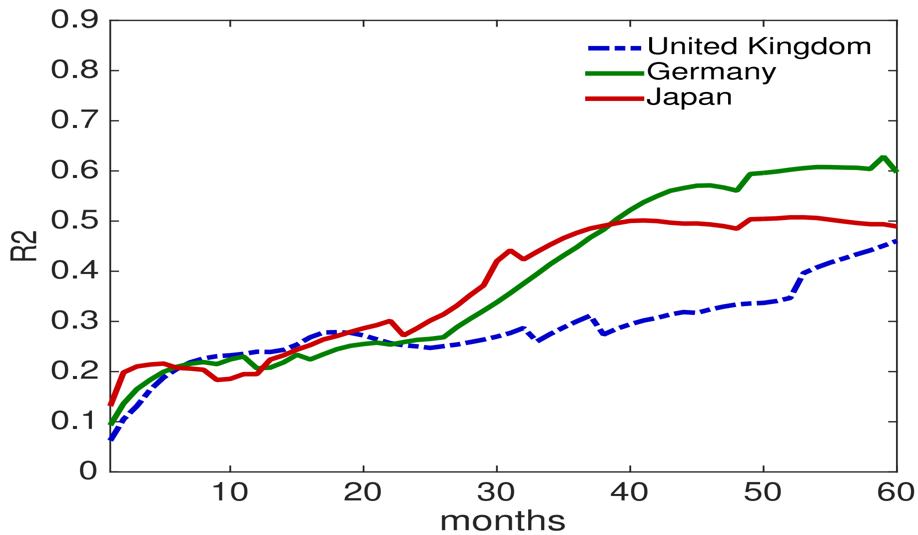


(c) Japan



C R2s as a function of the number of months for which we take the differences

Figure 9: R2s as a function of the number of months for which we take the differences, four commodities (best fit), 1960–2014



D Regression coefficients

Table 12: Regression coefficients: United Kingdom

	1960-2014	1960-1972	1973-1985	1986-1998	1999-2014
			(a) Level		
Oil		-0.317***		-0.033	
Fish				-0.189***	
Meat	0.078		-0.204***		-0.035
Aluminium	-0.113**	-0.045	-0.198**		
Copper				-0.178***	-0.186***
Gold			-0.250***		0.228
Wheat				0.254***	-0.122***
Maize	0.120***	-0.280***			
Timber	-0.230***	-0.010			
Cotton			0.055		
			(b) Four-year differences		
Oil				-0.012	
Fish	-0.213***	-0.252***		-0.218***	-0.162**
Meat		-0.176**	-0.322***	-0.457***	-0.139***
Aluminium	-0.197***	-0.456***			
Copper			0.402***	-0.127***	-0.196***
Gold			-0.329***		0.290***
Wheat					
Maize	0.121**				
Timber			-0.211***		
Cotton	0.027	-0.439***			

Note: Superscripts *, **, and *** denote statistical significance at 10%, 5%, and 1%, respectively.

Table 13: Regression coefficients: Germany

	1960-2014	1960-1972	1973-1985	1986-1998	1999-2014
			(a) Level		
Oil		-0.338***			
Fish	-0.234***				
Meat	0.095			0.218***	0.193**
Aluminium		0.575***	-0.612***		
Copper		-0.085***		-0.150**	-0.151***
Gold			0.120***	-0.063	-0.033
Wheat			-0.449***		-0.106**
Maize	0.139		0.302*		
Timber	-0.450***	0.088*		-0.138***	
Cotton					
			(b) Four-year differences		
Oil					
Fish	-0.319***	-0.095***	-0.194**		-0.375***
Meat			-0.005		
Aluminum			-0.541***		
Copper				-0.231***	
Gold				-0.138**	-0.186***
Wheat	-0.137*		-0.233***		-0.279***
Maize	0.223*	-0.032		0.252***	0.316**
Timber	-0.314***	0.282***		-0.149***	
Cotton		-0.537***			

Note: Superscripts *, **, and *** denote statistical significance at 10%, 5%, and 1%, respectively.

Table 14: Regression coefficients: Japan

	1960-2014	1960-1972	1973-1985	1986-1998	1999-2014
			(a) Level		
Oil		-0.331***	0.018		
Fish					0.393***
Meat		-0.313**	-0.090*	0.299***	
Aluminium		0.932***	-0.375***		
Copper		-0.124**		-0.137**	
Gold	-0.238***			-0.122**	-0.327***
Wheat	0.289**				0.200***
Maize	0.282***		0.239***		
Timber	-0.529***			-0.371***	
Cotton					-0.171***
			(b) Four-year differences		
Oil	0.173**			0.270***	
Fish			-0.318***	0.241***	0.346***
Meat			-0.027		
Aluminium	-0.170**		-0.434***	-0.260***	
Copper		-0.029***	0.109		
Gold		-0.290***			-0.532***
Wheat					0.205***
Maize		-0.244***			
Timber	-0.396***			-0.352***	
Cotton	-0.124*	-0.318***			-0.130***

Note: Superscripts *, **, and *** denote statistical significance at 10%, 5%, and 1%, respectively.

E Selecting three commodities (best fit)

Table 15: R^2 with three commodities (best fit)

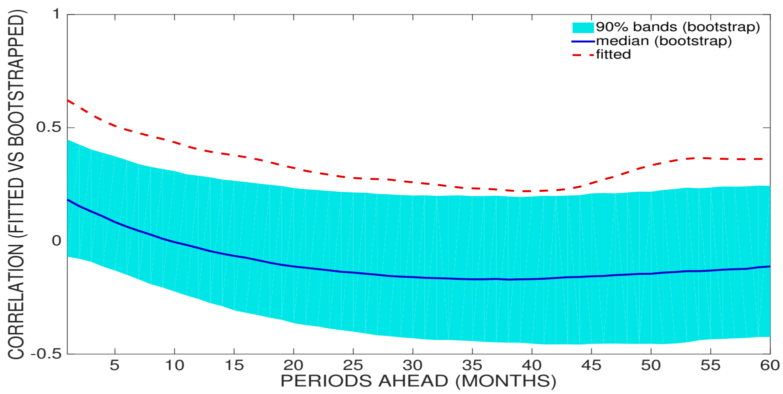
	1960–2014	1960–1972	1973–1985	1986–1998	1999–2014
			(a) Level		
United Kingdom	0.37	0.66	0.70	0.51	0.51
Germany	0.54	0.66	0.81	0.37	0.65
Japan	0.78	0.78	0.42	0.66	0.38
			(b) Four-year differences		
United Kingdom	0.24	0.68	0.78	0.63	0.39
Germany	0.53	0.84	0.80	0.72	0.72
Japan	0.46	0.81	0.58	0.73	0.69

Table 16: Bootstrap distributions of R^2 under the null hypothesis of orthogonality, with three commodities (best fit) in four-year differences

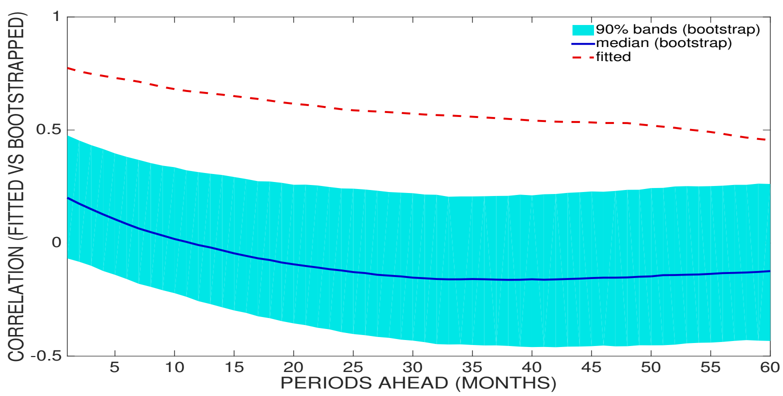
		Percentiles distribution of R^2				
	$\hat{R}^2$	Median	75	90	95	$\Pr(R^2 \geq \hat{R}^2)$
United Kingdom						
1960-2014	0.24	0.09	0.16	0.23	0.28	0.088
1960-1972	0.68	0.48	0.64	0.75	0.80	0.187
1973-1985	0.78	0.32	0.47	0.60	0.67	0.006
1986-1998	0.63	0.30	0.45	0.57	0.63	0.050
1999-2014	0.39	0.24	0.36	0.47	0.54	0.203
Germany						
1960-2014	0.53	0.10	0.17	0.24	0.29	0.000
1960-1972	0.84	0.37	0.52	0.64	0.70	0.002
1973-1985	0.80	0.31	0.47	0.60	0.66	0.005
1986-1998	0.72	0.33	0.48	0.61	0.67	0.024
1999-2014	0.72	0.23	0.36	0.49	0.57	0.005
Japan						
1960-2014	0.46	0.10	0.17	0.24	0.29	0.003
1960-1972	0.81	0.43	0.59	0.71	0.77	0.024
1973-1985	0.58	0.32	0.48	0.61	0.67	0.130
1986-1998	0.73	0.30	0.44	0.56	0.63	0.011
1999-2014	0.69	0.26	0.40	0.52	0.59	0.012

Figure 10: Fitted correlations and bootstrap bands under the null hypothesis of orthogonality (with three commodities, best fit)

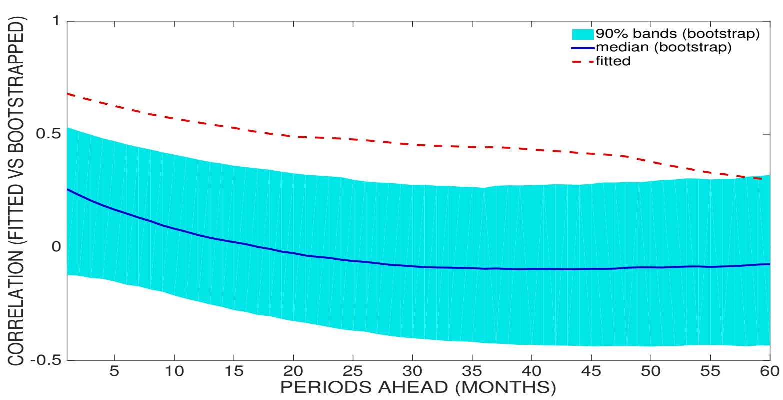
(a) United Kingdom



(b) Germany



(c) Japan



F Multiple representations of the real exchange rate

In this section we show that the model admits many representations of the bilateral real exchange rate. All of these representations hold simultaneously in equilibrium and are derived by substituting different equilibrium conditions into the definition of the real exchange rate. To make the discussion as simple as possible, we develop our arguments using an economy with one non-traded final good, Y_t, one traded intermediate good, q_t, two commodities, $x_t(1)$ and $x_t(2)$, and one type of labor. In terms of the previous notation, $J = N = 1$ and $H = 2$. We use the symbol "$\sim$" to represent variables from an arbitrary foreign country.

The nontraded final good is produced with a Cobb-Douglas technology

$$Y_t = \bar{\alpha} Z_t^y \left(n_t^y\right)^{\alpha} \left(q_t\right)^{1-\alpha},$$

where $\bar{\alpha} = \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)}$. The technology to produce intermediate goods is

$$Q_t = \bar{\beta} Z_t^q \left(n_t^q\right)^{\beta_n} x_t(1)^{\beta_1} x_t(2)^{\beta_2},$$

where $\bar{\beta} = \beta_n^{-\beta_n} \beta_1^{-\beta_1} \beta_2^{-\beta_2}$ and $\beta_n + \beta_1 + \beta_2 = 1$. Finally, commodity $h = 1, 2$ is produced with the production function

$$X_t(h) = Z_t^x(h) \left(n_t^x(h)\right)^{\eta_h} E(h)^{1-\eta_h}.$$

As mentioned above, we set $E(h) = 0$ if the economy is unable to produce commodity h.

Competitive markets imply that nominal prices equal marginal costs. In the final and intermediate goods sectors these conditions are, respectively,

$$P_t = (W_t)^{\alpha} (P_t^q)^{1-\alpha} / Z_t^y \tag{16}$$

$$P_t^q = (W_t)^{\beta_n} \left(P_t^x(1)\right)^{\beta_1} \left(P_t^x(2)\right)^{\beta_2} / Z_t^q, \tag{17}$$

where $P_t^x(h)$ denotes the nominal price of commodity $h = 1, 2$.

The two commodities and the intermediate good can be internationally traded, possibly with some frictions represented by trade taxes; these could be actual taxes or any other implicit impediment to trade that drives a wedge between home and foreign prices. If we let S_t denote the nominal exchange rate, defined as foreign currency per unit of domestic currency, the laws of one price in the intermediate goods and commodities sectors are, respectively,

$$\begin{aligned} S_t P_t^x(1)\, T_t^x(1) &= \tilde{P}_t^x(1) \qquad (18) \\ S_t P_t^x(2)\, T_t^x(2) &= \tilde{P}_t^x(2) \qquad (19) \\ S_t P_t^q T_t^q &= \tilde{P}_t^q, \qquad (20) \end{aligned}$$

where T_t^q and $T_t^x(h)$ for $h = 1, 2$ are (gross) taxes on foreign trade.

We now consider two representations of the bilateral real exchange rate in terms of commodity prices and nominal wages measured in a common currency. The two representations differ in whether we replace the law of one price in intermediate goods (20) in the price indexes (16). In particular, introducing the intermediate good price index (17) into the final good price index (16) and taking logs gives[27]

$$p_t = (\alpha + (1-\alpha)\,\beta_n) w_t + (1-\alpha)\,\beta_1 p_t^x(1) + (1-\alpha)\,\beta_2 p_t^x(2) - (1-\alpha)\, z_t^q - z_t^y.$$

Using the equivalent expression for the foreign country and the laws of one price (18) and (19), we derive the first representation of the bilateral real exchange rate in terms of commodity prices and nominal wages,

$$\begin{aligned} p_t + s_t - \tilde{p}_t &= [(1-\alpha)\,\beta_1 - (1-\tilde{\alpha})\,\tilde{\beta}_1] p_t^x(1) + [(1-\alpha)\,\beta_2 - (1-\tilde{\alpha})\,\tilde{\beta}_2] p_t^x(2) \\ & \quad [\alpha + (1-\alpha)\,\beta_n]\, w_t - [\tilde{\alpha} + (1-\tilde{\alpha})\,\tilde{\beta}_n]\,(\tilde{w}_t - s_t) + \varepsilon_{1t}, \qquad (21) \end{aligned}$$

where ε_{1t} is a bundle of productivity shocks and trade taxes defined as

$$\varepsilon_{1t} = \tilde{z}_t^y - z_t^y + (1-\tilde{\alpha})\,\tilde{z}_t^q - (1-\alpha)\, z_t^q - (1-\tilde{\alpha})\,[\tilde{\beta}_1 t_t^x(1) + \tilde{\beta}_2 t_t^x(2)].$$

Equation (21) is one representation of the real exchange rate in terms of commodity prices and wages. We can also use the law of one price (17) to obtain a similar representation of the real exchange rate. In particular, using equations (17), (18), and (19), the law of one price for intermediate goods (20) implies the following relation between home and foreign wages:

$$\beta_n w_t + (\beta_1 - \tilde{\beta}_1) p_t^x(1) + (\beta_2 - \tilde{\beta}_2) p_t^x(2) + t_t^q - z_t^q = \tilde{\beta}_n(\tilde{w}_t - s_t) + \tilde{\beta}_1 t_t^x(1) + \tilde{\beta}_2 t_t^x(2) - \tilde{z}_t^q.$$

[27] We use lowercase letters to denote the natural logarithm of the corresponding uppercase letters. In particular, $p_t = \log P_t$, $t_t^x(h) = \log T_t^x(h)$, and so forth.

Therefore, equation (21) can also be written as

$$
\begin{aligned}
p_t + s_t - \tilde{p}_t \;\; &= \;\; (\tilde{\alpha}\tilde{\beta}_1 - \alpha\beta_1)p_t^x(1) + (\tilde{\alpha}\tilde{\beta}_2 - \alpha\beta_2)p_t^x(2) \\
&\quad + \alpha(1-\beta_n)w_t - \tilde{\alpha}(1-\tilde{\beta}_n)(\tilde{w}_t - s_t) + \varepsilon_{2t},
\end{aligned} \tag{22}
$$

where ε_{2t} is given by

$$
\varepsilon_{2t} = \tilde{z}_t^y - z_t^y + \alpha z_t^q - \tilde{\alpha}\tilde{z}_t^q + \tilde{\alpha}\tilde{\beta}_1 t_t^x(1) + \tilde{\alpha}\tilde{\beta}_2 t_t^x(2) - t_t^q.
$$

The two representations (21) and (22) emphasize the relation between the bilateral real exchange rate, nominal wages measured in a common currency, productivity shocks, and trade taxes. Yet, the interpretation of the parameters multiplying wages and commodities changes depending on what equation we use to represent the real exchange rate.

It is also possible to express the real exchange rate in terms of commodity prices, allocations, shocks, and trade taxes. Suppose first that the home and foreign countries produce the same commodity, let us say $x_t(1)$. The first-order conditions for the optimal choice of labor in the commodity sector $h = 1$ in the home and foreign countries are given, respectively, by

$$
\begin{aligned}
w_t = &p_t^x(1) + z_t^x(1) - (1-\eta_1)\log n_t^x(1) + \log(\eta_1 E(1)^{1-\eta_1}) \\
\tilde{w}_t = &\tilde{p}_t^x(1) + \tilde{z}_t^x(1) - (1-\tilde{\eta}_1)\log \tilde{n}_t^x(1) + \log(\tilde{\eta}_1 \tilde{E}(1)^{1-\tilde{\eta}_1}).
\end{aligned}
$$

Using the law of one price (18) in the foreign first-order condition and replacing the two wage equations in (22) delivers the following expression for the real exchange rate in terms of commodity prices, labor allocations, productivity shocks, and trade taxes:

$$
p_t + s_t - \tilde{p}_t = (\alpha\beta_2 - \tilde{\alpha}\tilde{\beta}_2)(p_t^x(1) - p_t^x(2)) + \varepsilon_{3t}, \tag{23}
$$

where ε_{3t} depends on productivity shocks, trade taxes, and labor allocations as follows:

$$
\begin{aligned}
\varepsilon_{3t} = &\varepsilon_{2t} + \alpha(1-\beta_n)[z_t^x(1) - (1-\eta_1)\log n_t^x(1) + \log(\eta_1 E(1)^{1-\eta_1})] \\
&- \tilde{\alpha}(1-\tilde{\beta}_n)[t_t^x(1) + \tilde{z}_t^x(1) - (1-\tilde{\eta}_1)\log \tilde{n}_t^x(1) + \log(\tilde{\eta}_1 \tilde{E}(1)^{1-\tilde{\eta}_1})].
\end{aligned}
$$

There is, of course, a symmetric expression if both countries produce the commodity $x_t(2)$.

Suppose instead that the home country produces commodity $x_t(1)$ and the foreign country produces commodity $x_t(2)$. A similar algebra delivers the following expression for the real exchange rate:

$$p_t + s_t - \tilde{p}_t = \left(\tilde{\alpha}\tilde{\beta}_1 + \alpha\left(1 - \beta_n - \beta_1\right)\right)\left(p_t^x\left(1\right) - p_t^x\left(2\right)\right) + \varepsilon_{4t}, \tag{24}$$

where ε_{4t} is now given by

$$\begin{aligned}\varepsilon_{4t} =& \varepsilon_{2t} + \alpha\left(1 - \beta_n\right)\left[z_t^x\left(1\right) - \left(1 - \eta_1\right)\log n_t^x\left(1\right) + \log(\eta_1 E\left(1\right)^{1-\eta_1})\right] \\ & - \tilde{\alpha}(1 - \tilde{\beta}_n)[t_t^x\left(2\right) + \tilde{z}_t^x\left(2\right) - \left(1 - \tilde{\eta}_2\right)\log \tilde{n}_t^x\left(2\right) + \log(\tilde{\eta}_2 \tilde{E}\left(2\right)^{1-\tilde{\eta}_2})].\end{aligned}$$

Again, there is a symmetric representation if the home country produces $x_t\left(2\right)$ and the foreign country produces $x_t\left(1\right)$.

In obtaining the real exchange rate representations (23) and (24), we introduced the labor first-order conditions in the commodities sector into the representation in terms of wages (22). We could have used instead the cost minimization conditions in the final good's sector,

$$\frac{W_t}{P_t^q} = \frac{\alpha}{1-\alpha}\frac{q_t}{n_t^y} \text{ and } \frac{\tilde{W}_t}{\tilde{P}_t^q} = \frac{\tilde{\alpha}}{1-\tilde{\alpha}}\frac{\tilde{q}_t}{\tilde{n}_t^y}, \tag{25}$$

to replace nominal wages in equation (22). Inserting these expressions into (22) gives a representation of the real exchange rate in terms of commodity prices, intermediate good prices measured in a common currency, productivity shocks, trade taxes, and allocations:

$$\begin{aligned}p_t + s_t - \tilde{p}_t =& (\tilde{\alpha}\tilde{\beta}_1 - \alpha\beta_1)p_t^x\left(1\right) + (\tilde{\alpha}\tilde{\beta}_2 - \alpha\beta_2)p_t^x\left(2\right) \\ & + \alpha\left(1 - \beta_n\right)p_t^q - \tilde{\alpha}(1 - \tilde{\beta}_n)\left(\tilde{p}_t^q - s_t\right) + \varepsilon_{5t},\end{aligned} \tag{26}$$

where

$$\begin{aligned}\varepsilon_{5t} =& \varepsilon_{2t} + \alpha\left(1 - \beta_n\right)\log\left(q_t/n_t^y\right) - \tilde{\alpha}(1 - \tilde{\beta}_n)\log\left(\tilde{q}_t/\tilde{n}_t^y\right) \\ & + \log\left(\frac{\alpha}{1-\alpha}\right)\left(\alpha\left(1 - \beta_n\right) - \tilde{\alpha}(1 - \tilde{\beta}_n)\right).\end{aligned}$$

Finally, we show a representation of the real exchange rate that is independent of commodity prices. The cost minimization conditions (25), the price index (17), and the law of one price in commodities allows us to write

$$\begin{aligned}w_t(1 - \beta_n) &= \beta_1 p_t^x\left(1\right) + \beta_2 p_t^x\left(2\right) - z_t^q + \log\left(\frac{\alpha}{1-\alpha}\frac{q_t}{n_t^y}\right) \\ \left(\tilde{w}_t - s_t\right)\left(1 - \tilde{\beta}_n\right) &= \tilde{\beta}_1\left[p_t^x\left(1\right) + t_t^x\left(1\right)\right] + \tilde{\beta}_2\left[p_t^x\left(2\right) + t_t^x\left(2\right)\right] - \tilde{z}_t^q + \log\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\frac{\tilde{q}_t}{\tilde{n}_t^y}\right).\end{aligned}$$

Replacing these expressions in (22) delivers the representation real exchange rate that is independent of commodity prices,

$$p_t + s_t - \tilde{p}_t = \tilde{z}_t^y - z_t^y - t_t^q + \alpha \log\left(q_t/n_t\right) - \tilde{\alpha} \log\left(\tilde{q}_t/\tilde{n}_t\right) + \kappa, \tag{27}$$

where $\kappa = \alpha \log\left(\alpha/\left(1-\alpha\right)\right) - \tilde{\alpha} \log\left(\tilde{\alpha}/\left(1-\tilde{\alpha}\right)\right)$ is a constant.

G General Technologies

In this section we show that the simple case with Cobb-Douglas technologies generalizes, in an appropriate fashion, to general constant returns to scale production functions. The equilibrium prices in the final good sector and the intermediate good sector can be represented by the marginal cost function

$$P_t = C^y\left(W_t, P_t^q\right)/Z_t^y \tag{28}$$

$$P_t^q = C^q\left(W_t, P_t^x\left(1\right), P_t^x\left(2\right)\right)/Z_t^q \tag{29}$$

Replacing (28) with (29) and using that the cost function is homogeneous of degree one in factor prices, we can write the price of the final good as

$$P_t = P_t^x\left(1\right) c\left(\frac{W_t}{P_t^x\left(1\right)}, \frac{P_t^x\left(2\right)}{P_t^x\left(1\right)}\right)/Z_t^y$$

with a similar expression for the foreign country,

$$\tilde{P}_t = \tilde{P}_t^x\left(1\right) \tilde{c}\left(\frac{\tilde{W}_t}{\tilde{P}_t^x\left(1\right)}, \frac{\tilde{P}_t^x\left(2\right)}{\tilde{P}_t^x\left(1\right)}\right)/\tilde{Z}_t^y.$$

The real exchange rate is then defined as

$$rer_t = \frac{c\left(\frac{W_t}{P_t^x(1)}, \frac{P_t^x(2)}{P_t^x(1)}\right)}{\tilde{c}\left(\frac{\tilde{W}_t}{\tilde{P}_t^x(1)}, \frac{\tilde{P}_t^x(2)}{\tilde{P}_t^x(1)}\right)} \frac{\tilde{Z}_t^y}{Z_t^y} \frac{P_t^x\left(1\right) S_t}{\tilde{P}_t^x\left(1\right)}.$$

Using the law of one price for the commodities,

$$rer_t = \frac{c\left(\frac{W_t}{P_t^x(1)}, \frac{P_t^x(2)}{P_t^x(1)}\right)}{\tilde{c}\left(\frac{\tilde{W}_t}{\tilde{P}_t^x(1)}, \frac{P_t^x(2)}{P_t^x(1)}\frac{T_t^x(2)}{T_t^x(1)}\right)} \frac{\tilde{Z}_t^y/Z_t^y}{T_t^x\left(1\right)}. \tag{30}$$

Let the technology to produce commodity $i = 1, 2$ be given by $X_t\left(i\right) = Z_t^x\left(i\right) F_i\left(n_t^x\left(i\right)\right)$. The first-order condition with respect to labor is then given by

$$W_t/P_t^x\left(i\right) = Z_t^x\left(i\right) F'\left(n_t^x\left(i\right)\right).$$

Suppose that both countries produce commodity $X_t\left(1\right)$. Then replacing the previous condition evaluated at $i = 1$ in (30) gives the first representation of the real exchange rate in terms of commodity prices and labor allocations,

$$rer_t = \frac{c\left(Z_t^x\left(1\right)F_1'\left(n_t^x\left(1\right)\right), \frac{P_t^x(2)}{P_t^x(1)}\right)}{\tilde{c}\left(\tilde{Z}_t^x\left(1\right)F_1'\left(\tilde{n}_t^x\left(1\right)\right), \frac{P_t^x(2)}{P_t^x(1)}\frac{T_t^x(2)}{T_t^x(1)}\right)}\frac{\tilde{Z}_t^y/Z_t^y}{T_t^x\left(1\right)}. \tag{31}$$

Suppose instead that the home country produces commodity $X\left(1\right)$ but the foreign country produces commodity $X\left(2\right)$. The real exchange rate can then be written as

$$rer_t = \frac{c\left(Z_t^x\left(1\right)F_1'\left(n_t^x\left(1\right)\right), \frac{P_t^x(2)}{P_t^x(1)}\right)}{\tilde{c}\left(\tilde{Z}_t^x\left(2\right)F_2'\left(\tilde{n}_t^x\left(2\right)\right)\frac{P_t^x(2)}{P_t^x(1)}, \frac{P_t^x(2)}{P_t^x(1)}\right)}\frac{\tilde{Z}_t^y/Z_t^y}{T_t^x\left(2\right)}, \tag{32}$$

which is our second representation of the real exchange rate.

To obtain our third representation of the real exchange rate, we use the cost minimization condition in the final goods' sector, which can be written as

$$\frac{W_t}{P_t^q} = h\left(\frac{n_t^y}{q_t}\right) \tag{33}$$

where h is a decreasing function. Using this expression in (30) delivers

$$rer_t = \frac{c\left(h\left(\frac{n_t^y}{q_t}\right)\frac{P_t^q}{P_t^x(1)}, \frac{P_t^x(2)}{P_t^x(1)}\right)}{\tilde{c}\left(\tilde{h}\left(\frac{\tilde{n}_t^y}{\tilde{q}_t}\right)\frac{\tilde{P}_t^q}{\tilde{P}_t^x(1)}, \frac{T_t^x(2)}{T_t^x(1)}\right)}\frac{\tilde{Z}_t^y/Z_t^y}{T_t^x\left(1\right)}. \tag{34}$$

This expression relates the real exchange rate to relative commodity prices, to the relative price of the intermediate good in terms of commodity $X\left(1\right)$, and to the allocation.

Finally, we obtain an expression independent of commodity prices. To that end, we use that the equilibrium price in the intermediate goods' sector (29) can be written as

$$\frac{P_t^q}{P_t^x\left(1\right)} = \phi\left(\frac{W_t}{P_t^x\left(1\right)}, \frac{P_t^x\left(2\right)}{P_t^x\left(1\right)}\right)/Z_t^q.$$

But (33) implies

$$\frac{W_t}{P_t^x\left(1\right)} = \frac{P_t^q}{P_t^x\left(1\right)}h\left(\frac{n_t^y}{q_t}\right),$$

which, using the previous expression, becomes

$$\frac{W_t}{P_t^x\left(1\right)} = \phi\left(\frac{W_t}{P_t^x\left(1\right)}, \frac{P_t^x\left(2\right)}{P_t^x\left(1\right)}\right)h\left(\frac{n_t^y}{q_t}\right)/Z_t^q.$$

Using the properties of the cost function,

$$1 = \phi\left(1, \frac{P_t^x(2)/P_t^x(1)}{W_t/P_t^x(1)}\right) h\left(\frac{n_t^y}{q_t}\right)\frac{1}{Z_t^q}.$$

This is an implicit function which delivers

$$\frac{W_t}{P_t^x(1)} = \frac{P_t^x(2)}{P_t^x(1)}\kappa\left(\frac{n_t^y}{q_t}, Z_t^q\right).$$

The equivalent expression for the foreign country (using the law of one price for the commodities) is

$$\frac{\tilde{W}_t}{\tilde{P}_t^x(1)} = \frac{P_t^x(2)}{P_t^x(1)}\frac{T_t^x(2)}{T_t^x(1)}\tilde{\kappa}\left(\frac{\tilde{n}_t^y}{\tilde{q}_t}, \tilde{Z}_t^q\right).$$

Using the latter two expressions in (30) gives the last representation of the real exchange rate,

$$rer_t = \frac{c\left(\frac{P_t^x(2)}{P_t^x(1)}\kappa\left(\frac{n_t^y}{q_t}, Z_t^q\right), \frac{P_t^x(2)}{P_t^x(1)}\right)}{\tilde{c}\left(\frac{P_t^x(2)}{P_t^x(1)}\frac{T_t^x(2)}{T_t^x(1)}\tilde{\kappa}\left(\frac{\tilde{n}_t^y}{\tilde{q}_t}, \tilde{Z}_t^q\right), \frac{P_t^x(2)}{P_t^x(1)}\frac{T_t^x(2)}{T_t^x(1)}\right)}\frac{\tilde{Z}_t^y/Z_t^y}{T_t^x(1)}.$$

The properties of the cost function imply that the relative prices $P_t^x(2)/P_t^x(1)$ disappear,

$$rer_t = \frac{c\left(\kappa\left(\frac{n_t^y}{q_t}, Z_t^q\right), 1\right)}{\tilde{c}\left(\frac{T_t^x(2)}{T_t^x(1)}\tilde{\kappa}\left(\frac{\tilde{n}_t^y}{\tilde{q}_t}, \tilde{Z}_t^q\right), \frac{T_t^x(2)}{T_t^x(1)}\right)}\frac{\tilde{Z}_t^y/Z_t^y}{T_t^x(1)}. \tag{35}$$

H Multiple Consumption Goods

In this section we assume that there are many different types of (nontraded) consumption goods. In particular, we assume that there are $k = 1, ..., K$ types of consumption goods. Households in each country value the basket of consumption goods

$$\prod_{k=1}^{K} C_t(k)^{\rho(k)},$$

in which the coefficients, $\rho(k)$, add up to one. Again, we allow for different coefficients, $\rho(k)$, across countries. Each consumption good is produced according to the technology

$$C_t(k) = Z_t^C(k) \left(\prod_{j=1}^{J} [n_t^{C(k)}(j)]^{\psi^C(k,j)} \right)^{\alpha} \left(\prod_{i=1}^{N} q_t^{C(k)}(i)^{\varphi(k,i)} \right)^{1-\alpha}.$$

The rest of the economy (the production of intermediate goods and commodities) has the same structure as in Section 5.

H.1 Prices

Let P_t denote the price of the basket of goods and $P_t^C(k)$ the price of the type-k consumption good. Households' cost-minimization problem over different types of consumption goods implies that, in equilibrium, the price of the basket satisfies

$$\ln P_t = \sum_{k=1}^{K} \rho(k) \ln P_t^C(k).$$

Next, we use the fact that the price of each type of consumption good will be Cobb-Douglas functions of factor prices (see Section 5.1). So we reach the equivalent of equation (6) for the case of multiple consumption goods:

$$\ln P_t = \sum_{k=1}^{K} \rho(k) \left[\begin{array}{l} \ln\left(\frac{\kappa^{C(k)}}{Z_t^C(k)}\right) + \alpha \sum_{j=1}^{J} \psi^C(k,j) \ln W_t(j) + \\ (1-\alpha) \sum_{i=1}^{N} \varphi(k,i) \ln P_t^Q(i) \end{array} \right],$$

which is now a sum over the different types of consumption goods. Therefore, we can follow the same steps as described in Section 5.1 in order to derive a relationship between bilateral real exchange rates and PCP that resembles equation (15). The only difference in the case of multiple consumption goods is that the relationship between PCP and real exchange rates will also depend on the difference in preferences over the types of consumption goods across countries, that is, differences in the coefficients $\rho(k)$.

I Sum of coefficients

In this appendix, we show that the sum of the coefficients in prices and wages in expression (8), repeated here for convenience,

$$\begin{aligned}\ln P_t &= \ln \frac{\kappa^C}{z_t^y} + (1-\alpha)\sum_{i=1}^{N}\varphi(i)\,\frac{\kappa^{Q(i)}}{z_t(i)} \\ &+ \sum_{j=1}^{J}\left[\alpha\psi^C(j) + \sum_{i=1}^{N}(1-\alpha)\,\varphi(i)\,\beta(i)\psi^Q(i,j)\right]\ln W_t(j) \\ &+ (1-\alpha)\sum_{i=1}^{N}\varphi(i)\,(1-\beta(i))\sum_{h=1}^{N}\gamma(i,h)\ln P_t(h),\end{aligned}$$

add up to 1. The sum of the coefficients is

$$\sum_{j=1}^{J}[\alpha\psi^C(j) + (1-\alpha)\sum_{i=1}^{N}\beta(i)\varphi(i)\psi^Q(i,j)] + (1-\alpha)\sum_{h=1}^{N}\sum_{i=1}^{N}(1-\beta(i))\varphi(i)\gamma(i,h),$$

which can be written

$$\alpha\sum_{j=1}^{J}\psi^C(j) + (1-\alpha)\sum_{i=1}^{N}\beta(i)\varphi(i)\sum_{j=1}^{J}\psi^Q(i,j) + (1-\alpha)\sum_{i=1}^{N}(1-\beta(i))\varphi(i)\sum_{h=1}^{N}\gamma(i,h)$$

or

$$\begin{aligned}&\alpha + (1-\alpha)\sum_{i=1}^{N}\beta(i)\varphi(i) + (1-\alpha)\sum_{i=1}^{N}(1-\beta(i))\varphi(i) \\ =\ &\alpha + (1-\alpha)\sum_{i=1}^{N}\beta(i)\varphi(i) + (1-\alpha) - (1-\alpha)\sum_{i=1}^{N}\beta(i)\varphi(i) = 1.\end{aligned}$$