# The U.S. Public Debt Valuation Puzzle*

Zhengyang Jiang
Northwestern Kellogg

Hanno Lustig
Stanford GSB, NBER, SIEPR

Stijn Van Nieuwerburgh
Columbia Business School, NBER, CEPR

Mindy Z. Xiaolan
UT Austin McCombs

October 2, 2021
First draft: March 2019

**Abstract**

The government budget constraint ties the market value of government debt to the expected present discounted value of fiscal surpluses. Bond investors fail to impose this no-arbitrage restriction in the U.S., resulting in a government debt valuation puzzle. Both cyclical and long-run dynamics of tax revenues and government spending make the surplus claim risky. In a realistic asset pricing model, this risk in surpluses creates a wedge of 2.5 times GDP between the value of debt and that of the surplus claim, and implies an expected return on the debt portfolio that far exceeds the observed yield on Treasuries.

*Key Words: bond pricing, fiscal policy, term structure, convenience yield.*

*Jiang: Finance Department, Kellogg School of Management, Northwestern University; zhengyang.jiang@kellogg.northwestern.edu. Lustig: Department of Finance, Stanford Graduate School of Business, Stanford CA 94305; hlustig@stanford.edu; https://people.stanford.edu/hlustig/. Van Nieuwerburgh: Department of Finance, Columbia Business School, Columbia University, 3022 Broadway, New York, NY 10027; svnieuwe@gsb.columbia.edu; Tel: (212) 854-1282. Xiaolan: McCombs School of Business, the University of Texas at Austin; mindy.xiaolan@mccombs.utexas.edu. The authors would like to thank Jules van Binsbergen (discussant), Philip Bond, Markus Brunnermeier (discussant), John Cochrane, Max Croce (discussant), Tetiana Davydiuk (discussant), Peter DeMarzo, John Donaldson, Ben Hebert, Chris Hrdlicka, Nobu Kyotaki, Ralph Koijen, Yang Liu, Ian Martin, John Moore, Christian Moser, Carolin Pflueger (discussant), Jean-Paul Renne, Lukas Schmid (discussant), Jesse Schreger, Pierre Yared, Steven Zeldes, and seminar and conference participants at the Joint Stanford-U.C. Berkeley finance seminar, Columbia University macro-economics, Kellogg finance, LSE, Chicago Booth finance, UT Austin finance, the Federal Reserve Board, the University of Washington, Stanford economics, Stanford finance, USC, UCLA Anderson, Shanghai Advanced Institute of Finance, the virtual finance workshop, the 2019 Society for Economic Dynamics meetings in St Louis, the Advances in Macro-Finance Tepper-LAEF Conference, NBER SI AP/MEFM, the Western Finance Association, the Midwest Finance Association, and the Vienna Symposium on Foreign Exchange Markets for insightful discussions. We gratefully acknowledge financial support from NSF award 2049260.

# 1 Introduction

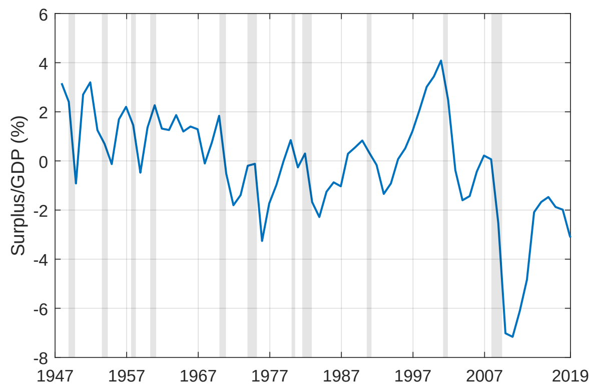
The U.S. Treasury is the largest borrower in the world. At the end of 2019, outstanding federal government debt held by the public was valued at $17 trillion. It doubled after the Great Financial Crisis to 78.4% of U.S. GDP. Before the GFC, there was widespread concern that the U.S. had embarked on an unsustainable fiscal path (see, e.g., Rubin, Orszag, and Sinai, 2004). Yet, recently, some economists have argued that the U.S. has ample debt capacity to fund additional spending by rolling over its debt because interest rates are below GDP growth rates (Blanchard, 2019). As a case in point, the massive spending increase in response to the covid-19 pandemic generated a deficit of 15% of GDP in 2020 and increased the debt to 100% of GDP. The $5 trillion debt increase has met with little resistance from bond markets so far.

The central idea in this paper is to price the entire portfolio of outstanding Treasury debt, rather than individual bond securities. In the absence of bubbles, the market value of outstanding debt should equal the present discounted value of current and future primary surpluses. By the same logic, the expected return on the debt portfolio has to reflect the risk profile of primary surpluses, consistent with the risk compensation in stocks and bonds. That is why this is a valuation equation, not an accounting identity. We find evidence of mispricing. The value of the bond portfolio exceeds the value of the surplus claim, a gap we label the government debt valuation puzzle, and that yields on the Treasury bond portfolio are lower than the relevant "interest rate" bond investors ought to be earning, the government debt risk premium puzzle.

To explain why, we use a stock pricing analogy. The price of a stock is the expected present discount value of future dividends. Risk-free interest rates are below dividend growth rates, yet the price of the stock is finite. Since the stock's dividend growth is pro-cyclical, its cash flows are low when the investor's marginal utility is high. The relevant "interest rate" for the stock contains a risk premium because of the risk exposure of its cash flow. Analogously, a portfolio strategy that buys all new Treasury issues and receives all Treasury coupon and principal payments has as its cash flow the primary surplus of the federal government. Primary surpluses are strongly pro-cyclical just like stock dividends, as shown in Figure 1. Spending by the federal government increases in recessions, while the progressive nature of the tax system produces sharply pro-cyclical revenue. In recessions, when marginal utility is high, surpluses are negative and net bond issuance is high. The Treasury portfolio cash flows have substantial business cycle risk. As explained below, tax revenue and spending also have substantial long-run risk due to cointegration with GDP. Taken together, the relevant "interest rate" for surpluses contains a substantial risk premium reflecting both short- and long-run risk exposures.

The value of the surplus claim is obtained as the difference between the value of a claim to future federal tax revenues, $P_t^{\tau}$, and the value of a claim to future federal spending excluding debt

Figure 1: U.S. Government Surplus




The figure plots the U.S. federal government primary surplus as a fraction of GDP. The construction of the primary surplus is detailed in Appendix D.1. The data source is NIPA Table 3.2. The sample period is from 1947 to 2019.

service, $P_t^g$. The pro-cyclicality of tax revenues makes the tax revenue claim risky; $P_t^\tau$ is low. The counter-cyclicality makes the spending claim safer; $P_t^g$ is high. The value of the surplus claim, $P_t^S = P_t^\tau - P_t^g$, is low.

Our contribution is to quantitatively evaluate the magnitude of the value of the surplus claim. We first do so in the familiar consumption-based asset pricing model. We then deploy a more realistic dynamic asset pricing model that matches a rich set of asset pricing moments for stocks and bonds. In both models, we find a large negative value for the surplus claim. The latter averages -2 times GDP in our main model. The market value of outstanding debt has averaged 38.23% of GDP over the same period. The wedge is almost 2.5 times GDP on average over our sample, and has widened dramatically in the last twenty years. The wedge quantifies the bond valuation puzzle. At the same time, the model predicts Treasury portfolio returns that are at least 3.00% too high –the government debt risk premium puzzle.

The surplus value measures the fiscal capacity of the U.S. government, that is, how much debt it can issue. As first pointed out by Bohn (1995), the surplus value can be decomposed as the present value of future surpluses, discounted using the risk-free term structure of interest rates, plus the covariance of future surpluses with the stochastic discount factor. Without aggregate risk, there is no covariance term and the fiscal capacity of the government is unbounded when the average risk-free rate is lower than the average growth rate of the economy. Much of the literature, including recent work, has ignored these covariance terms. However, in the presence of priced aggregate risk, the covariance term will typically bound the government's fiscal capacity because surpluses move with the business cycle in the short run and are co-integrated with output in the long run. Our work is the first to estimate and quantify the covariance term in a standard consumption CAPM and in a more realistic dynamic asset pricing model. When we insist that our model be consistent with moments of asset prices, we find that fiscal capacity is much lower than conventionally thought, even lower than the market value of outstanding debt.

The above argument relies on a realistic model of quantities and prices of risk. When modeling the quantity of risk in fiscal cash flows, adequately capturing the dynamics of government spending and tax revenue is crucial. We model the growth rates of tax revenues-to-GDP and government spending-to-GDP in a VAR alongside macro-economic and financial variables. This structure allows us to capture the cyclical properties of fiscal cash-flows. A second important feature of fiscal cash flows is that tax revenues and spending are co-integrated with GDP, so that revenues, spending, and GDP adjust when revenue-to-GDP or spending-to-GDP are away from their long-run relationship. This imposes a form of long-run automatic stabilization. With cointegration, GDP innovations permanently alter future surpluses. A deep recession not only raises current government spending and lowers current tax revenue as a fraction of GDP, it also lowers future spending and raises future revenue as a fraction of future GDP. Both the spending and the revenue claims are exposed to the same long-run risk as GDP. We include the debt/GDP ratio in the VAR since it might contain relevant information about future surpluses.

When modeling the price of risk, we posit a state-of-the-art stochastic discount factor (SDF) model. Rather than committing to a specific utility function, we use a flexible SDF that accurately prices the nominal and real term structure of Treasury bond yields. The model also closely matches stock prices and generates an equity risk premium. The SDF model's rich implications for the term structure of risk allow it to adequately price short- and long-run risk to spending and tax revenue.

Combining features from both quantities and prices of risk, the long-run discount rates on claims to tax revenues, spending, and GDP must all be equal. A claim to GDP is akin to an unlevered equity claim. In any reasonable asset pricing model with a large permanent component in the SDF, the unlevered equity risk premium exceeds the yield on a long-term government bond (Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). The discount rate for revenues and spending is high. Because of the dynamic government budget constraint, the relevant "interest rate" on the portfolio of government debt must also be high. Treasury investors seem willing to purchase government debt at low yields. The historical return on the U.S. government debt portfolio is only 1.16% in excess of the T-bill rate.

An important consequence is that the risk-free rate cannot be the right discount rate for future surpluses and hence for government debt. While one can roll over a constant dollar amount at the risk-free rate, one cannot roll over a cash flow stream that is pro-cyclical and co-integrated with GDP at the risk-free rate. The latter cash flow stream carries a substantial risk premium. Yet, it is commonplace in the literature to discount government surpluses at the one-period risk-free rate.

In the last part of the paper, we study several potential resolutions of the government bond valuation and risk premium puzzles. First, the valuation gap can be interpreted as a violation of the transversality condition (TVC) in the Treasury market, due to a rational bubble. However, in the presence of substantial long-run output risk premia, i.e., in models that resolve the equity risk premium, the TVC is likely to hold, as we explain. In addition, rational bubbles in government debt imply rational bubbles in any long-lived asset whose cash flows are cointegrated with aggregate output. Rational bubbles are unlikely in the presence of long-lived investors unless there are severe limits to arbitrage.

Second, the U.S. Treasury earns convenience yield on the debt it issues, making Treasury yields lower than the risk-free rate. Convenience yields generate an additional source of revenue which increase the surplus. Furthermore, convenience yields are counter-cyclical and hence reduce the riskiness of the surplus stream. Despite their theoretical appeal, we find that convenience yields only help modestly to explain the puzzle, because accounting for convenience yields also increases risk-free rates. Higher surpluses due to convenience are discounted at a higher rate only to result in a similar valuation for the surplus claim. The convenience yields needed to close the wedge are 6% per year, an order of magnitude larger than traditional estimates of convenience yield. Our work is the first to quantify the effect of convenience yields on the fiscal capacity of the U.S.

Third, we explore the possibility of a future large fiscal correction that is absent from our sample, but present in the minds of investors who value the surplus claim. We back out from the market value of debt what annual probability investor assign to such an austerity event. We obtain a probability of radical austerity of 24% on average which rises to 36% at the end of the sample. The high probability we infer belies the nature of a peso event, and is not consistent with rational expectations. Repeating the analysis in the model with convenience yields results in similar austerity probabilities.

Fourth, allowing fiscal shocks that are orthogonal to stock, bond prices and output growth to be priced helps to close the gap but implausibly requires that the stand-in investor experiences lower marginal utility growth when spending increases during recessions. This also results in implausibly large maximum Sharpe ratios, and it worsens the government risk premium puzzle. Finally, missing government assets are too small to resolve the puzzle. Future liabilities from Social Security, Medicare, and Medicaid obligations make our estimates of the wedge conservative. As a result, we conclude that the aggregate value of U.S. Treasurys is hard to square with reasonable estimates of future surpluses, especially in the past three decades, possibly because investors have been too optimistic about future surpluses.

**Related Literature** There is a growing literature that seeks to understand the riskiness of bonds as an asset class and relate it to other macro-economic risks in the economy (see Baele, Bekaert, and Inghelbrecht, 2010; David and Veronesi, 2013; Duffee, 2018; Campbell, Pflueger, and Viceira, 2020; Du, Pflueger, and Schreger, 2020; van Binsbergen, 2020). Our paper contributes to this literature by adding novel no-arbitrage restrictions on the aggregate Treasury portfolio, in addition to the no-arbitrage restrictions on individual bonds. The asset pricing model combines a vector auto-regression model for the state variables as in Campbell (1990); Campbell et al. (1993); Campbell (1996) with a no-arbitrage model for the (SDF) as in Duffie and Kan (1996); Dai and Singleton (2000); Ang and Piazzesi (2003). Lustig, Van Nieuwerburgh, and Verdelhan (2013) study the properties of the price-dividend ratio of a claim to aggregate consumption, the wealth-consumption ratio, and Gupta and Van Nieuwerburgh (2019) evaluate the performance of private equity funds in similar settings.

Our paper contributes to the literature on the fiscal capacity of the government (see D'Erasmo, Mendoza, and Zhang, 2016, for a recent review). One strand derives general time-series restrictions on the government revenue and spending processes that enforce the government's intertemporal budget constraint (Hamilton and Flavin, 1985; Trehan and Walsh, 1988, 1991; Hansen, Roberds, and Sargent, 1991; Bohn, 2007). Many authors in this literature use the risk-free rate as the discount rate for surpluses. They test the joint null hypothesis that the budget constraint holds and that the debt is risk-free so that surpluses can be priced off the risk-free yield curve. Our paper argues that risk premia on the surplus claim and hence on the government bond portfolio are not zero. It infers large risk premia on government debt when no-arbitrage restrictions on bond and stock markets are imposed.

Bohn (1995) was the first to study fiscal capacity in a world with aggregate risk and to introduce the covariance terms between the intertemporal marginal rate of substitution and the surplus. Our main new qualitative insight is that the overall government bond portfolio is a risky asset since the government must issues debt in high marginal utility states of the world. In other words, the covariance term is negative, reducing fiscal capacity. The main new quantitative result is that this covariance is large. Fiscal capacity is much smaller due to this covariance term. The presence of a large amount of permanent risk in output, and by virtue of cointegration, in tax revenues, spending, and debt, is crucial for the quantitative result. There is a parallel literature in asset pricing which tests the present value equation for stocks and other long-lived assets, starting with the seminal work by Shiller (1981); LeRoy and Porter (1981); Campbell and Shiller (1988). That work starts from the definition of a stock return to derive a testable relationship between stock prices and expected discounted dividend growth rates. Similarly, we start from the definition of the government budget constraint and derive a testable relationship between the market value of the government debt portfolio and expected discounted future surpluses. However, we insist that the discount rates for surpluses be consistent with those for other securities. While the prices of stocks appear excessively volatile relative to their fundamentals, government debt is fundamentally different: its valuation does not seem volatile enough relative to the fundamentals.

There is a large literature on rational bubbles in asset markets, starting with the seminal work by Samuelson (1958); Diamond (1965); Blanchard and Watson (1982). One interpretation of our puzzle is as a violation of the transversality condition in Treasury markets, consistent with the existence of a rational bubble. In economies with aggregate risk, however, the transversality condition for debt is likely to be satisfied, even if the risk-free interest rate is below the growth rate of the economy, since the relevant discount rate for debt in the far future contains a risk premium that reflects the long-run risk in output. When debt and output are cointegrated, debt inherits that output risk. While Bohn (1995) recognized this conceptually, we show that the risk premium on debt is actually large enough to make the TVC hold; the economy is dynamically efficient. In recent work, Barro (2020) shows that the TVC for government debt holds in a calibrated model with disaster risk. Sustaining rational bubbles requires severe limits to arbitrage (Shleifer and Vishny, 1997). Giglio, Maggiori, and Stroebel (2016) devise a model-free test for bubbles in housing markets. Our test is not model-free, but the results hold in a large class of models where permanent shocks to the pricing kernel are an important driver of risk premia.

Our work connects to the large literature on the convenience yield of U.S. government bonds (Longstaff, 2004; Krishnamurthy and Vissing-Jorgensen, 2012; Fleckenstein, Longstaff, and Lustig, 2014; Nagel, 2016; Van Binsbergen, Diamond, and Grotteria, 2019). Greenwood, Hanson, and Stein (2015) study the government debt's optimal maturity in the presence of such a premium, and Du, Im, and Schreger (2018); Jiang, Krishnamurthy, and Lustig (2021a); Koijen and Yogo (2019) study this premium in international finance. In recent work, Brunnermeier, Merkel, and Sannikov (2020) and Reis (2021) analyze models in which government debt helps agents smooth idiosyncratic income risk, and earns convenience yields as a result. We tackle the question of how expensive a portfolio of all Treasuries is relative to the underlying collateral, a claim to surpluses. Using the standard convenience yield estimates of Krishnamurthy and Vissing-Jorgensen (2012), we find that our puzzle remains. Even when we use the larger convenience yield estimates due to Jiang et al. (2021a); Koijen and Yogo (2019), we cannot close the gap.

Our approach is to estimate processes for government spending and revenue growth from the data, and to study its implications for the riskiness of the government debt portfolio in a model with realistic asset prices. A large literature following Barro (1979) and Lucas and Stokey (1983) analyzes optimal fiscal policy in settings with distortionary taxation. Karantounias (2018) and Bhandari, Evans, Golosov, Sargent, et al. (2017) bring a richer asset pricing model to this literature and study the optimal maturity structure of government debt.

We contribute to a recent literature at the intersection of asset pricing and public finance. Chernov, Schmid, and Schneider (2020); Pallara and Renne (2019) argue that higher CDS premia for U.S. Treasuries since the financial crisis are related to the underlying fiscal fundamentals. Our puzzle holds in the presence of default: the value of defaultable sovereign debt is still be backed by future surpluses. Liu, Schmid, and Yaron (2020) argue that increasing safe asset supply can be risky as more government debt increases corporate default risk premia despite providing more convenience. Croce, Nguyen, Raymond, and Schmid (2019) study cross-sectional differences in firms' exposure to government debt. Corhay, Kind, Kung, and Morales (2018) study how quantitative easing affects inflation by changing the maturity structure of government debt.

The rest of the paper is organized as follows. Section 2 presents theoretical results. Section 3 describes the data. Section 4 illustrates the valuation puzzle in a simple consumption CAPM. Section 5 sets up and solves the quantitative model. Section 6 documents the government risk premium puzzle in that model. Section 7 discusses potential resolutions of the puzzle. Section 8 concludes. The appendix presents proofs of the propositions, and details of model derivation and estimation.

## 2 Theoretical Results

We derive two theoretical results which are general in that they rely on the absence of arbitrage opportunities and two weak assumptions on government cash flows. The first assumption concerns the long run: tax revenues and government spending are cointegrated with GDP; they share a stochastic trend. The second assumption concerns the short-run: spending is counter-cyclical spending and tax revenues are pro-cyclical.

### 2.1 Valuation of Government Debt

Let $G_t$ denote nominal government spending before interest expenses on the debt, $T_t$ denote nominal government tax revenue, and $S_t = T_t - G_t$ denote the nominal primary surplus. Let $P_t^{\$}(h)$ denote the price at time $t$ of a nominal zero-coupon bond that pays \$1 at time $t+h$, where $h$ is the maturity. There exists a multi-period stochastic discount factor (SDF) $M_{t,t+h}^{\$} = \prod_{k=0}^{h} M_{t+k}^{\$}$ is the product of the adjacent one-period SDFs, $M_{t+k}^{\$}$. By no arbitrage, bond prices satisfy $P_t^{\$}(h) = \mathbb{E}_t\left[M_{t,t+h}^{\$}\right] = \mathbb{E}_t\left[M_{t+1}^{\$} P_{t+1}^{\$}(h-1)\right]$. By convention $P_t^{\$}(0) = M_{t,t}^{\$} = M_t^{\$} = 1$ and $M_{t,t+1}^{\$} = M_{t+1}^{\$}$. The government bond portfolio is stripped into zero-coupon bond positions $Q_t^{\$}(h)$, where $Q_t^{\$}(h)$ denotes the outstanding face value at time $t$ of the government bond payments due at time $t+h$. $Q_{t-1}^{\$}(1)$ is the total amount of debt payments that is due today. The outstanding debt reflects all past bond issuance decisions, i.e., all past primary deficits. Let $D_t$ denote the nominal market value of the outstanding government debt portfolio.

**Proposition 1** (**Value Equivalence**). In the absence of arbitrage opportunities and subject to a transversality condition, the market value of the outstanding government debt portfolio equals the expected present discounted value of current and future primary surpluses:

$$D_t \equiv \sum_{h=0}^{H} P_t^{\$}(h) Q_{t-1}^{\$}(h+1) \quad = \quad \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}^{\$} (T_{t+j} - G_{t+j}) \right] \equiv P_t^{\tau} - P_t^{g}, \tag{1}$$

where the cum-dividend value of the tax claim and value of the spending claim are defined as:

$$P_t^{\tau} = \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}^{\$} T_{t+j} \right], \quad P_t^{g} = \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}^{\$} G_{t+j} \right].$$

The proof is given in Appendix A. The proof relies only on the existence of a SDF, i.e., the absence of arbitrage opportunities, not on the uniqueness of the SDF, i.e., complete markets. It imposes a transversality condition (TVC) that rules out a government debt bubble: $\mathbb{E}_t \left[ M_{t,t+T} D_{t+T} \right] \to 0$ as $T \to \infty$. The market value of debt is the difference between the value of a claim to tax revenue and the value of a claim to government spending. Imposing the TVC rules out rational bubbles. We return to possible violations of the TVC in Section 7.1.

Even if the transversality condition holds, this valuation equation is not an accounting identity. The bond portfolio can be mispriced, just like a stock can be over- or under-valued. Equation (1) requires that the same SDF which prices individual government bonds and stocks also prices a claim to surpluses, i.e., the entire bond portfolio. Even when the SDF correctly prices individual bonds and stocks, this entire bond portfolio could be mis-priced, for example, because agents have misspecified beliefs about future surpluses. This equation is an accounting identity only when we do not impose any restrictions on discount rates.

When the government runs a deficit in a future date and state, it will need to issue new bonds to the investing public. If those dates and states are associated with a high value of the SDF for the representative bond investor, that debt issuance occurs at the "wrong" time. The representative investors who buys all debt issues and participates in all redemptions need to be induced by low prices (high yields) to absorb that new debt. To see this, we can rewrite the intertemporal budget constraint, with finite horizon $T$, as:

$$D_t = \sum_{j=0}^{T} P_t^{\$}(j) \mathbb{E}_t \left[ S_{t+j} \right] + \sum_{j=0}^{T} \mathrm{Cov}_t \left( M_{t,t+j}^{\$}, T_{t+j} \right) - \sum_{j=0}^{T} \mathrm{Cov}_t \left( M_{t,t+j}^{\$}, G_{t+j} \right) + \mathbb{E}_t \left[ M_{t,t+T} D_{t+T} \right] \tag{2}$$

The first term on the right-hand side is the present discounted value of all expected future surpluses, using the term structure of risk-free bond prices. It is the PDV for a risk-neutral investor. If the SDF is constant, this is the only term on the right-hand side. Then, the government's fiscal capacity is constrained by its ability to generate current and future surpluses. The second and third terms encode the riskiness of the government debt portfolio, and arise in the presence of time-varying discount rates. If tax revenues tend to be high when times are good ($M_{t,t+j}$ is low), then the second term is negative. If government spending tends to be high when times are bad ($M_{t,t+j}$ is high), then the third term is positive. If both are true, then the difference between the two covariance terms is negative. The covariance terms lower the government's fiscal capacity. Put differently, the risk-neutral present-value of future surpluses will need to be higher by an amount equal to the absolute value of the covariance terms to support a given, positive amount of government debt $D_t$. The covariance terms were first highlighted by Bohn (1995). Our paper is the first to quantify these terms in a realistic model of risk and return that is not subject to the equity risk premium puzzle. The covariance terms not only have the hypothesized sign, but they are also quantitatively important.

Discounting future surpluses using the term structure of risk-free interest rates, as typically done in the literature, is inappropriate. In fact, as $T \to \infty$, the first term will not converge if the average risk-free rate is lower than the average growth rate. Even when the debt is risk-free, the last term will not converge to zero if we discount at the risk-free rate.

The valuation equation (1) holds ex-ante both in nominal and in real terms.[1] The same valuation equation holds when we allow for sovereign default: the valuation of government debt is still backed by the value of future surpluses. Bond prices adjust to reflect the possibility of default. The proof is given in Appendix A.[2]

### 2.2 Discount Rates

As tax revenue and government spending may have very different cyclicality properties, their discount rates can be different and have first-order impact on the present value of the government surplus in (1).

We define the holding period returns on the bond portfolio, the tax claim, and the spending

[1] Ex-post, the government can erode the real value of outstanding debt by creating surprise inflation. Hilscher, Raviv, and Reis (2021) shows that this channel is not very powerful in practice. See Hall and Sargent (2011); Berndt, Lustig, and Yeltekin (2012) for a decomposition of the forces driving the U.S. debt/GDP ratio including inflation. Cochrane (2019a,b) explores the connection between inflation and the value of government debt without imposing no arbitrage restrictions.

[2] Bond prices satisfy $P_t^{\$}(h) = \mathbb{E}_t\left[M_{t,t+h}^{\$}(1-\chi_{t,t+h})\right]$, where $\chi_{t,t+h}$ is an indicator variable that is one when the government defaults between $t$ and $t+h$. We assume full default to keep the proof simple, but this is without loss of generality. Chernov et al. (2020) and Pallara and Renne (2019) study the response of CDS spreads to news about the fiscal surplus.

claim as:

$$R^d_{t+1} = \frac{\sum_{h=1}^{\infty} P^\$_{t+1}(h-1)Q^\$_t(h)}{\sum_{h=1}^{\infty} P^\$_t(h)Q^\$_t(h)}, \quad R^\tau_{t+1} = \frac{P^\tau_{t+1}}{P^\tau_t - T_t}, \quad R^g_{t+1} = \frac{P^g_{t+1}}{P^g_t - G_t}.$$

The expected returns on these three assets are connected as follows:

**Proposition 2 (Risk Premium Equivalence).** Under the same assumptions of Proposition 1, we have:

$$\mathbb{E}_t\left[R^d_{t+1}\right] = \frac{P^\tau_t - T_t}{D_t - S_t}\mathbb{E}_t\left[R^\tau_{t+1}\right] - \frac{P^g_t - G_t}{D_t - S_t}\mathbb{E}_t\left[R^g_{t+1}\right]. \tag{3}$$

where $D_t - S_t = (P^\tau_t - T_t) - (P^g_t - G_t)$.

The proof is given in Appendix A. The average discount rate on government debt is equal to the average discount rate on government assets, a claim to primary surpluses. Since the primary surpluses are tax revenues minus government spending, the discount rate on government debt equals the difference between the discount rates of tax revenues and government spending, appropriately weighted.

By subtracting the risk-free rate on both sides, we can express the relationship in terms of expected excess returns, or risk premia. To develop intuition, consider two simple scenarios. First, if the expected returns on tax revenue and spending claims are identical, then the risk premium on government debt is given by:

$$\mathbb{E}_t\left[R^d_{t+1} - R^f_t\right] = \mathbb{E}_t\left[R^\tau_{t+1} - R^f_t\right] = \mathbb{E}_t\left[R^g_{t+1} - R^f_t\right]. \tag{4}$$

Second, if the tax revenue claim is riskier than the spending claim and earns a higher risk premium, then the risk premium on government debt exceeds that on the revenue and the spending claims:

$$\mathbb{E}_t\left[R^d_{t+1} - R^f_t\right] > \mathbb{E}_t\left[R^\tau_{t+1} - R^f_t\right] > \mathbb{E}_t\left[R^g_{t+1} - R^f_t\right]. \tag{5}$$

We show below that the revenue claim is indeed riskier than the spending claim. The risk premium equivalence then implies that the portfolio of government debt ought to carry a positive risk premium. The right discount rate for government debt, given by (3), cannot be the risk-free rate.

To understand the riskiness of the debt claim, we study the short-run and long-run risk properties of the $T$- and $G$-claim. To do so, we study spending and revenue strips. A spending strip is a claim that pays off $G_{t+j}$ at time $t+j$ and nothing at other times. A revenue strip similarly pays off $T_{t+j}$. Let $R^{g,j}_{t,t+j}$ and $R^{\tau,j}_{t,t+j}$ be the holding period returns on these strips.

At the short end of the maturity spectrum (business cycle frequencies $j$ of 1—3 years), the risk premium on the revenue strip exceeds that on the corresponding-maturity spending strip:

$$\mathbb{E}_t\left[R^{\tau,j}_{t,t+j} - R^f_t\right] > \mathbb{E}_t\left[R^{g,j}_{t,t+j} - R^f_t\right]. \tag{6}$$

The reason is that tax revenue is highly pro-cyclical while government spending is counter-cyclical. Since government debt investors have a long position in a riskier claim and a short position in a safer claim, the short end contributes to a positive risk premium on the government debt portfolio.

At the long end of the strip curve, we study the limit of the strip returns as $j \to \infty$. We denote log returns by lowercase letters. We distinguish two cases in terms of the time series properties of government spending and tax revenues.

**Proposition 3** (**Long-run Discount Rates**). If the log of government spending $G$ and of tax revenue $T$ is stationary in levels (after removing a deterministic time trend), then the long-run expected log return on spending and revenue strips equals the yield on a long-term government bond as the payoff date approaches maturity.

$$\lim_{j\to\infty} \mathbb{E}_t\left[r^{g,j}_{t,t+j}\right] = y^{\$}_t(\infty), \quad \lim_{j\to\infty} \mathbb{E}_t\left[r^{\tau,j}_{t,t+j}\right] = y^{\$}_t(\infty),$$

where $y^{\$}_t(\infty)$ is the yield at time $t$ on a nominal government bond of maturity $+\infty$.

The proof is given in Appendix A. The result builds on work by Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Borovička et al. (2016); Backus et al. (2018), among others. Under this assumption on cash flows, the proposition implies that long-run $T$- and $G$-strips can be discounted off the term-structure for zero coupon bonds. In this case, the long-run discount rate on government debt is the yield on a long-term risk-free bond. However, the underlying assumption on cash flows is highly problematic. If there are no permanent shocks to $T$ or $G$, then it is imperative to assume that GDP and aggregate consumption are not subject to permanent shocks either. But if there are no permanent shocks to marginal utility, then the long bond is the riskiest asset in economy. That clearly is counterfactual (Alvarez and Jermann, 2005). The gap between the long-run discount rates on strips and the long bond yield is governed by the riskiness of the permanent component of the pricing kernel. Explaining the high returns on risky assets such as stocks requires permanent risk to be large, not zero (e.g., Borovička et al., 2016). Next we consider the more realistic case of permanent shocks to output and cointegration between spending (tax revenue) and GDP.

**Corollary 1.** If the log of government spending/output ratio $G/GDP$ (revenue/output ratio $T/GDP$)

is stationary in levels, then the long-run expected log excess return on long-dated spending (revenue) strips equals that on GDP strips:

$$\lim_{j\to\infty} \mathbb{E}_t \left[ r_{t,t+j}^{g,j} \right] = \lim_{j\to\infty} \mathbb{E}_t \left[ r_{t,t+j}^{\tau,j} \right] = \mathbb{E}_t \left[ r_{t,t+n}^{GDP,\infty} \right] \gg y_t^{\$}(\infty). \tag{7}$$

We show below that government spending and tax revenue are cointegrated with GDP in the data; their ratio is stationary in levels. Under this realistic assumption on cash flows, expected returns on long-dated spending and tax revenue strips tend to the expected return on a long-dated GDP strip. A claim to GDP can be thought of as an unlevered equity claim. In the presence of permanent shocks to marginal utility, the long-run discount rate on GDP (unlevered equity) is much higher than the yield on long-term risk-free bonds. This corollary implies that government bond investors have a net long position in a claim that is exposed to the same long-run risk as the GDP claim. It follows immediately from this discount rate argument that the value of the long-run spending minus revenue strips will be smaller than what would be obtained when discounting with long-term bond yields.

Combining the properties of short-run and long-run discount rates, theory predicts that

$$\mathbb{E}_t \left[ R_{t+1}^{d} - R_t^{f} \right] > \mathbb{E}_t \left[ R_{t+1}^{\tau} - R_t^{f} \right] > \mathbb{E}_t \left[ R_{t+1}^{g} - R_t^{f} \right]. \tag{8}$$

To summarize, a model of asset prices will have to confront two forces that push up the equilibrium returns on government debt. First, there is short-run cash flow risk that pushes the expected return on the revenue claim above the expected return on the spending claim. Second, the long-run discount rates are higher than the yield on a long-maturity bond, because of the long-run cash flow risk in the spending and revenue claims equals that of long-run GDP risk. Government debt investors have a net long position in a claim that is exposed to the same long-run cash flow risk as GDP. The excess returns on government debt will tend to be much higher than those on long-maturity bonds. As a result of these two forces, government debt investors earn a larger risk premium on the long end than what they pay on the short end, which increases the fair expected return on the debt claim.

The low observed interest rate, or equivalently the high observed value, of the government debt portfolio represents a challenge to standard dynamic asset pricing models in light of the fundamental risk of the cash flows backing that debt. Our paper is the first to highlight this tension.

An important implication of (3) is that, if the government wants to reduce the riskiness and hence expected return on government debt, it would need to make the tax claim safer. This would require counter-cyclical tax revenues and hence tax rates. The latter is strongly at odds with the behavior of observed fiscal policy (Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2020).

## 3 Data

We conduct our analysis at annual frequency, which is a better frequency to study cash flow risk in fiscal revenues and outlays. We focus on the period from 1947 until 2019.

Nominal federal tax revenue and government spending before interest expense are from the Bureau of Economic Analysis, as is nominal GDP. Constant-maturity Treasury yields are from Fred. Stock price and dividend data are from CRSP; we use the CRSP value-weighted total market to represent the U.S. stock market. Dividends are seasonally adjusted. Details are provided in Appendix D.

As was shown in Figure 1, the surpluses expressed as a fraction of GDP are strongly pro-cyclical. Non-discretionary spending, including Social Security, Medicare and Medicaid, food stamps, and unemployment benefits, accounts for at least two-thirds of government spending. Many of these transfer payments rise automatically in recessions. In addition, the government often temporarily increases transfer spending in recessions, e.g., the extension of unemployment benefits in 2009 and 2020. On the revenue side, the progressive nature of the tax code generates strongly pro-cyclical variation in tax revenue as a fraction of GDP.

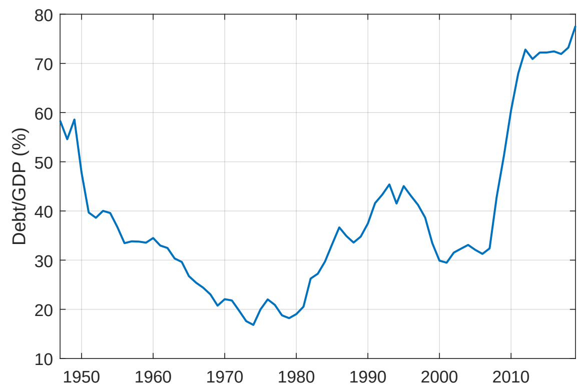
We construct the market value and the total returns of the marketable government bond portfolio using cusip-level data from the CRSP Treasuries Monthly Series. At the end of each period, we multiply the nominal price of each cusip by its total amount outstanding (normalized by the face value), and sum across all issuances (cusips). We exclude non-marketable debt which is mostly held in intra-governmental accounts.[3] Marketable debt includes the Treasury holdings of the Federal Reserve Bank. Hence, we choose not to consolidate the Fed and the Treasury, which would add reserves and subtract the Fed's Treasury holdings on the left hand side of (1). Doing so would mainly tilt the duration of the bond portfolio.

Following Hall and Sargent (2011) and extending their sample, we construct zero coupon bond (strip) positions from all coupon-bearing Treasury bonds (all cusips) issued in the past and outstanding in the current period. This is done separately for nominal and real bonds. Since zero-coupon bond prices are also observable, we can construct the left-hand side of eq. (1) as the market

[3] The largest holders of non-marketable debt are the Social Security Administration (SSA) and the federal government's defined benefit pension plan. Consolidating the SSA and the government DB plans with the Treasury department leads one to include the revenues and spending from the SSA/govt DB plan in the consolidated government revenue and spending numbers, and leads one to net out the SSA holdings of Treasuries, since they are an asset of one part of the consolidated government and a liability of the other part. Hence our treatments of debt and cash flows are mutually consistent.

value of outstanding marketable U.S. government debt.[4] Figure 2 plots its evolution over time, scaled by the U.S. GDP. It shows a large and persistent increase in the outstanding debt starting in 2008.

Figure 2: The Market Value of Outstanding Debt to GDP




The figure plots the ratio of the nominal market value of outstanding government debt divided by nominal GDP. GDP Data are from the Bureau of Economic Analysis. The market value of debt is constructed as follows. We multiply the nominal price (bid/ask average) of each cusip by its total amount outstanding (normalized by the face value), and then sum across all issuance (cusip). The series is annual from 1947 until 2019. Data Source: CRSP U.S. Treasury Database, BEA, authors' calculations.

Turning to returns, Table 1 reports summary statistics for the overall Treasury bond portfolio in Panel A and for individual bonds in Panel B. The excess returns on the entire Treasury portfolio realized by an investor who buys all of the new issuances and collects all of the coupon and principal payments is 1.16% per annum, on average. The portfolio has an average duration of 3.62 years. Given the secular decline in interest rates over the past forty years, the observed average realized return on the bond portfolio is, if anything, an over-estimate of investors' expected return.

Table 1: Summary Statistics for Government Bond Portfolio

| | Panel A | | | | | | | Panel B | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $R^d$ Nominal | $R^d$ Real | $R^d - R^f$ | $R^f$ | Duration | $\log(1+R^d)$ Nominal | $\log(1+R^d)$ Real | 1 Yr | 5 Yr | 10 Yr | 20 Yr |
| Mean | 5.38 | 2.20 | 1.16 | 4.22 | 3.62 | 5.15 | 2.18 | 4.93 | 5.71 | 5.87 | 6.64 |
| Std. Errors | [0.54] | [0.54] | [0.41] | [0.38] | [0.12] | [0.50] | [0.52] | [0.44] | [0.71] | [0.93] | [1.24] |
| Std. | 4.61 | 4.57 | 3.51 | 3.25 | 1.06 | 4.31 | 4.44 | 3.79 | 6.03 | 7.95 | 10.58 |
| Sharpe Ratio | 0.33 | | | | | | | 0.42 | 0.27 | 0.21 | 0.23 |

Panel A reports summary statistics for the holding period return on the aggregate government bond portfolio: the mean, the standard errors, and the standard deviation of the holding period return, $R^d$, the excess return, $R^d - R^f$, the three-month Tbill rate, $R^f$, the nominal and real log bond portfolio return $\log(R^d)$, and the weighted average Macaulay duration. Panel B reports the mean and the standard deviation of the holding period returns T-bonds with time-to-maturity of one year, five years, ten years and twenty years. All returns are expressed as annual percentage points. Duration is expressed in years. Data source: CRSP Treasuries Monthly Series. The sample period is from 1947 to 2019.

[4]Since the model fits nominal bond prices very well, as shown below, we can equivalently use model-implied bond prices. Similarly, we can use model-implied prices for real zero-coupon bonds.

# 4 Consumption-CAPM Model

To develop intuition, we start with a stylized version of the consumption-based asset pricing model (Breeden, 1979; Lucas, 1978). This stylized model only has one aggregate shock and a small number of parameters, but it illustrates the bond valuation puzzle.

The representative investor has CRRA preferences with risk aversion $\gamma$ and time discount factor $\beta$. The log of the real stochastic discount factor is given by:

$$m_{t,t+1} = -\beta - \gamma \Delta y_{t+1},$$

where log output growth is i.i.d. with Gaussian innovations:

$$\Delta y_{t+1} = \mu + \sigma_y \varepsilon^y_{t+1}.$$

In this simple model, we do not distinguish between output and consumption risk.

Spending and tax revenue are co-integrated with GDP. Specifically, we assume the log tax-to-output ratio $\tau_t = \log(T_t / Y_t)$ follows an AR(1) process:

$$\tau_{t+1} = \theta \tau_t + (1-\theta)\bar{\tau} + \sigma_\tau \varepsilon^y_{t+1} + \eta_\tau \varepsilon^\tau_{t+1},$$

whose innovation depends on the output growth shock $\varepsilon^y_{t+1}$ and a tax shock $\varepsilon^\tau_{t+1}$. This stationarity property guarantees that tax revenues and GDP are cointegrated.

We guess and verify (in Appendix B) that the value of a tax strip of maturity $j$ equals:

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+j}T_{t+j}\right] &= \mathbb{E}_t[\exp(m_{t,t+j})\exp(y_{t+j}+\tau_{t+j})] = Y_t \exp(a^\tau_j + b^\tau_j(\tau_t - \bar{\tau})), \textit{ where} \\
a^\tau_j &= (1-\gamma)\mu - \beta + a^\tau_{j-1} + \frac{1}{2}((1-\gamma)\sigma_y + b^\tau_{j-1}\sigma_\tau)^2 + \frac{1}{2}(b^\tau_{j-1}\eta_\tau)^2 \\
b^\tau_j &= b^\tau_{j-1}\theta.
\end{aligned}$$

The present value of the tax claim can then be computed as the sum of all strip values:

$$P^\tau_t = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}T_{t+j}\right] = Y_t \sum_{j=0}^{\infty} \exp(a^\tau_j + b^\tau_j(\tau_t - \bar{\tau})).$$

Similarly, we assume that the log of government spending to output follows an AR(1) process. By the same logic, the value of a spending strip is also exponentially affine in the log spending-to-output ratio. Then, by Proposition 1, the market value of debt equals the difference between the present value of tax claim $P^\tau_t$ and the present value of the spending claim, $P^g_t$.

**Calibration and Estimation** We calibrate three parameters and estimate the remainder. We set risk aversion to $\gamma = 10$, the volatility of output growth to $\sigma_y = 5\%$, and the subjective time discount factor $\beta$ such that we match the average annual real risk-free rate of 1.29% in the post-war data. We need a high $\gamma$ and high $\sigma_y$ to match the equity premium (Mehra and Prescott, 1985).[5] These parameter values deliver a maximum Sharpe ratio of $std(M_{t,t+1})/\mathbb{E}(M_{t,t+1}) \approx \gamma\sigma_y = 0.50$ per annum, accommodating the observed Sharpe ratio on U.S. equities (0.44 per annum in our sample).

We estimate the remaining parameters by GMM to fit the first and second moments of output growth, the tax-to-output ratio, and the spending-to-output ratio. The moment conditions are reported in Appendix B. Table B.1 reports the estimated parameter values and shows that the tax/output ratio is pro-cyclical ($\sigma_\tau > 0$), while the spending/output ratio is counter-cyclical ($\sigma_g < 0$). As a result, risk-averse investors use a significantly higher discount rate for the tax claim than for the spending claim. For example, the risk premium on the first period's tax strip is 3.0% (see Eq. (B.1)), whereas the risk premium on the first period's spending strip is 1.3%. In the long-term, as tax and spending are cointegrated with the GDP, the risk premia on tax and spending strips will converge to that of the GDP strip, which is about 2.4%, consistent with eqn. (8).

Panel A of Figure 3 shows the results. While the U.S. tax and spending levels are close to each other, as shown in the left panel, the valuation of the tax claim is well below that of the spending claim, as shown in the right panel, because of the discount rate gap. As a result, the market's valuation of future surpluses is negative, at around -350% of GDP. Panel B of Figure 3 reports the present value of surpluses normalized by GDP, as well as the one- and two-standard deviation bootstrapped confidence intervals.[6] The figure shows that the present value of government surpluses is below zero for nearly the entire sample in 95% of simulations. The right panel plots the difference of the debt/output ratio and the value of the surplus claim/output ratio, which we call the Wedge/GDP ratio. The Wedge/GDP ratio is around 4, with confidence intervals that are wide. However, we can reject the null hypothesis that the wedge is zero at the 5% statistical significance level.

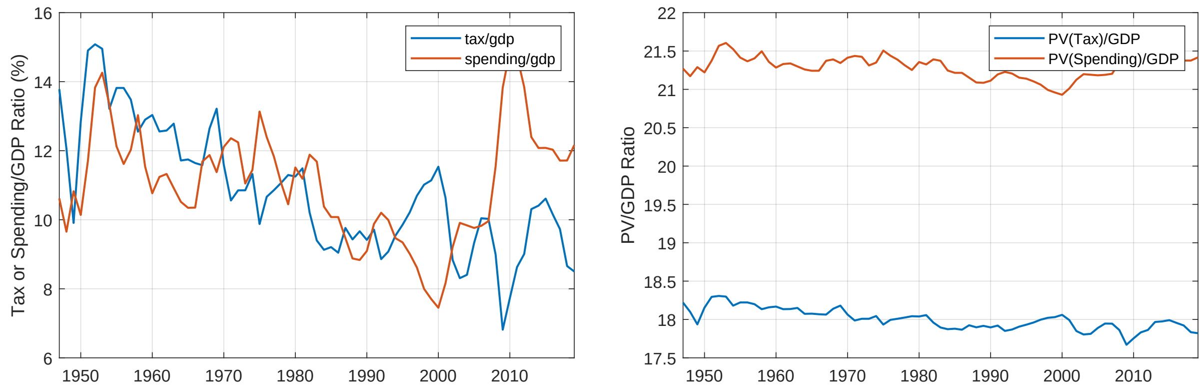
In this model, the expected real return on the tax claim is 4.1%, which, as we know from the inequality in eqn. (5), puts a lower bound on the return on Treasurys. The realized return on

[5]We choose an artificially low $\beta$ to avoid the risk-free rate puzzle (Weil, 1989). Alternatively, we could have used Epstein and Zin (1989) preferences and chosen the elasticity of intertemporal substitution to match the risk-free rate. Given that output growth is i.i.d., this is mathematically equivalent to freeing up the $\beta$ parameter in a setting with CRRA utility (Kocherlakota, 1996).

[6]The confidence interval is obtained by bootstrapping the ten parameters in Table B.1 from a normal distribution with mean equal to the point estimates and the variance-covariance matrix equal to the estimated one. For each parameter draw, we solve the valuation ratios of the tax and spending claims, and then compute the present value of government surpluses using the observed tax and spending series. We repeat this procedure 10,000 times. We drop the parameter draws that violate the transversality condition, which, for example, can happen when the drawn persistence parameters $\theta_\tau$ and $\theta_g$ are above 1.

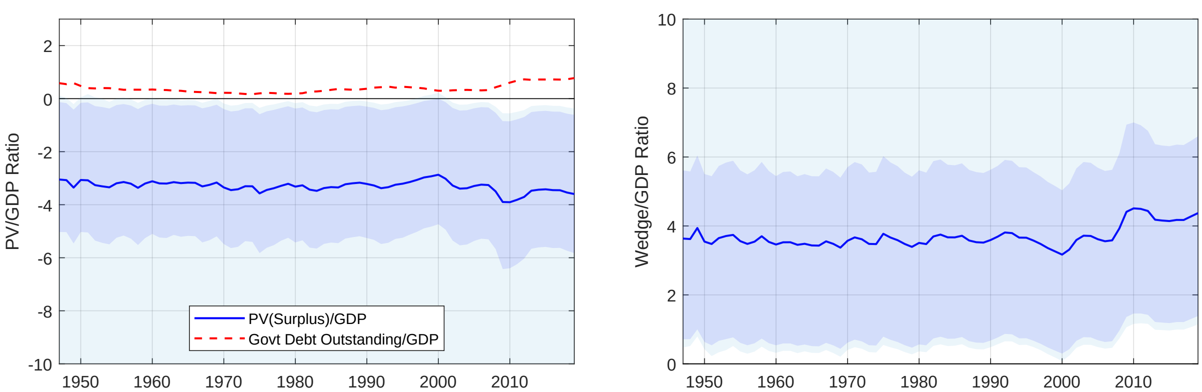
Figure 3: CCAPM Valuation of U.S. Government Debt

Panel A: Taxes and Spending.




The actual U.S. tax and spending are on the left. The present values of the tax claim and the spending claim based on the CRRA model are on the right. All time series are normalized by the concurrent U.S. GDP. The sample is annual, 1947—2019.

Panel B: Present Value of Government Surpluses and the Debt Valuation Gap.




The left panel plots the present value of government surpluses and the market value of debt as fractions of the current GDP. We plot the one- and two-standard-error confidence intervals based on 10,000 bootstrap iterations. The right panel plots the wedge between the market value of debt and the present value of government surpluses.

Treasurys is only 2.20%. This gap is the government bond risk premium puzzle. Really, the only way to generate a positive value of debt when the government runs deficits is to increase the valuation ratio of the tax claim, but its higher risk premium (2.81%) compared to the spending claim (2.22%) pushes in the other direction.

Bohn (1995)'s insight about the SDF covariance terms when valuing government surpluses is quantitatively important when permanent output (and consumption) shocks earn large risk premia. However, the CCAPM model is too stylized. The model has a constant risk-free rate, a flat yield curve, and a constant equity risk premium. In addition, we did not allow for feedback from the debt/output ratio to taxes and spending. Next, we estimate a full-fledged Dynamic Asset Pricing Model that remedies those shortcomings, but we ultimately arrive at similar estimates for the wedge. We show that our results are quite robust.

# 5 Quantitative Dynamic Asset Pricing Model

In order to quantify the value of the claims to tax revenue and government spending in (1), we need to (i) take a stance on the time-series properties of revenue and spending, and (ii) a stochastic discount factor $M_{t,t+j}$ to discount these cash flows.

## 5.1 Cash Flow Dynamics

We start by describing the cash flow dynamics.

**State Variables** We assume that the $N \times 1$ vector of state variables $\boldsymbol{z}$ follows a Gaussian first-order VAR:

$$\boldsymbol{z}_t = \boldsymbol{\Psi} \boldsymbol{z}_{t-1} + \boldsymbol{u}_t = \Psi \boldsymbol{z}_{t-1} + \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\varepsilon}_t, \tag{9}$$

with $N \times N$ companion matrix $\Psi$ and homoscedastic innovations $\boldsymbol{u}_t \sim i.i.d. \mathcal{N}(0, \Sigma)$. The Cholesky decomposition of the covariance matrix, $\boldsymbol{\Sigma} = \Sigma^{\frac{1}{2}} \left(\Sigma^{\frac{1}{2}}\right)'$, has non-zero elements on and below the diagonal. In this way, shocks to each state variable $u_t$ are linear combinations of its own structural shock $\boldsymbol{\varepsilon}_t$, and the structural shocks to the state variables that precede it in the VAR, with $\boldsymbol{\varepsilon}_t \sim i.i.d. \mathcal{N}(0, I)$. These state variables are defined in Table 2, in order of appearance of the VAR. The vector $\boldsymbol{z}$ contains the state variables demeaned by their respective sample averages.

Table 2: State Variables

| Position | Variable | Mean | Description |
|---|---|---|---|
| 1 | $\pi_t$ | $\pi_0$ | Log Inflation |
| 2 | $x_t$ | $x_0$ | Log Real GDP Growth |
| 3 | $y_t^{\$}(1)$ | $y_0^{\$}(1)$ | Log 1-Year Nominal Yield |
| 4 | $yspr_t^{\$}$ | $\underline{yspr}_0^{\$}$ | Log 5-Year Minus Log 1-Year Nominal Yield Spread |
| 5 | $pd_t$ | $\overline{pd}$ | Log Stock Price-to-Dividend Ratio |
| 6 | $\Delta d_t$ | $\mu_d$ | Log Stock Dividend Growth |
| 7 | $\Delta \log \tau_t$ | $\mu_\tau$ | Log Tax Revenue-to-GDP Growth |
| 8 | $\log \tau_t$ | $\log \tau_0$ | Log Tax Revenue-to-GDP Level |
| 9 | $\Delta \log g_t$ | $\mu_g$ | Log Spending-to-GDP Growth |
| 10 | $\log g_t$ | $\log g_0$ | Log Spending-to-GDP Level |
| 11 | $\Delta \log b_t$ | $\mu_b$ | Log Debt-to-GDP Growth |
| 12 | $\log b_t$ | $\log b_0$ | Log Debt-to-GDP Level |

This approach takes spending and tax policy as given, rather than being optimally determined.

By including spending and taxes in the state vector, we assume that the government commits a tax and spending policy that is affine in the state vector. Both policies are allowed to depend on a rich set of state variables with dependencies that are estimated from 73 years of data. The VAR includes $\Delta \log \tau_t$ and $\Delta \log g_t$, the log change in tax revenue-to-GDP and the log change in government spending-to-GDP in its seventh and eight rows. It also includes the log level of revenue-to-GDP, $\tau_t$, and spending-to-GDP, $g_t$, in its ninth and tenth rows. This fiscal cash flow structure has three important features.

First, our approach allows spending and revenue growth to depend not only on its own lag, but also on a rich set of macroeconomic and financial variables. Lagged inflation, GDP growth, interest rates, the slope of the term structure, the stock price-dividend ratio, and dividend growth all predict future revenue and spending growth. In addition, we allow innovations to the fiscal variables to be correlated with contemporaneous innovations in these macro-finance variables.

Second, we include the level variables $\tau_t$ and $g_t$. When there is a positive shock to spending, spending tends to revert back to its long-run trend with GDP. Similarly, after a negative shock to tax revenue, future revenues tend to increase back to their long-run level relative to GDP. This mean reversion captures the presence of automatic stabilizers and of corrective fiscal action, as pointed out by Bohn (1998). By having spending-to-GDP growth $\Delta \log g_t$ (revenue-to-GDP $\Delta \log \tau_t$) depend on lagged spending $g_t$ (lagged revenue-to-GDP $\tau_t$) with a negative coefficient, our VAR captures this mean reversion. Mean reversion is further amplified when spending-to-GDP growth $\Delta \log g_t$ ($\Delta \log \tau_t$) depends on lagged revenue-to-GDP $\tau_t$ ($g_t$) with a positive sign.

Formally, the inclusion of the levels of spending and tax revenue relative to GDP in the VAR is motivated by a cointegration analysis; the system becomes a vector error correction model. Appendix E.1 performs Johansen and Phillips-Ouliaris cointegration tests. The results support two cointegration relationships, one between log tax revenue and log GDP and one between log spending and log GDP.[7] In the absence of cointegration, all shocks to spending and tax revenues would be permanent rather than mean-reverting. Importantly, we are being conservative about future fiscal rectitude by imposing cointegration.

Third, based on prior findings that highlight a fiscal response to the level of debt (Bohn (1998); Cochrane (2019a,b)), we include the log debt-to-gdp ratio as a predictor variable in the state vector, and allow spending and revenue growth to depend on the lagged debt/output ratio. However, we do not impose the no-arbitrage condition (1) on the debt. Imposing that condition is equivalent to assuming that the government commits to a policy for the debt/output ratio. In our approach, we assume that the government commits to a tax and spending policy. The government cannot

[7] The coefficients estimates of the cointegration relationships tend to vary across sample periods. As a result, we take an a priori stance that the tax-to-GDP ratio $\log \tau$ and the spending-to-GDP ratio $\log g$ are stationary. That is, we assume cointegration coefficients of $(1, -1)$ for both relationships.

simultaneously commit to a debt, tax, and spending policy (see Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2021b, for a complete analysis).[8]

Cochrane (2019a,b) includes debt/GDP in the VAR and argues that this affects the dynamics of the surplus in important ways. In particular, a negative shock to GDP (or a negative shock to the surplus) leads to a deficit on impact. The deficit not only reverts back to zero in subsequent periods, but turns into a surplus. It is these S-shaped surplus dynamics, he argues, that makes government debt risk-free. We allow for these dynamics in our VAR. Our empirical approach does not rule out risk-free zero-beta debt.[9]

**Estimation** Two empirical issues require further discussion. First, the US tax/GDP ratio trends down in our sample, while the spending/GDP ratio has a slight upward trend. The sample average of $\Delta \log \tau_t$ is $\widehat{\mu}^{\tau} = -0.7\%$ and the sample average of $\Delta \log g_t$ is $\widehat{\mu}^{g} = 0.2\%$. Because we impose cointegration on the log tax-to-GDP ratio and the log spending-to-GDP ratio, the true unconditional growth rates of the tax-to-GDP ratio and the spending-to-GDP ratio have to be zero ($\mu_0^{\tau} = \mu_0^{g} = 0$).

To avoid biased estimates of the VAR coefficients, we cannot include trending variables in the VAR. Hence, when we estimate the dynamics of the state variables—and only then,—we remove the sample averages of the growth rates. We reconstruct the log tax-to-GDP and log spending-to-GDP ratios that enter in the VAR as follows:

$$\log \tau_t = \log \tau_1 + \sum_{k=1}^{t} (\Delta \log \tau_k - \widehat{\mu}^{\tau}), \quad \log g_t = \log g_1 + \sum_{k=1}^{t} (\Delta \log g_k - \widehat{\mu}^{g}),$$

where the initial level $\log g_1$ is the the actual log spending-to-GDP ratio at the start of our sample in 1947, while $\log \tau_1$ is chosen so that the resulting average surplus-to-GDP ratio is the same as in the unadjusted data. Importantly, when we price assets and value claims to spending and tax revenues, we always evaluate the state vector at the actual values of $\tau$ and $g$, not the de-trended ones. This approach is conservative, because the actual tax/GDP ratio (spending/GDP) is well below (slightly above) the detrended one. Hence, the model's cash flow forecasts imply larger tax revenue increases (spending declines) in the future than we would obtain if we had used the detrended variables instead.

Second, the log debt/GDP ratio, $\log b_t$, is highly persistent. Its first-order autocorrelation is 0.925. We include both the first-difference and the level of the log debt/GDP ratio in the VAR and

[8] Jiang et al. (2021b) show that the value of the debt implied by the model, divided by GDP, cannot be affine in the state vector, when the government commits to a tax and spending policy that is affine in the state vector.

[9] Risk-free debt imposes additional measurability restrictions, which can be tested. The results are available upon request. Note that the Hansen et al. (1991) analysis of risk-free debt does not extend to stochastically growing economies with permanent output risk.

impose the same error-correction dynamics as we did for spending/GDP and revenues/GDP. Furthermore, we allow for a structural break in the debt/output ratio in 2007. The Chow test for structural breakpoints rejects the null hypothesis of no structural break at the 1% level in 2007 and at no other date. Following the approach for stocks in Lettau and Van Nieuwerburgh (2008), demean the log debt/output ratio before 2007 with the pre-2007 sample mean (-1.167) and the log debt/output ratio after 2007 with the post-2007 sample mean (-0.411). The structural break introduces a -0.755 log point permanent increase in the debt/output ratio. The persistence of the resulting series is lower at 0.903.

This way of incorporating debt in the VAR results not only in a better behaved time series but also in more realistic predictions for future debt and surplus dynamics. It is conservative in that it results in a stronger response of surpluses to an increase in the debt/GDP ratio.

We estimate the VAR system in equation (9) using OLS. The point estimates of $\mathbf{\Psi}$ are reported in Table 3. Lagged macro-finance variables affect fiscal variables, and vice versa. Consistent with the error correction dynamics imposed by cointegration, we find that the response of the tax-to-GDP growth to the lagged tax-to-GDP level (i.e., $\Psi_{[7,8]}$) and the response of the spending-to-GDP growth to the lagged spending-to-GDP level (i.e., $\Psi_{[9,10]}$) are negative. In addition, the tax-to-GDP growth is also increasing in the lagged debt-to-GDP level, and the spending-to-GDP growth is decreasing in the lagged debt-to-GDP level.

The dynamics of $\log \tau_t$, $\log g_t$, and $\log b_t$ in rows 8, 10, and 12 of the VAR are implied by the corresponding dynamics of their first differences $\Delta \log \tau_t$, $\Delta \log g_t$, and $\Delta \log b_t$ in rows 7, 9, and 11, respectively, with the exception of the autoregressive coefficient which is 1 minus the corresponding coefficient. Likewise, there is no independent innovation to these level variables.

Table 3 also reports the estimate of $\Sigma^{\frac{1}{2}}$, the Cholesky decomposition of the residual variance-covariance matrix. The innovation in tax revenue-to-GDP growth is positively correlated with the GDP growth rate innovation, while the spending-to-GDP growth shock is negatively correlated with the GDP growth shock. In other words, tax revenues are strongly pro-cyclical and government spending is strongly counter-cyclical, as anticipated by our earlier discussion.

**Implied Revenue and Spending Dynamics** Figure 4 plots the impulse-response functions (IRFs) of the tax revenue-to-GDP ratio ($\tau_t$, left panels), government spending-to-GDP ratio ($g_t$, middle panels), and surplus-to-GDP ratio ($s_t$, right panels) to a GDP shock (top row), a revenue shock (middle row), and a spending shock (bottom row). The shocks are calibrated such that the log GDP growth decreases by 1%, the revenue-to-GDP ratio goes down by 1%, and the spending-to-GDP ratio goes up by 1%. The top row shows that the tax revenue-to-GDP ratio declines and the government spending-to-GDP ratio increases in response to a negative GDP shock. The surplus-

Table 3: VAR Estimate $\boldsymbol{\Psi}$

| | $\pi_{t-1}$ | $x_{t-1}$ | $y^{\$}_{t-1}(1)$ | $yspr^{\$}_{t-1}$ | $pd_{t-1}$ | $\Delta d_{t-1}$ | $\Delta\log\tau_{t-1}$ | $\log\tau_{t-1}$ | $\Delta\log g_{t-1}$ | $\log g_{t-1}$ | $\Delta\log b_{t-1}$ | $\log b_{t-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_t$ | **0.380** | -0.076 | 0.026 | -0.319 | -0.008 | 0.033 | **0.064** | **-0.036** | -0.015 | **-0.015** | 0.002 | **-0.036** |
| $x_t$ | -0.090 | 0.210 | **0.391** | 0.272 | *0.021* | 0.081 | -0.038 | *-0.059* | 0.067 | **0.078** | -0.037 | **0.042** |
| $y^{\$}_t(1)$ | 0.048 | 0.043 | **0.899** | 0.048 | 0.005 | 0.046 | -0.005 | -0.035 | -0.005 | 0.022 | -0.019 | -0.004 |
| $yspr^{\$}_t$ | -0.073 | **-0.092** | -0.050 | **0.485** | **-0.008** | **-0.031** | 0.017 | *0.016* | 0.008 | **-0.024** | 0.018 | -0.007 |
| $pd_t$ | **-2.697** | -1.347 | 0.354 | 2.909 | **0.769** | -0.219 | -0.020 | 0.207 | 0.089 | -0.260 | -0.129 | -0.032 |
| $\Delta d_t$ | 0.477 | 0.453 | -0.395 | *-1.913* | **0.063** | **0.276** | *-0.178* | **-0.173** | -0.105 | 0.122 | **0.263** | *0.066* |
| $\Delta\log\tau_t$ | -0.683 | 0.488 | 0.642 | **-3.876** | **0.106** | 0.121 | **0.258** | **-0.535** | 0.164 | **0.258** | 0.131 | **0.092** |
| $\log\tau_t$ | -0.683 | 0.488 | 0.642 | **-3.876** | **0.106** | 0.121 | **0.258** | **0.465** | 0.164 | **0.258** | 0.131 | **0.092** |
| $\Delta\log g_t$ | -0.451 | 0.054 | **-2.189** | -1.198 | **-0.170** | *-0.235* | *0.188* | *0.129* | **0.266** | **-0.536** | 0.119 | **-0.215** |
| $\log g_t$ | -0.451 | 0.054 | **-2.189** | -1.198 | **-0.170** | *-0.235* | *0.188* | *0.129* | **0.266** | **0.464** | 0.119 | **-0.215** |
| $\Delta\log b_t$ | 0.154 | -0.931 | 0.773 | **4.527** | -0.025 | -0.306 | 0.109 | 0.001 | -0.042 | 0.019 | -0.072 | -0.041 |
| $\log b_t$ | 0.154 | -0.931 | 0.773 | **4.527** | -0.025 | -0.306 | 0.109 | 0.001 | -0.042 | 0.019 | -0.072 | 0.959 |

We report our estimate of the VAR transition matrix $\boldsymbol{\Psi}$. Numbers in bold have $t$-statistics in excess of 1.96 in absolute value. Numbers in italics have t-statistics in excess of 1.645 but below 1.96.

VAR Estimate $100 \times \boldsymbol{\Sigma}^{\frac{1}{2}}$

| | $\varepsilon^{\pi}_t$ | $\varepsilon^{x}_t$ | $\varepsilon^{y^{\$}(1)}_t$ | $\varepsilon^{yspr^{\$}}_t$ | $\varepsilon^{pd}_t$ | $\varepsilon^{\Delta d}_t$ | $\varepsilon^{\Delta\log\tau}_t$ | | $\varepsilon^{\Delta\log g}_t$ | | $\varepsilon^{\Delta\log b}_t$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_t$ | 0.92 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_t$ | 0.55 | 1.84 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $y^{\$}_t(1)$ | 0.36 | 0.49 | 1.21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $yspr^{\$}_t$ | -0.10 | -0.19 | -0.27 | 0.43 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $pd_t$ | -3.41 | -2.08 | 1.17 | 0.78 | 14.79 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta d_t$ | -0.14 | 1.02 | 0.91 | -0.80 | -0.72 | 4.23 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta\log\tau_t$ | 2.32 | 1.77 | 0.10 | -0.49 | 0.92 | 0.82 | 4.47 | 0 | 0 | 0 | 0 | 0 |
| $\log\tau_t$ | 2.32 | 1.77 | 0.10 | -0.49 | 0.92 | 0.82 | 4.47 | 0.00 | 0 | 0 | 0 | 0 |
| $\Delta\log g_t$ | -0.90 | -2.17 | -1.30 | -0.40 | 0.05 | -1.31 | 0.14 | 0.00 | 3.28 | 0 | 0 | 0 |
| $\log g_t$ | -0.90 | -2.17 | -1.30 | -0.40 | 0.05 | -1.31 | 0.14 | 0.00 | 3.28 | 0.00 | 0 | 0 |
| $\Delta\log b_t$ | -2.50 | -3.24 | -1.30 | -0.32 | 2.73 | 2.04 | -0.86 | 0.00 | 0.37 | 0.00 | 6.35 | 0 |
| $\log b_t$ | -2.50 | -3.24 | -1.30 | -0.32 | 2.73 | 2.04 | -0.86 | 0.00 | 0.37 | 0.00 | 6.35 | 0.00 |

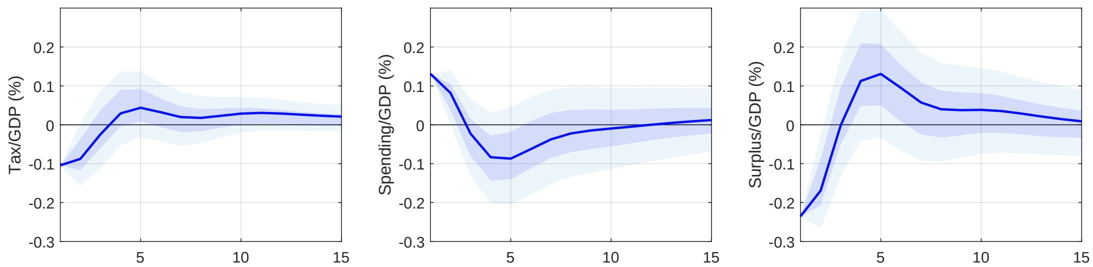
We report our estimate of the VAR innovation matrix $\Sigma^{\frac{1}{2}}$, multiplied by 100 for readability.

to-GDP is pro-cyclical. In addition, mean reversion in spending and revenues brings their responses to their own shocks back to zero within a few years. The instantaneous response of the surplus to all three shocks is negative. There is some evidence of an S-shaped response as the deficits turn into small surpluses after 3 years. In the case of tax and spending shocks these surpluses are short-lived. The confidence intervals on the IRFs are wide, so that for all three shocks, even the peak surplus response after 4-5 years is not significantly different from zero. All responses revert to zero in the long run, because of cointegration between spending and GDP and between tax revenues and GDP.

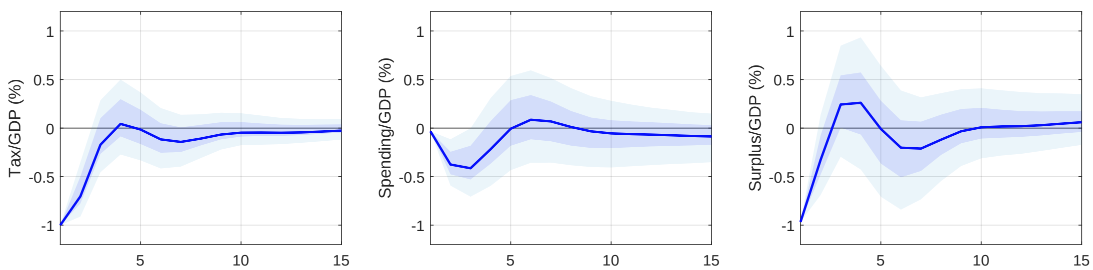
Figure 5 adds further credibility to the cash-flow projections by plotting expected cumulative spending and revenue growth over the next one, five, and ten years against realized future spending and revenue growth. To assess predictive accuracy, we compare the prediction of the benchmark annual VAR to that of the best linear forecaster at that horizon. By design, the VAR prediction is the best linear forecaster at the one-year horizon, but not at the five- and ten-year

Figure 4: Fiscal Impulse Responses

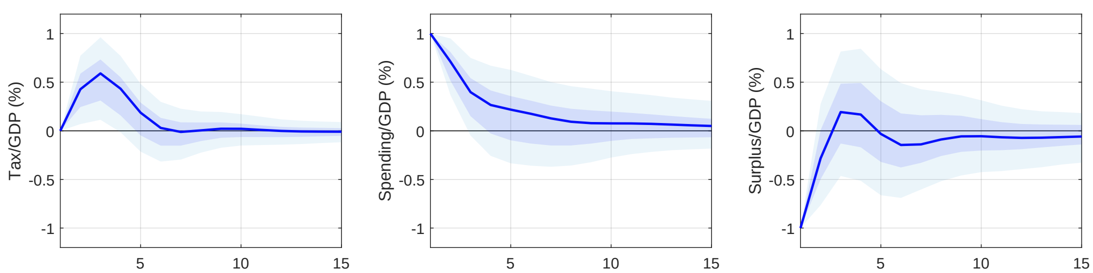
Panel A: $-1\%$ Shock to GDP Growth.




Panel B: $-1\%$ Shock to Tax-to-GDP.




Panel C: 1% Shock to Spending-to-GDP.




Solid blue line shows the impulse responses for the benchmark VAR. The impulse in the top row is a $-1$ percentage point shock to GDP growth $x_t$. The impulse in the middle row is a $-1$ percentage point shock to tax revenues. The impulse in the bottom row is a +1 percentage point shock to spending growth. We plot the one- and two-standard-deviation confidence intervals based on bootstrapping over 10,000 rounds.

horizons.[10] Predictive accuracy of the VAR is similar to that of the best linear forecast. The graph shows that the VAR implies reasonable behavior of long-run fiscal cash flows. Note how the long-run tax revenue forecasts from the VAR at the end of the sample are on the high side, while the spending forecasts are on the low side. This implies that the VAR predicts, if anything, too much mean reversion in the surplus compared to the data. This is conservative in that this will result in

[10]Since we use the actual tax/GDP and spending/GDP series to compute the VAR predictions but the companion matrix is estimated using the detrended series, the VAR series has a higher RMSE than the OLS prediction at the one-year horizon.

a higher present value of future surpluses.[11]

### 5.2 Asset Pricing

We take a pragmatic approach and choose a flexible SDF model that only assumes no arbitrage, and prices the term structure of interest rates as well as stocks well. This approach guarantees that our debt valuation is consistent with observed Treasury bond prices. It also results in an SDF that has enough permanent risk to account for the equity risk premium. This model extends the SDF from Section 4 to allow for additional priced sources of risk beyond GDP growth risk.

Motivated by the no-arbitrage term structure literature, we specify an exponentially affine SDF. The nominal SDF $M^{\$}_{t+1} = \exp(m^{\$}_{t+1})$ is conditionally log-normal:

$$m^{\$}_{t+1} = -y^{\$}_t(1) - \frac{1}{2}\Lambda'_t\Lambda_t - \Lambda'_t\varepsilon_{t+1}, \tag{10}$$

The real SDF is $M_{t+1} = \exp(m_{t+1}) = \exp(m^{\$}_{t+1} + \pi_{t+1})$, which is also conditionally Gaussian. The priced sources of risk are the structural innovations in the state vector $\varepsilon_{t+1}$ from equation (9). These aggregate shocks are associated with a $N \times 1$ market price of risk vector $\Lambda_t$ of the affine form:

$$\Lambda_t = \Lambda_0 + \Lambda_1 z_t,$$

The $N \times 1$ vector $\Lambda_0$ collects the average prices of risk while the $N \times N$ matrix $\Lambda_1$ governs the time variation in risk premia. Asset pricing in this model amounts to estimating the market prices of risk in $\Lambda_0$ and $\Lambda_1$. All asset pricing results are proven in Appendix C.

**Bond Pricing** We use $y^{\$}_t(h)$ to denote the nominal bond yield of maturity $h$, which is affine in the state vector:

$$y^{\$}_t(h) = -\frac{A^{\$}(h)}{h} - \frac{B^{\$}(h)'}{h} z_t;$$

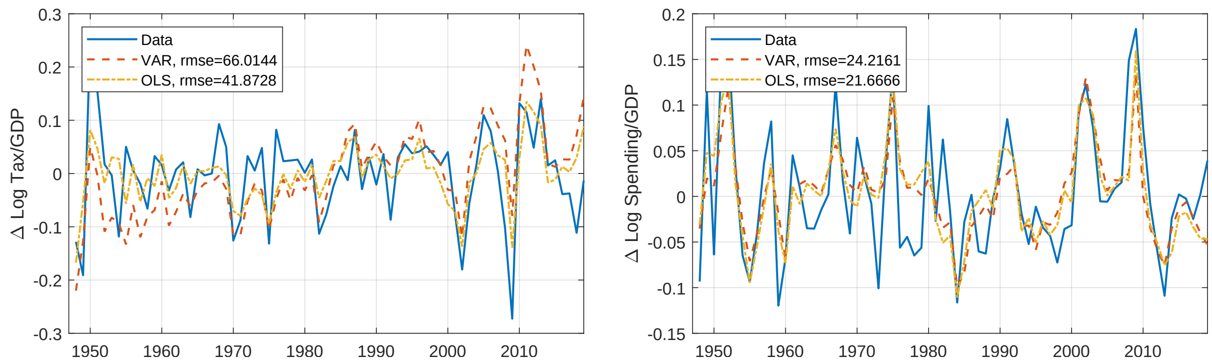
the scalar $A^{\$}(h)$ and the vector $B^{\$}(h)$ follow ordinary difference equations that depend on the properties of the state vector and of the market prices of risk. There is a similar formula for real bonds. We use this pricing equation to calculate the real interest rate, real bond risk premia, and inflation risk premia on bonds of various maturities.

Since both the nominal short rate ($y^{\$}_t(1)$) and the slope of the term structure ($y^{\$}_t(5) - y^{\$}_t(1)$) are included in the VAR, internal consistency requires the SDF model to price these bonds closely. The nominal short rate is matched automatically; it does not identify any market price of risk

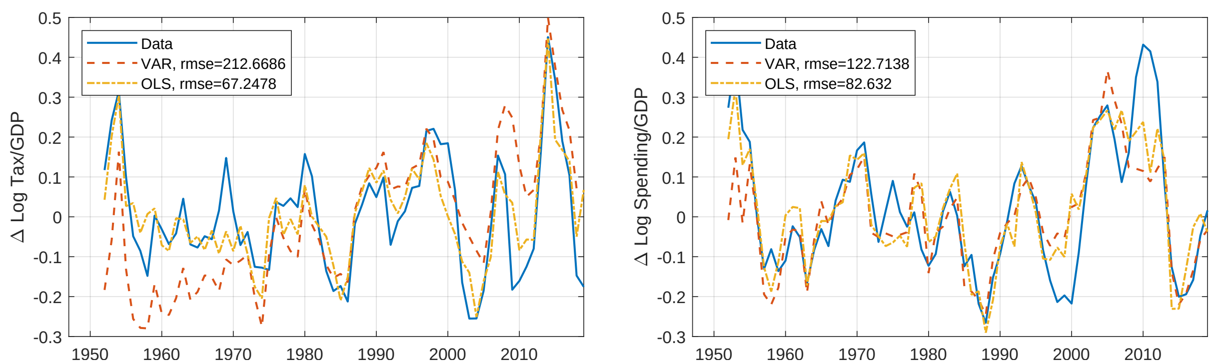
[11] This occurs because we evaluate these forecasts at the actual value of the tax/GDP and spending/GDP ratios. The former is well below its long-run mean towards the end of the sample, while the latter is above its mean. The error correction dynamics resulting from co-integration result in higher future tax revenue/GDP and lower future spending/GDP estimates.

Figure 5: Cash Flow Forecasts

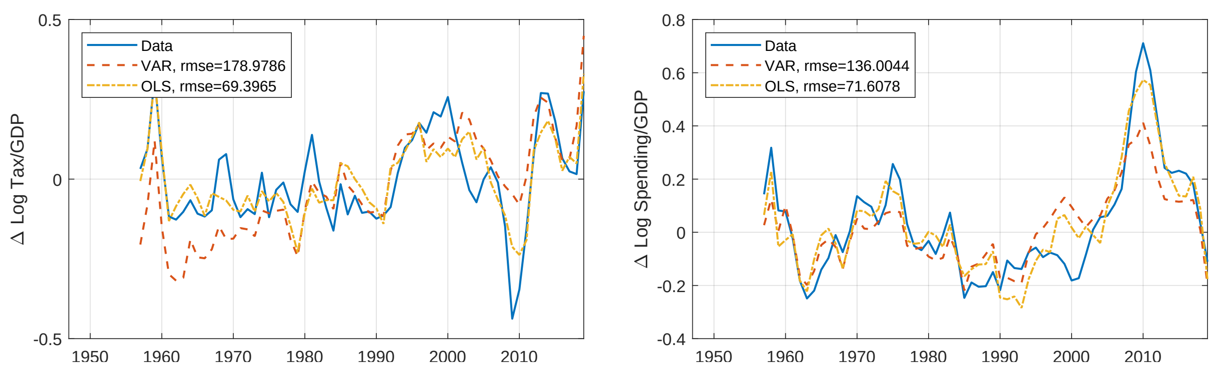
Panel A: Forecast of 1-Year Growth in Log Tax/GDP and Log Spending/GDP.




Panel B: Forecast of 5-Year Growth in Log Tax/GDP and Log Spending/GDP.




Panel C: Forecast of 10-Year Growth in Log Tax/GDP and Log Spending/GDP.




We plot the actual log tax and spending growth rates over 1-year, 5-year and 10-year rolling windows in solid blue lines. The value at each year represents the $k$-year growth rates that end at that year. We also plot these rates as forecasted by our VAR model in dashed red lines and these rates as forecasted by the OLS model in dash-dotted yellow lines. The value at each year represents the $k$-year growth rates condition on the information $k$ years ago.

parameters. Matching the slope of the yield curve generates $N+1$ parameter restrictions:

$$\begin{aligned} -A^{\$}(5)/5 &= y_0^{\$}(1) + yspr_0^{\$} = y_0^{\$}(5) \\ -B^{\$}(5)/5 &= e^{y1} + e^{yspr} \end{aligned}$$

They pin down the fourth element of $\Lambda_0$ and the fourth row of $\Lambda_1$. We also allow for a non-zero third element of $\Lambda_0$ and two non-zero elements in the third row of $\Lambda_1$. We pin down these elements by matching bond yields of maturities 2, 10, 20, and 30 years in each year $t \in 1, \cdots, T$. Since they represent $T \times 4$ moments for only 3 parameters, there are $T \times 4 - 3$ over-identifying restrictions. Since the behavior of long-term interest rates is important for our valuation results—recall the discussion on long-term bond yields in Section 2,—we impose extra weight on matching the 30-year bond yields.

We also price the yields on real bonds (Treasury inflation-index securities) for maturities 5, 7, 10, 20, and 30 years. They are available over a shorter sample of $T_2$ years. This adds $T_2 \times 5$ over-identifying restrictions. Again, we overweight matching the 30-year maturity.

**Equity Pricing** The VAR includes both log dividend growth and the log price-dividend ratio. The two time-series imply a time series for stock returns. We impose that the expected excess return time series implied by the VAR matches the equity risk premium time series in the model. The latter depends on the covariance of the SDF with stock returns and hence on the market price of risk parameters. The equity risk premium conditions pin down the sixth element of $\Lambda_0$ and the sixth row of $\Lambda_1$.

Let $PD_t^m(h)$ denote the price-dividend ratio of the dividend strip with maturity $h$ (Wachter, 2005; Van Binsbergen, Brandt, and Koijen, 2012). Then, the aggregate price-to-dividend ratio can be expressed as

$$PD_t^m = \sum_{h=0}^{\infty} PD_t^m(h). \tag{11}$$

In this SDF model, log price-dividend ratios on dividend strips are affine in the state vector:

$$pd_t^m(h) = \log\left(PD_t^m(h)\right) = A^m(h) + (B^m(h))' z_t.$$

Since the log price-dividend ratio on the stock market in part of the state vector, it is affine in the state vector by assumption; see the left-hand side of (12):

$$\exp\left(\overline{pd} + (e^{pd})' z_t\right) = \sum_{h=0}^{\infty} \exp\left(A^m(h) + (B^m(h))' z_t\right), \tag{12}$$

Equation (12) rewrites the present-value relationship (11), and articulates that it implies a restriction on the coefficients $A^m(h)$ and $(B^m(h))'$. Matching the time series for the price-dividend ratio in model and data provides $T \times 1$ additional over-identifying restrictions.

**Good Deal Bounds and Regularity Conditions** We impose good deal bounds on the standard deviation of the log SDF in the spirit of Cochrane and Saa-Requejo (2000). Specifically, we impose a penalty for annual Sharpe ratios in excess of 1.5.

Second, we impose regularity conditions on (unobserved) nominal and real bond yields of maturities of 100 to 4000 years. Specifically, we impose that yields stabilize and that nominal yields remain above real yields by at least long-run expected inflation. This is tantamount to a weak positivity restriction on the long-run inflation risk premium.

Third, we impose that bond return volatilities on very long-maturity bonds are bounded from below by 20%.

### 5.3 Estimation

We estimate the model's risk prices by minimizing the distance between the aforementioned bond and stock price moments in model and data. Appendix E reports the point estimates for the market price of risk parameters. Appendix F shows that the model matches the asset prices in the data closely. It provides a tight fit for the entire time series of nominal bond yields of the various maturities. It also shows a reasonable fit for real bond yields. The model closely matches the dynamics of the nominal bond risk premium, and generates reasonable behavior on nominal and real yields at very long horizons. Finally, the model produces reasonable equity risk premium level and dynamics, and provides a close fit to the time-series of the price-dividend ratio. Because it is able to generate an expected equity return that fits the data well, and is large compared to the long-term real rate, the SDF has a large permanent component. Having formulated and estimated a realistic SDF, we now turn to our main exercise.

## 6 Government Debt Valuation in DAPM

### 6.1 Surplus Pricing

With the VAR dynamics and the SDF in hand, we can calculate the expected present discounted value of the primary surplus:

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} S_{t+j}\right] = \sum_{j=0}^{\infty} \mathbb{E}_t\left[M^{\$}_{t,t+j} T_{t+j}\right] - \sum_{j=0}^{\infty} \mathbb{E}_t\left[M^{\$}_{t,t+j} G_{t+j}\right] = P_t^{\tau} - P_t^{g}, \tag{13}$$

where $P_t^\tau$ is the cum-dividend value of a claim to future nominal tax revenues and $P_t^g$ is the cum-dividend value of a claim to future nominal government spending. The following proposition shows how to price the government cash flows.

**Proposition 4 (Pricing Government Cash Flows).** The price-dividend ratios on the tax claim and the spending claim are the sum of the price-dividend ratios of their strips, whose logs are affine in the state vector $z_t$:

$$PD_t^\tau = \frac{P_t^\tau}{T_t} = \sum_{h=0}^{\infty} \exp(A^\tau(h) + B^\tau(h)' z_t), \tag{14}$$

$$PD_t^g = \frac{P_t^g}{G_t} = \sum_{h=0}^{\infty} \exp(A^g(h) + B^g(h)' z_t). \tag{15}$$

The proof is in Appendix C.4. The coefficients $B^g(h)$ and $B^\tau(h)$ in (14)-(15) measure the risk exposure of spending and tax revenue strips to each state variable, generalizing the univariate expressions of Section 4.

Combining these equations, note that the value of debt is given by :

$$T_t PD_t^\tau - G_t PD_t^g.$$

When the government runs deficits, we need to a larger valuation ratio for the tax claim: $PD_t^\tau > PD_t^g$ to get a positive valuation of the debt. However, the tax claim is riskier, which will tend to push the valuation ratio of the tax claim below that of the spending claim. This needs to be offset by generating higher expected growth of tax revenue in the short run when the government runs deficits. As we show, this effect is not strong enough in the data.

## 6.2 Main Results

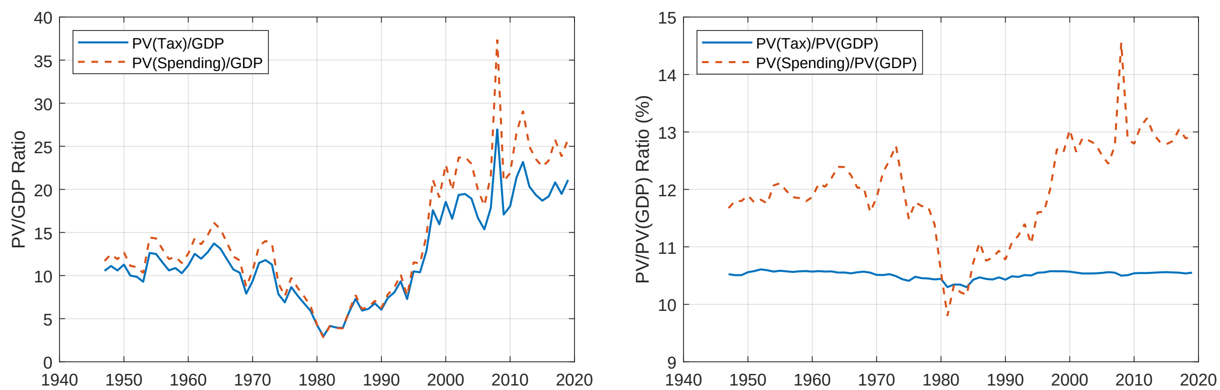
### 6.2.1 The Valuation Puzzle

The left panel of Figure 6 plots the present value of tax revenue normalized by GDP, $P_t^\tau / GDP_t$. The time-series average of this ratio is 12.19. In other words, the representative investor is willing to pay 12.19 times the annual GDP on average for the right to receive all current and future tax revenues. The value of the tax claim displays substantial time-variation. A pronounced V-shape arises, which is inherited from the inverse V-shape of long-term real interest rate. Real rates are high in the mid-1970s to mid-1980s and low at the beginning and end of the sample. Discounting future tax revenues by a low (high) long-term real rate results in a high (low) valuation ratio.

The time-series average of the present value of government spending normalized by GDP, $P_t^g / GDP_t$, is 14.27. The spending claim is more valuable than the revenue claim, which (in part) reflects the counter-cyclicality of government spending in the short run. Its value shows the same inverse V-shaped dynamics as the revenue claim.

The right panel of Figure 6 plots the same present values of tax revenue and government spending, but now normalized by the present value of GDP—namely, $P_t^{\tau}/P_t^{GDP}$ and $P_t^{g}/P_t^{GDP}$. This alternative scaling is a ratio of two stocks, rather than a ratio of a stock to a flow. It expresses the values of tax and spending claims relative to total wealth in society. Current and future tax revenues represents on average 10.52% of total wealth, while current and future spending averages to 11.99% of total wealth. We also find that the present value ratios of tax revenue and GDP and of spending and GDP are more stable over time. This reflects the common long-run dynamics of T, G, and GDP implied by cointegration.

Figure 6: DAPM Valuations of Taxes and Spending




(a) Normalized by GDP (b) Normalized by the Present Value of GDP

The figure plots the cum-dividend present values of tax revenues and of government spending. Both time series are scaled by the current U.S. GDP in the left panel, and by the present value of GDP in the right panel. The sample is 1947 until 2019. The unit is percentage points.

Now we are in a position to evaluate the claim to future government surpluses as the tax claim minus the spending claim, the right-hand side of equation (13). Figure 7 plots the present value of government surpluses as the solid blue line. The market value of the US government debt is plotted as the dashed red line. The unconditional average present value of the government surplus is -207.88% of GDP, far below the average market value of outstanding government debt, 38.23% of GDP.

The valuation wedge measures the difference between the market value government debt and the present value of surpluses. It quantifies the government debt valuation puzzle. The wedge/GDP ratio is 246% on average. In the time series, the gap widens dramatically in the last 20 years of the sample, as the level of government debt rises to 52.0% of the GDP and the valuation of the surplus claim increases dramatically in absolute value to 444% of GDP. In other words, the

U.S. government has been issuing government debt while simultaneously decreasing the expected surpluses to back up the debt. The result has been a widening of the valuation gap to 552% of the GDP at the end of 2019. The puzzle will deepen further with the large deficits incurred in the wake of the coronavirus crisis of 2020.

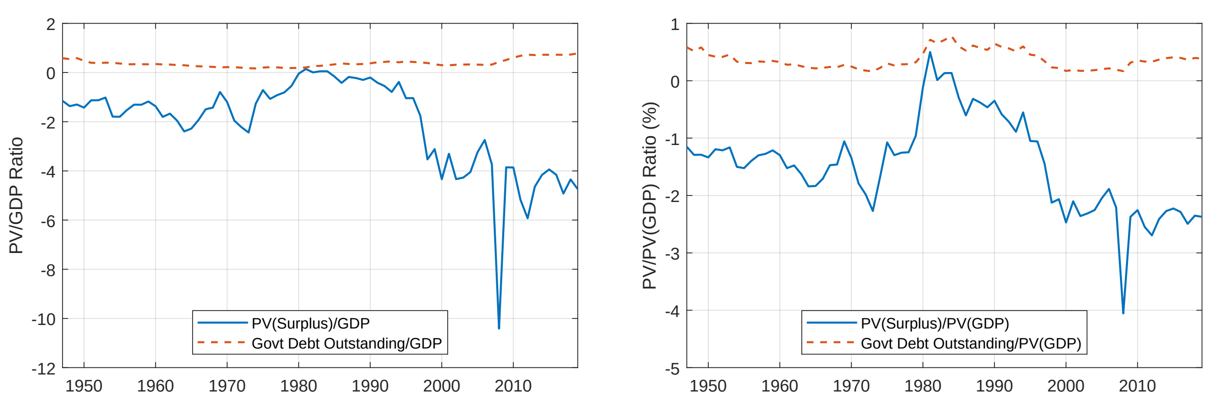
Equation (2) lets us interpret the puzzle further. The first term on the right-hand side, the risk-neutrally discounted present value of surpluses, is just about zero since the average primary surplus is about zero in our sample. Therefore, the entire wedge of 246% of GDP stems from the differential riskiness of the revenue and the spending claims.

Lastly, we note that imposing cointegration between tax revenues and GDP and spending and GDP is not only imperative to accurately describe fiscal dynamics but also leads to conservative estimates for the wedge/GDP ratio. Without the error correction dynamics present in our VAR system, an increase in government spending following a recession is not offset by future reductions in spending or future increases in tax revenues, but rather becomes permanent. As a result, the spending claim would be much safer and the tax revenue claim much riskier, leading to a much more negative present value of government surpluses and a much larger valuation wedge.

#### 6.2.2 Risk Premia on Tax Revenue and Spending Strips

Figure 8 plots the risk premia on revenue and spending strips for maturities from 1 to 20 years. For comparison, it also plots the risk premia on GDP strips. At the short end of the maturity spectrum, risk premia on spending strips are very low ($-1\%$ at the one-year horizon), as shown in eqn. 6. Because spending is counter-cyclical, these strips are a hedge. In contrast, short-maturity

Figure 7: Present Value of Government Surpluses and Market Value of Government Debt

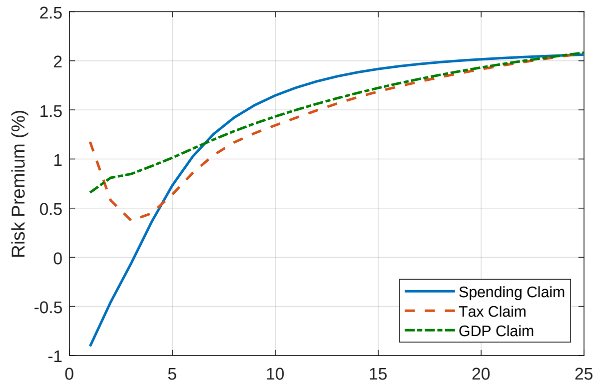



(a) Normalized by Current GDP (b) Normalized by the Present Value of GDP

The figure plots the cum-dividend present values of the government surplus and the market value of government debt. Both time series are scaled by the current U.S. GDP in the left panel, and by the present value of GDP in the right panel. The sample is 1947 until 2019. The unit is percentage points.

tax revenue strips have high risk premia (+1%) because tax revenues are low in high marginal utility times, making the tax claim a risky asset.

Figure 8: Term Structure of Risk Premia on the T-Claim and the G-Claim




This figure plots the term structures of risk premia on the spending claim, the tax claim, equity, and the GDP claim in our benchmark model. Each point is an annualized holding-period risk premium, as derived in equation (C.13) of the Appendix.

As we move to long maturities, risk premia on revenue and spending strips converge towards each other. As noted in eqn. (7), since tax and spending are cointegrated with the GDP, their risk premia also converge towards the risk premium on a GDP strip. Claims to GDP are like unlevered equity claims. They have risk premia well in excess of real bond risk premia but below (levered) equity risk premia. By horizon of 20 years, most of this convergence in risk premia has taken place.[12]

In our sample, the average one-year nominal interest rate is $y_0^{\$}(1)$= 4.5% whereas the unconditional average one-year nominal GDP growth rate is $x_0 + \pi_0$= 6.2%. The risk-free interest rate is on average below the growth rate, as emphasized by Blanchard (2019). However, government tax and spending processes are sufficiently risky. The average nominal discount rates, $r_0^{\tau} = 7.08\%$ and $r_0^{g} = 6.99\%$, are similar to the unconditional nominal discount rate for the GDP claim, $r_0^{x} = 7.09\%$, and above the average nominal GDP growth rate.[13] If we use this average nominal discount rate in a simple Gordon growth model $PD^g = \frac{1}{r_0^g - (x_0 + \pi_0)}$, then this delivers an average valuation ratio for the spending claim of 130.97, very close to the actual average valuation ratio for the spending claim of 131.47 in the full model.

These discount rates have implausible implications for bond returns on the entire portfolio.

[12]Not shown in the graph is the term structure of dividend strip

[13]As derived in Appendix C.4, $r_0^g = x_0 + \pi_0 + \kappa_0^g - \overline{pg}(1 - \kappa_1^g)$, where $\overline{pg}$ is the long-run mean of the log price-dividend ratio on the G-claim, and $\kappa_0^g$ and $\kappa_1^g$ are linearization constants, with similar expressions for the T-claim.

The expected returns in levels on the tax claim is given by $\mathbb{E}[R^{\tau}] = 8.46\%$.[14] From the inequality in eqn. (8), it follows that the expected nominal return on Treasurys is bounded below by 8.46%, or 5.28% in real terms. The actual realized real return on the entire portfolio of marketable Treasurys in the sample is only 2.20%, even though the realized return is enhanced by the secular decline in real rates in the last part of the sample (e.g., van Binsbergen, 2020). This 3.08% gap is the government risk premium puzzle. This gap is six times the standard error on the measured real return on Treasurys.

We generate these discount rates while maintaining an excellent fit for the term structure of Treasury yields. The claim to surpluses reflects the risk of the government's future debt issuance strategy. Future net debt issuances at inopportune (high SDF) times make the overall bond portfolio riskier than individual Treasury bonds. Therefore, even if risk-free interest rates are below growth rates, the risk premia on government tax and spending processes are large enough to generate a finite valuation for the surplus claim. We recall from Section 2 that when the unconditional expected returns on T- and G-claims are similar, the unconditional expected return on the government debt portfolio is equal to the expected return on the G- and T-claims. That is, the debt portfolio is highly risky.

# 7 Alternative Explanations

We discuss five alternative explanations for the U.S. government valuation puzzle but find that, ultimately, all of them fall short.

## 7.1 Bubbles and Limits to Arbitrage

The valuation gap can be interpreted as violation of the transversality condition (TVC) in Treasury markets, consistent with the presence of a rational bubble (Samuelson, 1958; Diamond, 1965; Blanchard and Watson, 1982). The TVC is violated if the value of debt in the far future does not converge to zero:

$$\lim_{T\to\infty} \mathbb{E}_t \left[ M_{t,t+T} \frac{D_{t+T}}{Y_{t+T}} Y_{t+T} \right] \neq 0.$$

Several pieces of evidence speak against this explanation. First, the key piece of evidence is the plot of GDP risk premia in Figure 8. In the long run, the GDP strip earn a risk premium of more than 2% above the real risk-free rate. Even if the debt is risk-free, as long as the debt is co-integrated with GDP, then we need to discount the claim to future at the risk-free rate plus at least 2%. This follows immediately from Corollary 1. The TVC is unlikely to be violated because

[14] See eqn. (C.19) in the Appendix for the mathematical expression.

the risk-adjusted discount rate on the portfolio of Treasury debt is higher than the growth rate of GDP. To develop intuition for this result, note that when the debt/output ratio is constant ($b_t = D_t/GDP_t = b$), the value of the debt in the far future is given by the price of an output strip:

$$\lim_{T\to\infty} \mathbb{E}_t \left[ M_{t,t+T} \frac{D_{t+T}}{Y_{t+T}} Y_{t+T} \right] = b \lim_{T\to\infty} \mathbb{E}_t \left[ M_{t,t+T} Y_{t+T} \right]$$

Put differently, if the TVC were violated, a claim to GDP would also have infinite value. Models that violate the TVC for debt typically produce violations of TVCs in all long-lived assets.

Second, Brock (1982); Tirole (1982); Milgrom and Stokey (1982); Santos and Woodford (1997) argue that rational bubbles are hard to sustain in the presence of long-lived investors absent other frictions. Third, as Figure 7 shows, the valuation gap is growing faster than GDP, which is inconsistent with rational bubbles. In rational bubble models, the debt/GDP ratio declines over time. Fourth, the rise in the sovereign CDS spread after the Great Financial Crisis, documented by Chernov et al. (2020); Pallara and Renne (2019), seems at odds with a rational bubble in Treasury debt.

### 7.2 Convenience Yields

U.S. government bonds occupy a privileged place in the world's financial system. They carry a "convenience yield" which makes Treasury yields lower than the safe rate of interest. The convenience yield produces an additional source of revenue, because the U.S. Treasury can sell its bonds for more than their fundamental value. The question is how far this explanations can go towards accounting for the bond valuation puzzle. We enrich the baseline model to account for convenience.

The convenience yield, $\lambda_t$, is the government bonds' expected returns that investors are willing to forgo under the risk-neutral measure. Assuming a uniform convenience yield across the maturity spectrum, the Euler equation for a Treasury bond with maturity $h+1$ is:

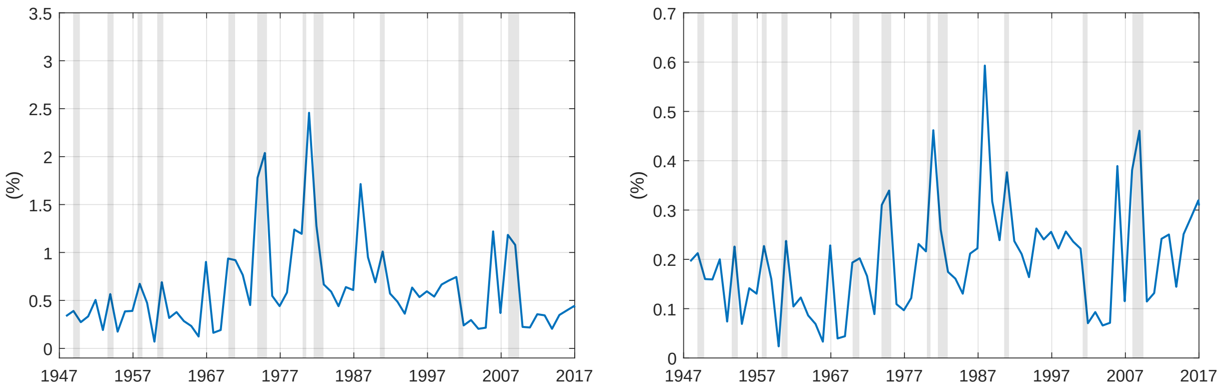
$$e^{-\lambda_t} = \mathbb{E}_t \left[ M_{t+1}^{\$} \frac{P_{t+1}^{\$}(h)}{P_t^{\$}(h+1)} \right].$$

**Proposition 5.** If the TVC holds, the value of the government debt portfolio equals the value of future surpluses plus the value of future seigniorage revenue:

$$\sum_{h=0}^{H} Q_{t-1,h+1}^{\$} P_t^{\$}(h) = E_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}^{\$} \left( T_{t+j} - G_{t+j} \right) \right] + E_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}^{\$} (1 - e^{-\lambda_{t+j}}) \sum_{h=1}^{H} Q_{t+j,h}^{\$} P_{t+j}^{\$}(h) \right] \quad (16)$$

where $\sum_{h=0}^{H} Q_{t-1}^{\$}(h+1) P_t^{\$}(h)$ on the left-hand side denotes the cum-dividend value of the gov-

Figure 9: Convenience Yield and Seigniorage Revenue




(a) Convenience Yield $\lambda_t$ (%) (b) Seigniorage Revenue/GDP Ratio (%)

The left panel reports the annual convenience yield time series time series for $\lambda_t$, computed as the weighted average of Aaa-Treasury and high-grade commercial papers-bills yield spreads. The right panel reports time series of the seigniorage revenue from convenience scaled by GDP, $(1 - e^{-\lambda_t})D_t/GDP$. The sample period is from 1947 until 2019. The shaded areas indicate NBER recessions.

ernment's debt portfolio at the start of period $t$, and $\sum_{h=1}^{H} Q^{\$}_{t+j}(h)P^{\$}_{t+j}(h)$ on the right-hand side denotes the ex-dividend value of the government's debt portfolio at the end of period $t + j$.

When there is no convenience yield (i.e., $\lambda_t = 0$), we end up back in the standard case of Proposition 1. If the quantity of current and future outstanding government debt is positive, then a positive convenience yield will always increase the value of government debt, acting as an additional source of revenue. This additional income is akin to seigniorage revenue and could potentially turn government fiscal deficits into (broadly defined) surpluses.

As an empirical strategy, we measure the convenience yield following Krishnamurthy and Vissing-Jorgensen (2012). To proxy for $\lambda_t$, we use the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread where the time series of weights are computed to match the duration of the government bond portfolio period by period. The left panel of Figure 9 shows the time series of the convenience yield. Over the sample period from 1947 to 2019, the average convenience yield is 0.60% per year, which implies average seigniorage revenue of $42.75 billion per year, or 0.20% of U.S. GDP, as shown in the right panel of Figure 9. The figure also illustrates the counter-cyclical nature of the convenience yield and seigniorage revenue.[15]

We rewrite equation (16) as:

$$\sum_{h=0}^{K} Q^{\$}_{t-1}(h+1)P^{\$}_{t}(h) = E_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}T_{t+j}K_{t+j}\right] - E_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}G_{t+j}\right],$$

[15]Appendix D shows that this convenience yield measure is close to other measures proposed in the literature.

where the variable $K$ represents the combined tax and seigniorage revenues as a fraction of the current tax revenue:

$$K_{t+j} \quad = \quad 1 + \frac{(1 - e^{-\lambda_{t+j}}) \sum_{h=1}^{H} Q^{\$}_{t+j}(h) P^{\$}_{t+j}(h)}{T_{t+j}}.$$

We call a claim to the combined tax and seigniorage revenues, $T_{t+j}K_{t+j}$, the modified tax claim. We introduce $\Delta \log K_t$ as an additional state variable in the VAR, with an unconditional mean of zero because $\log K_t$ is stationary. The augmented state vector is $\tilde{z}_t = [z_t, \Delta \log K_t]$. We then re-estimate the VAR dynamics and the market prices of risk, and use the same method as in Proposition 4 to price the modified tax claim. The pricing formula for the modified revenue claim is:

$$\mathbb{E}_t \left[ \sum_{j=0}^{\infty} M^{\$}_{t,t+j} T_{t+j} K_{t+j} \right] \quad = \quad T_t K_t \cdot PD^k_t,$$

where $PD^k_t$ is a function of the state variables $\tilde{z}_t$.

The left panel of Figure 10 reports the present value of government surpluses plus seigniorage revenues in the green dashed line and the present value of government surpluses (without seigniorage) in dashed red line. The convenience yield increases the model-implied value of the debt. On average, the present value of government surpluses and seigniorage revenues is -130.1% while the present value of government surpluses alone is -183.9%.

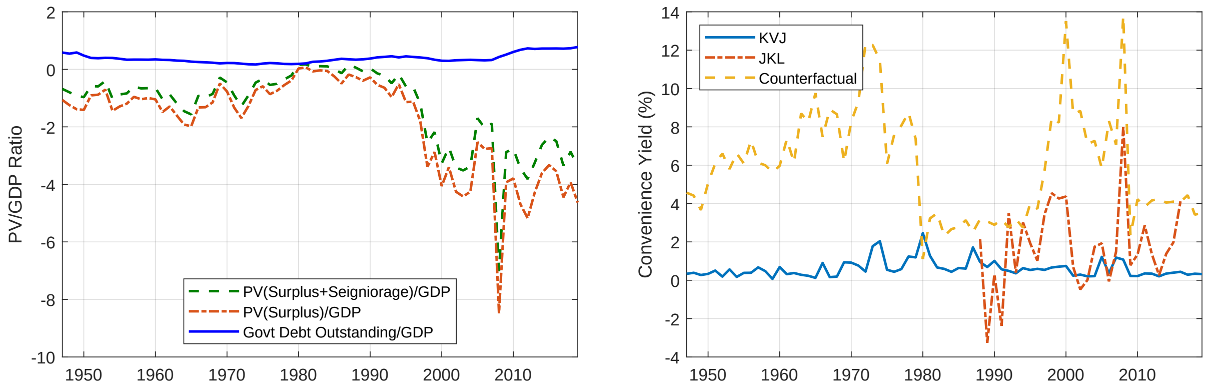
Although the effect of convenience is sizable, it only closes about one quarter of the gap between the market value of U.S. government debt and the present value of surpluses. This may be a surprising result given the large perceived convenience yield on Treasuries. There are two offsetting effects at work. On the one hand, there is positive seigniorage revenue which increases the surplus and its present value. On the other hand, the convenience yield raises the true risk-free rate given observed bond yields. Higher safe rates increase the discount rate of future revenues and spending, lowering the present value of surpluses. The positive cash flow effect is partly offset by the negative discount rate effect, weakening the power of convenience yields as a resolution to the puzzle.

How large does the convenience yield need to be to resolve the puzzle? To answer this question, we fix the VAR and market price of risk parameters, and solve for the counter-factual convenience yield that makes the government's valuation equation hold:

$$T_t K_t^{cf} PD_t^{k,cf} - G_t PD_t^g = \sum_{h=0}^{H} Q^{\$}_{t-1}(h+1) P^{\$}_t(h),$$

where the convenience yield impacts both the tax revenue multiplier $K_t$ and the price-dividend

Figure 10: Present Value of Government Surpluses and Implied Convenience Yield




(a) Valuation/GDP Ratio (%)

(b) Convenience Yield (%)

The left panel plots the present value of government surpluses with and without seigniorage revenue, scaled by GDP. The right panel plots the actual and the counterfactual convenience yields $\lambda_t$ and $\widetilde{\lambda}_t$. The 'Counterfactual' convenience yield is defined as the convenience yield reverse-engineered to enforce the government's intertemporal budget constraint. The KVJ convenience yield is the Krishnamurthy and Vissing-Jorgensen (2012) measure, while the 'JKL' is the Jiang et al. (2021a) measure.

ratio $PD_t^k$ for the modified tax claim. The right panel of Figure 10 reports the resulting counterfactual convenience yield process $\lambda_t^{cf}$ in the dashed yellow line alongside the actual convenience yield $\lambda_t$ from Krishnamurthy and Vissing-Jorgensen (2012) in the solid blue line. The convenience yield needed to match the present values of government surpluses and seigniorage revenues to the actual debt value is 6.00% on average, an order of magnitude larger than the actual average of 0.60%.

Some have argued that the convenience yields are larger than implied by the AAA-Treasury spread. For example, Jiang et al. (2021a) argue that foreigners earn convenience yields from a broad range of dollar assets, including investment-grade corporate bonds. Subtracting Treasury from U.S. AAA corporate yields removes that dollar safety premium. The right panel of Figure 10 also plots the convenience yields from the JKL paper. They average to 1.75% per annum.[16] This is a generous estimate of the convenience yield since it assumes that all holders of Treasury bonds assign the same convenience yield as foreign investors. Nevertheless, it still substantially below the counterfactual convenience yields. As the supply of safe assets increases, convenience yields may decline (Krishnamurthy and Vissing-Jorgensen, 2012, 2015), or even disappear altogether if the U.S. dollar were to lose its privileged role in the world financial system (Farhi, Gourinchas, and Rey, 2011; Farhi and Maggiori, 2018; He, Krishnamurthy, and Milbradt, 2019). The above analysis ignores this possibility, which leads us to possibly overstate the importance of convenience.

[16] In related work, Koijen and Yogo (2019) obtain a 2.10% average convenience yield on Treasuries earned by foreigners.

### 7.3 Austerity as a Peso Event

Next, we consider a model in which bond investors price in the possibility of a major government spending cut. However, such radical austerity never occurs in our 73-year sample. How large does the spending cut probability need to be in order to equate the market value of government debt to the present value of surpluses?[17]

More precisely, we consider a permanent spending cut that lowers today's and future government spending by 40%. For reference, defense spending accounted for 16% of the federal budget, Social Security for 23%, and Medicare for 15% in 2019. For a typical year with an average spending-to-GDP ratio of 11%, the cut lowers it to 6.6%. Moreover, this spending cut is permanent. That is, we assume the long-run mean of spending-to-GDP, $g_0$, falls by 40%. The dynamics of the demeaned state variables, including the demeaned log spending-to-GDP ratio, from that point forward are still given by the benchmark VAR. As a result, the price of the G-claim scaled by GDP is simply scaled by a factor of $\ell = 1 - 40\% = 60\%$ when the peso event happens. Moreover, the peso event itself is not priced; we do not change the market prices of risk $\Lambda_t$.[18]

Under this simple setting, we ask how likely the spending cut should be in each year to precisely match the present value of government surpluses to the market value of debt. We denote this probability of the spending cut that closes the valuation gap by $\phi_t$, which should then satisfy

$$D_t = T_t PD_t^{\tau} - G_t PD_t^{g}(1 - \phi_t \ell), \quad \forall t.$$

Figure 11 reports the resulting time series of $\phi_t$. The average gap between the market value of debt and the present value of surpluses under the benchmark model exceeds two hundred percent of GDP and grows in magnitude in the last several decades of the sample. To match such a large gap, the probability of the spending cut has to be large and growing. The spending cut probability is 24.19% on average and rises to 36.69% at the end of the sample. Such a large probability is at odds with the notion of a peso event that never happens in a 70-year sample. We interpret this result as a restatement rather than a resolution of the puzzle.

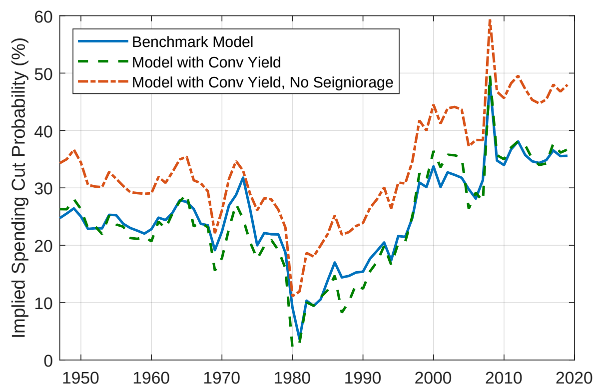
Next, we repeat the austerity analysis in the model with convenience yields. The dashed green line in Figure 11 shows that the implied austerity probabilities that resolve the valuation puzzle are very similar to those in the benchmark model without convenience yield. The model in which U.S. Treasury debt enjoys convenience has a higher risk-free rate. Absent the actual seigniorage revenue, this discount rate effect increases the implied austerity probability, as shown by the dash-

[17] The possibility of a large future increase in tax revenues is an alternative way to engineer a fiscal correction. We have confirmed that the results are similar.

[18] If the fiscal correction took place in high marginal utility states, as in a rare disaster model, the implied probability of these fiscal corrections would likely be smaller. But that strikes us as implausible. Governments do not suddenly switch to running large primary surpluses in bad states of the world.

dotted red line. The cash flow effects from convenience revenues are offset by the discount rate effect, resulting in a similar time series for the austerity probability.

Figure 11: Probabilities of Spending Cut Implied by Debt-to-GDP Ratio




This figure reports the time series of probabilities of spending cuts implied by the debt-to-GDP ratio, $\phi_t$.

### 7.4 Priced Orthogonal Fiscal Shocks

The benchmark model assumes that the fiscal shocks which are orthogonal to all macro-economic and financial variables (that precede it in the VAR) are not priced. While that seems like a reasonable assumption, we nevertheless explore the possibility of non-zero market price of risk for orthogonal spending shocks.[19] We choose the price of spending innovations to minimize the gap. The details are reported in Appendix G. We find that this extended model can largely close the wedge, but only by making the spending claim very risky, i.e. by assigning a very large and positive Sharpe ratio of 1.89 to the orthogonal spending shocks. To render the spending claim risky, the stand-in investor is forced to experience much lower marginal utility growth when there are large positive innovations to spending that are not correlated with stocks, bonds, or GDP growth. This naturally leads to a much higher maximum Sharpe ratio for the model.

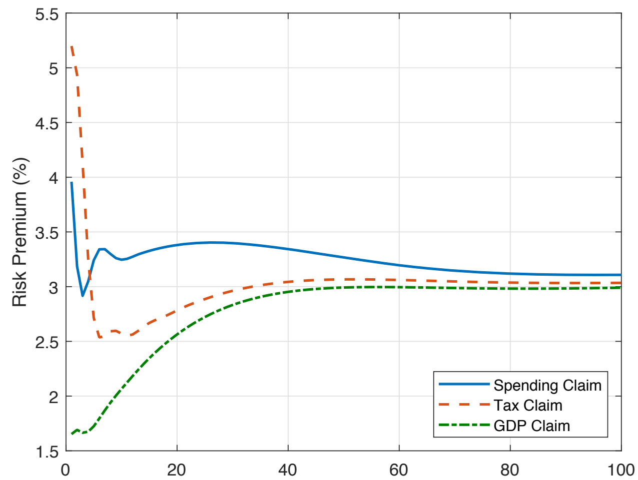
In this version of the model, government spending processes is very risky. The average nominal discount rates, $r_0^g = 8.05\%$, or 4.87% in real terms, is now higher than the discount rate for the GDP claim 7.83%.This spending discount rate is 1% point higher than the corresponding discount rate in the baseline model. The discount rate for taxes is 7.86% in nominal terms, or 4.68% in real terms.

Figure 12 plots the term structure of risk premia. We have distorted the risk premium for the spending claim by more than 5% points at the one-year horizon relative to the benchmark

[19]The results are similar for orthogonal tax revenue shocks.

model, and by about 1% point at longer horizons. The risk premia on the spending claim are much higher than those on the GDP claim at all maturities, even though U.S. spending is clearly counter-cyclical. That makes little economic sense.[20] In addition, the government's spending policy inflicts a lot more risk onto taxpayers in this model by spending more in low marginal utility states, which seems equally implausible.

Figure 12: Term Structure of Risk Premia on the T-Claim and the G-Claim with Priced Orthogonal Spending Shocks




This figure plots the term structures of risk premia on the spending claim, the tax claim, equity, and the GDP claim in the model where fiscal shocks that are orthogonal to GDP growth, inflation, bond yields, an stock prices and dividend growth, are priced. this model is estimated in Appendix G. Each point on the graph is an annualized holding-period risk premium, as derived in equation (C.13) of the Appendix.

What is the implied expected return on Treasurys? The model implies an expected return in levels on the spending claim of 9.48% or 6.3% in real terms. Assuming that the expected return on the tax claim still exceeds the expected return on the spending claim, based on the fundamental risk properties, the model-implied real return on Treasurys is bounded below by 6.3%. The actual realized excess return on the entire portfolio of marketable Treasurys in the sample is only 2.20%, 4.1 % less than the model-implied real return on Treasurys. This 4.1% gap is 7.5× the standard error on the mean Treasury return. In sum, while the discount rate distortion helps to close the valuation gap, it worsens the government risk premium puzzle.

[20] In fact, if we introduce another commonly used state variable that moves at business cycle frequencies, the spread between the valuation ratios of value and growth stocks, we find that this renders the tax claim riskier and the spending claim safer, worsening the valuation puzzle.

### 7.5 Other Government Assets and Liabilities

The government owns various assets, including outstanding student loans and other credit transactions, cash balances, and various financial instruments. Based on Congressional Budget Office data, the total value of these government assets is 8.8% of the GDP as of 2018. While these assets bring the net government debt held by the public from 77.8% to 69.1% of GDP, the bulk of the government debt valuation puzzle remains.[21]

Other significant sources of government revenues and outlays are those associated with the Social Security Administration (SSA). Based on the CBO data, net flows from the SSA are close to 0 as of 2019, but will turn into a deficit of 0.7% of GDP per annum from 2020 to 2029. As the SSA turns from a net contributor of primary surpluses into a net contributor to the deficit in 2020 and beyond, the government will need to issue additional debt to the public. Absent new spending cuts or tax increases, this will deepen the puzzle.

## 8 Conclusion

Because government deficits tend to occur in recessions, times when bond investors face high marginal utility, governments must tap debt markets at inopportune times. This consideration imposes a novel no-arbitrage restriction which affects inference on the riskiness of the overall government debt portfolio. The government debt portfolio is a risky claim whose expected return far exceeds risk-free bond yields. We quantify that the increase in riskiness lowers the government's fiscal capacity by 2.5 times GDP. The negative effects of the 2020 covid pandemic on current and future primary surpluses will add to this number. The pricing of U.S. Treasury debt violates the no-arbitrage restrictions implied by the government budget constraint, a violation we call the government debt valuation puzzle. We show that the valuation of debt cannot be reconciled with rational expectations, provided that a no-bubble condition holds. Conventional estimates of convenience yields cannot explain it either. These findings are robust to changes in model specification. Perhaps investors expect an unprecedented fiscal correction. If so, we show that they have been expecting a correction for a long time, and have been assigning ever-increasing probability to the event, in violation of rational expectations.

[21] Bansal, Croce, Kiao, and Rosen (2019) study reallocation of resources towards government investment in times of high uncertainty.

## References


Alvarez, Fernando, and Urban J Jermann, 2005, Using asset prices to measure the persistence of the marginal utility of wealth, *Econometrica* 73, 1977–2016.

Ang, Andrew, and Monika Piazzesi, 2003, A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables, *Journal of Monetary Economics* 50, 745–787.

Backus, David, Nina Boyarchenko, and Mikhail Chernov, 2018, Term structures of asset prices and returns, *Journal of Financial Economics* 129, 1–23.

Baele, Lieven, Geert Bekaert, and Koen Inghelbrecht, 2010, The determinants of stock and bond return comovements, *Rev. Financ. Stud.* 23, 2374–2428.

Bansal, Ravi, Mariano Massimiliano Croce, Wenxi Kiao, and Samuel Rosen, 2019, Uncertainty-induced reallocations and growth, CEPR Discussion Paper No. DP13964.

Barro, Robert J, 1979, On the determination of the public debt, *Journal of Political Economy* 87, 940–971.

Barro, Robert J, 2020, r minus g, NBER Working Paper.

Berndt, Antje, Hanno Lustig, and Şevin Yeltekin, 2012, How does the us government finance fiscal shocks?, *American Economic Journal: Macroeconomics* 4, 69–104.

Bhandari, Anmol, David Evans, Mikhail Golosov, Thomas Sargent, et al., 2017, The optimal maturity of government debt, Technical report, Working Paper.

Blanchard, Olivier, 2019, Public debt and low interest rates, *American Economic Review* 109, 1197–1229.

Blanchard, Olivier J, and Mark W Watson, 1982, Bubbles, rational expectations and financial markets, *NBER Working Paper* .

Bohn, Henning, 1995, The sustainability of budget deficits in a stochastic economy, *J. Money Credit Bank.* 27, 257–271.

Bohn, Henning, 1998, The behavior of us public debt and deficits, *Quarterly Journal of Economics* 113, 949–963.

Bohn, Henning, 2007, Are stationarity and cointegration restrictions really necessary for the intertemporal budget constraint?, *Journal of Monetary Economics* 54, 1837–1847.

Borovička, Jaroslav, Lars Peter Hansen, and José A Scheinkman, 2016, Misspecified recovery, *Journal of Finance* 71, 2493–2544.

Breeden, Douglas, 1979, An intertemporal asset pricing model with stochastic consumption and investment opportunities, *Journal of Financial Economics* 7, 265–296.

Brock, William A, 1982, Asset prices in a production economy, in *The Economics of Information and Uncertainty*, 1–46 (University of Chicago Press).

Brunnermeier, Markus, Sebastian Merkel, and Yuliy Sannikov, 2020, The fiscal theory of price level with a bubble.

Campbell, John Y, 1990, A variance decomposition for stock returns, Technical report, National Bureau of Economic Research.

Campbell, John Y, 1996, Understanding risk and return, *Journal of Political Economy* 104, 298–345.

Campbell, John Y, et al., 1993, Intertemporal asset pricing without consumption data, *American Economic Review* 83, 487–512.

Campbell, John Y, Carolin Pflueger, and Luis M Viceira, 2020, Macroeconomic drivers of bond and equity risks, *J. Polit. Econ.* 128, 3148–3185.

Campbell, John Y, and Robert J Shiller, 1988, The dividend-price ratio and expectations of future dividends and discount factors, *The Review of Financial Studies* 1, 195–228.

Campbell, John Y., and Robert J. Shiller, 1991, Yield spreads and interest rate movements: A bird's eye view, *Review of Economic Studies* 58, 495–514.

Chernov, Mikhail, Lukas Schmid, and Andres Schneider, 2020, A macrofinance view of us sovereign cds premiums, *Journal of Finance* 75, 2809–2844.

Cochrane, John H, 2019a, The fiscal roots of inflation, Technical report, National Bureau of Economic Research.

Cochrane, John H, 2019b, The value of government debt, Technical report, National Bureau of Economic Research.

Cochrane, John H, and Jesus Saa-Requejo, 2000, Beyond arbitrage: Good-deal asset price bounds in incomplete markets, *Journal of Political Economy* 108, 79–119.

Corhay, Alexandre, Thilo Kind, Howard Kung, and Gonzalo Morales, 2018, Discount rates, debt maturity, and the fiscal theory, *Unpublished Manuscript* .

Cox, J., J. Ingersoll, and S. Ross, 1985, An intertemporal general equilibrium model of asset pricing, *Econometrica* 53, 363–384.

Croce, Mariano M, Thien T Nguyen, Steve Raymond, and Lukas Schmid, 2019, Government debt and the returns to innovation, *Journal of Financial Economics* 132, 205–225.

Dai, Qiang, and Kenneth J. Singleton, 2000, Specification analysis of affine term structure models, *Journal of Finance* 55, 1943–1978.

David, Alexander, and Pietro Veronesi, 2013, What ties return volatilities to price valuations and fundamentals?, *J. Polit. Econ.* 121, 682–746.

D'Erasmo, Pablo, Enrique G. Mendoza, and Jing Zhang, 2016, What is a sustainable public debt?, in *Handbook of Macroeconomics*, volume 2, chapter 32, 2493–2597 (Elsevier).

Diamond, Peter A, 1965, National debt in a neoclassical growth model, *American Economic Review* 55, 1126–1150.

Du, Wenxin, Joanne Im, and Jesse Schreger, 2018, The us treasury premium, *J. Int. Econ.* 112, 167–181.

Du, Wenxin, Carolin E Pflueger, and Jesse Schreger, 2020, Sovereign debt portfolios, bond risks, and the credibility of monetary policy, *J. Finance* 75, 3097–3138.

Duffee, Gregory R, 2018, Expected inflation and other determinants of treasury yields: Expected inflation and treasury yields, *J. Finance* 73, 2139–2180.

Duffie, Darrell, and Rui Kan, 1996, A yield-factor model of interest rates, *Mathematical Finance* 6, 379–406.

Epstein, L. G., and S. Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Farhi, Emmanuel, Pierre-Olivier Gourinchas, and Hélène Rey, 2011, *Reforming the international monetary system* (CEPR).

Farhi, Emmanuel, and Matteo Maggiori, 2018, A model of the international monetary system, *Quarterly Journal of Economics* 133, 295–355.

Fleckenstein, Matthias, Francis A Longstaff, and Hanno Lustig, 2014, The tips-treasury bond puzzle, *the Journal of Finance* 69, 2151–2197.

Giglio, Stefano, Matteo Maggiori, and Johannes Stroebel, 2016, No-bubble condition: Model-free tests in housing markets, *Econometrica* 84, 1047–1091.

Greenwood, Robin, Samuel G Hanson, and Jeremy C Stein, 2015, A comparative-advantage approach to government debt maturity, *Journal of Finance* 70, 1683–1722.

Gupta, Arpit, and Stijn Van Nieuwerburgh, 2019, Valuing private equity strip by strip, Technical report, National Bureau of Economic Research.

Hall, George J, and Thomas J Sargent, 2011, Interest rate risk and other determinants of post-wwii us government debt/gdp dynamics, *American Economic Journal: Macroeconomics* 3, 192–214.

Hamilton, James D, and Marjorie A Flavin, 1985, On the limitations of government borrowing: A framework for empirical testing, Technical report, National Bureau of Economic Research.

Hansen, Lars Peter, William Roberds, and Thomas J Sargent, 1991, Time series implications of present value budget balance and of martingale models of consumption and taxes, *Rational Expectations Econometrics* 121–61.

Hansen, Lars Peter, and José A Scheinkman, 2009, Long-term risk: An operator approach, *Econometrica* 77, 177–234.

He, Zhiguo, Arvind Krishnamurthy, and Konstantin Milbradt, 2019, A model of safe asset determination, *American Economic Review* 109, 1230–62.

Hilscher, Jens, Alon Raviv, and Ricardo Reis, 2021, Inflating away the public debt? an empirical assessment, *Review of Financial Studies* .

Jiang, Zhengyang, Arvind Krishnamurthy, and Hanno Lustig, 2021a, Foreign safe asset demand and the dollar exchange rate, *The Journal of Finance* n/a.

Jiang, Zhengyang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan, 2020, Manufacturing risk-free government debt, NBER Working Paper No. 27786.

Jiang, Zhengyang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan, 2021b, Quantifying U.S. treasury investor optimism.

Karantounias, Anastasios G, 2018, Optimal fiscal policy with recursive preferences, *Review of Economic Studies* 85, 2283–2317.

Kocherlakota, Narayana R, 1996, The equity premium: It's still a puzzle, *J. Econ. Lit.* 34, 42–71.

Koijen, Ralph S J, and Motohiro Yogo, 2019, Exchange rates and asset prices in a global demand system.

Krishnamurthy, Arvind, and Annette Vissing-Jorgensen, 2012, The aggregate demand for treasury debt, *Journal of Political Economy* 120, 233–267.

Krishnamurthy, Arvind, and Annette Vissing-Jorgensen, 2015, The impact of treasury supply on financial sector lending and stability, *Journal of Financial Economics* 118, 571–600.

LeRoy, Stephen F, and Richard D Porter, 1981, The present-value relation: Tests based on implied variance bounds, *Econometrica* 555–574.

Lettau, Martin, and Stijn Van Nieuwerburgh, 2008, Reconciling the return predictability evidence: The review of financial studies: Reconciling the return predictability evidence, *Review of Financial Studies* 21, 1607–1652.

Liu, Yang, Lukas Schmid, and Amir Yaron, 2020, The risks of safe assets.

Longstaff, Francis A, 2004, The flight-to-liquidity premium in us treasury bond prices, *Journal of Business* 77.

Lucas, Robert, 1978, Asset prices in an exchange economy, *Econometrica* 46, 1429–54.

Lucas, Robert E, and Nancy L Stokey, 1983, Optimal fiscal and monetary policy in an economy without capital, *Journal of Monetary Economics* 12, 55–93.

Lustig, Hanno, Stijn Van Nieuwerburgh, and Adrien Verdelhan, 2013, The wealth-consumption ratio, *Review of Asset Pricing Studies* 3(1), 38–94., Review of Asset Pricing Studies.

Mehra, Rajnish, and Edward Prescott, 1985, The equity premium: A puzzle, *Journal of Monetary Economics* 15, 145–161.

Milgrom, Paul, and Nancy Stokey, 1982, Information, trade and common knowledge, *Journal of Economic Theory* 26, 17–27.

Nagel, Stefan, 2016, The liquidity premium of near-money assets, *Quarterly Journal of Economics* 131, 1927–1971.

Pallara, Kevin, and Jean-Paul Renne, 2019, Fiscal limits and sovereign credit spreads, *Available at SSRN* .

Reis, Ricardo, 2021, The constraint on public debt when r¡g but g¡m, Working Paper London School of Economics.

Rubin, Robert E, Peter R Orszag, and Allen Sinai, 2004, Sustained budget deficits: the risk of financial and fiscal disarray, in *AEA-NAEFA Joint Session, Allied Social Science Associations Annual Meetings*.

Samuelson, Paul A, 1958, An exact consumption-loan model of interest with or without the social contrivance of money, *Journal of Political Economy* 66, 467–482.

Santos, Manuel S, and Michael Woodford, 1997, Rational asset pricing bubbles, *Econometrica* 19–57.

Shiller, Robert J, 1981, Do stock prices mmove too much to be justified by subsequent changes in dividends?, *American Economic Review* .

Shleifer, Andrei, and Robert W Vishny, 1997, The limits of arbitrage, *Journal of Finance* 52, 35–55.

Tirole, Jean, 1982, On the possibility of speculation under rational expectations, *Econometrica* 1163–1181.

Trehan, Bharat, and Carl E Walsh, 1988, Common trends, the government's budget constraint, and revenue smoothing, *Journal of Economic Dynamics and Control* 12, 425–444.

Trehan, Bharat, and Carl E Walsh, 1991, Testing intertemporal budget constraints: Theory and applications to us federal budget and current account deficits, *Journal of Money, Credit and banking* 23, 206–223.

Van Binsbergen, Jules, Michael Brandt, and Ralph Koijen, 2012, On the timing and pricing of dividends, *American Economic Review* 102, 1596–1618.

van Binsbergen, Jules H, 2020, Duration-Based stock valuation: Reassessing stock market performance and volatility.

Van Binsbergen, Jules H, William F Diamond, and Marco Grotteria, 2019, Risk-free interest rates, Technical report, National Bureau of Economic Research.

Wachter, Jessica A, 2005, Solving models with external habit, *Finance Research Letters* 2, 210–226.

Weil, Philippe, 1989, The equity premium puzzle and the risk-free rate puzzle, *Journal of Monetary Economics* 24, 401–424.

Online Appendix for The U.S. Public Debt Valuation Puzzle

# A Proofs of Propositions

### Proposition 1

*Proof.* All objects in this appendix are in nominal terms but we drop the superscript $^{\$}$ for ease of notation. The government faces the following one-period budget constraint:

$$G_t - T_t + Q^{\$}_{t-1}(1) = \sum_{h=1}^{H} (Q^{\$}_t(h) - Q^{\$}_{t-1}(h+1))P^{\$}_t(h),$$

where $G_t$ is total nominal government spending, $T_t$ is total nominal government revenue, $Q^{\$}_t(h)$ is the number of nominal zero-coupon bonds of maturity $h$ outstanding in period $t$ each promising to pay back \$1 at time $t+h$, and $P^{\$}_t(h)$ is today's price for a $h$-period zero-coupon bond with \$1 face value. A unit of $h+1$-period bonds issued at $t-1$ becomes a unit of $h$-period bonds in period $t$. That is, the stock of bonds evolves of each maturity evolves according to $Q^{\$}_t(h) = Q^{\$}_{t-1}(h+1) + \Delta Q^{\$}_t(h)$. Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons.

The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit $G - T$ and one-period debt from last period that is now maturing. The right hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equalling total income:

$$G_t + Q^{\$}_{t-1}(1) + \sum_{h=1}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = T_t + \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h),$$

We can now iterate the budget constraint forward. The period $t$ constraint is given by:

$$\begin{aligned} T_t - G_t \quad = \quad & Q^{\$}_{t-1}(1) - Q^{\$}_t(1)P^{\$}_t(1) + Q^{\$}_{t-1}(2)P^{\$}_t(1) - Q^{\$}_t(2)P^{\$}_t(2) + Q^{\$}_{t-1}(3)P^{\$}_t(2) - Q^{\$}_t(3)P^{\$}_t(3) \\ & + \cdots - Q^{\$}_t(H)P^{\$}_t(H) + Q^{\$}_{t-1}(H+1)P^{\$}_t(H). \end{aligned}$$

Consider the period-$t+1$ constraint,

$$\begin{aligned} T_{t+1} - G_{t+1} \quad = \quad & Q^{\$}_t(1) - Q^{\$}_{t+1}(1)P^{\$}_{t+1}(1) + Q^{\$}_t(2)P^{\$}_{t+1}(1) - Q^{\$}_{t+1}(2)P^{\$}_{t+1}(2) + Q^{\$}_t(3)P^{\$}_{t+1}(2) - Q^{\$}_{t+1}(3)P^{\$}_{t+1}(3) \\ & + \cdots - Q^{\$}_{t+1}(H)P^{\$}_{t+1}(H) + Q^{\$}_t(H+1)P^{\$}_{t+1}(H). \end{aligned}$$

multiply both sides by $M^{\$}_{t+1}$, and take expectations conditional on time $t$:

$$\begin{aligned} \mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1} - G_{t+1})\right] \quad = \quad & Q^{\$}_t(1)P^{\$}_t(1) - \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] + Q^{\$}_t(2)P^{\$}_t(2) - \mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] + Q^{\$}_t(3)P^{\$}_t(3) \\ & - \mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] + \cdots + Q^{\$}_t(H)P^{\$}_t(H) \\ & - \mathbb{E}_t[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)] + Q^{\$}_t(H+1)P^{\$}_t(H+1), \end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t[M^{\$}_{t+1}] = P^{\$}_t(1)$, $\mathbb{E}_t[M^{\$}_{t+1}P^{\$}_{t+1}(1)] = P^{\$}_t(2)$, $\cdots$, $\mathbb{E}_t[M^{\$}_{t+1}P^{\$}_{t+1}(H-1)] = P^{\$}_t(H)$, and $\mathbb{E}_t[M^{\$}_{t+1}P^{\$}_{t+1}(H)] = P^{\$}_t(H+1)$.

Consider the period $t+2$ constraint, multiplied by $M^{\$}_{t+1}M^{\$}_{t+2}$ and take time-t expectations:

$$\begin{aligned} \mathbb{E}_t\left[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2} - G_{t+2})\right] \quad = \quad & \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] - \mathbb{E}_t[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)] + \mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] \\ & - \mathbb{E}_t[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)] + \mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] - \cdots \\ & + \mathbb{E}_t[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)] - \mathbb{E}_t[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)] + \mathbb{E}_t[Q^{\$}_{t+1}(H+1)M^{\$}_{t+1}P^{\$}_{t+1}(H+1)], \end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M^{\$}_{t+2}] = P^{\$}_{t+1}(1)$, $\mathbb{E}_{t+1}[M^{\$}_{t+2}P^{\$}_{t+2}(1)] = P^{\$}_{t+1}(2)$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at $t$, $t+1$, and $t+2$ we get:

$$\begin{aligned} T_t - G_t + \mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1} - G_{t+1})\right] + \mathbb{E}_t\left[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2} - G_{t+2})\right] = \textstyle\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) + \\ - \mathbb{E}_t[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)] - \mathbb{E}_t[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)] - \cdots - \mathbb{E}_t[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)]. \end{aligned}$$

Similarly consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$ , we get:

$$\sum_{h=0}^{H} Q_{t-1}^{\$}(h+1)P_t^{\$}(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{J} M_{t,t+j}^{\$}(T_{t+j}-G_{t+j})\right]+\mathbb{E}_t\left[M_{t,t+J}^{\$}\sum_{h=1}^{H} Q_{t+J}^{\$}(h)P_{t+J}^{\$}(h)\right]$$

where we used the cumulate SDF notation $M_{t,t+j}^{\$}=\prod_{i=0}^{j} M_{t+i}^{\$}$ and by convention $M_{t,t}^{\$}=M_t^{\$}=1$ and $P_t^{\$}(0)=1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next $J$ years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J\to\infty$:

$$\sum_{h=0}^{H} Q_{t-1}^{\$}(h+1)P_t^{\$}(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$}(T_{t+j}-G_{t+j})\right]+\lim_{J\to\infty}\mathbb{E}_t\left[M_{t,t+J}^{\$}\sum_{h=1}^{H} Q_{t+J}^{\$}(h)P_{t+J}^{\$}(h)\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j}-G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J\to\infty}\mathbb{E}_t\left[M_{t,t+J}^{\$}\sum_{h=1}^{H} Q_{t+J}^{\$}(h)P_{t+J}^{\$}(h)\right]=0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, $D_t$, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t=\sum_{h=0}^{H} Q_{t-1}^{\$}(h+1)P_t^{\$}(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$}(T_{t+j}-G_{t+j})\right].$$

This is equation (1) in the main text.

## Case with Default

*Proof.* We consider only full default, without loss of generality. Alternatively, we can write the budget constraint that obtains in case of no default at $t$:

$$G_t+Q_{t-1}^{\$}(1)+\sum_{h=1}^{H} Q_{t-1}^{\$}(h+1)P_t^{\$}(h)=T_t+\sum_{h=1}^{H} Q_t^{\$}(h)P_t^{\$}(h),$$

and, in case of default at $t$, the one-period budget constraint is given by:

$$G_t=T_t+\sum_{h=1}^{H} Q_t^{\$}(h)P_t^{\$}(h).$$

We can now iterate the budget constraint forward. In case of no default, the period $t$ constraint is given by:

$$\begin{aligned}T_t-G_t \quad = \quad & Q_{t-1}^{\$}(1)-Q_t^{\$}(1)P_t^{\$}(1)+Q_{t-1}^{\$}(2)P_t^{\$}(1)-Q_t^{\$}(2)P_t^{\$}(2)+Q_{t-1}^{\$}(3)P_t^{\$}(2)-Q_t^{\$}(3)P_t^{\$}(3)\\ & +\cdots-Q_t^{\$}(H)P_t^{\$}(H)+Q_{t-1}^{\$}(H+1)P_t^{\$}(H).\end{aligned}$$

In case of default, the period $t$ constraint is given by:

$$T_t-G_t \quad = \quad -Q_t^{\$}(1)P_t^{\$}(1)-Q_t^{\$}(2)P_t^{\$}(2)-Q_t^{\$}(3)P_t^{\$}(3)-Q_t^{\$}(H)P_t^{\$}(H)$$

First, consider the period-$t+1$ constraint in case of no default,

$$\begin{aligned}T_{t+1}-G_{t+1} \quad = \quad & Q_t^{\$}(1)-Q_{t+1}^{\$}(1)P_{t+1}^{\$}(1)+Q_t^{\$}(2)P_{t+1}^{\$}(1)-Q_{t+1}^{\$}(2)P_{t+1}^{\$}(2)+Q_t^{\$}(3)P_{t+1}^{\$}(2)-Q_{t+1}^{\$}(3)P_{t+1}^{\$}(3)\\ & +\cdots-Q_{t+1}^{\$}(H)P_{t+1}^{\$}(H)+Q_t^{\$}(H+1)P_{t+1}^{\$}(H).\end{aligned}$$

Second, consider the period-$t+1$ constraint in case of default,

$$T_{t+1} - G_{t+1} \quad = \quad -Q^{\$}_{t+1}(1)P^{\$}_{t+1}(1) - Q^{\$}_{t+1}(2)P^{\$}_{t+1}(2) - Q^{\$}_{t+1}(3)P^{\$}_{t+1}(3) - Q^{\$}_{t+1}(H)P^{\$}_{t+1}(H).$$

We use $\chi_t$ as an indicator variable for default. To simplify, we consider only full default with zero recovery. This is without loss of generality. Next, multiply both sides of the no default constraint by $(1-\chi_{t+1})M^{\$}_{t+1}$, and take expectations conditional on time $t$:

$$\begin{aligned}
\mathbb{E}_t\left[M^{\$}_{t+1}(1-\chi_{t+1})(T_{t+1}-G_{t+1})\right] \quad = \quad & Q^{\$}_t(1)\mathbb{E}_t\left[M^{\$}_{t+1}(1-\chi_{t+1})\right] - \mathbb{E}_t[Q^{\$}_{t+1}(1)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)] + E_t[(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)]Q^{\$}_t(2) \\
& -\mathbb{E}_t[Q^{\$}_{t+1}(2)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(2)] + E_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(2)]Q^{\$}_t(3) \\
& -\mathbb{E}_t[Q^{\$}_{t+1}(3)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(3)] + \cdots + Q^{\$}_t(H)E_t[M^{\$}_{t+1}(1-\chi)P^{\$}_{t+1}(H-1)] \\
& -\mathbb{E}_t[Q^{\$}_{t+1}(H)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(H)] + Q^{\$}_t(H+1)E_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)],
\end{aligned}$$

and multiply both sides of the default constraint by $M^{\$}_{t+1}\chi_{t+1}$

$$\begin{aligned}
\mathbb{E}_t\left[M^{\$}_{t+1}\chi_{t+1}(T_{t+1}-G_{t+1})\right] \quad = \quad & -\mathbb{E}_t[Q^{\$}_{t+1}(1)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(1)] - \mathbb{E}_t[Q^{\$}_{t+1}(2)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(2)] \\
& -\mathbb{E}_t[Q^{\$}_{t+1}(3)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(3)] - \cdots - \mathbb{E}_t[Q^{\$}_{t+1}(H)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(H)].
\end{aligned}$$

By adding these 2 constraints, we obtain the following expression:

$$\begin{aligned}
\mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1}-G_{t+1})\right] \quad = \quad & Q^{\$}_t(1)\mathbb{E}_t\left[M^{\$}_{t+1}(1-\chi_{t+1})\right] - \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] + E_t[(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)]Q^{\$}_t(2) \\
& -\mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] + E_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(2)]Q^{\$}_t(3) \\
& -\mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] + \cdots + Q^{\$}_t(H)E_t[M_{t+1}(1-\chi)P^{\$}_{t+1}(H-1)] \\
& -\mathbb{E}_t[Q^{\$}_{t+1}(H)M_{t+1}P^{\$}_{t+1}(H)] + Q^{\$}_t(H+1)E_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)].
\end{aligned}$$

This can be restated as:

$$\begin{aligned}
\mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1}-G_{t+1})\right] \quad = \quad & Q^{\$}_t(1)P^{\$}_t(1) - \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] + Q^{\$}_t(2)P^{\$}_t(2) - \mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] + Q^{\$}_t(3)P^{\$}_t(3) \\
& -\mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] + \cdots + Q^{\$}_t(H)P^{\$}_t(H) - \mathbb{E}_t[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)] + Q^{\$}_t(H+1)P^{\$}_t(H+1),
\end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t\left[M^{\$}_{t+1}(1-\chi_{t+1})\right] = P^{\$}_t(1)$, $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(1)] = P^{\$}_t(2)$, $\cdots$, $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H-1)] = P^{\$}_t(H)$, and $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)] = P^{\$}_t(H+1)$.

The rest of the proof is essentially unchanged. Consider the period $t+2$ constraint, multiplied by $M^{\$}_{t+1}M^{\$}_{t+2}(1-\chi_{t+2})$ in the no-default case, and $M^{\$}_{t+1}M^{\$}_{t+2}(\chi_{t+2})$ for the default case, and take time-t expectations (after adding default and no-default states):

$$\begin{aligned}
\mathbb{E}_t\left[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2}-G_{t+2})\right] \quad = \quad & \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] - \mathbb{E}_t[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)] + \mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] \\
& -\mathbb{E}_t[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)] + \mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] - \cdots \\
& +\mathbb{E}_t[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)] - \mathbb{E}_t[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)] + \mathbb{E}_t[Q^{\$}_{t+1}(H+1)M^{\$}_{t+1}P^{\$}_{t+1}(H+1)],
\end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M^{\$}_{t+2}(1-\chi_{t+2})] = P^{\$}_{t+1}(1)$, $\mathbb{E}_{t+1}[M^{\$}_{t+2}(1-\chi_{t+2})P^{\$}_{t+2}(1)] = P^{\$}_{t+1}(2)$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at $t$, $t+1$, and $t+2$ we get:

$$\begin{gathered}
T_t - G_t + \mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1}-G_{t+1})\right] + \mathbb{E}_t\left[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2}-G_{t+2})\right] = \textstyle\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) + \\
-\mathbb{E}_t[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)] - \mathbb{E}_t[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)] - \cdots - \mathbb{E}_t[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)].
\end{gathered}$$

Similarly consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{J} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right] + \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right]$$

where we used the cumulate SDF notation $M^\$_{t,t+j} = \prod_{i=0}^{j} M^\$_{t+i}$ and by convention $M^\$_{t,t} = M^\$_t = 1$ and $P^\$_t(0) = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next $J$ years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J \to \infty$:

$$\sum_{h=0}^{H} Q^\$_{t-1}(h+1)P^\$_t(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^\$_{t,t+j}(T_{t+j} - G_{t+j})\right] + \lim_{J\to\infty} \mathbb{E}_t\left[M^\$_{t,t+J}\sum_{h=1}^{H} Q^\$_{t+J}(h)P^\$_{t+J}(h)\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j} - G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J\to\infty} \mathbb{E}_t\left[M^\$_{t,t+J}\sum_{h=1}^{H} Q^\$_{t+J}(h)P^\$_{t+J}(h)\right] = 0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, $D_t$, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^{H} Q^\$_{t-1}(h+1)P^\$_t(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^\$_{t,t+j}(T_{t+j} - G_{t+j})\right].$$

This is equation (1) in the main text.

□

## Proposition 2

From the time-$t$ budget constraint, we get that the primary surplus

$$-S_t = -Q^\$_{t-1}(1) + \sum_{h=1}^{H}(Q^\$_t(h) - Q^\$_{t-1}(h+1))P^\$_t(h).$$

It follows that

$$D_t - S_t = \sum_{h=0}^{H} Q^\$_{t-1}(h+1)P^\$_t(h) - Q^\$_{t-1}(1) + \sum_{h=1}^{H}(Q^\$_t(h) - Q^\$_{t-1}(h+1))P^\$_t(h) = \sum_{h=1}^{H} Q^\$_t(h)P^\$_t(h).$$

We obtain equation (3) in the main text.

$$\begin{aligned} r^d_{t+1}(D_t - S_t) &= \sum_{h=0}^{\infty} P^\$_{t+1}(h)Q^\$_{t,h+1} = D_{t+1} = P^\tau_{t+1} - P^g_{t+1} \\ &= (P^\tau_t - T_t)r^\tau_{t+1} - (P^g_{t+1} - G_t)r^g_{t+1}. \end{aligned}$$

□

## Proposition 3

*Proof.* We follow the proof in the working paper version of Backus et al. (2018) on page 16 (Example 5). Hansen and Scheinkman (2009) consider the following equation:

$$\mathbb{E}_t[M_{t,t+1}v_{t+1}] = \nu v_t, \tag{A.1}$$

where $\nu$ is the dominant eigenvalue and $v_t$ is the eigenfunction. Claims to stationary cash flows earn a return equal to the yield on the long bond. We consider the following decomposition of the pricing kernel:

$$M^1_{t,t+1} = M_{t,t+1}v_{t+1}/\nu v_t, \tag{A.2}$$

$$M^2_{t,t+1} = \nu v_t/v_{t+1}. \tag{A.3}$$

By construction, $\mathbb{E}_t[M^1_{t,t+1}] = 1$. The long yields converge to $-\log\nu$. The long-run bond return converges to $\lim_{n\to\infty} R^n_{t,t+1} = \frac{1}{M^2_{t,t+1}} = v_{t+1}/\nu v_t$. This implies that $\mathbb{E}[\log R^\infty_{t,t+1}] = -\log\nu$.

To value claims to uncertain cash flows with one-period growth rate $g_{t,t+1}$, we define $\widehat{p}^n_t$ to denote the price of a strip that pays off $d_{t,t+n}$, $n$ periods from now.

$$\widehat{p}^n_t = \mathbb{E}_t[M_{t,t+1}g_{t,t+1}\widehat{p}^{n-1}_{t+1}] = \mathbb{E}_t[\widehat{M}_{t,t+1}\widehat{p}^{n-1}_{t+1}],$$

where $\widehat{M}_{t,t+1} = M_{t,t+1}g_{t,t+1}$. Consider the problem of finding the dominant eigenvalue:

$$\mathbb{E}_t[\widehat{M}_{t,t+1}\widehat{v}_{t+1}] = \nu\widehat{v}_t. \tag{A.4}$$

If the cash flows are stationary, then the same $\nu$ that solves this equation for $M_{t,t+1}$ in eqn. A.1 solves the one for $\widehat{M}_{t,t+1}$. Hence, if $(\nu, v_t)$ solves eqn. A.1, then $(\nu, v_t/d_t)$ solves the hat equation eqn. A.4. □

**Proposition 4** The proof of proposition 4 is in Appendix C.4.

## Proposition 5

*Proof.* Start from government budget constraint:

$$T_t - G_t = Q^{\$}_{t-1}(1) + \sum_{k=1}^{K-1} Q^{\$}_{t-1}(k+1)P^{\$}_t(k) - \sum_{k=1}^{K} Q^{\$}_t(k)P^{\$}_t(k)$$

We assume these bond prices contain the same convenience yield $\lambda_t$:

$$\begin{aligned} E_t[M^{\$}_{t+1}] &= P^{\$}_t(1)e^{-\lambda_t}, \\ E_t[M^{\$}_{t+1}P^{\$}_{t+1}(1)] &= P^{\$}_t(2)e^{-\lambda_t}, \\ E_t[M^{\$}_{t+1}P^{\$}_{t+1}(K)] &= P^{\$}_t(K+1)e^{-\lambda_t}. \end{aligned}$$

Consider the period-$t+1$ constraint, multiplied by $M_{t+1}$, and take expectations conditional at time $t$:

$$\begin{aligned} & E_t\left[M^{\$}_{t+1}(T_{t+1} - G_{t+1})\right] \\ = \; & E_t[M^{\$}_{t+1}Q^{\$}_t(1) + \sum_{k=1}^{K-1} M^{\$}_{t+1}Q^{\$}_t(k+1)P^{\$}_{t+1}(k) - \sum_{k=1}^{K} M^{\$}_{t+1}Q^{\$}_{t+1}(k)P^{\$}_{t+1}(k)] \\ = \; & P^{\$}_t(1)e^{-\lambda_t}Q^{\$}_t(1) + \sum_{k=1}^{K-1} P^{\$}_t(k+1)e^{-\lambda_t}Q^{\$}_t(k+1) - E_t[M^{\$}_{t+1}\sum_{k=1}^{K} Q^{\$}_{t+1}(k)P^{\$}_{t+1}(k)]. \end{aligned}$$

Consider the period-$t+2$ constraint, multiplied by $M^{\$}_{t,t+2}$:

$$\begin{aligned} & E_t\left[M^{\$}_{t,t+2}(T_{t+2} - G_{t+2})\right] \\ = \; & E_t[M^{\$}_{t,t+2}Q^{\$}_{t+1}(1) + \sum_{k=1}^{K-1} M^{\$}_{t,t+2}Q^{\$}_{t+1}(k+1)P^{\$}_{t+2}(k) - \sum_{k=1}^{K} M^{\$}_{t,t+2}Q^{\$}_{t+2}(k)P^{\$}_{t+2}(k)] \\ = \; & E_t[M^{\$}_{t+1}P^{\$}_{t+1}(1)e^{-\lambda_{t+1}}Q^{\$}_{t+1}(1)] + E_t[M^{\$}_{t+1}\sum_{k=1}^{K-1} P^{\$}_{t+1}(k+1)e^{-\lambda_{t+1}}Q^{\$}_{t+1}(k+1)] - E_t[M^{\$}_{t,t+2}\sum_{k=1}^{K} Q^{\$}_{t+2}(k)P^{\$}_{t+2}(k)], \end{aligned}$$

where we have used that

$$E_t[M^{\$}_{t,t+2}Q^{\$}_{t+1}(1)] = E_t[M^{\$}_{t,t+1}Q^{\$}_{t+1}(1)E_{t+1}M^{\$}_{t+1,t+2}] = E_t[M^{\$}_{t,t+1}Q^{\$}_{t+1}(1)e^{-\lambda_{t+1}}P^{\$}_{t+1}(1)],$$

and, similarly, that:

$$E_t[M^{\$}_{t,t+2}Q^{\$}_{t+1}(k)P^{\$}_{t+2}(k)] = E_t[M^{\$}_{t,t+1}Q^{\$}_{t+1}(k)E_{t+1}M^{\$}_{t+1,t+2}P^{\$}_{t+2}(k)] = E_t[M^{\$}_{t,t+1}Q^{\$}_{t+1}(k)e^{-\lambda_{t+1}}P^{\$}_{t+1}(k+1)].$$

By adding up the $t$, $t+1$ and $t+2$ constraint, we get that $E_t[T_t - G_t + M^{\$}_{t+1}(T_{t+1} - G_{t+1}) + M^{\$}_{t,t+2}(T_{t+2} - G_{t+2})]$ equals:

$$\begin{aligned} = \; & Q^{\$}_{t-1}(1) + \sum_{k=1}^{K-1} Q^{\$}_{t-1}(k+1)P^{\$}_t(k) \\ + \; & P^{\$}_t(1)(e^{-\lambda_t} - 1)Q^{\$}_t(1) + \sum_{k=1}^{K-1} P^{\$}_t(k+1)(e^{-\lambda_t} - 1)Q^{\$}_t(k+1) \\ + \; & E_t[M^{\$}_{t+1}P^{\$}_{t+1}(1)(e^{-\lambda_{t+1}} - 1)Q^{\$}_{t+1}(1)] + E_t[M^{\$}_{t+1}\sum_{k=1}^{K-1} P^{\$}_{t+1}(k+1)(e^{-\lambda_{t+1}} - 1)Q^{\$}_{t+1}(k+1)] - E_t[M^{\$}_{t,t+2}\sum_{k=1}^{K} Q^{\$}_{t+2}(k)P^{\$}_{t+2}(k)]. \end{aligned}$$

Next, consider the period-$t+3$ constraint, multiplied by $M^{\$}_{t,t+3}$

$$
\begin{aligned}
& E_t\left[M^{\$}_{t,t+3}(T_{t+3}-G_{t+3})\right] \\
= \quad & E_t[M^{\$}_{t,t+3}Q^{\$}_{t+2}(1)+\sum_{k=1}^{K-1}M^{\$}_{t,t+3}Q^{\$}_{t+2}(k+1)P^{\$}_{t+2}(k)-\sum_{k=1}^{K}M^{\$}_{t,t+3}Q^{\$}_{t+2}(k)P^{\$}_{t+2}(k)] \\
= \quad & E_t[M^{\$}_{t,t+2}P^{\$}_{t+2}(1)e^{-\lambda_{t+2}}Q^{\$}_{t+2}(1)]+E_t[M^{\$}_{t,t+2}\sum_{k=1}^{K-1}P^{\$}_{t+2}(k+1)e^{-\lambda_{t+2}}Q^{\$}_{t+2}(k+1)]-E_t[M^{\$}_{t,t+3}\sum_{k=1}^{K}Q^{\$}_{t+3}(k)P^{\$}_{t+3}(k)],
\end{aligned}
$$

where we use:

$$E_t[M^{\$}_{t,t+3}Q^{\$}_{t+2}(1)]=E_t[M^{\$}_{t,t+2}Q^{\$}_{t+2}(1)E_{t+2}M^{\$}_{t+2,t+3}]=E_t[M^{\$}_{t,t+2}Q^{\$}_{t+2}(1)e^{-\lambda_{t+2}}P^{\$}_{t+2}(1)],$$

and, similarly, that:

$$E_t[M^{\$}_{t,t+3}Q^{\$}_{t+2}(k)P^{\$}_{t+3}(k)]=E_t[M^{\$}_{t,t+2}Q^{\$}_{t+2}(k)E_{t+1}M^{\$}_{t+2,t+3}P^{\$}_{t+3}(k)]=E_t[M^{\$}_{t,t+2}Q^{\$}_{t+2}(k)e^{-\lambda_{t+2}}P^{\$}_{t+2}(k+1)].$$

Iterating forward, and aggregating the discounted surpluses $(T_{t+j}-G_{t+j})$, we obtain:

$$E_t\left[\sum_{j=0}^{\infty}M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right]+E_t\left[\sum_{j=0}^{\infty}M^{\$}_{t,t+j}(1-e^{-\lambda_{t+j}})\sum_{k=1}^{K}Q^{\$}_{t+j}(k)P^{\$}_{t+j}(k)\right] \quad = \quad \sum_{k=0}^{K}Q^{\$}_{t-1}(k+1)P^{\$}_{t}(k)-\lim_{\tau\to\infty}E_t[M^{\$}_{t,t+\tau}\sum_{k=1}^{K}Q^{\$}_{t+\tau}(k)P^{\$}_{t+\tau}(k)].$$

Let $D_t(t+j)$ denote the time-$t$ value of the government's debt portfolio at $t+j$. We can restate the previous equation as follows:

$$E_t\left[\sum_{j=0}^{\infty}M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right]+E_t\left[\sum_{j=0}^{\infty}M^{\$}_{t,t+j}(1-e^{-\lambda_{t+j}})D_{t+j}(t+j)\right] \quad = \quad D_t(t-1)-\lim_{\tau\to\infty}E_t[M^{\$}_{t,t+\tau}\sum_{k=1}^{K}Q^{\$}_{t+\tau}(k)P^{\$}_{t+\tau}(k)].$$

If the discounted value of distant future bond portfolio is 0,

$$\lim_{\tau\to\infty}E_t[M^{\$}_{t,t+\tau}\sum_{k=1}^{K}Q^{\$}_{t+\tau}(k)P^{\$}_{t+\tau}(k)] \quad = \quad 0,$$

then debt value is the present value of future surpluses and future seignorage revenue from issuing bonds that earn convenience yields:

$$E_t\left[\sum_{j=0}^{\infty}M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right]+E_t\left[\sum_{j=0}^{\infty}M^{\$}_{t,t+j}(1-e^{-\lambda_{t+j}})D_{t+j}(t+j)\right]=D_t(t-1).$$

□

# B Derivation of the Consumption-Based Model

The asset pricing equation for tax strip is

$$\mathbb{E}_t[\exp(m_{t,t+j})\exp(y_{t+j}+\tau_{t+j})] = Y_t PD_t^{\tau}(j)$$

with

$$PD_t^{\tau}(j) = \mathbb{E}_t[\exp(m_{t,t+1})\frac{Y_{t+1}}{Y_t}PD_{t+1}^{\tau}(j-1)]$$

Conjecture

$$PD_t^{\tau}(j) = \exp(a_j^{\tau} + b_j^{\tau}(\tau_t - \bar{\tau}))$$

where $a_0^{\tau} = \bar{\tau}$ and $b_0^{\tau} = 1$.
Then

$$\begin{aligned}
& \exp(a_j^{\tau} + b_j^{\tau}(\tau_t - \bar{\tau})) \\
= \; & \mathbb{E}_t[\exp(m_{t,t+1} + \Delta y_{t+1})\exp(a_{j-1}^{\tau} + b_{j-1}^{\tau}(\tau_{t+1} - \bar{\tau}))] \\
= \; & \mathbb{E}_t[\exp(-\beta + (1-\gamma)\mu + (1-\gamma)\sigma_y\varepsilon_{t+1}^{c} + a_{j-1}^{\tau} + b_{j-1}^{\tau}(\theta_\tau(\tau_t - \bar{\tau}) + \sigma_\tau\varepsilon_{t+1}^{c} + \eta_\tau\varepsilon_{t+1}^{\tau}))] \\
= \; & \exp(-\beta + (1-\gamma)\mu + a_{j-1}^{\tau} + b_{j-1}^{\tau}\theta_\tau(\tau_t - \bar{\tau}) + \frac{1}{2}((1-\gamma)\sigma_y + b_{j-1}^{\tau}\sigma_\tau)^2 + \frac{1}{2}(b_{j-1}^{\tau}\eta_\tau)^2)
\end{aligned}$$

So

$$\begin{aligned}
a_j^{\tau} &= (1-\gamma)\mu - \beta + a_{j-1}^{\tau} + \frac{1}{2}((1-\gamma)\sigma_y + b_{j-1}^{\tau}\sigma_\tau)^2 + \frac{1}{2}(b_{j-1}^{\tau}\eta_\tau)^2 \\
b_j^{\tau} &= b_{j-1}^{\tau}\theta_\tau
\end{aligned}$$

The moment conditions are

$$\begin{aligned}
\mathbb{E}_t[\Delta y_{t+1} - \mu] &= 0 \\
\mathbb{E}_t[\tau_{t+1} - \bar{\tau}] &= 0 \\
\mathbb{E}_t[(\Delta y_{t+1} - \mu)(\tau_{t+1} - \theta_\tau\tau_t - (1-\theta_\tau)\bar{\tau}) - \sigma_y\sigma_\tau] &= 0 \\
\mathbb{E}_t[(\tau_{t+1} - \theta_\tau\tau_t - (1-\theta_\tau)\bar{\tau})^2 - \sigma_\tau^2 - \eta_\tau^2] &= 0 \\
\mathbb{E}_t[(\tau_{t+1} - \bar{\tau})(\tau_t - \bar{\tau}) - \theta_\tau(\tau_t - \bar{\tau})^2] &= 0 \\
\mathbb{E}_t[g_{t+1} - \bar{g}] &= 0 \\
\mathbb{E}_t[(\Delta y_{t+1} - \mu)(g_{t+1} - \theta_g g_t - (1-\theta_g)\bar{g}) - \sigma_y\sigma_g] &= 0 \\
\mathbb{E}_t[(g_{t+1} - \theta_g g_t - (1-\theta_g)\bar{g})^2 - \sigma_g^2 - \eta_g^2] &= 0 \\
\mathbb{E}_t[(g_{t+1} - \bar{g})(g_t - \bar{g}) - \theta_g(g_t - \bar{g})^2] &= 0
\end{aligned}$$

Table B.1: Parameter Value Estimates

| | Estimate | Std. Error | t value | Pr(>\|t\|) |
|---|---|---|---|---|
| $\mu$ | 0.03 | 0.00 | 11.72 | 0.00 |
| $\bar{\tau}$ | $-2.24$ | 0.02 | $-117.13$ | 0.00 |
| $\sigma_\tau$ | 0.02 | 0.01 | 2.76 | 0.01 |
| $\eta_\tau$ | 0.08 | 0.01 | 9.40 | 0.00 |
| $\theta_\tau$ | 0.88 | 0.08 | 11.22 | 0.00 |
| $\bar{g}$ | $-2.21$ | 0.02 | $-125.79$ | 0.00 |
| $\sigma_g$ | $-0.02$ | 0.00 | $-3.86$ | 0.00 |
| $\eta_g$ | $-0.07$ | 0.01 | $-12.11$ | 0.00 |
| $\theta_g$ | 0.90 | 0.05 | 16.59 | 0.00 |

This table reports the GMM estimates of the parameters in the CRRA model.

The log risk-free rate is

$$r_t^f \quad = \quad \beta + \gamma\mu - \frac{1}{2}(\gamma\sigma_y)^2$$

and the log risk-premium on the tax strip at $\tau_t = \bar{\tau}$ is

$$\mathbb{E}_t[\log\left(\frac{Y_{t+1}}{Y_t}\frac{PD^\tau_{t+1}(j-1)}{PD^\tau_t(j)}\right) - r_t^f] \quad = \quad \mu + a^\tau_{j-1} - a^\tau_j - r_t^f = -\frac{1}{2}((1-\gamma)\sigma_y + b^\tau_{j-1}\sigma_\tau)^2 - \frac{1}{2}(b^\tau_{j-1}\eta_\tau)^2 + \frac{1}{2}(\gamma\sigma_y)^2 \tag{B.1}$$

**Risk Premia of the Fiscal Claims** Next, we consider the return of the claim to current and future spending strips. We log-linearize the return around $g_t = \bar{g}$:

$$r^g_{t+1} = \kappa^g_0 + \Delta\log G_{t+1} + \kappa^g_1 pd^g_{t+1} - pd^g_t.$$

where $pd^g = \log\left(PD^g_t - 1\right)$. The unconditional mean log return of the G claim is $r^g_0 = E[r^g_t]$.

We obtain $\overline{pg}$ from the precise valuation formula at $g_t = \bar{g}$. We define linearization constants $\kappa^g_0$ and $\kappa^g_1$ as:

$$\kappa^g_1 = \frac{e^{\overline{pg}}}{e^{\overline{pg}}+1} < 1 \ \text{ and } \ \kappa^g_0 = \log\left(e^{\overline{pg}}+1\right) - \frac{e^{\overline{pg}}}{e^{\overline{pg}}+1}\overline{pg}.$$

Then, under a log-linear approximation, the unconditional expected return is:

$$r^g_0 = \kappa^g_0 + \mu - \overline{pg}(1-\kappa^g_1)$$

The log ex-dividend price-dividend ratio on the entire spending claim is affine in the state vector and verify the conjecture by solving the Euler equation for the claim.

$$pg_t = \overline{pg} + \overline{b}^g(g_t - \bar{g})$$

This allows us to write the return as:

$$\begin{aligned} r^g_{t+1} &= \kappa^g_0 + g_{t+1} - g_t + \mu + \sigma_y\varepsilon^y_{t+1} + \kappa^g_1(\overline{pg} + \overline{b}^g(g_{t+1}-\bar{g})) - (\overline{pg} + \overline{b}^g(g_t - \bar{g})) \\ &= r^g_0 + \sigma_y\varepsilon^y_{t+1} + (1+\kappa^g_1\overline{b}^g)(g_{t+1}-\bar{g}) - (1+\overline{b}^g)(g_t-\bar{g}) \end{aligned}$$

Starting from the Euler equation:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[\exp\{m_{t+1} + r^g_{t+1}\}\right] \\ &= \mathbb{E}_t[\exp\{-r^f - \frac{1}{2}(\gamma\sigma_y)^2 + r^g_0 + (-\gamma\sigma_y + \sigma_y + (1+\kappa^g_1\overline{b}^g)\sigma_g)\varepsilon^y_{t+1} \\ &+ ((1+\kappa^g_1\overline{b}^g)\theta_g - (1+\overline{b}^g))(g_t-\bar{g}) + (1+\kappa^g_1\overline{b}^g)\eta_\tau\varepsilon^g_{t+1}\}] \end{aligned}$$

which implies the level of risk premium is

$$r^g_0 - r^f + Jensen \quad = \quad \gamma\sigma_y(\sigma_y + (1+\kappa^g_1\overline{b}^g)\sigma_g)$$

and

$$\overline{b}^g \quad = \quad \frac{\theta_g - 1}{1-\kappa^g_1\theta_g}$$

Following this calculation, the level of risk premium for the spending strip is 2.22% and that for the tax strip is 2.81%.

# C Asset Pricing Model

## C.1 Risk-free rate

The real short yield $y_t(1)$, or risk-free rate, satisfies $\mathbb{E}_t[\exp\{m_{t+1} + y_t(1)\}] = 1$. Solving out this Euler equation, we get:

$$y_t(1) \quad = \quad y^\$_t(1) - E_t[\pi_{t+1}] - \frac{1}{2}(e^\pi)'\Sigma e^\pi + (e^\pi)'\Sigma^{\frac{1}{2}}\Lambda_t$$

$$
\begin{aligned}
&= y_0(1) + \left[(e^{yn})' - (e^{\pi})'\Psi + (e^{\pi})'\Sigma^{\frac{1}{2}}\Lambda_1\right] z_t. && \text{(C.1)}\\
y_0(1) &\equiv y_0^{\$}(1) - \pi_0 - \frac{1}{2}(e^{\pi})'\Sigma e^{\pi} + (e^{\pi})'\Sigma^{\frac{1}{2}}\Lambda_0. && \text{(C.2)}
\end{aligned}
$$

where we used the expression for the real SDF

$$
\begin{aligned}
m_{t+1} &= m_{t+1}^{\$} + \pi_{t+1}\\
&= -y_t^{\$}(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + \pi_0 + (e^{\pi})'\Psi z_t + (e^{\pi})'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\\
&= -y_t(1) - \frac{1}{2}(e^{\pi})'\Sigma e^{\pi} + (e^{\pi})'\Sigma^{\frac{1}{2}}\Lambda_t - \frac{1}{2}\Lambda_t'\Lambda_t - \left(\Lambda_t' - (e^{\pi})'\Sigma^{\frac{1}{2}}\right)\varepsilon_{t+1}
\end{aligned}
$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

## C.2 Nominal and real term structure

**Proposition 6.** Nominal bond yields are affine in the state vector:

$$
y_t^{\$}(h) = -\frac{A^{\$}(h)}{h} - \frac{(B^{\$}(h))'}{h} z_t,
$$

where the coefficients $A^{\$}(h)$ and $B^{\$}(h)$ satisfy the following recursions:

$$
\begin{aligned}
A^{\$}(h+1) &= -y_0^{\$}(1) + A^{\$}(h) + \frac{1}{2}(\left(B^{\$}(h)\right))'\Sigma\left(B^{\$}(h)\right) - \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_0, && \text{(C.3)}\\
\left(B^{\$}(h+1)\right)' &= \left(B^{\$}(h)\right)'\Psi - (e^{yn})' - \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_1, && \text{(C.4)}
\end{aligned}
$$

initialized at $A^{\$}(0) = 0$ and $B^{\$}(0) = \mathbf{0}$.

*Proof.* We conjecture that the $t+1$-price of a $\tau$-period bond is exponentially affine in the state:

$$
\log(P_{t+1}^{\$}(h)) = A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}
$$

and solve for the coefficients $A^{\$}(h+1)$ and $B^{\$}(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$
\begin{aligned}
P_t^{\$}(h+1) &= E_t[\exp\{m_{t+1}^{\$} + \log\left(P_{t+1}^{\$}(h)\right)\}]\\
&= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}\}]\\
&= \exp\{-y_0^{\$}(1) - (e^{yn})'z_t - \frac{1}{2}\Lambda_t'\Lambda_t + A^{\$}(h) + \left(B^{\$}(h)\right)'\Psi z_t\} \times\\
&\quad E_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right].
\end{aligned}
$$

We use the log-normality of $\varepsilon_{t+1}$ and substitute for the affine expression for $\Lambda_t$ to get:

$$
\begin{aligned}
P_t^{\$}(h+1) &= \exp\left\{-y_0^{\$}(1) - (e^{yn})'z_t + A^{\$}(h) + \left(B^{\$}(h)\right)'\Psi z_t + \frac{1}{2}\left(B^{\$}(h)\right)'\Sigma\left(B^{\$}(h)\right)\right.\\
&\quad \left. - \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}(\Lambda_0 + \Lambda_1 z_t)\right\}.
\end{aligned}
$$

Taking logs and collecting terms, we obtain a linear equation for $\log(p_t(h+1))$:

$$
\log\left(P_t^{\$}(h+1)\right) = A^{\$}(h+1) + \left(B^{\$}(h+1)\right)' z_t,
$$

where $A^{\$}(h+1)$ satisfies (C.3) and $B^{\$}(h+1)$ satisfies (C.4). The relationship between log bond prices and bond yields is given by $-\log\left(P_t^{\$}(h)\right)/\tau = y_t^{\$}(h)$. □

Define the one-period return on a nominal zero-coupon bond as:

$$r^{b,\$}_{t+1}(h) = \log\left(P^{\$}_{t+1}(h)\right) - \log\left(P^{\$}_{t}(h+1)\right)$$

The nominal bond risk premium on a bond of maturity $\tau$ is the expected excess return corrected for a Jensen term, and equals negative the conditional covariance between that bond return and the nominal SDF:

$$\begin{aligned} E_t\left[r^{b,\$}_{t+1}(h)\right] - y^{\$}_t(1) + \frac{1}{2}V_t\left[r^{b,\$}_{t+1}(h)\right] &= -Cov_t\left[m^{\$}_{t+1}, r^{b,\$}_{t+1}(h)\right] \\ &= \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_t \end{aligned}$$

*Real* bond yields, $y_t(h)$, denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$A(h+1) = -y_0(1) + A(h) + \frac{1}{2}(B(h))'\Sigma(B(h)) - (B(h))'\Sigma^{\frac{1}{2}}\left(\Lambda_0 - \Sigma^{\frac{1}{2}\prime}e^{\pi}\right), \tag{C.5}$$

$$(B(h+1))' = -(e^{yn})' + (e^{\pi} + B(h))'\left(\Psi - \Sigma^{\frac{1}{2}}\Lambda_1\right). \tag{C.6}$$

For $\tau = 1$, we recover the expression for the risk-free rate in (C.1)-(C.2).

## C.3 Stocks

### C.3.1 Aggregate Stock Market

We define the real return on the aggregate stock market as $R^m_{t+1} = \frac{P^m_{t+1}+D^m_{t+1}}{P^m_t}$, where $P^m_t$ is the ex-dividend price on the equity market. A log-linearization delivers:

$$r^m_{t+1} = \kappa^m_0 + \Delta d^m_{t+1} + \kappa^m_1 pd^m_{t+1} - pd^m_t. \tag{C.7}$$

The unconditional mean log real stock return is $r^m_0 = E[r^m_t]$, the unconditional mean real dividend growth rate is $\mu^m = E[\Delta d^m_{t+1}]$, and $\overline{pd^m} = E[pd^m_t]$ is the unconditional average log price-dividend ratio on equity. The linearization constants $\kappa^m_0$ and $\kappa^m_1$ are defined as:

$$\kappa^m_1 = \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}}+1} < 1 \text{ and } \kappa^m_0 = \log\left(e^{\overline{pd^m}}+1\right) - \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}}+1}\overline{pd^m}. \tag{C.8}$$

Our state vector $z$ contains the (demeaned) log real dividend growth rate on the stock market, $\Delta d^m_{t+1} - \mu^m$, and the (demeaned) log price-dividend ratio $pd^m - \overline{pd^m}$.

$$\begin{aligned} pd^m_t(h) &= \overline{pd^m} + (e^{pd})'z_t, \\ \Delta d^m_t &= \mu^m + (e^{divm})'z_t, \end{aligned}$$

where $(e^{pd})'$ $(e^{divm})$ is a selector vector that has a one in the fifth (sixth) entry, since the log pd ratio (log dividend growth rate) is the fifth (sixth) element of the VAR.

We *define* the log return on the stock market so that the log return equation holds exactly, given the time series for $\{\Delta d^m_t, pd^m_t\}$. Rewriting (C.7):

$$\begin{aligned} r^m_{t+1} - r^m_0 &= \left[(e^{divm} + \kappa^m_1 e^{pd})'\Psi - (e^{pd})'\right]z_t + \left(e^{divm} + \kappa^m_1 e^{pd}\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}. \\ r^m_0 &= \mu^m + \kappa^m_0 - \overline{pd^m}(1-\kappa^m_1). \end{aligned}$$

The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

$$\begin{aligned} 1 &= E_t\left[M_{t+1}\frac{P^m_{t+1}+D^m_{t+1}}{P^m_t}\right] = E_t\left[\exp\{m^{\$}_{t+1} + \pi_{t+1} + r^m_{t+1}\}\right] \\ &= E_t\left[\exp\left\{-y^{\$}_{t,1} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + \pi_0 + (e^{\pi})'z_{t+1} + r^m_0 + (e^{divm} + \kappa^m_1 e^{pd})'z_{t+1} - (e^{pd})'z_t\right\}\right] \\ &= \exp\left\{-y^{\$}_0(1) - \frac{1}{2}\Lambda_t'\Lambda_t + \pi_0 + r^m_0 + \left[(e^{divm} + \kappa^m_1 e^{pd} + e^{\pi})'\Psi - (e^{pd})' - (e^{yn})'\right]z_t\right\} \\ &\quad \times E_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(e^{divm} + \kappa^m_1 e^{pd} + e^{\pi}\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right] \end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{ r_0^m + \pi_0 - y_0^{\$}(1) + \left[ (e^{divm} + \kappa_1^m e^{pd} + e^{\pi})'\Psi - (e^{pd})' - (e^{yn})' \right] z_t \right\} \\
&\quad \times \exp\left\{ \frac{1}{2} \left( e^{divm} + \kappa_1^m e^{pd} + e^{\pi} \right)' \Sigma \left( e^{divm} + \kappa_1^m e^{pd} + e^{\pi} \right) - \left( e^{divm} + \kappa_1^m e^{pd} + e^{\pi} \right)' \Sigma^{\frac{1}{2}} \Lambda_t \right\}
\end{aligned}
$$

Taking logs on both sides delivers:

$$
\begin{aligned}
r_0^m + \pi_0 - y_0^{\$}(1) + \left[ (e^{divm} + \kappa_1^m e^{pd} + e^{\pi})'\Psi - (e^{pd})' - (e^{yn})' \right] z_t \qquad & \qquad (C.9) \\
+\frac{1}{2} \left( e^{divm} + \kappa_1^m e^{pd} + e^{\pi} \right)' \Sigma \left( e^{divm} + \kappa_1^m e^{pd} + e^{\pi} \right) &= \left( e^{divm} + \kappa_1^m e^{pd} + e^{\pi} \right)' \Sigma^{\frac{1}{2}} \Lambda_t \\
E_t \left[ r_{t+1}^{m,\$} \right] - y_{t,1}^{\$} + \frac{1}{2} V_t \left[ r_{t+1}^{m,\$} \right] &= -Cov_t \left[ m_{t+1}^{\$}, r_{t+1}^{m,\$} \right]
\end{aligned}
$$

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

$$
\begin{aligned}
E_t \left[ r_{t+1}^m \right] - y_{t,1} + \frac{1}{2} V_t \left[ r_{t+1}^m \right] &= -Cov_t \left[ m_{t+1}, r_{t+1}^m \right] \\
r_0^m - y_0(1) &+ \left[ (e^{divm} + \kappa_1^m e^{pd} + e^{\pi})'\Psi - (e^{pd})' - (e^{yn})' - (e^{\pi})'\Sigma^{1/2}\Lambda_1 \right] z_t \\
+\frac{1}{2}(e^{divm} + \kappa_1^m e^{pd})'\Sigma(e^{divm} + \kappa_1^m e^{pd}) &= \left( e^{divm} + \kappa_1^m e^{pd} \right)' \Sigma^{1/2} (\Lambda_t - \left( \Sigma^{1/2} \right)' e^{\pi})
\end{aligned}
$$

We combine the terms in $\Lambda_0$ and $\Lambda_1$ on the right-hand side and plug in for $y_0(1)$ from (C.2) to get:

$$
\begin{aligned}
& r_0^m + \pi_0 - y_{0,1}^{\$} + \frac{1}{2}(e^{\pi})'\Sigma e^{\pi} \\
& + \frac{1}{2}(e^{divm} + \kappa_1^m e^{pd})'\Sigma(e^{divm} + \kappa_1^m e^{pd}) + (e^{\pi})'\Sigma \left( e^{divm} + \kappa_1^m e^{pd} \right) \\
& + \left[ (e^{divm} + \kappa_1^m e^{pd} + e^{\pi})'\Psi - (e^{pd})' - (e^{yn})' \right] z_t \\
& = \left( e^{divm} + \kappa_1^m e^{pd} \right)' \Sigma^{1/2} \Lambda_t + (e^{\pi})'\Sigma^{\frac{1}{2}}\Lambda_0 + (e^{\pi})'\Sigma^{1/2}\Lambda_1 z_t
\end{aligned}
$$

This recovers equation (C.9).

### C.3.2 Dividend Strips

**Proposition 7.** Log price-dividend ratios on dividend strips are affine in the state vector:

$$
pd_t^d(h) = A^m(h) + (B^m(h))'z_t,
$$

where the coefficients $A^m(h)$ and $B^m(h)$ follow recursions:

$$
\begin{aligned}
A^m(h+1) &= A^m(h) + \mu^m - y_0(1) + \frac{1}{2} \left( e^{divm} + B^m(h) \right)' \Sigma \left( e^{divm} + B^m(h) \right) \\
&\quad - \left( e^{divm} + B^m(h) \right)' \Sigma^{\frac{1}{2}} \left( \Lambda_0 - \Sigma^{\frac{1}{2}\prime} e^{\pi} \right), \qquad (C.10) \\
B^{m\prime}(h+1) &= \left( e^{divm} + e^{\pi} + B^m(h) \right)' \Psi - (e^{yn})' - \left( e^{divm} + e^{\pi} + B^m(h) \right)' \Sigma^{\frac{1}{2}} \Lambda_1, \qquad (C.11)
\end{aligned}
$$

initialized at $A_0^m = 0$ and $B_0^m = 0$.

*Proof.* We conjecture the affine structure and solve for the coefficients $A^m(h+1)$ and $B^m(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$
\begin{aligned}
PD_t^d(h+1) &= \mathbb{E}_t \left[ M_{t+1} PD_{t+1}^d(h) \frac{D_{t+1}^m}{D_t^m} \right] = E_t \left[ \exp\{ m_{t+1}^{\$} + \pi_{t+1} + \Delta d_{t+1}^m + pd_{t+1}^d(h) \} \right] \\
&= \mathbb{E}_t \left[ \exp\{ -y_{t,1}^{\$} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + \pi_0 + (e^{\pi})'z_{t+1} + \mu^m + (e^{divm})'z_{t+1} + A^m(h) + B(h)^{m\prime} z_{t+1} \} \right] \\
&= \exp\{ -y_0^{\$}(1) - (e^{yn})'z_t - \frac{1}{2}\Lambda_t'\Lambda_t + \pi_0 + (e^{\pi})'\Psi z_t + \mu^m + (e^{divm})'\Psi z_t + A^m(h) + B(h)^{m\prime}\Psi z_t \}
\end{aligned}
$$

$$\times \mathbb{E}_t \left[ \exp\{-\Lambda_t' \varepsilon_{t+1} + \left(e^{divm} + e^{\pi} + B^m(h)\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right].$$

We use the log-normality of $\varepsilon_{t+1}$ and substitute for the affine expression for $\Lambda_t$ to get:

$$\begin{aligned} pd_t^d(h+1) &= -y_0^\$(1) + \pi_0 + \mu^m + A^m(h) + \left[\left(e^{divm} + e^{\pi} + B^m(h)\right)' \Psi - (e^{yn})'\right] z_t \\ &\quad + \frac{1}{2}\left(e^{divm} + e^{\pi} + B^m(h)\right)' \Sigma \left(e^{divm} + e^{\pi} + B^m(h)\right) \\ &\quad - \left(e^{divm} + e^{\pi} + B^m(h)\right)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t) \end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^d(h+1)$:

$$pd_t^d(h+1) = A^m(h+1) + B^m(h+1)' z_t,$$

where:

$$\begin{aligned} A^m(h+1) &= A^m(h) + \mu^m - y_0^\$(1) + \pi_0 + \frac{1}{2}\left(e^{divm} + e^{\pi} + B^m(h)\right)' \Sigma \left(e^{divm} + e^{\pi} + B^m(h)\right) \\ &\quad - \left(e^{divm} + e^{\pi} + B^m(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_0, \\ B^m(h+1)' &= \left(e^{divm} + e^{\pi} + B^m(h)\right)' \Psi - (e^{yn})' - \left(e^{divm} + e^{\pi} + B^m(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_1. \end{aligned}$$

We recover the recursions in (C.10) and (C.11) after using equation (C.2). □

We define the dividend strip risk premium as:

$$\begin{aligned} \mathbb{E}_t\left[r_{t+1}^{d,\$}(h)\right] - y_{t,1}^\$ + \frac{1}{2} V_t\left[r_{t+1}^{d,\$}(h)\right] &= -Cov_t\left[m_{t+1}^\$, r_{t+1}^{d,\$}(h)\right] \\ &= \left(e^{divm} + e^{\pi} + B^m(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_t \end{aligned}$$

## C.4 Government Spending and Tax Revenue Claims

This appendix computes $P_t^T$, the value of a claim to future tax revenues, and $P_t^G$, the value of a claim to future government spending. It contains the proof for Proposition 5.

### C.4.1 Spending Claim

Nominal government spending growth equals

$$\Delta \log G_{t+1} = \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^g + \left(e^{\Delta g} + e^x + e^{\pi}\right)' z_{t+1}. \tag{C.12}$$

We conjecture the log price-dividend ratios on spending strips are affine in the state vector:

$$pd_t^g(h) = \log\left(PD_t^g(h)\right) = A^g(h) + (B^g(h))' z_t.$$

We solve for the coefficients $A^g(h+1)$ and $B^g(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} PD_t^g(h+1) &= \mathbb{E}_t\left[M_{t+1} PD_{t+1}^g(h) \frac{G_{t+1}}{G_t}\right] = \mathbb{E}_t\left[\exp\{m_{t+1}^\$ + \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} + pd_{t+1}^g(h)\}\right] \\ &= \exp\{-y_0^\$(1) - (e^{yn})' z_t - \frac{1}{2}\Lambda_t'\Lambda_t + \mu^g + x_0 + \pi_0 + (e^{\Delta g} + e^x + e^{\pi} + B^g(h))' \Psi z_t + A^g(h)\} \\ &\quad \times \mathbb{E}_t\left[\exp\{-\Lambda_t' \varepsilon_{t+1} + \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right]. \end{aligned}$$

We use the log-normality of $\varepsilon_{t+1}$ and substitute for the affine expression for $\Lambda_t$ to get:

$$\begin{aligned} PD_t^G(h+1) &= \exp\{-y_0^{\$}(1) + \mu^g + x_0 + \pi_0 + (e^{\Delta g} + e^x + e^{\pi} + B^g(h))'\Psi - (e^{yn})')z_t + A^g(h) \\ &\quad + \frac{1}{2}\left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right)' \Sigma \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right) \\ &\quad - \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\} \end{aligned}$$

Taking logs and collecting terms, we obtain

$$\begin{aligned} A^g(h+1) &= -y_0^{\$}(1) + \mu^g + x_0 + \pi_0 + A^g(h) + \frac{1}{2}\left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right)' \Sigma \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right) \\ &\quad - \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_0, \\ B^g(h+1)' &= (e^{\Delta g} + e^x + e^{\pi} + B^g(h))'\Psi - (e^{yn})' - \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_1, \end{aligned}$$

and the price-dividend ratio of the cum-dividend spending claim is

$$\sum_{h=0}^{\infty} \exp(A^g(h+1) + B^g(h+1)'z_t)$$

**Derivation of Risk Premium** We note that the 1-period holding return on a spending strip is

$$\exp(r_{t+1}^g(h)) = \exp\{\Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} + pd_{t+1}^g(h) - pd_t^g(h+1)\}$$

so that the Euler equation is $\mathbb{E}_t[\exp(m_{t+1}^{\$} + r_{t+1}^g(h))] = 1$.

We can express the expected return as

$$\begin{aligned} \mathbb{E}_t[r_{t+1}^G(h)] &= -\mathbb{E}_t[m_{t+1}^{\$}] - \frac{1}{2}var_t(m_{t+1}^{\$}) - \frac{1}{2}var_t(r_{t+1}^g(h)) - cov_t(m_{t+1}^{\$}, r_{t+1}^g(h)) \\ &= y_t^{\$}(1) - \frac{1}{2}var_t(r_{t+1}^g(h)) - cov_t(m_{t+1}^{\$}, r_{t+1}^g(h)) \end{aligned}$$

and the risk premium is

$$\begin{aligned} \mathbb{E}_t[r_{t+1}^g(h)] - y_t^{\$}(1) &= -\frac{1}{2}var_t(r_{t+1}^g(h)) + cov_t(\Lambda_t'\varepsilon_{t+1}, r_{t+1}^g(h)) \\ &= \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t) - \frac{1}{2}\left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right)' \Sigma \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h)\right) \end{aligned}$$

To evaluate the risk premium for the entire duration of the strip, we define the holding-period risk premium as

$$\frac{1}{h}\sum_{k=0}^{h-1} \mathbb{E}_t[r_{t+k+1}^g(h-k) - y_{t+k}^{\$}(1)]$$

when the state variable is at $z_t = 0$, the expected holding-period risk premium simplifies to

$$\frac{1}{h}\sum_{k=0}^{h-1} \left(e^{\Delta g} + e^x + e^{\pi}\right)' \Sigma^{\frac{1}{2}} \Lambda_0 - \frac{1}{2}\left(e^{\Delta g} + e^x + e^{\pi} + B^g(h-k)\right)' \Sigma \left(e^{\Delta g} + e^x + e^{\pi} + B^g(h-k)\right) \tag{C.13}$$

**Entire Spending Claim** Next, we define the (nominal) return on the claim as $R_{t+1}^g = \frac{P_{t+1}^g}{P_t^g - G_t} = \frac{P_{t+1}^{g,ex} + G_{t+1}}{P_t^{g,ex}}$, where $P_t^g$ is the cum-dividend price on the spending claim and $P_t^{g,ex}$ is the ex-dividend price. We log-linearize the return around $z_t = 0$:

$$r_{t+1}^g = \kappa_0^g + \Delta \log G_{t+1} + \kappa_1^g pd_{t+1}^g - pd_t^g. \tag{C.14}$$

where $pd_t^g \equiv \log\left(\frac{P_t^{g,ex}}{G_t}\right) = \log\left(\frac{P_t^g}{G_t} - 1\right)$. The unconditional mean log return of the G claim is $r_0^g = E[r_t^g]$.

We obtain $\overline{pg}$ from the precise valuation formula (15) at $z_t = 0$. We define linearization constants $\kappa_0^g$ and $\kappa_1^g$ as:

$$\kappa_1^g = \frac{e^{\overline{pg}}}{e^{\overline{pg}} + 1} < 1 \text{ and } \kappa_0^g = \log\left(e^{\overline{pg}} + 1\right) - \frac{e^{\overline{pg}}}{e^{\overline{pg}} + 1}\overline{pg}. \tag{C.15}$$

Then, under a log-linear approximation, the unconditional expected return is:

$$r_0^g = x_0 + \pi_0 + \kappa_0^g - \overline{pg}(1 - \kappa_1^g). \tag{C.16}$$

The log ex-dividend price-dividend ratio on the entire spending claim is affine in the state vector and verify the conjecture by solving the Euler equation for the claim.

$$pg_t = \overline{pg} + (\bar{B}^g)' z_t \tag{C.17}$$

This allows us to write the return as:

$$r_{t+1}^g = r_0^g + \left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' z_{t+1} - (\bar{B}^g)' z_t. \tag{C.18}$$

*Proof.* Starting from the Euler equation:

$$\begin{aligned}
1 &= \mathbb{E}_t\left[\exp\{m_{t+1}^\$ + r_{t+1}^g\}\right] \\
&= \exp\{-y_0^\$(1) - (e^{yn})' z_t - \frac{1}{2}\Lambda_t'\Lambda_t + r_0^g + [\left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' \Psi - (\bar{B}^g)'] z_t\} \\
&\quad \times \mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right].
\end{aligned}$$

We use the log-normality of $\varepsilon_{t+1}$ and substitute for the affine expression for $\Lambda_t$ to get:

$$\begin{aligned}
1 &= \exp\{r_0^g - y_0^\$(1) + [\left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' \Psi - (\bar{B}^g)' - (e^{yn})'] z_t \\
&\quad + \underbrace{\frac{1}{2}\left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' \Sigma \left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)}_{Jensen} - \left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\}
\end{aligned}$$

Taking logs and collecting terms, we obtain the following system of equations:

$$r_0^g - y_0^\$(1) + Jensen = \left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' \Sigma^{\frac{1}{2}} \Lambda_0 \tag{C.19}$$

and

$$\left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' \Psi - (\bar{B}^g)' - (e^{yn})' = \left(e^{\Delta g} + e^x + e^\pi + \kappa_1^g \bar{B}^g\right)' \Sigma^{\frac{1}{2}} \Lambda_1 \tag{C.20}$$

The left-hand side of this equation is the unconditional expected excess log return with Jensen adjustment. The right hand side is the unconditional covariance of the log SDF with the log return. This equation describes the unconditional risk premium on the claim to government spending. Equation (C.20) describes the time-varying component of the government spending risk premium. Given $\Lambda_1$, the system of $N$ equations (C.20) can be solved for the vector $\bar{B}^g$:

$$\bar{B}^g = \left(I - \kappa_1^g \left(\Psi - \Sigma^{\frac{1}{2}} \Lambda_1\right)'\right)^{-1} \left[\left(\Psi - \Sigma^{\frac{1}{2}} \Lambda_1\right)' \left(e^{\Delta g} + e^x + e^\pi\right) - e^{yn}\right]. \tag{C.21}$$

□

## C.4.2 Revenue Claim

Nominal government revenue growth equals

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^\tau + \left(e^{\Delta\tau} + e^x + e^\pi\right)' z_{t+1}. \tag{C.22}$$

where $\tau_t = T_t / GDP_t$ is the ratio of government revenue to GDP. Note that this ratio is assumed to have a long-run growth rate of zero. This imposes cointegration between government revenue and GDP. The growth ratio in this ratio can only temporarily deviate from zero.

The remaining proof exactly mirrors the proof for government spending, with

$$pd_t^\tau \equiv \log\left(\frac{P_t^{\tau,ex}}{T_t}\right) = \log\left(\frac{P_t^\tau}{T_t} - 1\right) = \overline{pd^\tau} + (B^\tau)' z_t \tag{C.23}$$

$$r_{t+1}^{\tau} = r_0^{\tau} + \left(e_{\Delta\tau} + e^{x} + e^{\pi} + \kappa_1^{g} B^{\tau}\right)' z_{t+1} - (B^{\tau})' z_t, \tag{C.24}$$

and

$$r_0^{\tau} = x_0 + \pi_0 + \kappa_0^{\tau} - \overline{p\tau}(1 - \kappa_1^{\tau}).$$

$$r_0^{\tau} - y_0^{\$}(1) + Jensen = \left(e^{\Delta\tau} + e^{x} + e^{\pi} + \kappa_1^{\tau} B^{\tau}\right)' \Sigma^{\frac{1}{2}} \Lambda_0. \tag{C.25}$$

# D Data Sources

## D.1 Primary Surpluses

The primary surpluses are constructed using NIPA Table 3.2 Federal Government Current Receipts and Expenditures from 1947 to 2019. All variables are seasonally adjusted.

The government revenue is the sum of the corporate and personal tax revenue, the net income from the rest of the world, and the federal government dividends income receipts on assets. The personal tax revenue is the total of the current personal tax receipts, the tax revenue from production and imports, the net income from the rest of the world, and surpluses from government-sponsored enterprise net of subsidies. The net income from the rest of the world includes the tax income from the rest of the world, the contributions from government social insurance from the rest of the world, the current transfer receipts from the rest of the world, net of the government transfer payments to the rest of the world and the interest payments to the rest of the world.

The government spending is the domestic net transfer payments before interest payments plus discretionary spending (i.e. consumption expenditures). The domestic net transfer is the domestic current transfer receipts net of the domestic contribution from government social insurances and the domestic current transfer receipts.

The primary surpluses are the government revenue minus the government spending before interest payments.

## D.2 State Variables

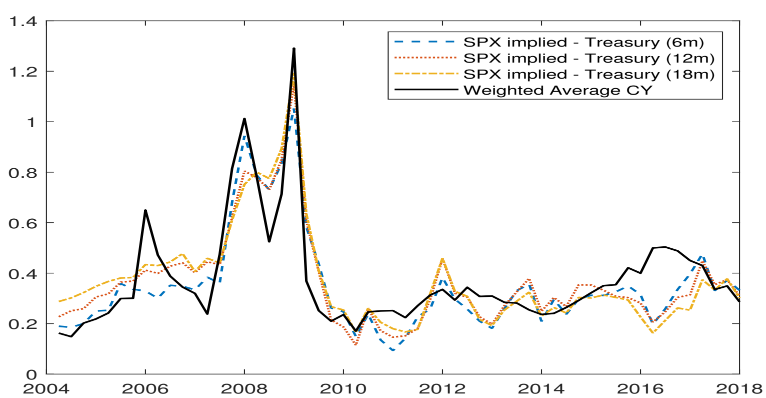
We obtain the time series of GDP from NIPA Table 1.1.5, and inflation is the change in the GDP price index from NIPA Table 1.1.4. The real GDP growth $x_t$ is nominal GDP growth minus inflation. The Treasury yields for all maturities are constant maturity yields from Fred. There are some periods where the 20-year bond was not issued and some periods where the 30 year bond was not issued. The log-price-dividend ratio and the log real dividend growth are computed using CRSP database. Dividends are seasonally adjusted and quarterly. We include the growth of both the government revenue to GDP ratio and the government spending to GDP ratio in the state vector. The government revenue and government spending are defined in Section 1.

## D.3 Other Measures of the Convenience Yield

In this section, we compare our measure of the convenience yield with the implied convenience yields from Van Binsbergen et al. (2019). Figure D.1 shows the 6-month, 12-month, and 18-month convenience yields from Van Binsbergen et al. (2019), which are spreads between the SPX option implied interest rates and government bond rates with corresponding maturities. All measures of the convenience yield exhibit similar time-series patterns over the sample period from 2004-01 to 2017-04.

Figure D.1: Measures of the Convenience Yield

The figure shows the time series of different measures of the convenience yield. The dashed blue line is the spread of 6-month zero coupon interest rates implied from SPX options with 6-month Treasury bill rate. The dotted red line is the spread of 12-month zero coupon interest rates implied from SPX options with 12-month Treasury bill rate. The dashed yellow line is the spread of 18-month zero coupon interest rates implied from SPX options with 18-month Treasury bond rate. The data is from Van Binsbergen et al. (2019). The solid black line is the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread. All yields are in the quarter frequency, and expressed in percentage per annum. The sample period is from 2014-01 to 2017-04.



# E Coefficient Estimates

## E.1 Cointegration Tests

We perform a Johansen cointegration test by first estimating the vector error correction model :

$$\Delta w_t \quad = \quad A(B' w_{t-1} + c) + D\Delta w_{t-1} + \varepsilon_t, \quad \text{where } w_t = \begin{pmatrix} \log T_t \\ \log G_t \\ \log GDP_t \end{pmatrix}.$$

Both the trace test and the max eigenvalue test do not reject the null of cointegration rank 2 (with $p$-values of 0.11), but reject the null of cointegration rank 0 and 1 (with $p$-values lower than 0.01). These results are in favor of two cointegration relationships between variables in $w_t$.

We also conduct the Phillips-Ouliaris cointegration test on the $\{w_t\}$ matrix with a truncation lag parameter of 2, and reject the null hypothesis that $w$ is not cointegrated with a $p$-value of 0.03.

## E.2 Market Prices of Risk

The constant market price of risk vector is estimated to be:

$$\Lambda_0' = [0.00, 0.16, -0.41, 0.05, 0.00, 1.58, 0, 0, 0, 0, 0, 0]$$

The time-varying market price of risk matrix is estimated at:

$$\Lambda_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 38.71 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12.99 & -88.61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4.13 & -17.34 & -5.84 & -0.27 & -0.67 & 1.80 & 2.74 & -4.91 & -0.38 & -2.24 & -5.82 & -3.42 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -41.74 & -13.08 & -18.04 & 56.82 & -4.31 & -0.00 & -3.12 & -0.24 & -0.79 & -3.45 & 3.25 & -0.21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

### E.2.1 Identification

The first four rows of the VAR govern the dynamics of the term structure block of the model. We allow for three of the first four innovations to carry non-zero risk prices, resulting in a three factor model for bond yields. Empirically, three factors explain more than 97% of the variation in bond yields across maturities. We allow for priced shocks to the level (short rate), the slope (five-year minus one-year yield), and real GDP growth. As explained below, matching the five-year bond yield pins down the fourth element of $\Lambda_0$ and the fourth row of $\Lambda_1$, given the elements in the first three rows.

Since the spread between the five-year bond yield and the one-year bond yield is the fourth element of the state vector, and the short rate is the third element of the state vector, the five-year bond yield can be written as:

$$y^{\$}_{t,5} = y^{\$}_{0,5} + (e_{yn} + e_{yspr})' z_t = -\frac{A^{\$}_5}{5} - \frac{B^{\$\prime}_5}{5} z_t$$

This restriction identifies one element in the constant $\Lambda_0$, specifically

$$y^{\$}_{0,1} + yspr_0 = -\frac{1}{5} A^{\$}_5$$

and $N$ elements in the time-varying market price of risk matrix $\Lambda_1$:

$$e'_{y1} + e'_{yspr} = -\frac{1}{5}\left(B^{\$}_5\right)'$$

The recursions for the coefficients in the affine term structure model are repeated here for convenience:

$$A^{\$}_{\tau+1} \quad = \quad -y^{\$}_{0,1} + A^{\$}_{\tau} + \frac{1}{2}\left(B^{\$}_{\tau}\right)' \Sigma \left(B^{\$}_{\tau}\right) - \left(B^{\$}_{\tau}\right)' \Sigma^{\frac{1}{2}} \Lambda_0, \tag{E.1}$$

$$\left(B^{\$}_{\tau+1}\right)' \quad = \quad \left(B^{\$}_{\tau}\right)' \Psi - e'_{yn} - \left(B^{\$}_{\tau}\right)' \Sigma^{\frac{1}{2}} \Lambda_1, \tag{E.2}$$

initialized at $A_0^\$ = 0$ and $B_0^\$ = 0$. Define $\widetilde{\Psi} = \Psi - \Sigma^{\frac{1}{2}}\Lambda_1$ to be the risk-neutral companion matrix. Then (C.4) can be written as:

$$\frac{-\left(B_{\tau+1}^\$\right)'}{5} = \frac{1}{5}e_{yn}'(I-\widetilde{\Psi})^{-1}(I-\widetilde{\Psi}^{\tau+1})$$

The restriction on $\Lambda_1$ can be written as:

$$e_{y1}' + e_{yspr}' = \frac{1}{5}e_{yn}'(I-\widetilde{\Psi}^{\tau+1})(I-\widetilde{\Psi})^{-1}$$

The left-hand side is a $N \times 1$ vector with a 1 in elements 3 and 4 and a 0 in the other four positions. Hence, the same must be true of the right-hand side. This imposes $N$ restrictions on $\Lambda_1$ which affects $\widetilde{\Psi}$, given $\Psi$. There is one restriction on each of the columns of $\Lambda_1$. It is a restriction on a linear combination of the elements in the first four rows of $\Lambda_1$ of that column. For example, if the elements in the 10th column and first three rows of $\Lambda_1$ are all zero, it is a simple restriction on the element in the fourth row and 10th column.

We allow for three non-zero elements in the first three rows of $\Lambda_1$: $\Lambda_1(2,2)$, $\Lambda_1(3,3)$, and $\Lambda_1(3,4)$. The price of GDP growth risk is allowed to depend on the level of GDP growth and the price of interest rate risk is allowed to depend on the level of the interest rate (as in Cox, Ingersoll, and Ross, 1985) and on the slope (as in Campbell and Shiller, 1991). The constant market prices of risk associated with GDP growth and the level factor are also allowed to differ from zero. These five parameters are identified off the cross-section of nominal and real bond yields; we include five additional nominal yield maturities beyond the 1- and 5-quarter yields, and five real bond maturities. We expect a positive price of risk since positive innovations to GDP growth are good news. As in the classic term structure models of Cox et al. (1985), we expect level risk to the term structure to carry a negative risk price. Level risk mostly reflects the risk to low-frequency changes to expected inflation, for example due to changes in the Central Bank's inflation target. We expect the shock to the yield spread that is orthogonal to the preceding three shocks to carry a positive risk price $\Lambda_0(4)$, as positive slopes indicate improving economic conditions. This risk price helps the model match the average slope of the term structure. For parsimony, unexpected inflation shocks are not priced.

The second block of the SDF controls the pricing of risk in the stock market, and is captured by the market prices of risk in the fifth and sixth rows of $\Lambda_t$. Recall that the VAR models the dynamics of the price-dividend ratio (row 5) and the dividend growth rate (row 6). The VAR implies an expected excess log stock return including a Jensen adjustment given by the left-hand side of the following equation:

$$\begin{aligned} & r_0^m + \pi_0 - y_{0,1}^\$ + \frac{1}{2}e_\pi'\Sigma e_\pi + \frac{1}{2}(e_{divm} + \kappa_1^m e_{pd})'\Sigma(e_{divm} + \kappa_1^m e_{pd}) + e_\pi'\Sigma\left(e_{divm} + \kappa_1^m e_{pd}\right) \\ & + \left[(e_{divm} + \kappa_1^m e_{pd} + e_\pi)'\Psi - e_{pd}' - e_{yn}'\right] z_t = \left(e_{divm} + \kappa_1^m e_{pd}\right)' \Sigma^{1/2}(\Lambda_0 + \Lambda_1' z_t) + e_\pi'\Sigma^{\frac{1}{2}}\Lambda_0 + e_\pi'\Sigma^{1/2}\Lambda_1 z_t \end{aligned} \tag{E.3}$$

The first term on the second line is the time-varying component of the expected excess stock return. This is just data, i.e. the companion matrix of the VAR $\Psi$, not asset pricing. The asset pricing is on the right-hand side of the above equation. It reports the equity risk premium, which is negative the conditional covariance of the the log stock return and the log SDF. It depends on the market prices of risk.

To replicate this time-variation, the sixth row of $\Lambda_1$ must be such that the time-varying components of the left- and right-hand sides of equation (E.3) are equalized:

$$(e_{divm} + \kappa_1^m e_{pd} + e_\pi)'\Psi - e_{pd}' - e_{yn}' = \left(e_{divm} + \kappa_1^m e_{pd}\right)' \Sigma^{1/2}\Lambda_1' + e_\pi'\Sigma^{1/2}\Lambda_1$$

This is a linear system of $N$ equations in $N$ unknowns, which uniquely pins down the $N$ elements in the sixth row of $\Lambda_1$. Given the structure of $\Psi$ and the structure of the market prices of risk in the first four rows of $\Lambda_1$, the sixth row of $\Lambda_1$ must have non-zero elements in all columns.

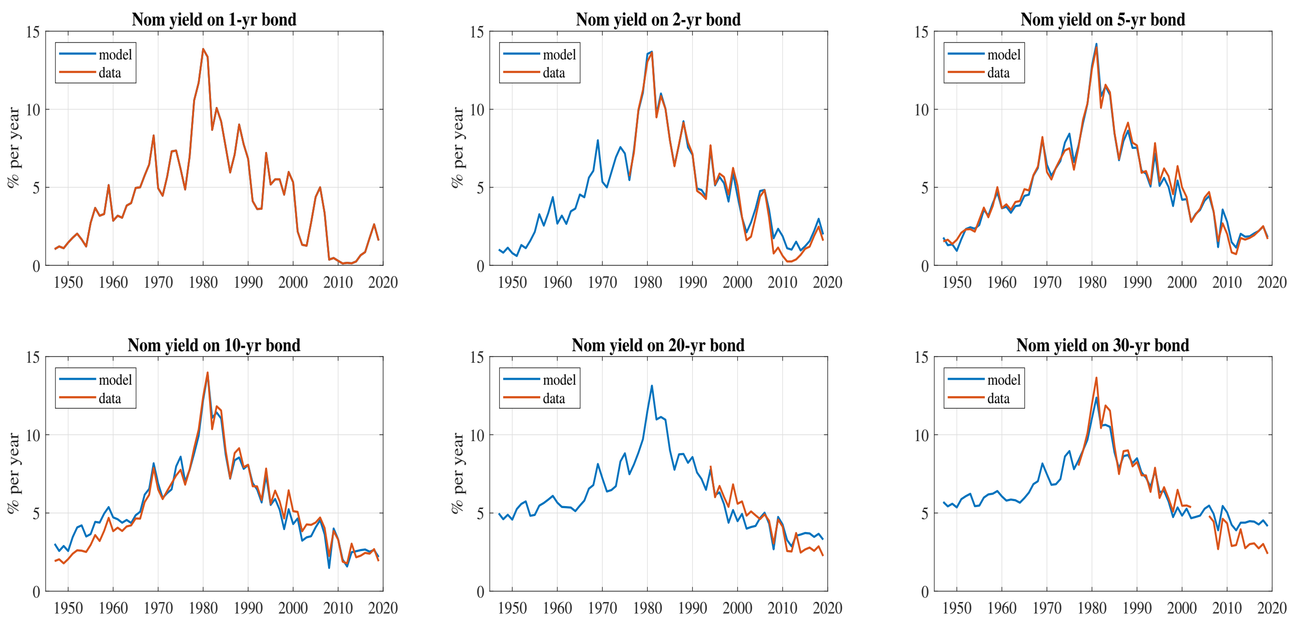
The identification of the sixth row of the market prices of risk is also aided by the moment that equates the price-dividend ratio in the data to the price-dividend ratio in the model, where the latter is calculated at the sum of the price-dividend ratios of the first 3600 dividend strips.

We expect and indeed estimate a positive risk price for innovations to dividend growth $\Lambda_0(6)$, since positive innovations in dividend growth are good news for the economy.

# F Model Fit

Figure F.1 shows that the model matches the nominal term structure in the data closely. The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. In the estimation of the market prices of risk, we overweigh matching the 5-year bond yield since it is included in the VAR and the 30-year bond yield since the behavior of long-term bond yields is important for the results.

Figure F.1: Dynamics of the Nominal Term Structure of Interest Rates




The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. Yields are measured at the end of the year. Data are from FRED and FRASER. the sample is 1947 until 2019.

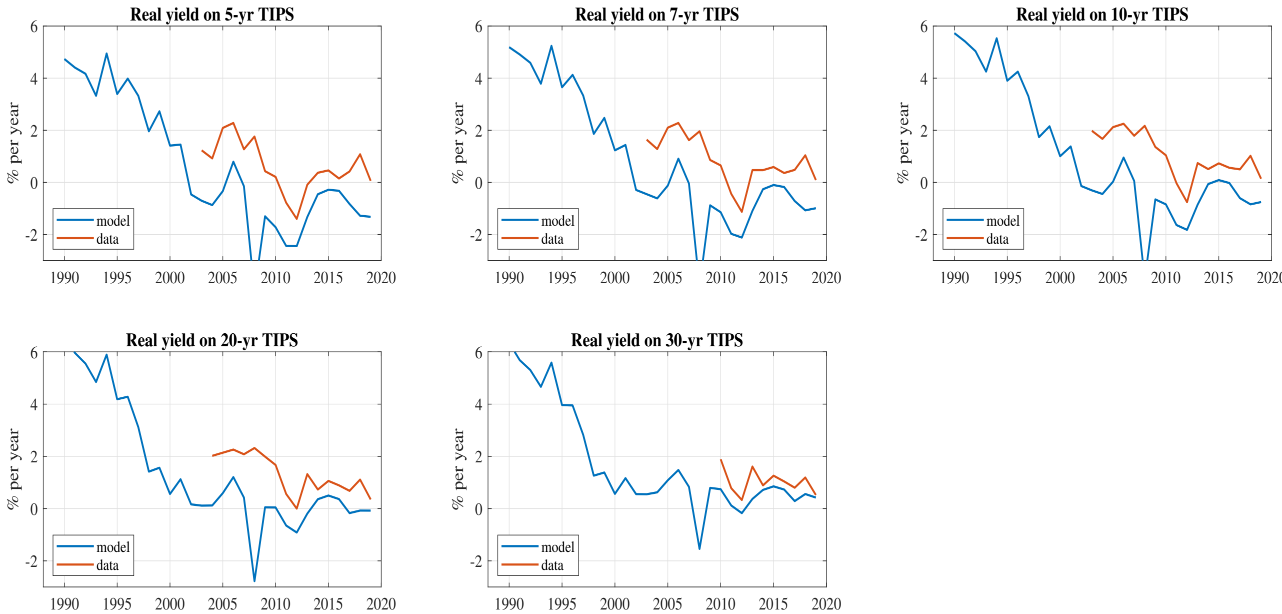
Figure F.2 shows that the model matches the real term structure in the data closely. The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real Treasury bond yields (Treasury Inflation Indexed securities). In the estimation of the market prices of risk, we overweigh matching the 30-year bond yield since the behavior of long-term bond yields is important for the results.

The top panels of Figure F.3 show the model's implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These yields are well behaved, with very long-run nominal (real) yields stabilizing at around 7.20% (4.01%) per year. We impose conditions that ensure that the nominal and real term structure are well behaved at very long maturities, for which we have no data. Specifically, we impose that average nominal (real) yields of bonds with maturities of 100, 500, 1000, 2000, 3000, and 4000 years remain above 6.23% (3.04%) per year, which is the long-run nominal (real) GDP growth rate $x_0 + \pi_0$ ($x_0$) observed in our sample. Second, we impose that nominal yields remain above real yields plus 3.18% expected inflation at those same maturities. This imposes that the inflation risk premium remain positive at very long maturities. Third, we impose that the nominal and real term structures of interest rates flatten out, with an average yield difference between 100- and 50-year yields that must not exceed 2% per year and between 200- and 100-year maturity that must not exceed 1% per year. These restrictions are satisfied at the optimum.

The bottom left panel of Figure F.3 shows that the model matches the dynamics of the nominal bond risk premium, defined as the expected excess return on the five-year nominal bond, quite well. Bond risk premia decline in the latter part of the sample, possibly reflecting the arrival of foreign investors who value U.S. Treasuries highly. The bottom right panel shows a decomposition of the nominal bond yield on a five-year bond into the five-year real bond yield, annual expected inflation inflation over the next five years, and the five-year inflation risk premium. On average, the 5.0% nominal bond yield is comprised of a 1.9% real yield, a 3.2% expected inflation rate, and a -0.1% inflation risk premium. The graph shows that the importance of these components fluctuates over time.

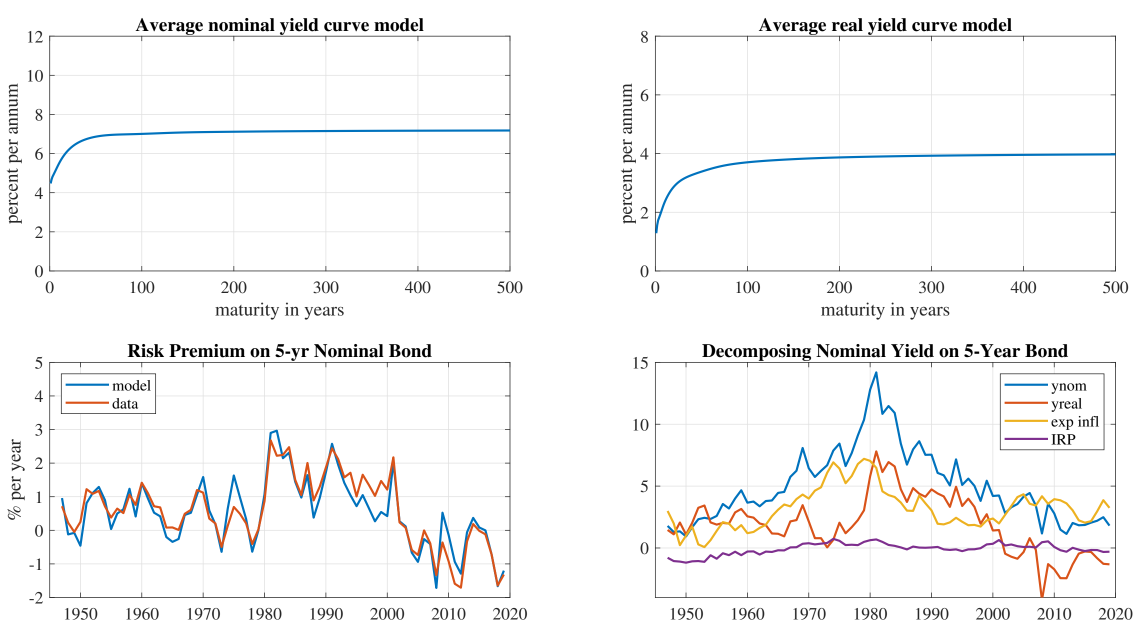
Figure F.4 shows the equity risk premium, the expected excess return, in the left panel and the price-dividend ratio in the right panel. The risk premia in the data are the expected equity excess return predicted by the VAR. Their dynamics are sensible, with low risk premia in the dot-com boom of 1999-2000 and very high risk premia in the Great Financial Crisis of 2008-09. The VAR-implied equity risk premium occasionally turns negative. The figure's right panel shows a tight fit for equity price levels. Hence, the model fits both the behavior of expected returns and stock price levels.

## Figure F.2: Dynamics of the Real Term Structure of Interest Rates



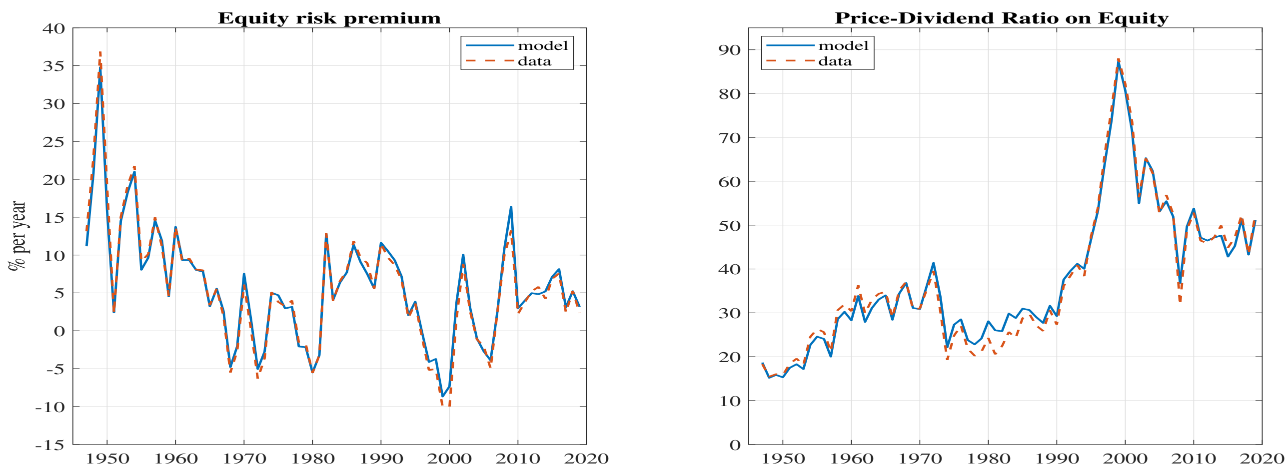

The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real bond yields. Data are from FRED and start in 2003. For ease of readability, we start the graph in 1990 but the model of course implies a real yield curve for the entire 1947-2019 period.

## Figure F.3: Long-term Yields and Bond Risk Premia




The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 to 500 years. The bottom left panel plots the nominal bond risk premium on the five year bond in model and data. The nominal bond risk premium is measured as the five year bond yield minus the expected one-year bond yield over the next five years. The bottom right panel decomposes the model's five-year nominal bond yield into the five-year real bond yield, the five year expected inflation, and the five-year inflation risk premium.

Figure F.4: Equity Risk Premium and Price-Dividend Ratio




The figure plots the observed and model-implied equity risk premium on the overall stock market in the left panel and the price-dividend ratio in the right panel. The price-dividend ratio is the price divided by the annualized dividend. Data are from 1947-2019. Monthly stock dividends are seasonally adjusted.

# G Model with Priced Fiscal Shock

This appendix explores a model in which we allow for a non-zero market price of risk for shocks to government spending growth that are orthogonal to all macro-economic and financial state variable innovations as well as to tax revenue/GDP growth shocks. We refer to these orthogonal shocks as "pure" spending shocks.

When estimating the market prices of risk, we re-estimate the first six elements of $\Lambda_0$ and the first six rows of $\Lambda_1$ (with starting values equal to their benchmark model values) and free up the market prices of risk associated with the pure spending shock: $\Lambda_0(9)$ and the ninth row of $\Lambda_1$. We add to the objective function that minimizes the distance between asset pricing moments in model and data an additional condition to minimize the sum of squared errors between the observed debt-to-GDP ratio and the model-implied present value of primary surpluses. Even though the pure spending shock does not affect the contemporaneous innovations to stock returns and bond yields by virtue of the ordering of the variables in the VAR (the Cholesky decomposition), it is contemporaneously correlated with the shock to debt/GDP growth. In addition, the purse spending shock raises $g_t$ and affects all asset prices in future periods through the dynamics of the VAR.

The constant market price of risk vector is estimated to be:

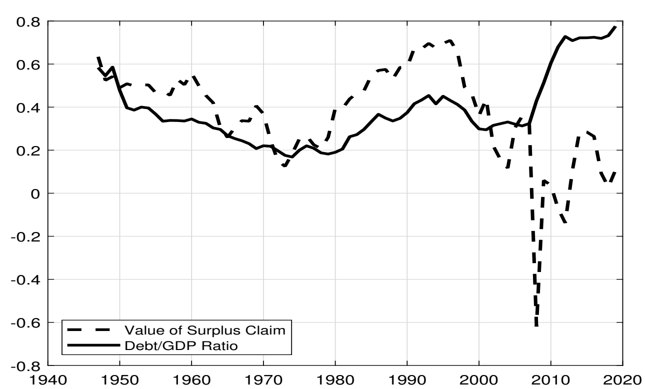
$$\tilde{\Lambda}_0' = [0, 0.30, -0.44, 0.05, 0, 1.41, 0, 0, 1.89, 0, 0, 0]$$

The time-varying market price of risk matrix is estimated at:

$$\tilde{\Lambda}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 14.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12.44 & -117.61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3.59 & -7.55 & -7.79 & -0.49 & -0.51 & 3.11 & 2.83 & -5.55 & -0.51 & -2.30 & -4.31 & -2.70 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42.53 & -20.51 & -19.61 & 28.03 & -4.46 & -0.00 & -2.45 & -0.28 & -0.55 & -3.22 & 2.08 & -0.24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15.03 & -41.53 & 58.97 & 79.51 & 4.37 & 2.29 & -1.86 & -1.02 & -10.93 & 16.59 & -2.91 & 5.04 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure G.1 plots the present value of government surpluses. This extended model brings the present value of surpluses closer to the market value of government debt, at least on average with an average wedge of -1.53% of GDP. The wedge remains substantial in the last 20 years of the sample. Pricing errors are similar to those in the benchmark model. Since the price of pure spending shocks is estimated to be positive 1.89, pure spending shock (unrelated to the broader economy) are good news for the representative agent (low SDF, low marginal utility growth states). This lowers $P_t^g$ and increases $P_t^s$. Put differently, the spending claim now becomes a much riskier assets, offsetting the hedging features from counter-cyclical government spending. In addition, the model with priced spending shocks results in a maximum Sharpe ratio that is substantially higher than the benchmark model.

Figure G.1: Present Value of Surpluses with Priced Pure Spending Shocks




The figure plots government debt-GDP ratio and present value of government surpluses with priced pure spending shocks.