SOVEREIGN DEBT CRISES AND LOW INTEREST RATES

G. BLOISE AND Y. VAILAKIS

Current version: May 2, 2022

Abstract

We revisit the occurrence of self-fulfilling crises in sovereign debt markets under time-varying interest rates and growth. We show that, when long-term interest rates exceed growth, insolvency is solely caused by the exhaustion of the sovereign's debt repayment capacity subject to limited commitment. Indeed, high interest rates impose discipline on market sentiments, because creditors necessarily become more optimistic about solvency when the sovereign reduces debt exposure. Creditors' beliefs respond instead ambiguously under low interest rates fluctuating around growth. As long as interest rates exceed growth, debt reduction alleviates the fiscal burden. However, the sovereign also benefits from the prospect of rolling over outstanding debt as long as interest rates remain below growth. Thus, creditors' sentiments might adjust adversely to fiscal consolidation. When the default punishment is not disproportionately severe, this mechanism sustains belief-driven debt crises even when fundamentals would otherwise ensure solvency.

Keywords: Sovereign default risk; Self-fulfilling crises; Low interest rates; Limited commitment.
JEL Classifications: F34, F41, H63.

1. Introduction

The recent generation of debt crises in the European Union has put sovereign debt markets front and center, challenged old conventional views, and motivated the exploration of

We are grateful to Manuel Amador and Herakles Polemarchakis for inspiring discussions at various stages of this project. We also thank Mark Aguiar, John Geanakoplos, Pietro Reichlin and Paolo Siconolfi for useful comments and suggestions, as well as participants to the 2018 SWET Workshop in Paris, the 2019 CRETA Workshop in Warwick and the 2019 SAET Conference in Ischia. Gaetano Bloise acknowledges the financial support of the Italian Ministry of Education (PRIN 2015). Yiannis Vailakis acknowledges the financial support of an ERC starting grant (FP7, Ideas specific program, Project 240983 DCFM) and of two ANR research grants (Projects Novo Tempus and FIRE), as well as the hospitality of Australian National University.

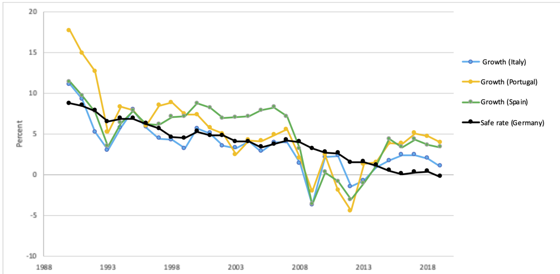
Gaetano Bloise, University of Rome II (gaetano.bloise@gmail.com).

Yiannis Vailakis, Adam Smith Business School, University of Glasgow (yiannis.vailakis@glasgow.ac.uk).

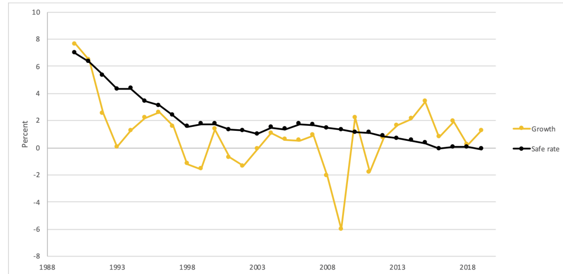
alternative mechanisms to account for belief-driven solvency crises. This paper fits into a growing literature that studies the vulnerability of international capital markets to self-sustaining market panic. Its aim is to show that low interest rates are a source of multiple equilibria. In a bad equilibrium investors' pessimism about debt repayment inflates risk premia making debt service more costly and raising the likelihood of default relative to a good equilibrium. This type of multiplicity does not occur when the interest rates are instead high.

Empirical evidence suggests that a low safe rate of interest (relative to growth) is more the historical norm than the exception for a broad set of developed and emerging countries, a trend that is expected to remain the case for the foreseeable future (Blanchard [15] and Jorda et al. [31]). In fact, as documented in Figure 1, growth rates of several borrowing countries have been fluctuating around the time-varying long-term safe interest rate over the last decades, with interest rate exceeding growth in some periods, and falling below growth in other periods. Debt roll-over is feasible when interest rates fall below the growth rate and large amounts of debt can be accumulated without requiring a consequent increase of the fiscal burden. We argue that, while alleviating the fiscal cost of debt, low interest rates also increase the vulnerability to debt crises. High interest rates impose a substantial discipline on creditors' beliefs about solvency. When this discipline disappears, sovereign debt is exposed to spontaneous revisions of market sentiments, unjustified by fundamentals, eventually triggering default episodes.

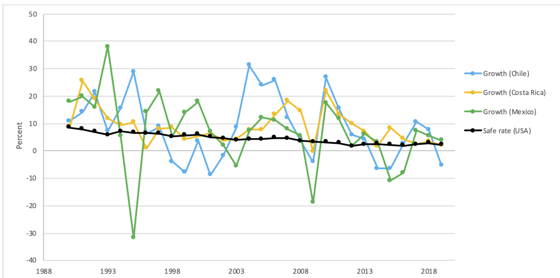
It is a common belief that sovereign debt markets are vulnerable to extemporaneous liquidity crises, largely caused by a static coordination failure among creditors. These sentiments-driven turmoils occur as sudden shifts along the Laffer curve (as in Calvo [21] and Lorenzoni and Werning [32]), when the sovereign is unable to adjust fiscal plans in front of poor auction outcomes, or as more traditional debt runs preventing the sovereign from rolling over a large outstanding debt (as in Cole and Kehoe [23]). The crisis might burst rapidly or might propagate slowly over time through sharply rising spreads and a faster debt accumulation, inducing default eventually. We instead consider a more canonical framework for sovereign debt markets and study the long-term sustainability of borrowing plans subject to limited commitment, as in Eaton and Gersovitz [27]. By ruling out any static miscoordination of creditors' beliefs, we unravel an autonomous role of low interest rates in increasing the volatility of market sentiments.



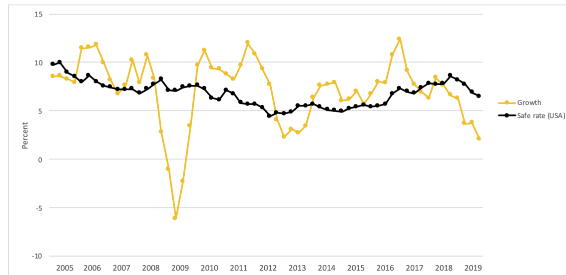
(a) European Union



(b) Japan



(c) Latin America



(d) Mexico

Figure 1. Figure 1a exhibits the time-pattern of the yield on German 10-year bonds relative to the nominal growth rates of Italy, Spain and Portugal in the European Monetary Union. Figure 1b illustrates the same comparison for Japan in its local currency. Figure 1c provides evidence about some Latin America countries. The rates of growth are measured in current US dollars and compared with the yield on US 10-year bonds. Finally, Figure 1d reports the local currency yield on US 10-year bonds relative to the nominal growth rate of Mexico in local currency. A proxy of the risk-free US bond yield in Mexican local currency is estimated correcting for the long-term forward premium, as in Du and Schreger [26].
Source: FRED, World Development Indicators, Global Economy Dataset and Du and Schreger [26].

Recent studies (*e.g.*, Aguiar and Amador [3] and Auclert and Rognlie [13]) have ascertained that equilibrium is unique in the traditional Eaton and Gersovitz [27]'s model with constant positive interest rate and no growth, the established paradigm for most of the recent theoretical and empirical research on sovereign debt markets (see Aguiar and Amador [2], Aguiar and Amador [5]). We provide a theory of self-fulfilling debt crises by deviating from this canonical framework: consistently with the mentioned empirical evidence, the rate of interest is time-varying and recurrently falls below the growth rate. We use otherwise conventional assumptions and, in particular, Ponzi games are ruled out in our framework: interest rates might be exceeding economic growth for a long phase with positive probability and the accumulation of debt would be explosive without recurrent repayments to creditors.[1]

Our first contribution consists in singling out the assumptions responsible for uniqueness of equilibrium in Eaton and Gersovitz [27]'s framework. We thus encompass time-varying interest rate, a diversified market for securities and growth of the sovereign's income.[2] None of these features plays any role: as long as interest rates remain above growth in the long-term, the equilibrium is unique and constrained efficient. Default episodes might still occur, but they reflect the effective debt repayment capacity of the sovereign, subject to limited commitment, and are therefore determined by fundamentals. Our proof of uniqueness reveals the intrinsic logic: when the sovereign reduces its debt exposure, this alleviates the burden of serving its debt at high interest rates and, so, unambiguously eases budget discipline; under rational expectations, creditors cannot become more pessimistic; hence, bad equilibria cannot be supported. This monotonicity in beliefs uniquely determines default events.

Interestingly, unnoticed by the previous literature, equilibrium uniqueness still obtains, independently of the level of the interest rate, provided that the default punishment is sufficiently severe, that is, when the sovereign cannot secure the autarkic value upon default. It is worth clarifying that such a condition is more restrictive than it might appear at a first glance: it requires default costs be incomparably large relative to a negligible debt exposure, thus preventing equilibrium default at low levels of debt. Instead, as documented by Trebesch and Zabel [36], historical evidence suggests that the disruptive effects of sovereign default vary in length and intensity proportionally with the amount of delinquent debt.

[1]To the best of our knowledge, our paper is the first to study the effects of low interest rates in the Eaton and Gersovitz [27]'s workhorse model of sovereign debt markets. A time-varying interest rate is not for the sake of generality. Indeed, with a constant interest rate persistently below growth, equilibrium would uninterestingly depend on some artificial upper bound on debt preventing Ponzi games. We instead consider a framework in which borrowing is bounded by a finite natural debt limit and creditors' beliefs justly reflect sovereign default incentives.

[2]A diversified market for securities is not only a theoretical speculation. Several emerging countries issue bonds in local and foreign currencies and with several maturities. In fact, the sovereign debt composition tends to be tilted toward foreign currency and, on average, around three-fourths of sovereign external debt is denominated in foreign currency, as documented by Ottonello and Perez [33]. Furthermore, governments do hold diversified assets and liabilities. In the European Union, for instance, the market value of government financial assets represented around one-third of the value of government liabilities before the last pandemic recession. From a solvency perspective, therefore, it seems important to take assets into account when assessing the sustainability of government debt.

Furthermore, as we illustrate by examples, low interest rates mitigate the costs of default, turning certain punishments lenient even though they would be severe under high interest rates.

Under low interest rates, and when low levels of debt entail low default costs, market sentiments are unrestricted to a large extent. Less sovereign debt implies a relaxation of the fiscal burden conditional on the interest rate exceeding growth. However, the interest rate might remain below growth for a long phase with some probability. The sovereign benefits from rolling over its debt conditional on low interest rates and, as a consequence, a permanent reduction of outstanding debt tightens, rather than relieves, the budget constraint. Due to these conflicting forces, creditors' beliefs might respond ambiguously to fiscal consolidation. More debt accumulation would be sustainable at the constrained efficient equilibrium, but investors' fears end up inflating bond yields and the sovereign's attempts of reducing debt exposure are ultimately interpreted as signals of future insolvency. Our main contribution is to show that this perverse mechanism results in full indeterminacy of equilibrium under low interest rates.

We supplement our analysis with a digression on long-term debt. In a recent paper, Aguiar and Amador [4] establish equilibrium multiplicity with long-term debt of a given fixed maturity. We argue that, when the sovereign is able to shorten the maturity structure by issuing additional one-period debt, the extent of such a multiplicity is limited under high interest rates: disparities in equilibrium beliefs about default are bounded by the value of long-term debt as an instrument to hedge against risk. Therefore, if no hedging motive exists (as in Aguiar et al. [7]), or else if other short-term securities provide a similar insurance, equilibrium is unique under high interest rates. This thought experiment reveals that our equilibrium multiplicity under low interest rates is distinct from Aguiar and Amador [4]'s multiplicity with long-term debt under high interest rates. Hence, our papers are complementary.

Our analysis exploits two methodological innovations: a dominant root theory for long-term interest rate and a planning approach to determine the constrained efficient equilibrium. The dominant root theory provides a tractable approach to time-varying interest rates. Dominant roots estimate bounds to long-term interest rates and govern the long-run tendencies of the debt-to-income ratio. The dominant root theory is developed in Bloise et al. [17, 18] (see also Appendix B) and well serves our purposes in this paper. The planning approach to sovereign default is inspired by Aguiar et al. [7] and provides a benchmark in which default events are belief-free and are triggered by fundamentals only. Under conditions ruling out Ponzi games, the planner program is well defined and delivers the constrained efficient equilibrium. Constrained inefficient equilibria might exist only under low interest rates.

The layout of the paper is as follows: in section 2 we discuss some related literature; in section 3 we present an extended Eaton-Gersovitz economy with time-varying interest rate and a diversified asset market; in section 4 we characterize the constrained efficient equilibrium by considering a planning program; in section 5 we show that, under high (relative to growth) interest rates, the equilibrium is unique and, therefore, efficient; in section 6 we present simple examples to clarify several aspects of the underlying mechanism and in section 7 we argue that low interest rates open the route in general to self-fulfilling crises; in section 8 we extend the baseline model to long-term bonds; and, finally, in section 9 we conclude briefly. All proofs are collected in Appendix A.

2. Related literature

Our adherence to the established Eaton and Gersovitz [27]'s paradigm helps the comparison with previous studies. A recent strand of literature considers long maturity bonds (Hatchondo and Martinez [28], Arellano and Ramanarayanan [12] and Chatterjee and Eyigungor [22]). As long-term bond prices reflect the occurrence of sovereign default in all future periods up to maturity, they introduce an additional pecuniary externality because the sovereign cannot commit to a plan of bond issuance in the future. Aguiar and Amador [4] exploit the feedback of future default on the current value of outstanding debt to show that multiple equilibria occur. Stangebye [35] reaches the same conclusion by introducing a sunspot that leads to self-fulfilling debt crises. Our short digression on long-term debt (section 8) provides a contribution to this issue that is of independent interest.

Another strand of the literature alters the timing and action space of the game and explores how self-fulfilling market panics emerge as coordination failures among investors. For instance, in Cole and Kehoe [23], the sovereign is allowed to default after observing the outcome of the current's period bond auction. This leads to a coordination problem that might support a bad equilibrium: offering a low price limits the ability to roll-over existing debt, therefore, triggering immediate default. This type of multiplicity has been explored quantitatively by Bocola and Dovis [19] and extended recently by Aguiar et al. [8] to account for 'desperate deals' that allow sovereigns to escape default by issuing a minimal amount of bonds at low prices. Lorenzoni and Werning [32] and Ayres et al. [14] show that a coordination problem can emerge due to a feedback between interest rates and debt burden as in Calvo [21]: a shift in investors' beliefs may lead to an increase in borrowing costs, forcing governments into a path of debt accumulation that can raise the likelihood of default, therefore validating the initial shift in beliefs.

Multiplicity of equilibria is a pervasive feature of dynamic games, because a large variety of outcomes is in general sustainable by the worst credible threat. In fact, recent studies find a continuum of equilibria in sovereign debt games. Passadore and Xandri [34] assume that the sovereign can issue debt but cannot save. Equilibrium uniqueness in Eaton and Gersovitz [27] depends on the ability of the sovereign to replicate the consumption pattern of an optimistic (high debt) equilibrium under the conditions of a pessimistic (low debt) equilibrium, yielding a contradiction. This arbitrage requires the accumulation of saved repayments to creditors under low debt, which is unfeasible when savings are precluded and, hence, multiple equilibria coexist. However, as proved by Auclert and Rognlie [13, Proposition 7], allowing for any small amount of savings will restore equilibrium uniqueness. Dovis [25]'s sovereign debt game encompasses production by means of imported intermediate goods and private information about domestic productivity shocks. Furthermore, in addition to the exclusion from future borrowing and saving, sovereign default entails a cost in terms of unexploited static gains from trade: the sovereign imposes the expropriation of imported intermediate goods and, given this extreme tariff, foreign countries supply no intermediate goods, so preventing mutual gains from international trade and domestic production. This static coordination failure sustains multiple equilibria.

Turning to a related literature on imperfect risk-sharing, our characterization might appear as a reminiscence of the First Welfare Theorem in competitive economies under limited commitment. As a matter of fact, Alvarez and Jermann [10, Proposition 4.6] establish that equilibrium with risk of default is constrained efficient when implied interest rates are high, that is, when the present value of the aggregate endowment is finite. As constrained efficient equilibria are (locally) unique, this rules out equilibrium multiplicity subject to high implied interest rates. The analogy is however deceptive: we obtain equilibrium indeterminacy under a finite present value of sovereign's future income, so as to prevent Ponzi schemes. In fact, as also clarified by Abel et al. [1, Section III], low safe interest rates are perfectly consistent with well-defined present values of future contingent claims.

We finally relate the contribution in this paper to our previous work on sustainable debt and sovereign default. In Bloise and Vailakis [16] we develop Aguiar and Amador [3]'s contraction mapping approach to Eaton and Gersovitz [27]'s sovereign debt equilibrium. When the long-term interest rate exceeds growth, equilibrium is unique and can be determined by means of a contracting operator, so providing an efficient computational tool. In this paper we also establish that equilibrium is unique under hight interest rates in a more general framework. However, we privilege economic intuition and exploit a more direct arbitrage argument: any consumption plan in an optimistic equilibrium with low debt can be replicated in a pessimistic equilibrium with high debt, thus contradicting the asserted pessimistic expectations of creditors about default. The purpose of this general characterization of conditions ensuring a unique equilibrium is to identify, as a complement, the space for multiple equilibria.

In Bloise et al. [18] we study the competitive equilibrium of an economy with limited commitment in which debts are not secured by collateral or legal sanctions. Default only entails a loss of the reputation for repayment, inducing a consequent inability to borrow in the future. We prove that debt is sustainable by the reputation for repayment alone, while Ponzi games are infeasible, contrary to Bulow and Rogoff [20]'s celebrated theorem. Incentives to repayments necessarily require low interest rates, relative to growth, because otherwise the borrower would find it profitable to default at the maximum debt exposure and to revert to a self-insurance policy, as established by Bulow and Rogoff [20] (see also Bloise et al. [17] for an extension to incomplete markets). In fact, each individual is indifferent between repaying the maximum sustainable debt and relying after default on self-insurance without borrowing in the future. It turns out that this is equivalent to establishing the existence of multiple sovereign debt equilibria *without* default, one with borrowing and one without borrowing, when default only precludes the issuance of future debt. In this paper we thoroughly study sovereign debt equilibria *with* default under a variety of other punishments. Furthermore, at a competitive equilibrium, interest rates are *endogenously* determined by market clearing, reflecting mutual gains to lenders and borrowers. As a result, the mere observation of low interest rates does not reveal the presence of multiple equilibria. In this paper, instead, we do establish the occurrence of multiple sovereign debt equilibria for all *exogenously* given processes of low interest rates.[3]

3. Equilibrium

The economy extends over an infinite horizon, $\mathbb{T} = \{0, 1, 2, \ldots, t, \ldots\}$, subject to uncertainty generated by a Markov process on the finite state space S with irreducible transition $P : S \to \Delta(S)$. When the process is initiated from a given Markov state s_0 in S, it generates a probability space $(\Omega, \mathcal{F}, \mu)$ and a filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ of finite partitions of Ω corresponding to partial histories of Markov states occurring with positive probability. To be parsimonious on notation, we describe all variables as stochastic processes. In particular, we let L be the space of all processes $f : \mathbb{T} \times \Omega \to \mathbb{R}$ adapted to the filtration $\mathcal{F}$, and let L_t be the space of random variables $f_t : \Omega \to \mathbb{R}$. As usual, L^+ denotes the (weakly) positive cone of L.

The sovereign's preferences over consumption plans c in L^+ are given by

$$U_t(c) = \mathbb{E}_t \sum_{s \in \mathbb{T}} \delta^s u(c_{t+s}),$$

where δ in $(0, 1)$ is the discount factor and $u : \mathbb{R}^+ \to \mathbb{R}$ is the per-period utility function. The sovereign's uncertain endowment, or income, is described by a process y in L^+. As our arguments apply independently of stationarity, we do not assume that the endowment obeys any Markov transition. In particular, we can accommodate a framework with stochastic growth (for instance, Aguiar and Gopinath [6]).

Assumption 1 (Utility)**.** The sovereign's utility $u : \mathbb{R}^+ \to \mathbb{R}$ is bounded, continuous and strictly increasing.

We consider a diversified asset market with multiple short-term (one-period) securities. In particular, a finite set J of securities is traded over time. In every period, security j in J is described by a possibly contingent payoff $R^j_{t,t+1}$ in L^+_{t+1} at the following period. A trading plan z in Z is an adapted process in L^J, where z_t in Z_t is the portfolio of securities held in period t in $\mathbb{T}$. Portfolios yield a contingent payoff according to the linear functional $R_{t,t+1}$:

[3] It is a common feature of competitive economies with limited enforcement of debt contracts to admit an autarkic equilibrium, in addition to an equilibrium in which lending and borrowing occur (*e.g.*, Alvarez and Jermann [10]). However, interest rates at the autarkic equilibrium will be distinct from those prevailing at the equilibrium with debt. In Eaton and Gersovitz [27]'s sovereign debt market, instead, a multiplicity of equilibria occurs at exogenously fixed safe interest rates.

$Z_t \to L_{t+1}$, that is, $R_{t,t+1}(z_t) = \sum_{j \in J} R^j_{t,t+1} z^j_t$. In the absence of default, securities are priced by a linear functional $Q_t : Z_t \to L_t$ given as

$$Q_t(z_t) = \frac{1}{\pi_t} \mathbb{E}_t \pi_{t+1} R_{t,t+1}(z_t),$$

where the strictly positive process π in L^+ is interpreted as the *stochastic discount factor*, or *pricing kernel*.

In most of the literature inspired by Eaton and Gersovitz [27] (see Aguiar and Amador [5]), the pricing kernel corresponds to the valuation of risk-neutral creditors who have access to international capital markets at a constant interest rate. We instead assume, as in some recent quantitative literature (for instance, Arellano [11, Section D], Arellano and Ramanarayanan [12, Section VI] and Hatchondo et al. [29]), a more general time-varying pricing kernel. This can still be interpreted as the marginal rate of substitution of a typical lender. Time variation reflects both the volatility in the temporal pattern for interest rate and changes in the exposure to default events of risk-averse lenders. The extent of risk premium on sovereign default will so depend on the covariation of the lenders' marginal rate of substitution with default events.

To rule out Ponzi-type debt schemes, we assume that the default-free valuation of the sovereign's endowment is finite. This condition mirrors Eaton and Gersovitz [27]'s and Bulow and Rogoff [20]'s original assumption of a positive interest rate, and it is ubiquitous in the literature on sovereign debt default.[4] Notice that the valuation of the sovereign's endowment is necessarily finite when the pricing kernel is interpreted as the marginal rate of substitution of an infinitely-lived lender with commitment ability.

Assumption 2 (Finite valuation)**.** The default-free valuation of the sovereign's endowment is finite, that is, process N in L^+ is finitely valued, where

$$N_t = \frac{1}{\pi_t} \mathbb{E}_t \sum_{s \in \mathbb{T}} \pi_{t+s} y_{t+s}. \tag{3.1}$$

At the beginning of each period, the government inherits some wealth w_t in L_t from previous transactions. This is a claim if positive and a debt if negative. When the government defaults, it receives an exogenous reservation (utility) value $\underline{V}_t$ in L_t, which entails all

[4] The only notable exception is Hellwig and Lorenzoni [30]: the value of the sovereign's endowment cannot be finite at any equilibrium with self-enforcing debt in their economy. In fact, debt can be rolled-over exactly in their economy and corresponds to a speculative bubble. This sort of Ponzi-type debt contracts can also occur in our economy when Assumption 2 is removed.

the consequences of insolvency. We assume that the adapted process $\underline{V}$ in L is bounded, though unrestricted otherwise.

Assumption 3 (Reservation values)**.** The process $\underline{V}$ in L of sovereign's reservation values is bounded.

Conditional on no default, instead, the government receives y_t in L_t^+ as endowment, consumes c_t in L_t^+ and purchases a portfolio of securities z_t in Z_t subject to the budget constraint

$$Q_t (z_t|g) + c_t \leq y_t + w_t,$$

where the pricing reflects the expectations on default. Creditors believe that the government will renege on its promises whenever its current liabilities exceed some state-contingent thresholds g in L^+. The portfolio held by the government is, thus, priced anticipating default, that is,

$$Q_t (z_t|g) = \frac{1}{\pi_t} \mathbb{E}_t \pi_{t+1} R_{t,t+1} (z_t) \chi_{t+1} (z_t|g),$$

where $\chi_{t+1} (z_t|g)$ is the indicator function of the repayment event $\{R_{t,t+1} (z_t) \geq -g_{t+1}\}$ in $\mathcal{F}_t$. Notice that default cannot occur on single securities: when the sovereign's declares bankruptcy, it repudiates its liabilities altogether and cannot claim payments on the other securities held in its portfolio.

The plan of the government is optimal conditional on creditors' expectations about its solvency. We let $V_t (w_t|g)$ in L_t^* be the (utility) value of debt repayment, so that the government defaults whenever $V_t (w_t|g) < \underline{V}_t$. This value is recursively determined by

$$V_t (w_t|g) = \sup_{(c_t, z_t) \in L_t^+ \times Z_t} u (c_t) + \delta \mathbb{E}_t \max \{V_{t+1} (w_{t+1}|g), \underline{V}_{t+1}\}$$

subject to

$$Q_t (z_t|g) + c_t \leq y_t + w_t,$$

with

$$w_{t+1} = R_{t,t+1} (z_t).$$

We use $\mathcal{V}$ to denote the space of all bounded-from-above adapted processes for the value function of the government.[5] A process V in $\mathcal{V}$ is optimal for the sovereign, conditional on default thresholds g in L^{+}, if it satisfies the above dynamic programming condition.

We finally impose boundary conditions to ensure that default is profitable when debt is sufficiently large. Notice that, as in Aguiar et al. [7, Assumption 1(iv)], we require a sort of Strong Inada Condition: $u(0)$ is supposed to be sufficiently negative, though finite by Assumption 1 in order to simplify our analysis. We remark that all the assumptions in this section are maintained throughout the remaining part of the paper.

Assumption 4 (Boundary conditions)**.** The value of default is restricted by

$$u(0) + \delta \mathbb{E}_t \bar{V}_{t+1} < \underline{V}_t \leq V_t(0|0),$$

where $\bar{V}$ in $\mathcal{V}$ is an upper bound on utility.

An *equilibrium* is a situation in which the beliefs of creditors about the sovereign's default are correct. In other terms, conditional on creditors' beliefs, default thresholds are optimal for the government, that is,

$$V_t(-g_t|g) = \underline{V}_t. \tag{3.2}$$

As the value function is increasing, this condition establishes that it is optimal for the government to repay any debt not exceeding the threshold, so confirming creditors' expectations.

4. Planning

We consider a planning program and show that it implements an equilibrium. The planner defaults whenever the reservation value exceeds the utility from consumption in the continuation and this is perfectly anticipated in the pricing of the securities. In other terms, the planner corresponds to a government that can commit to a consumption plan (or a plan of debt issuances) but cannot commit not to default. The consumption plan is then chosen optimally subject to a default-adjusted budget constraint. This planning program delivers

[5] Conventionally, $V_t(w_t|g) = -\infty$ when the feasible set is empty. Consistently, L^{*} denotes adapted processes with value in the extended reals $\mathbb{R}^{*}$. The value function is bounded from above because of our Assumption 1.

the efficient outcome when creditors concede debt up to the sovereign's credible repayment capacity.[6]

The planner cannot commit: once a temporal pattern for consumption is announced, default cannot be prevented when profitable. Thus, given a consumption plan c in L^{+}, the utility of the planner is determined as

$$U_t^*(c) = u(c_t) + \delta \mathbb{E}_t \max\left\{U_{t+1}^*(c), \underline{V}_{t+1}\right\}. \tag{4.1}$$

As $u : \mathbb{R}^{+} \to \mathbb{R}$ is a bounded continuous map, this utility function is well-defined and continuous.[7] The objective of the planner is to maximize this default-adjusted utility function subject to a default-adjusted sequential budget constraint. We now explain how this budget set is constructed.

Given a consumption plan c in L^{+}, default occurs if and only if $U_t^*(c) < \underline{V}_t$. Let $\chi_t(c)$ be the indicator function of the repayment event $\{U_t^*(c) \geq \underline{V}_t\}$ in $\mathcal{F}_t$. Prices of securities are determined so as to reflect the expectations of default, that is,

$$Q_t(z_t|c) = \frac{1}{\pi_t}\mathbb{E}_t \pi_{t+1} R_{t.t+1}^{+}(z_t) - \frac{1}{\pi_t}\mathbb{E}_t \pi_{t+1} R_{t.t+1}^{-}(z_t)\, \chi_{t+1}(c).$$

Notice that, in this formula, default can only occur if $R_{t,t+1}(z_t) < 0$, that is, when the sovereign has accumulated a debt. This is consistent with the assumption that default thresholds g in L^{+} are positive and, so, saving cannot be forced.

The planner is subject to a budget constraint, where securities are priced under default risk. More precisely, a consumption plan c in L^{+} is in the *default-adjusted budget set* if and only if, at any contingency before default occurs,

$$Q_t(z_t|c) + c_t \leq y_t + w_t \tag{4.2}$$

and

$$-\frac{1}{\pi_t}\mathbb{E}_t \pi_{t+1} N_{t+1} \leq Q_t(z_t|c), \tag{4.3}$$

[6] A minimum-expenditure planning approach is developed by Aguiar et al. [7]. Precedents of the more direct planning program in this section appear in the previous literature on competitive equilibrium under limited commitment (for instance, in Alvarez and Jermann [10]). In a recent paper, Aguiar and Amador [3, Section 4] present a planning formulation similar to the one we study in this section.

[7] Notice that the plan specifies consumption levels even after default. This is motivated by notational convenience with no effective implication, because the specific continuation after default does not affect utility values before default.

with

$$w_{t+1} = R_{t,t+1}(z_t).$$

The role of condition (4.3) is to prevent Ponzi schemes. It imposes an upper bound on the revenue that the sovereign can raise by selling securities. As any debt exceeding the process N in L^+ would not be repayable out of the endowment, the planner cannot credibly borrow more than the value of this future claim.

The optimal plan subject to the default-adjusted budget constraint is in fact an equilibrium. This observation allows us to establish, at the same time, existence and efficiency. Optimality is achieved because creditors' expectations about default are determined by the effective debt repayment capacity of the sovereign. In principle, other equilibria might exist which are supported by self-fulfilling pessimistic beliefs about solvency.

Proposition 4.1 (Existence)**.** *The value function V^* in $\mathcal{V}$ of the planning program, along with the induced default thresholds $g^* \leq N$ in L^+, is an equilibrium.*

We finally show that any other equilibrium, if it exists, is dominated by the equilibrium generated by the planning program. Thus, the planning program effectively provides an efficient benchmark. As we argue in the following analysis, this is the only equilibrium under high implied interest rates.

Proposition 4.2 (Efficiency)**.** *If V in $\mathcal{V}$ is an equilibrium with default thresholds $g \leq N$ in L^+, then $V \leq V^*$, where V^* in $\mathcal{V}$ is the value function of the planning program.*

5. Uniqueness revisited

We show that, when interest rates are high, equilibrium is unique and thus necessarily efficient. Creditors' beliefs about solvency cannot be more optimistic than those implied by the planning program, as this would be inconsistent with the no Ponzi game condition. In turn, an equilibrium with more pessimistic beliefs is unsustainable: the sovereign would be issuing lower levels of debt due to pessimistic beliefs, while creditors' beliefs would necessarily be more optimistic due to reduced debt exposure. As a result, expectations will faithfully reflect the repayment capacity of the sovereign.

We further show that, even under low interest rates, equilibrium is unique when the sovereign cannot secure the autarkic value after defaulting on the outstanding debt, provided the utility function is concave.[8] Any adverse shift of creditors' beliefs about sovereign solvency might be compensated by a proportional contraction in debt exposure and consumption levels. The residual part of income is unnecessary to serve the reduced amount of debt and can be diverted to increase consumption. Under concavity this ensures more than autarkic utility, so confuting pessimistic beliefs.

We first consider a situation with high interest rates. To identify the exact condition on the pricing kernel, we consider the canonical (default-free) *valuation functional* $\Pi : L \to L$ defined as

$$\Pi_t\left(b_{t+1}\right) = \inf_{z_t \in Z_t} Q_t\left(z_t\right) \tag{5.1}$$

subject to

$$b_{t+1} \leq R_{t,t+1}\left(z_t\right).$$

This operator computes the current cost of meeting future obligations, conditional on available securities. Under incomplete markets, the valuation functional is only sublinear, as some contingent claims might not be tradable and some contingent obligations cannot be fulfilled exactly. We also consider the space of all contingent claims growing no faster than the sovereign's endowment, that is,

$$L\left(y\right) = \left\{b \in L : |b| \leq \lambda y \text{ for some } \lambda > 0\right\}.$$

Thus, we estimate the long-term interest rate, net of growth, by considering the Perron-Frobenius dominant root of the valuation functional. A fully developed theory is provided by Bloise et al. [17, 18] (see also Appendix B). Here we only present the relevant restriction on the pricing kernel and an informal discussion.

Assumption 5 (High implied interest rates)**.** There exists an interior b in $L^+\left(y\right)$ such that, for some ρ in $(0,1) \subset \mathbb{R}^+$,

$$\rho b_t = \Pi_t\left(b_{t+1}\right).$$

Furthermore, the default-free valuation of the sovereign's endowment N lies in $L^+\left(y\right)$.

The dominant root ρ in $(0,1) \subset \mathbb{R}^+$ can be interpreted as (the reciprocal of) the greatest rate of growth of an investment fund relative to the endowment. Indeed, as the valuation

[8] In our framework with discounted expected utility, concavity is enforced by a decreasing marginal utility ($u'' < 0$), an acceptable and widespread assumption. Nonetheless, it worth noticing that concavity of the overall utility function is a rather restrictive property under recursive preferences or non-expected utility.

functional is sublinear,

$$\Pi_t\left(\rho^{-1}b_{t+1}\right) = b_t.$$

This formula shows that, given the current value of the fund b_t in L_t^+, an active investment policy in available securities will ensure a value $\rho^{-1}b_{t+1}$ in L_{t+1}^+ of the fund in the following period. Thus, relative to the sovereign's endowment, the value of the investment fund grows over time at the rate $\rho^{-1} - 1 > 0$. This can be interpreted as a sort of long-term interest rate, that is, as the greatest rate of growth on savings, relative to the endowment, which can be ensured permanently. Consistently, the economic interpretation of Assumption 5 is that, over the long-term, the interest rate exceeds the growth rate of the endowment.

For a better understanding, it is worth considering a simplified framework in which the risk-free bond is the only security and its price varies over time. Assume that the economy is not growing, or declining, and that fluctuations in the price of the risk-free bond are governed by a simple Markov chain on state space S with strictly positive transitions (a rather restrictive property, also for an empirical application of the theory). In such a simple situation,

$$\rho = \max_{s\in S} q_s,$$

where q_s in $\mathbb{R}^{++}$ denotes the bond price in state s in S. In fact, although unlikely, the economy might remain in the state corresponding to the greatest bond price forever, with the investment fund growing according to

$$\rho b_{t+1} = b_t.$$

This is the only growth rate of the investment fund that can be guaranteed.

The arbitrage argument ruling out multiple equilibria is fundamentally inspired by Bulow and Rogoff [20] and Auclert and Rognlie [13]. To gain intuition about the role of high implied interest rates for uniqueness, consider two equilibria V and $\tilde{V}$ in $\mathcal{V}$, and let g and $\tilde{g}$ in L^+ be their associated default thresholds satisfying

$$V_t\left(-g_t|g\right) = \underline{V}_t = \tilde{V}_t\left(-\tilde{g}_t|\tilde{g}\right).$$

To simplify, suppose that these two equilibria can be ordered in terms of the amount of debt that can be credibly issued by the sovereign, that is, $\tilde{g} < g$. As Ponzi games are ruled out, the distance between the optimistic and the pessimistic thresholds can be bounded (up to a

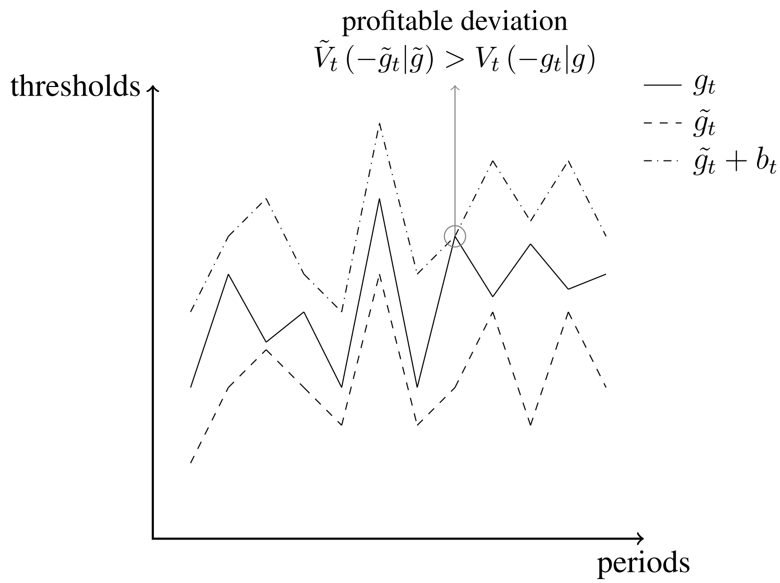


Figure 2. Replication policy under high interest rates

proportionality constant, if needed) by the process in Assumption 5, that is,

$$-\tilde{g}_t \leq -g_t + b_t.$$

This condition ensures that any borrowing plan that is feasible under optimistic beliefs g in L^+ can be replicated under pessimistic beliefs $\tilde{g}$ in L^+ along with a saving fund b in L^+. The saving fund will be alimented by the unnecessary repayments to creditors arising from a reduced initial debt exposure of the sovereign in the low-debt equilibrium. As an interest will be accruing on such a saving, consumption could be even increased. Consequently, the sovereign would be better off under the tighter default thresholds, contradicting pessimistic beliefs of creditors, as illustrated by Figure 2. The role of Assumption 5 is to provide a suitable replication portfolio when the interest rate is time-varying and multiple securities are traded in the asset market. We remark that such a replication portfolio would not exist in general when Assumption 5 is violated.

Proposition 5.1 (Uniqueness)**.** *Under high implied interest rates (Assumption 5), equilibrium with default thresholds $g \leq N$ in L^+ is unique.*

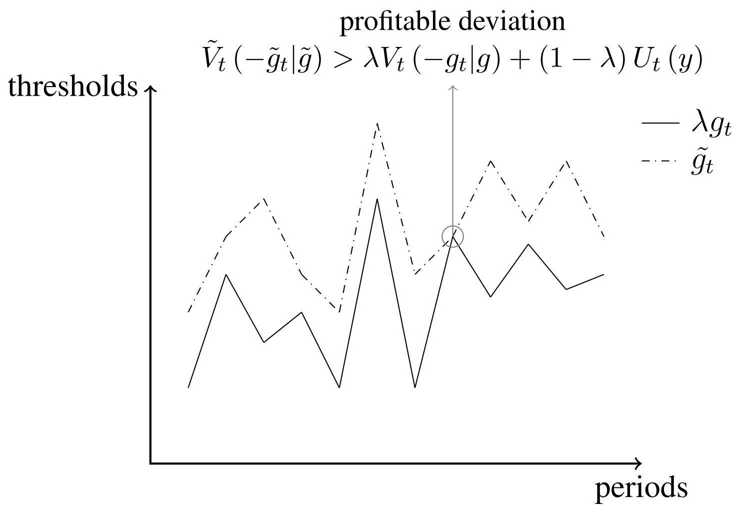


Figure 3. Replication policy under severe punishment

The insistence on a pure arbitrage approach inspired by Bulow and Rogoff [20] might have obscured the presence of a more straight approach to uniqueness when utility is concave and the default punishment is severe. Indeed, under supposedly pessimist beliefs, the sovereign might contract the borrowing plan carried out under optimistic beliefs and save the residual part of income for consumption. Although this adjustment is not a pure arbitrage, under a concave utility, and when the punishment is harsher than autarky, it ensures a utility gain, thus contradicting creditors' pessimistic expectations of default. The logic of this construction is illustrated by Figure 3.[9]

Assumption 6 (Severe punishment)**.** For some sufficiently small ϵ in $\mathbb{R}^{++}$,

$$\underline{V}_t < u\left(y_t - \epsilon N_t\right) + \delta \mathbb{E}_t \underline{V}_{t+1},$$

where the default-free valuation of the sovereign's endowment N lies in $L^+(y)$

Proposition 5.2 (Uniqueness)**.** *Under severe punishment (Assumption 6), and decreasing marginal utility ($u'' < 0$), equilibrium with default thresholds $g \leq N$ in L^+ is unique.*

To conclude, uniqueness has a natural economic interpretation: under high implied interest rates, reducing debt exposure has an unambiguous positive effect on creditors' beliefs,

[9] The approach bears similarities with the *compressed mimicking* argument of Auclert and Rognlie [13, Section 4.1] when saving is subject to an exogenous upper bound.

because it also cuts down the burden of serving the debt. As a consequence, pessimism cannot be supported as a rational belief. Furthermore, when the autarkic value cannot be secured upon default, the sovereign is able to accommodate any adverse shift in creditors' beliefs by means of contraction of borrowing plans. This discipline on equilibrium, however, ceases when low interest rates mitigate the disruptive effects of default. Creditors might become more pessimistic when the sovereign downsizes its debt, this lowers the sovereign's ability to raise revenues and ultimately justifies creditors' increased fears for solvency. Self-fulfilling pessimism creates no logical inconsistency because a reduced debt exposure produces ambiguous effects: it alleviates the debt burden when interest rate exceeds growth, whereas it reduces the gains from debt roll-over when interest rate falls below growth. We turn next to the study of self-fulfilling crises when the hypothesis of high implied interest rates (Assumption 5) fails, beginning with a few examples.

6. ILLUSTRATIVE EXAMPLES

6.1. **Autarkic reversion.** We consider a minimal environment in which equilibrium indeterminacy arises. The state space is $S = \{h, l\}$, and each state occurs with equal probability in every period. We assume that the sovereign's income is constantly y in $\mathbb{R}^+$ and the sovereign's utility $u : \mathbb{R}^{++} \to \mathbb{R}^+$ is smoothly strictly concave, subject to Inada condition $\lim_{c \to 0} u(c) = -\infty$. Markov states only affect the default-free price of the bond, and this is the only security issued by the sovereign. In particular, bond prices satisfy

$$\delta < q_l < 1 < q_h, \tag{6.1}$$

where δ in $(0, 1) \subset \mathbb{R}^+$ is the sovereign's discount factor. Thus, the sovereign is more impatient than international lenders, so as to provide an incentive to borrow. As the interest rate is recurrently negative, $q_h > 1$, the hypothesis of high implied interest rates (Assumption 5) is violated. However, as the interest rate is also recurrently positive, $q_l < 1$, finite valuation is enforced by an appropriate choice of the stochastic discount factor and, consequently, Ponzi schemes are infeasible (Assumption 2).

No borrowing is allowed after default. As the sovereign is more impatient than the market, saving is not optimal after default and, thus, autarky is the default value. We show that, under mild restrictions, the efficient equilibrium involves debt and is so distinct from the most pessimistic equilibrium (autarky). In particular, as the efficient equilibrium obtains through a planning program (section 4), it is sufficient to argue that the default-adjusted budget set of the planner contains a borrowing plan yielding more than reservation utility in all periods. Indeed, although this plan might not be optimal, it would ensure that the value of the planning program exceeds autarky and, by Propositions 4.1-4.2, this value coincides with the most optimistic equilibrium. This observation remarkably simplifies our arguments.

To our purpose, we consider contingent debt issuances (g_h, g_l) in $\mathbb{R}^+ \times \mathbb{R}^+$, with $g_l < g_h$, and estimate the implied gain in the sovereign's utility. In particular, we pose

$$W_h(g_h) = u(y - g_h + q_h g_l) + \delta \left(\frac{W_h(g_l) + W_l(g_l)}{2} \right), \tag{6.2}$$

$$W_h(g_l) = u(y - g_l + q_h g_l) + \delta \left(\frac{W_h(g_l) + W_l(g_l)}{2} \right), \tag{6.3}$$

$$W_l(g_l) = u(y - g_l + q_{lh} g_h) + \delta \left(\frac{W_h(g_h) + \underline{V}_l}{2} \right), \tag{6.4}$$

where $q_{lh} < q_l$ in $\mathbb{R}^{++}$ is the price of the bond issued in state l in S conditional on repayment only if state h in S occurs in the next period. In other terms, we conjecture that, when the initial liability is g_l in $\mathbb{R}^+$ in state l in S, the sovereign issues debt up to threshold g_h in $\mathbb{R}^+$, defaulting if state l in S occurs in the following period. At the same time, when the initial liability is g_h in $\mathbb{R}^+$ in state h in S, the sovereign issues debt only up to threshold g_l in $\mathbb{R}^+$, so that default will not occur in the following period. To verify that this plan is indeed profitable under low interest rates, we study the system of equations (6.2)-(6.4).

Default thresholds are consistent only if $W_h(g_h) = \underline{V}_h$ and $W_l(g_l) = \underline{V}_l$. Furthermore, equation (6.4) is certainly satisfied if the ratio of default thresholds is given by $g_l = q_{lh} g_h$, so that the sovereign can roll over its debt in state l in S. Therefore, subject to the fixed ratio of default thresholds, restrictions (6.2)-(6.4) amount to satisfy a single condition (equation (6.2)),

$$\Phi_h(g_l) = u\left(y + \left(q_h - \frac{1}{q_{lh}} \right) g_l \right) + \delta \left(\frac{W_h(g_l) + \underline{V}_l}{2} \right) = \underline{V}_h,$$

where $W_h(g_l)$ in $\mathbb{R}^+$ is fully determined by equation (6.3) subject to $W_l(g_l) = \underline{V}_l$. Direct computation reveals that $\Phi'_h(0) > 0$ if and only if

$$\left(\frac{1}{q_{lh}} - q_h \right) < \delta \left(1 - \frac{\delta}{2} \right)^{-1} (q_h - 1). \tag{6.5}$$

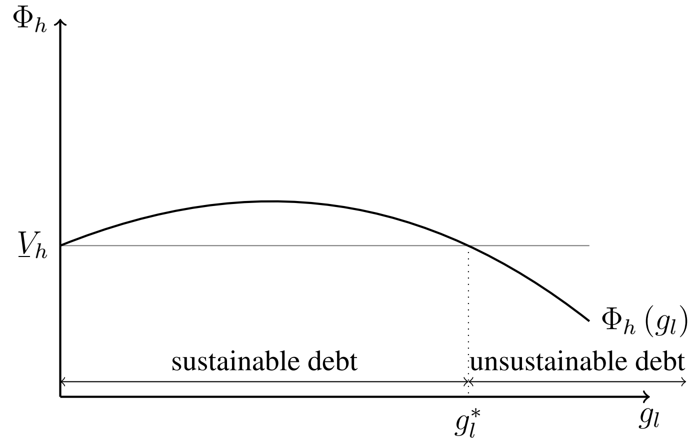


Figure 4. Welfare gains
(subject to $g_l = q_{lh} g_h$ and default in state l in S)

The left-hand side is the cost of serving the debt in state h in S avoiding default in the next period. The right-hand side is the benefit of rolling over the debt at a discount as long as state h in S occurs repeatedly. Thus, inequality (6.5) imposes that, at autarky, the marginal benefit of the given debt issuances exceeds its marginal cost, as illustrated in Figure 4. Instances of bond prices robustly ensuring such welfare gains under the no Ponzi game condition (Assumption 2) are plotted in Figure 5.[10]

6.2. **Secured debt.** We now assume that the sovereign is able to secure a small amount of debt d_* in $\mathbb{R}^{++}$ by means of legal arrangements before as well as after default. By impatience, the value of default is $\underline{V}_s(d_*) = W_s(d_*)$, where

$$W_s(d) = u(y + (q_s - 1)d) + \delta\left(\frac{W_h(d) + W_l(d)}{2}\right).$$

Indeed, after default the borrower will be repaying the amount of secured debt and reissuing debt subject to collateral restrictions indefinitely. In a pessimistic equilibrium only secured debt will be traded and, under high interest rates, this would be the *only* equilibrium. Under

[10] It should be noticed that the left-hand side of condition (6.5) cannot be negative under our assumption of no Ponzi games: otherwise the government would be able to roll over any large amount of debt without repayments, defaulting as soon as state l in S occurs. To enforce Assumption 2, we consider a stationary stochastic discount factor π in $\mathbb{R}^{S\times S}$, consistent with the given bond prices, and impose that its Perron-Frobenius dominant root be less than unity, thus ensuring discounting.

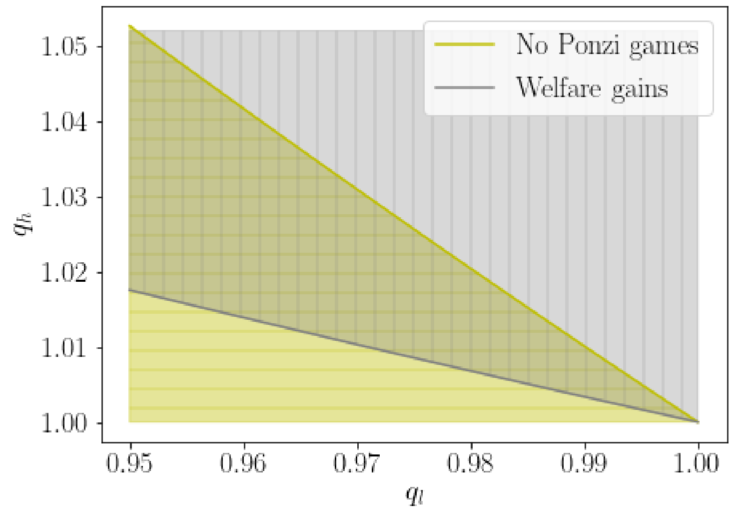


Figure 5. Bond prices
(subject to $\delta = 1$, $\pi_{hh} = 0$, $\pi_{ll} = 0$)

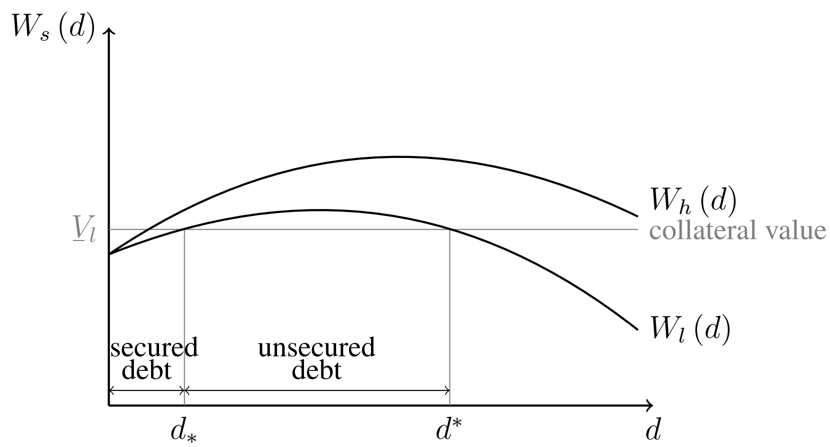


Figure 6. Secured borrowing

low interest rates, instead, a larger amount of unsecured debt can be sustained by optimistic beliefs, as we explain next.

Consider the value $W_s(d)$ in $\mathbb{R}$ of a plan consisting in maintaining a constant amount of debt d in $\mathbb{R}^+$ over time. In order to ensure sustainability of unsecured debt, we assume that

$W_s'(0) > 0$, which amounts to fulfill the restriction

$$\delta(q_h - 1) > 2(1-\delta)(1-q_l) + \delta(1-q_l)\,. \tag{6.6}$$

In other terms, the marginal gain from rolling over debt at a negative interest rate exceeds the marginal cost of serving the debt at a positive interest rate. As illustrated by Figure 6, a constant-debt plan yields a value exceeding reservation utilities and, hence, the planning program returns a higher value of debt repayment. By Propositions 4.1-4.2, this reveals the existence of an optimistic equilibrium with additional unsecured debt.

6.3. **Reentry after default.** We finally show that our theory extends whenever the sovereign is able to reenter the market after defaulting on the outstanding debt. A complication arises because the value of default itself is to be determined endogenously at equilibrium, and we need to amend our general framework so as to encompass this circumstance. In fact, accounting for reentry, the reservation utility is given by

$$\underline{V}_t(\mu_t, g) = u(y_t - \epsilon_t) + \delta\mu_t \mathbb{E}_t V_{t+1}(0|g) + \delta(1-\mu_t)\mathbb{E}_t \underline{V}_{t+1}(\mu_{t+1}, g)$$

where μ in $(0,1) \subset \mathbb{R}^+$ is the probability of reentering the market after default. We assume that this probability is uniformly distributed over a small closed interval around a mean $\bar{\mu}$ in $(0,1) \subset \mathbb{R}^+$, and that this probability is known at the beginning of the period before default choice. Furthermore, as it is common in the literature, we postulate that the sovereign incurs a contingent output drop ϵ in $[0, y] \subset L^+$ upon default. Adapting the approach developed in Aguiar and Amador [3, Section 6], the existence of an equilibrium can be established by means of a *conditional* planning program, along with a fixed point theorem.[11]

Fixing default thresholds g in $[0, N] \subset L^+$ artificially, we obtain well-defined reservation values $\underline{V}(g)$ in L and, hence, a suitable planning program conditional on such reservation values. By Proposition 4.1, the conditional planning program determines revised default thresholds (Tg) in L. This adjustment process defines a map $T : [0, N] \to [0, N]$, and a fixed point of this map corresponds to an equilibrium with reentry. Notice, however, that not all equilibria need be rest points of such a revision process.

We now turn back to the study of the example with two Markov states and first consider a situation without output drop. By impatience (condition (6.1)), we immediately verify

[11] The additional small randomness on the reentry probability is needed to avoid a discontinuity with respect to reservation values, due to the maintained assumption that the sovereign will repay the outstanding debt when indifferent. Indeed, a slight change in reservation values might affect default beliefs dramatically, thus inducing an immediate drop in the bond price. The added randomness prevents such a discontinuity.

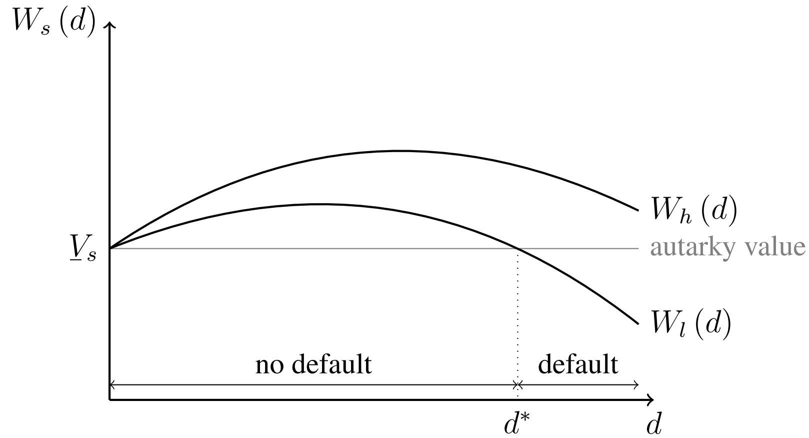


Figure 7. Reentry without output drop

that autarky is a pessimistic equilibrium without debt. To ascertain the existence of another equilibrium with debt, consider again the value $W_s(d)$ in $\mathbb{R}$ of a plan consisting in maintaining a constant amount of debt d in $\mathbb{R}^+$ over time, subject to condition (6.6). As illustrated by Figure 7, additional debt is sustainable given the autarkic reservation value and, as a consequence, the conditional planning program returns higher adjusted reservation values. As at least one equilibrium must be stable for the adjustment process based on the conditional planning program (*i.e.*, it must be a fixed point of $T : [0, N] \rightarrow [0, N]$), this uncovers the existence of another equilibrium with debt.

We now argue that the basic multiplicity with reentry persists when default involves a further cost in terms of output drop. In particular, we constructively prove that an equilibrium with low levels of borrowing remains, while another equilibrium with high levels of debt exists because of our previous arguments based on the conditional planning program. To this end, we preliminarily notice that conditions (6.1) and (6.6) can be satisfied when interest rates are arbitrarily close to zero (*i.e.*, $q_s^0 = 1$). Hence, we confine attention to a neighborhood of zero interest rate and, for a more transparent illustration of the argument, we neglect the randomness on the probability of reentry.

We consider an uncontingent small default threshold d in $\mathbb{R}^{++}$. By impatience, borrowing up to this uncontingent limit without defaulting is the optimal plan. Default cannot occur because it would imply no revenue raised by debt issuance, as repayment is expected either in both states or in no state. Consistently, the value of default is given by

$$\underline{V}_s(d) = u(y) + \delta\bar{\mu}\left(\frac{W_h^0(d) + W_l^0(d)}{2}\right) + \delta(1-\bar{\mu})\left(\frac{\underline{V}_h(d) + \underline{V}_l(d)}{2}\right),$$

where the reentry utility is computed as

$$W_s^0(d) = u(y + q_s d) + \delta\left(\frac{W_h(d) + W_l(d)}{2}\right).$$

By direct inspection, we immediately verify that, in a neighborhood of zero interest rates, $W_s(d) < \underline{V}_s(d)$. Hence, subject to this inequality, we calibrate contingent output drops so as to ensure indifference between repayment and default.

Let ϵ in $[0,y] \times [0,y] \subset \mathbb{R}^+ \times \mathbb{R}^+$ be contingent output drops. The default values vary continuously with these output drops, according to

$$\underline{V}_s(d,\epsilon) = u(y - \epsilon_s) + \delta\bar{\mu}\left(\frac{W_h^0(d) + W_l^0(d)}{2}\right) + \delta(1-\bar{\mu})\left(\frac{\underline{V}_h(d,\epsilon) + \underline{V}_l(d,\epsilon)}{2}\right).$$

We thus consider a continuous map $f : [0,y] \to [0,y]$ such that, letting $\epsilon_h = f(\epsilon_l)$,

$$W_h(d) - \underline{V}_h(d,\epsilon) = W_l(d) - \underline{V}_l(d,\epsilon).$$

The latter restriction can always be satisfied because states occur with equal probabilities and $\lim_{c\to 0} u(c) = -\infty$. By the Intermediate Value Theorem, given the established boundary condition, there exist output drops such that $W_s(d) = \underline{V}_s(d,\epsilon)$. This shows that the construction is indeed an equilibrium with low debt issuances. Furthermore, an alternative equilibrium with large amounts of debt still exists by our previous arguments.

7. Self-fulfilling crises

We argue that, under low implied interest rates, self-fulfilling debt crises do occur. We construct a benchmark economy in which debt roll-over (a Ponzi game) is feasible and use a perturbation method. In particular, we establish that a low-debt equilibrium coexists with a high-debt equilibrium. Furthermore, any belief about the sovereign's solvency varying from the most pessimistic to the most optimistic can be sustained as an equilibrium with rational expectations. Hence, equilibrium is fully indeterminate.

We consider a Markov economy governed by an irreducible transition $P : S \to \Delta(S)$ on a finite state space S. Furthermore, to simplify our arguments, we straightly assume that some low level of debt can be issued by the sovereign under creditors' pessimistic beliefs and argue that more optimistic beliefs are also sustainable. Though we leave the default punishment undefined for tractability, a strict interpretation relies on the sovereign's ability to issue, even after default, a limited amount of debt secured by collateral or other legal arrangements. As a particular instance, we encompass a situation in which no further debt can be credibly issued upon default and, so, a trivial equilibrium without debt always exists: when creditors expect default unconditionally, the sovereign cannot raise any revenue by issuing debt and, hence, is effectively restricted by a no borrowing constraint.[12]

Assumption 7 (Reservation values)**.** For some *given* Markov default thresholds $\underline{g}$ in $L^+(y)$, the reservation value upon default is

$$\underline{V}_t = V_t\left(-\underline{g}_t|\underline{g}\right).$$

We construct a benchmark situation in which the sovereign can roll-over its debt because interest rates are never positive, net of growth, and, so, debt can be refinanced at no burden. Furthermore, as interest rates occasionally exceed growth, a debt roll-over plan even yields extra resources for consumption. Default is never profitable when debt can be rolled over in such conditions, but the value of the sovereign's endowment is necessarily infinite, so violating our fundamental Assumption 2. We then perturb the benchmark pricing kernel and consider any economy in a neighborhood of the debt roll-over regime falling under the domain of our Assumption 2. Though debt roll-over is not anymore feasible in this perturbed economy, an approximated roll-over plan is still available to the sovereign and would not induce default. As a consequence, when creditors' beliefs are optimistic, a larger amount of debt is sustainable. By Assumption 7, however, another equilibrium with pessimist beliefs always exists and sustains low levels of debt.

Using the valuation operator $\Pi : L \to L$, long-term tendencies of the interest rate (net of growth) are captured by Perron-Frobenius eigenvalues. The upper dominant root ρ in $\mathbb{R}^{++}$ satisfies, for some Markov process b in the interior of $L^+(y)$,

$$\rho b_t = \Pi_t\left(b_{t+1}\right).$$

[12]It is worth remarking that the reservation value $\underline{V}_t = V_t(0|0)$ is natural in a game-theoretical approach, as in Dovis [25] and Passadore and Xandri [34]. Indeed, in a sovereign debt game, a subgame perfect equilibrium is sustained only by the threat of reverting to the worst subgame perfect equilibrium of the *same* policy game, without any further punishment in the event of default. An Eaton-Gersovitz equilibrium outcome is thus a subgame perfect equilibrium of an associated sovereign debt game when interdiction from borrowing is the default value.

Similarly, the lower dominant root γ in $\mathbb{R}^{++}$ satisfies, for some Markov process b in the interior of $L^{+}(y)$,

$$\gamma b_t = -\Pi_t(-b_{t+1}).$$

By sublinearity of the valuation operator, $\rho \geq \gamma$. The upper dominant root accounts for the pessimistic growth of the asset-to-income ratio, whereas the lower dominant root for the pessimistic growth of the debt-to-income ratio. The basic elements of this approach are presented in Appendix B. Relevantly, finite valuation (Assumption 2) is ensured only if $\gamma < 1$. It is worth remarking that, in a simple Markov economy with strictly positive transition (a rather restrictive feature in empirical applications), dominant roots are given by

$$\rho = \max_{s \in S} q_s \text{ and } \gamma = \min_{s \in S} q_s,$$

where q_s in $\mathbb{R}^{++}$ is the price of the risk-free bond in state s in S.

We show the existence of another equilibrium in which lenders generously extend credit to the sovereign. The argument unfolds in two steps. First, we characterize debt roll-over in a benchmark economy. Second, we perturb the pricing kernel, so that debt cannot be rolled over, and establish that another equilibrium with borrowing emerges. We begin with the roll-over regime in the benchmark economy.

Assumption 8 (Benchmark economy)**.** In the benchmark economy, $\hat{\rho} > 1$ and $\hat{\gamma} = 1$.

Lemma 7.1 (Debt roll-over)**.** *In the benchmark economy, there exist Markoviav thresholds $\hat{g}$ in L^{+} such that $\hat{g} \geq \underline{g}$ and*

$$\hat{V}_t(-\hat{g}_t|\hat{g}) > \underline{V}_t, \tag{7.1}$$

where $\hat{V}$ in $\mathcal{V}$ is the value function in the benchmark economy.

We next show that, for any perturbation of the pricing kernel in the benchmark economy, condition (7.1) continues being satisfied. In fact, although debt roll-over is not feasible anymore, the plan only requires low repayments in some states and it still improves upon the low-borrowing regime. The pricing kernel is varied, subject to Markov measurability, so as to satisfy, for some sufficiently small $\epsilon > 0$,

$$\left| \frac{\pi_{t+1}}{\pi_t} - \frac{\hat{\pi}_{t+1}}{\hat{\pi}_t} \right| < \epsilon.$$

As a consequence of this variation, both the upper and the lower dominant roots are slightly affected.[13]

Lemma 7.2 (Approximated debt roll-over). *For any slight perturbation of the benchmark economy, thresholds $\hat{g}$ in L^+ still satisfy*

$$V_t\left(-\hat{g}_t|\hat{g}\right) > \underline{V}_t. \tag{7.2}$$

We can now establish the existence of another equilibrium with large borrowing in any perturbed economy falling under the domain of our Assumption 2 and, so, such that $\gamma < 1$. To see that large borrowing is sustainable, notice that condition (7.2) reveals that debt above thresholds $\hat{g}$ in L^+ is sustainable when creditors' beliefs are optimistic. In fact, our argument uses the planning program developed in section 4: though it is not an equilibrium, the optimal plan subject to default thresholds $\hat{g}$ in L^+ is in the planner's default-adjusted budget set. This simple observation immediately shows that the efficient equilibrium involves larger borrowing and, therefore, must be distinct from the pessimistic equilibrium.

Proposition 7.1 (Optimistic equilibrium). *In any perturbation of the benchmark economy satisfying Assumption 2, there exists a distinct equilibrium with efficient default thresholds $\bar{g}$ in L^+, that is, such that*

$$V_t\left(-\bar{g}_t|\bar{g}\right) = \underline{V}_t.$$

We finally discuss the occurrence of slow-moving crises in which creditors anticipate future solvency turbulences and immediately increase the risk-premium on sovereign bonds. Equilibrium is indeed fully indeterminate: any current default threshold between the most optimistic and the most pessimistic equilibrium can be sustained by sufficiently pessimistic beliefs about the sovereign's future solvency. This sort of indeterminacy resembles the logic of hyper-inflationary equilibria in traditional overlapping generations economies. However, due to non-convexity arising from the default option, dynamics might be richer than in a simple overlapping generations economy.

Proposition 7.2 (Indeterminacy). *In any perturbation of the benchmark economy satisfying Assumption 2, there exists an equilibrium with default thresholds g in L^+ for any arbitrary choice of g_0 in $\left(\underline{g}_0, \bar{g}_0\right) \subset L_0^+$.*

[13] A simple perturbation of the pricing kernel would be given by $\pi_t = \gamma^t \hat{\pi}_t$ for some sufficiently large $\gamma < 1$.

It is worth describing our analytical argument, as it is innovative and of potential applicability to other dynamic frameworks exhibiting ordered equilibria. Consider a truncated planning program in which, at some future period n in $\mathbb{T}$, the planner is artificially restricted by an *ad hoc* continuation value

$$V_{n+1}^{\alpha,n}\left(w_{n+1}\right)=\alpha V_{n+1}\left(w_{n+1}|\bar{g}\right)+\left(1-\alpha\right)V_{n+1}\left(w_{n+1}|\underline{g}\right),$$

where α lies in $[0,1]\subset\mathbb{R}^{+}$. Let $V^{\alpha,n}$ in $\mathcal{V}$ be the value of this truncated program. When $\alpha=0$, then the planner cannot credibly issue any debt above the pessimistic thresholds in the continuation and, so, by backward induction, $V_0^{0,n}\left(-g_0\right)<\underline{V}_0$. On the other side, when $\alpha=1$, the additional restrictions are ineffective, as the planner would already optimally default on any debt exceeding those thresholds, and therefore $V_0^{1,n}\left(-g_0\right)>\underline{V}_0$. By the Intermediate Value Theorem, we obtain $V_0^{\alpha(n),n}\left(-g_0\right)=\underline{V}_0$ for some $\alpha\left(n\right)$ in $(0,1)\subset\mathbb{R}^{+}$. This is not in general an equilibrium because the additional restrictions might be effective in the continuation, distorting the planner's incentives to default. However, we can relax the truncation by taking the limit with respect to n in $\mathbb{T}$, and show that the limit is indeed an equilibrium. This thought experiment uncovers full indeterminacy of equilibrium, although it remains silent on the dynamical features of creditors' pessimistic beliefs.

8. LONG-TERM DEBT

We finally enlarge our framework in order to deal with long-term debt. A richer set of maturities allows the sovereign to better insure against uncertainty. It also permits to capture a more interesting propagation mechanism, because expectations of future default are immediately reflected by current long-term bond prices as risk premia. The stance of the recent literature inspired by Eaton and Gersovitz [27] is that, under high implied interest rates, self-fulfilling debt crises occur when the sovereign issues long-term debt, a view that is most prominently advocated by Aguiar and Amador [3].[14] We argue that, as long as short-term debt is still traded, the extent of equilibrium multiplicity with long-term debt is limited, or even absent, under high interest rates. In particular, equilibrium multiplicity

[14] In a recent paper, DeMarzo et al. [24] argue that Aguiar and Amador [3]'s multiplicity is fragile. In particular, in continuous time with linear utility, they establish that equilibrium is unique because gains from trade are entirely dissipated when the government lacks commitment ability and is more impatient than lenders. We are unable to encompass this feature in our analysis. As our simple example illustrates, welfare gains might be substantial even without commitment. Furthermore, DeMarzo et al. [24, Proposition 2]'s claim of no gains from trade holds true for any arbitrarily low reservation value upon default and, hence, even when the absence of limited commitment seems less relevant.

is only driven by the role of long-term debt as an insurance against income risk. Thus, if other short-term securities provide similar insurance, or else no hedging motive exists, the equilibrium is unique under high interest rates even in the presence of long-term debt. This reinforces our claim that low interest rates are an independent and autonomous cause of sovereign debt market instability.

To encompass different maturities in our framework, we introduce an updating rule for portfolios. The enlarged budget constraint becomes

$$Q_t\left(z_t, z_t^- | g\right) + c_t \leq y_t + w_t,$$

where

$$w_{t+1} \leq R_{t,t+1}\left(z_t\right) \text{ and } z_{t+1}^- = \Phi_{t,t+1}\left(z_t\right).$$

The sovereign inherits a legacy portfolio z_t^- in Z_t from the past and adjusts its holdings of all securities, so determining a new portfolio z_t in Z_t. The price $Q_t\left(z_t, z_t^- | g\right)$ in L_t depends on both the legacy portfolio and the portfolio formed after trading in the current period. In fact, the price only reflects the market value of transactions corresponding to portfolio changes $\Delta z_t = z_t - z_t^-$ in Z_t. The (linear) operation $\Phi_{t,t+1} : Z_t \to Z_{t+1}$ revises maturities of all securities in the portfolio after one period. The construction is simple: an n-maturity bond becomes an $(n-1)$-maturity bond in the following period. Details are omitted as they are, though straightforward, notationally intensive.[15]

We assume that some short-term (one-period) securities are still available. When z_t in Z_t consists only of short-term securities, $\Phi_{t,t+1}\left(z_t\right) = 0$ in Z_{t+1}, because all such securities will reach maturity in the following period. Similarly, when the sovereign adjusts its legacy portfolio z_t^- in Z_t only in short-term securities, remaining inactive in long-term securities, the adjustment $\Delta z_t = z_t - z_t^-$ in Z_t yields $\Phi_{t,t+1}\left(\Delta z_t\right) = 0$ in Z_{t+1} in the following period. In fact, when the sovereign is always inactive in long-term securities, the legacy portfolio evolves according to

$$z_{t+1}^- = \Phi_{t,t+1}\left(z_t^-\right).$$

All trades, in such a case, occur in short-term securities.

Default thresholds now depend on legacy debt. In fact, creditors' belief is that default will occur whenever

$$w_t < -g_t\left(z_t^-\right).$$

[15] An even more general approach is taken in Aguiar et al. [7], where the sovereign actually chooses a temporal pattern for future payments in a given abstract feasible set.

The sovereign is otherwise regarded as solvent and can credibly issue debt.[16] Because of long-term maturities, however, thresholds do not fully identify default-adjusted prices of securities. The price of a long-term bond depends on the expectation of default up to maturity of the bond, which is in turn affected by the evolution of the sovereign liabilities. Fortunately, for the analysis in this paper, we do not need to enter into the full specification of default-adjusted prices. As a matter of fact, we only impose a minimal consistency requirement: If $\Phi_{t,t+1}\left(z_t - z_t^-\right) = 0$, then

$$Q_t\left(z_t, z_t^- | g\right) = \frac{1}{\pi_t} \mathbb{E}_t \pi_{t+1} R_{t,t+1}\left(z_t - z_t^-\right) \chi_{t+1}\left(z_t | g\right), \tag{8.1}$$

where the repayment event is

$$\left\{R_{t,t+1}\left(z_t\right) \geq -g_{t+1}\left(\Phi_{t,t+1}\left(z_t\right)\right)\right\} \in \mathcal{F}_{t+1}.$$

This condition says that, when the sovereign only modifies short-term positions, the cost of the portfolio adjustment equals the expected payment on new issuances conditional on no default in the following period. Trades in long-term securities, instead, might induce complicated effects on the market prices, due to the revision of beliefs about future default. Differently form Aguiar et al. [7], we remain silent about the determination of such feedback effects, so that our formulation encompasses that in their paper as a particular case.

As in our previous analysis, equilibrium requires that default thresholds be consistent with the sovereign's incentives to default, that is,

$$V_t\left(z_t^-, -g_t\left(z_t^-\right)\right) = \underline{V}_t.$$

Though we are parsimonious in notation, it should be clear that the value $V_t\left(z_t^-, w_t\right)$ in L_t^* depends on a given default-adjusted pricing rule that is only restricted by the minimal condition (8.1). In fact, the correct pricing of short-term securities (condition (8.1)) is the only discipline we impose on equilibrium, thus leaving room to a potential large multiplicity of beliefs.

To compare equilibria in this economy, we consider an environment in which, although long-term securities are available, the sovereign is constrained to remain inactive in such

[16] Notice that default thresholds need not be positive anymore. When the sovereign has accumulated large amounts of long-term debt in the past, solvency might require a positive level of current liquid resources w_t in L_t^+.

trades and does not modify its legacy portfolio in long-term maturities. Basically, this amounts to impose the additional constraint

$$\Phi_{t,t+1}\left(z_t - z_t^-\right) = 0,$$

so that all changes in the legacy portfolio reach their maturity in the following period. The value of this restricted program is denoted by $V_t^{\text{short}}\left(z_t^-, w_t\right)$ in L_t^*, whereas the value of the unrestricted program is $V_t\left(z_t^-, w_t\right)$ in L_t^*. The discrepancy reflects the advantage of possibly using securities of longer maturities for insurance purposes. We estimate these gains by posing

$$V_t^{\text{short}}\left(z_t^-, \Delta g_t\left(z_t^-\right) - g_t\left(z_t^-\right)\right) = V_t\left(z_t^-, -g_t\left(z_t^-\right)\right). \tag{8.2}$$

In other terms, $\Delta g_t\left(z_t^-\right)$ in L_t^+ is an upfront value compensating the sovereign for not adjusting long-term securities in the future. Finally, we only consider equilibria in which, when positions in long-term securities are not modified over time, default thresholds remain bounded relative to the endowment, that is,

$$\left(g_t\left(z_t^-\right)\right)_{t\in\mathbb{T}} \in L\left(y\right), \text{ where } z_{t+1}^- = \Phi_{t,t+1}\left(z_t^-\right).$$

Assumption 9 (High implied interest rates strengthened)**.** There exists an interior b in $L^+\left(y\right)$ such that, for some ρ in $(0,1) \subset \mathbb{R}^+$,

$$\rho b_t = \Pi_t^{\text{short}}\left(b_{t+1}\right),$$

where $\Pi_t^{\text{short}} : L_{t+1} \to L_t$ is the valuation operator restricted to short-term securities only.

Proposition 8.1 (Default thresholds)**.** *Under high implied interest rates (Assumption 9), (bounded) default thresholds of distinct equilibria with value functions V and $\tilde{V}$ in $\mathcal{V}$ satisfy*

$$g_t\left(z_t^-\right) \leq \tilde{g}_t\left(z_t^-\right) + b_t,$$

where the process b in $L^+\left(y\right)$ in Assumption 9 is scaled so that $\Delta g_t\left(z_t^-\right) \leq (1-\rho)\, b_t$, at every t in $\mathbb{T}$, and the legacy portfolio passively evolves according to

$$z_{t+1}^- = \Phi_{t,t+1}\left(z_t^-\right).$$

Although Proposition 8.1 does not rule out the occurrence of multiple equilibria, it uncovers the fundamental role of the insurance motive. Indeed, $\Delta g_t\left(z_t^-\right)$ in L_t^+ measures the insurance value of long-term securities and bounds the disparity of default thresholds at distinct equilibria. When long-term securities are not needed for insurance, they are redundant and add no further value, so that $\Delta g_t\left(z_t^-\right) = 0$. In this case, Proposition 8.1 shows that multiple equilibria cannot occur. When long-term securities provide insurance, instead, pessimistic beliefs of creditors might be effective at equilibrium. However, at worst, they induce a welfare drop which is still above the most optimistic equilibrium with inactive long-term securities.

Arellano and Ramanarayanan [12] argue that the optimal maturity of debt is determined by balancing the incentive effect of short-term debt with the hedging benefit of long-term debt. The price of long-term debt is more sensitive to default risk because the sovereign cannot commit to future debt issuances. Short-term debt is so a better instrument to provide incentives to repayment, as sharply established by Aguiar et al. [7]. However, exactly because of this higher sensitivity, the value of outstanding long-term debt falls relatively more in bad states than in good states, thus permiting a better hedge against risk. In periods of high default probability, the incentives to repayment provided by short-term debt are most valuable, and thus the optimal portfolio shifts toward short-term debt. Condition (8.2) effectively estimates the hedging benefit in the proximity of default. Arellano and Ramanarayanan [12]'s analysis suggests that the incentive effect of short-term debt is dominant and, hence, that long-term debt is dispensable. As a repercussion of this, the extent of multiplicity due to long-maturity debt seems limited when short-term debt remains available.

9. Conclusion

We have shown that low interest rates expose sovereign debt markets to self-fulfilling crises. At the origin of this vulnerability lies the inability of the sovereign to commit to future debt issuances. When interest rates are high, however, this mechanism cannot operate, because rational beliefs about future solvency always respond favourably when outstanding debt is reduced. Low interest rates, on the contrary, release market sentiments, and fiscal adjustments might be interpreted by creditors as signals of increased likelihood of future insolvency.

References

[1] A. B. Abel, G. N. Mankiw, L. H. Summers, and R. J. Zeckhauser. Assessing dynamic efficiency: Theory and evidence. *Review of Economic Studies*, 56(1):1–19, 1989.

[2] M. Aguiar and M. Amador. Sovereign debt. In G. Gopinath, E. Helpman, and K. Rogoff, editors, *Handbook of International Economics*, volume 4, chapter 11, pages 647–687. Elsevier, 2014.

[3] M. Aguiar and M. Amador. A contraction for sovereign debt models. *Journal of Economic Theory*, 183:842–875, 2019.

[4] M. Aguiar and M. Amador. Self-fulfilling debt dilution: Maturity and multiplicity in debt models. *American Economic Review*, 110(9):2783–2818, 2020.

[5] M. Aguiar and M. Amador. *The Economics of Sovereign Debt and Default*. Princeton University Press, 2021.

[6] M. Aguiar and G. Gopinath. Defaultable debt, interest rates, and the current account. *Journal of International Economics*, 69(1):64–83, 2006.

[7] M. Aguiar, M. Amador, H. Hopenhayn, and I. Werning. Take the short route: Equilibrium default and debt maturity. *Econometrica*, 87(2):423–462, 2019.

[8] M. Aguiar, S. Chatterjee, H. Cole, and Z. Stangebye. Self-fulfilling debt crises, revisited: The art of the desperate deal. Forthcoming, *Journal of Political Economy*, NBER Working Paper No. 23312, December 2020.

[9] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis*. Berlin: Springer, 3rd edition, 2006.

[10] F. Alvarez and U. J. Jermann. Efficiency, equilibrium, and asset pricing with risk of default. *Econometrica*, 68(4):775–797, 2000.

[11] C. Arellano. Default risk and income fluctuations in emerging economies. *American Economic Review*, 98(3):690–712, 2008.

[12] C. Arellano and A. Ramanarayanan. Default and the maturity structure in sovereign bonds. *Journal of Political Economy*, 120(2):187–232, 2012.

[13] A. Auclert and M. Rognlie. Unique equilibrium in the Eaton-Gersovitz model of sovereign debt. *Journal of Monetary Economics*, 84:134–146, 2016.

[14] J. Ayres, N. Gaston, N. Junapa, and T. Pedro. Sovereign defaults: The role of expectations. *Journal of Economic Theory*, 175:803–812, 2018.

[15] O. J. Blanchard. Public debt and low interest rates. *American Economic Review*, 109 (4):1197–1229, 2018.

[16] G. Bloise and Y. Vailakis. On sovereign default with time-varying interest rates. *Review of Economic Dynamics*, 44:211–224, 2022.

[17] G. Bloise, H. Polemarchakis, and Y. Vailakis. Sovereign debt and default incentives with uninsurable risks. *Theoretical Economics*, 12(3):1121–1154, 2017.

[18] G. Bloise, H. Polemarchakis, and Y. Vailakis. Sustainable debt. *Theoretical Economics*, 16(4):1513–1555, 2021.

[19] L. Bocola and A. Dovis. Self-fulfilling debt crises: A quantitative analysis. *American Economic Review*, 109(12):4343–4377, 2019.

[20] J. Bulow and K. Rogoff. Sovereign debt: Is to forgive to forget? *American Economic Review*, 79(1):43–50, 1989.

[21] G. A. Calvo. Servicing the public debt: The role of expectations. *American Economic Review*, 78(4):647–661, 1988.

[22] S. Chatterjee and B. Eyigungor. Maturity, indebtedness, and default risk. *American Economic Review*, 102(6):2674–2699, 2012.

[23] H. L. Cole and T. J. Kehoe. Self-fulfilling debt crises. *Review of Economic Studies*, 67:91–116, 2000.

[24] P. M. DeMarzo, Z. He, and F. Tourre. Sovereign debt ratchets and welfare destruction. NBER Working Paper 28599, March 2021.

[25] A. Dovis. Efficient sovereign default. *Review of Economic Studies*, 86:282–312, 2019.

[26] W. Du and J. Schreger. Local currency sovereign risk. *Journal of Finance*, LXXI(3): 1027–1070, 2016.

[27] J. Eaton and M. Gersovitz. Debt with potential repudiation: Theoretical and empirical analysis. *Review of Economic Studies*, 48(2):289–309, 1981.

[28] J. C. Hatchondo and L. Martinez. Long-duration bonds and sovereign defaults. *Journal of International Economics*, 79(1):117–125, 2009.

[29] J. C. Hatchondo, L. Martinez, and C. Sosa-Padilla. Debt dilution and sovereign default risk. *Journal of Political Economy*, 124(5):1383–1422, 2016.

[30] C. Hellwig and G. Lorenzoni. Bubbles and self-enforcing debt. *Econometrica*, 77(4): 1137–1164, 2009.

[31] O. Jorda, K. Knoll, D. Kuvshinov, M. Schularick, and A. M. Taylor. The rate of return on everything: 1870-2015. *Quarterly Journal of Economics*, 134(3):1225–1298, 2019.

[32] G. Lorenzoni and I. Werning. Slow moving debt crises. *American Economic Review*, 109(9):3229–3263, 2019.

[33] P. Ottonello and D. J. Perez. The currency composition of sovereign debt. *American Economic Journal: Macroeconomics*, 11(3):174–208, 2019.
[34] J. Passadore and J. P. Xandri. Robust predictions in dynamic policy games. Working paper, November 2021.
[35] Z. R. Stangebye. Belief shocks and long-maturity sovereign debt. Forthcoming, *Journal of International Economics*, https://doi.org/10.1016/j.jinteco.2020.103381.
[36] C. Trebesch and M. Zabel. The output costs of hard and soft sovereign default. *European Economic Review*, 92:416–432, 2017.

APPENDIX A. PROOFS

We collect here some basic properties of the sovereign's value function for given default thresholds. In particular, we establish that the program is well-defined and admits a unique solution. We also show that, at equilibrium, default thresholds must satisfy a minimal consistency requirement (A.2). The left hand-side inequality ensures that the budget set is non-empty (*i.e.*, consumption is positive) at the default threshold, because otherwise the sovereign could not secure the reservation value. The right hand-side inequality simply overestimates the revenue raised by the sovereign by issuing debt, given that no revenue will obtain at a future contingency when debt exceeds the default threshold.

Proposition A.1 (Continuity)**.** *Given default thresholds g in L^+, the value function V in $\mathcal{V}$ of the sovereign is uniquely identified and, at every t in $\mathbb{T}$, V_t is increasing and upper semicontinuous (on the extended reals). Furthermore, continuity occurs whenever*

$$V_t\left(w_t|g\right) > u\left(0\right) + \delta\mathbb{E}_t\bar{V}_{t+1}, \tag{A.1}$$

where $\bar{V}$ in $\mathcal{V}$ is the upper bound on utility. Finally, at equilibrium, default thresholds satisfy

$$g_t \leq y_t - \inf_{z_t\in Z_t} Q_t\left(z_t|g\right) \leq y_t + \frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}g_{t+1}. \tag{A.2}$$

Proof of Proposition A.1. At equilibrium, default thresholds satisfy condition (A.2) because otherwise the budget set would be empty, so implying a strict incentive to default. We develop the rest of the proof only in its unconventional parts.

Consider the Bellman operator $T : \mathcal{V} \to \mathcal{V}$ defined as

$$\left(TV\right)_t\left(w_t\right) = \sup u\left(c_t\right) + \delta\mathbb{E}_t\max\left\{V_{t+1}\left(w_{t+1}\right), \underline{V}_{t+1}\right\}$$

subject to

$$Q_t(z_t|g) + c_t \leq y_t + w_t,$$

with

$$w_{t+1} \leq R_{t,t+1}(z_t).$$

As usual, the convention is that the value is negative infinity when the feasible set is empty. To establish uniqueness, consider the related operator $\hat{T} : \hat{\mathcal{V}} \to \hat{\mathcal{V}}$ given by

$$\hat{T}\left(\hat{V}\right) = \max\left\{T\left(\hat{V}\right), \underline{V}\right\},$$

where $\hat{\mathcal{V}}$ is the space of all bounded elements of $\mathcal{V}$, endowed with the supremum norm given by

$$\left\|\hat{V}\right\|_\infty = \sup_{t\in\mathbb{T}} \sup_{w_t\in L_t} \sup_{\omega\in\Omega} \left|\hat{V}_t(\omega)(w_t)\right|.$$

This operator satisfies Blackwell's Conditions and, so, is a contraction. As a consequence, a unique fixed point exists $\hat{V}^*$ in $\hat{\mathcal{V}}$ exists. It can be verified that $V^* = T\left(\hat{V}^*\right)$ in $\mathcal{V}$ is a fixed point of the original operator. In addition, if V^* in $\mathcal{V}$ is a fixed point of the original operator, then $\hat{V}^* = \max\{V^*, \underline{V}\}$ in $\hat{\mathcal{V}}$ is a fixed point of the transformed operator, because

$$\hat{T}(\max\{V^*, \underline{V}\}) = \max\{T(\max\{V^*, \underline{V}\}), \underline{V}\} = \max\{T(V^*), \underline{V}\} = \max\{V^*, \underline{V}\}.$$

This delivers existence and uniqueness of the value function for given default thresholds g in L^+.

The value function is certainly increasing. To verify upper semicontinuity, notice that operator $\hat{T} : \hat{\mathcal{V}} \to \hat{\mathcal{V}}$ maps upper semicontinuous functions into upper semicontinuous functions, as the feasible set is given by an upper hemicontinuous correspondence. So, $\hat{V}^*$ is upper semicontinuous and so is $V^* = T\left(\hat{V}^*\right)$. To verify continuity, notice that, under condition (A.1), the optimal plan necessarily involves a strictly positive consumption, for otherwise the value would not be attained. Given this optimal plan, for any sufficiently small ϵ in $\mathbb{R}$, we obtain

$$V_t(w_t + \epsilon) \geq V_t(w_t) + (u(c_t + \epsilon) - u(c_t)).$$

This is true because the original plan remains budget feasible when consumption is adjusted so as to compensate for the change in initial wealth. Hence,

$$\begin{aligned}\liminf_{\epsilon\to 0} V_t\left(w_t+\epsilon\right) &\geq V_t\left(w_t\right)+\liminf_{\epsilon\to 0}\left(u\left(c_t+\epsilon\right)-u\left(c_t\right)\right)\\ &= V_t\left(w_t\right)\\ &\geq \limsup_{\epsilon\to 0} V_t\left(w_t+\epsilon\right),\end{aligned}$$

where upper semicontinuity is used in the last inequality. This establishes continuity on the restricted domain. □

Let $C_0\left(w_0\right)$ be the set of default-adjusted budget-feasible consumption plans given initial wealth w_0 in L_0. The value function of the planner is

$$V_0^*\left(w_0\right)=\sup_{c\in C_0\left(w_0\right)} U_0^*\left(c\right).$$

As the economy can be restarted at any future contingency, the value function is also well-defined in every other period, conditional on available information, that is,

$$V_t^*\left(w_t\right)=\sup_{c\in C_t\left(w_t\right)} U_t^*\left(c\right).$$

We preliminary show that the planning program identifies exactly default thresholds.

Lemma A.1 (Default thresholds)**.** *A process of default thresholds $g^*\leq N$ in L^+ exists such that*

$$V_t^*\left(-g_t^*\right)=\underline{V}_t.$$

Proof of Lemma A.1. Notice that $C_0\left(w_0\right)$ is compact in the product topology.[17] Indeed, as long as the budget set is non-empty, conditional on no default (that is, $\chi_1\left(c\right)\cdot\cdots\cdot\chi_t\left(c\right)=1$), consumption is limited by the upper bound on debt, as (4.2)-(4.3) yield

$$c_t\leq y_t+w_t^+ +\frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}N_{t+1}.$$

[17] Remember that, in defining the budget set $C_0\left(w_0\right)$, we assume that $c_t=0$ after default, that is, when $\chi_1\left(c\right)\cdot\cdots\cdot\chi_t\left(c\right)=0$. A minor point concerns the existence of redundant securities, and so portfolios might not be bounded. This, however, bears no relevant implication for feasible consumption because redundant securities are priced by the default-free pricing kernel.

Furthermore, when default does not occur (that is, $\chi^t(c) = 1$), the upper bound on debt is given by

$$w_t^- \leq y_t + \frac{1}{\pi_t}\mathbb{E}_t \pi_{t+1} N_{t+1} = N_t.$$

Therefore, using the budget constraint and the pricing rule,

$$y_t + w_t^+ \geq Q_t(z_t|c) \geq \frac{1}{\pi_t}\mathbb{E}_t \pi_{t+1} w_{t+1}^+ - \frac{1}{\pi_t}\mathbb{E}_t \pi_{t+1} w_{t+1}^- \wedge N_{t+1},$$

which imposes an upper bound on the evolution of claims. The set $C_0(w_0)$ is also closed as the pricing is lower semicontinuous due to continuity of utility. Thus, it exists a lower bound $\underline{w}_0$ in L_0 such that $C_0(w_0)$ is a non-empty compact set for every $w_0 \geq \underline{w}_0$.

As the optimal plan under no borrowing is budget feasible, and induces no default by Assumption 4, $\underline{w}_0 \leq 0$. By the Maximum Theorem, the value of the program is upper semicontinuous on the restricted domain $\{w_0 \in L_0 : w_0 \geq \underline{w}_0\}$, as the budget set is upper hemicontinuous in the initial wealth. Furthermore, condition (3.1) reveals that no consumption is affordable when $w_0 < -N_0$ and, therefore, $\underline{w}_0 \geq -N_0$. We now show that default thresholds exist by an adaptation of the Intermediate Value Theorem.

By upper semicontinuity, $\{w_0 \in L_0 : V_0^*(w_0) \geq \underline{V}_0\}$ is closed (and non-empty by Assumption 4). Let $g_0^* \leq N_0$ in L_0^+ be given by

$$-\min\{w_0 \in L_0 : V_0^*(w_0) \geq \underline{V}_0\}.$$

Observe that the optimal plan c in $C_0(-g_0^*)$ necessarily involves $c_0 > 0$, because otherwise

$$\underline{V}_0 \leq V_0^*(-g_0^*) \leq u(0) + \delta \mathbb{E}_t \bar{V}_{t+1},$$

so violating Assumption 4. As a consequence, for every sufficiently small ϵ in $\mathbb{R}$,

$$V_0^*(-g_0^* + \epsilon) \geq V_0^*(-g_0^*) + (u(c_0 + \epsilon) - u(c_0)).$$

This implies that $V_0^*(-g_0^*) = \underline{V}_0$, for otherwise a slight increase in initial debt would not trigger default. This suffices to prove the claim at the initial contingency as well as at any other future contingency. □

Proof of Proposition 4.1. Given default thresholds g^* in L^+ generated by the planning program (Lemma A.1), consider operator $T : \mathcal{V} \to \mathcal{V}$ defined as

$$(TV)_t(w_t) = \sup u(c_t) + \delta \mathbb{E}_t \max\{V_{t+1}(w_{t+1}), \underline{V}_{t+1}\}$$

subject to

$$Q_t(z_t|g^*) + c_t \leq y_t + w_t \tag{A.3}$$

and

$$w_{t+1} \leq R_{t,t+1}(z_t). \tag{A.4}$$

We show that the value function of the planner V^* in $\mathcal{V}$ is a fixed point of this operator and, so, an equilibrium. This requires a minimal adaptation of canonical arguments.

We first show that $(TV^*)_t(w_t) \geq V_t^*(w_t)$, which is only meaningful when $V_t^*(w_t) > -\infty$. To this purpose, observe that there exists a budget feasible consumption plan c in $C_t(w_t)$ such that

$$V_t^*(w_t) = u(c_t) + \delta \mathbb{E}_t \max\left\{U_{t+1}^*(c), \underline{V}_{t+1}\right\}.$$

This plan satisfies constraint (4.2). However, notice that

$$\begin{aligned} \chi_{t+1}(c) = 1 \quad &\text{only if} \quad U_{t+1}^*(c) \geq \underline{V}_{t+1} \\ &\text{only if} \quad w_{t+1} \geq -g_{t+1}^* \\ &\text{only if} \quad \chi_{t+1}(z_t|g^*) = 1. \end{aligned}$$

Therefore, constraint (A.3) is also satisfied because $\chi_{t+1}(c) = 1$ when $R_{t,t+1}(z_t) \geq 0$. This shows that the plan is also feasible in the recursive program and, thus, $(TV^*)_t(w_t) \geq V_t^*(w_t)$.

We now verify that the opposite inequality $(TV^*)_t(w_t) \leq V_t^*(w_t)$ holds true as well. Just to simplify notation, assume that the inequality is violated in the initial period, that is, $(TV^*)_0(w_0) > V_0^*(w_0)$. The previous argument ensures that, at the optimal policy in the recursive program,

$$(TV^*)_t(w_t) \leq u(c_t) + \delta \mathbb{E}_t \max\left\{(TV^*)_{t+1}(w_{t+1}), \underline{V}_{t+1}\right\}. \tag{*}$$

Thus, the consumption plan c in L^+ generated by the optimal policy in the recursive program satisfies

$$V_0^*(w_0) < (TV^*)_0(w_0) \leq U_0^*(c).$$

Furthermore, at any other non-initial period, $(TV^*)_t(w_t) \leq U_t^*(c)$. These conclusions follow from (*) along with the fact that the default-adjusted utility of the planner is the fixed point of the traditional contraction mapping. In particular, the latter allows us to establish that

$$\begin{aligned}\chi_{t+1}\left(z_t|g^*\right)=1 \quad &\text{only if} \quad R_{t,t+1}\left(z_t\right)\geq -g^*_{t+1}\\ &\text{only if} \quad w_{t+1}\geq -g^*_{t+1}\\ &\text{only if} \quad V^*_{t+1}\left(w_{t+1}\right)\geq \underline{V}_{t+1}\\ &\text{only if} \quad \left(TV^*\right)_{t+1}\left(w_{t+1}\right)\geq \underline{V}_{t+1}\\ &\text{only if} \quad U^*_{t+1}\left(c\right)\geq \underline{V}_{t+1}\\ &\text{only if} \quad \chi_{t+1}\left(c\right)=1.\end{aligned}$$

Thus, the consumption plan is also in the default-adjusted budget set of the planner, a contradiction. □

Proof of Proposition 4.2. As the argument can be reproduced beginning from any contingency, we show that $V_0\left(w_0\right)\leq V^*_0\left(w_0\right)$. Assuming that $V_0\left(w_0\right)$ is finite, an optimal plan exists and satisfies, before default,

$$V_t\left(w_t\right)=u\left(c_t\right)+\delta\mathbb{E}_t\max\left\{V_{t+1}\left(w_{t+1}\right),\underline{V}_{t+1}\right\}.$$

The claim is true since the planner can implement the same consumption plan, because it defaults only at contingencies when the government does, *i.e.*,

$$\begin{aligned}\chi_{t+1}\left(z_t|g\right)=1 \quad &\text{only if} \quad R_{t,t+1}\left(z_t\right)\geq -g_{t+1}\\ &\text{only if} \quad w_{t+1}\geq -g_{t+1}\\ &\text{only if} \quad V_{t+1}\left(w_{t+1}\right)\geq \underline{V}_{t+1}\\ &\text{only if} \quad U_{t+1}\left(c\right)\geq \underline{V}_{t+1}\\ &\text{only if} \quad \chi_{t+1}\left(c\right)=1.\end{aligned}$$

This proves the claim. □

Proof of Proposition 5.1. Suppose that $\tilde{g}\leq N$ in L^+ are alternative equilibrium default thresholds. At no loss of generality, process b in the interior of $L^+\left(N\right)$ can be scaled up or down so as to satisfy $g\leq\tilde{g}+b$ and $g_0>\tilde{g}_0+\rho b_0$. The first condition holds true, possibly reversing the roles of g and $\tilde{g}$ in $L^+\left(N\right)$, because b lies in the interior of $L^+\left(N\right)$. When the second condition fails at all contingencies, process b in $L^+\left(N\right)$ can be further scaled down, a contradiction. It is only to simplify notation that we assume that the second condition is satisfied in the initial period $t=0$.

Notice that, by Assumption 5, there exists a portfolio plan Δz in Z such that, at every t in $\mathbb{T}$,

$$Q_t(\Delta z_t) \leq \rho b_t < b_t \text{ and } b_{t+1} \leq R_{t,t+1}(\Delta z_t).$$

This is true because, under our maintained assumptions, a cost-minimizing portfolio solving (5.1) certainly exists at all contingencies. We use this portfolio strategy to replicate the optimal plan in the benchmark equilibrium under the conditions of the alternative equilibrium. As replication permits higher utility, this contradicts self-enforcing condition (3.2).

Consider the optimal plan in the benchmark equilibrium beginning with initial debt g_0 in L_0^+. Such a plan exists and satisfies, at every t in $\mathbb{T}$ before default,

$$V_t(w_t) = u(c_t) + \delta \mathbb{E}_t \max\left\{V_{t+1}(R_{t,t+1}(z_t)), \underline{V}_{t+1}\right\}$$

subject to budget constraint

$$Q_t(z_t|g) + c_t = y_t + w_t,$$

with

$$w_{t+1} = R_{t,t+1}(z_t).$$

Adding the replication portfolio, we obtain

$$Q_t(z_t|g) + Q_t(\Delta z_t) + c_t \leq y_t + (w_t + b_t) - (1-\rho) b_t.$$

Notice that

$$\begin{aligned} V_{t+1}(R_{t,t+1}(z_t)) \geq \underline{V}_{t+1} \quad & \text{only if} \quad R_{t,t+1}(z_t) \geq -g_{t+1} \\ & \text{only if} \quad R_{t,t+1}(z_t + \Delta z_t) \geq b_{t+1} - g_{t+1} \\ & \text{only if} \quad R_{t,t+1}(z_t + \Delta z_t) \geq -\tilde{g}_{t+1} \\ & \text{only if} \quad \tilde{V}_{t+1}(R_{t,t+1}(z_t + \Delta z_t)) \geq \underline{V}_{t+1}, \end{aligned}$$

where we use the fact that value functions are weakly increasing. This condition establishes that, when default does not occur in the benchmark equilibrium, neither does under the alternative conditions when the plan is translated. Also observe that

$$R^+_{t,t+1}(z_t + \Delta z_t) \leq R^+_{t,t+1}(z_t) + R_{t,t+1}(\Delta z_t)$$

and

$$-R^-_{t,t+1}(z_t + \Delta z_t) \leq -R^-_{t,t+1}(z_t).$$

This implies that

$$\begin{aligned}
Q_t\left(z_t+\Delta z_t|\tilde{g}\right) &= \frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}R^+_{t,t+1}\left(z_t+\Delta z_t\right) \\
&\quad -\frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}R^-_{t,t+1}\left(z_t+\Delta z_t\right)\chi_{t+1}\left(z_t+\Delta z_t|\tilde{g}\right) \\
&\leq \frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}R^+_{t,t+1}\left(z_t+\Delta z_t\right)-\frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}R^-_{t,t+1}\left(z_t+\Delta z_t\right)\chi_{t+1}\left(z_t|g\right) \\
&\leq \frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}R^+_{t,t+1}\left(z_t\right)+\frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}R_{t,t+1}\left(\Delta z_t\right) \\
&\quad -\frac{1}{\pi_t}\mathbb{E}_t\pi_{t+1}R^-_{t,t+1}\left(z_t\right)\chi_{t+1}\left(z_t|g\right) \\
&= Q_t\left(z_t|g\right)+Q_t\left(\Delta z_t\right).
\end{aligned}$$

Thus, we obtain, at every t in $\mathbb{T}$ before default occurs in the benchmark equilibrium,

$$\tilde{V}_t\left(w_t+b_t\right)\geq u\left(c_t\right)+\delta\max\left\{\tilde{V}_{t+1}\left(R_{t,t+1}\left(z_t+\Delta z_t\right)\right),\underline{V}_{t+1}\right\}$$

subject to budget constraint

$$Q_t\left(z_t+\Delta z_t|\tilde{g}\right)+c_t\leq y_t+\left(w_t+b_t\right)-\left(1-\rho\right)b_t.$$

This establishes that $\tilde{V}_0\left(-g_0+\rho b_0\right)\geq V_0\left(-g_0\right)$ and, therefore,

$$\underline{V}_0\geq\tilde{V}_0\left(-\tilde{g}_0\right)>\tilde{V}_0\left(-g_0+\rho b_0\right)\geq V_0\left(-g_0\right)\geq\underline{V}_0,$$

so revealing a contradiction. □

Proof of Proposition 5.2. Consider the thresholds $g\leq N$ in L^+ associated with the efficient equilibrium, and suppose that $\tilde{g}\leq g$ in L^+ are alternative equilibrium default thresholds. At no loss of generality, there exists λ in $[0,1]\subseteq\mathbb{R}^+$ such that $\lambda g\leq\tilde{g}$ and $\lambda g_0\geq\tilde{g}_0-(1-\lambda)\,\epsilon N_0$. When the second condition fails at all contingencies, thresholds g in $L^+\left(N\right)$ can be further scaled down, a contradiction. It is only to simplify notation that we assume that the second condition is satisfied in the initial period $t=0$.

Consider the optimal plan in the efficient equilibrium beginning with initial debt g_0 in L^+_0. Such a plan exists and satisfies, at every t in $\mathbb{T}$ before default,

$$V_t\left(w_t\right)=u\left(c_t\right)+\delta\mathbb{E}_t\max\left\{V_{t+1}\left(R_{t,t+1}\left(z_t\right)\right),\underline{V}_{t+1}\right\}$$

subject to budget constraint

$$Q_t(z_t|g) + c_t = y_t + w_t,$$

with

$$w_{t+1} = R_{t,t+1}(z_t).$$

Contracting the portfolio by λ in $[0,1] \subset \mathbb{R}^+$, and noticing that $\lambda g \leq \tilde{g}$, we obtain

$$Q_t(\lambda z_t|\tilde{g}) + \lambda c_t + (1-\lambda) y_t \leq y_t + \lambda w_t.$$

This in turn implies

$$\tilde{V}_t(\lambda w_t - (1-\lambda)\epsilon N_t) \geq \lambda u(c_t) + (1-\lambda) u(y_t - \epsilon N_t) + \delta \max\left\{\tilde{V}_{t+1}(\lambda w_{t+1}), \underline{V}_{t+1}\right\}.$$

Developing this inequality forward, and exploiting Assumption 6, we conclude that

$$\tilde{V}_0(-\lambda g_0 - (1-\lambda)\epsilon N_0) > \lambda V_0(-g_0) + (1-\lambda)\underline{V}_0.$$

Observing that $\tilde{V}_0(-\tilde{g}_0) \geq \tilde{V}_0(-\lambda g_0 - (1-\lambda)\epsilon N_0)$, we obtain a contradiction. □

Proof of Lemma 7.1. Let $\hat{b}$ in L^+ be the lower dominant eigenprocess in the benchmark economy (see Claim B.2). As a consequence, there exists a Markovian trading plan Δz in Z such that

$$\hat{Q}_t(-\Delta z_t) \leq -\hat{b}_t \text{ and } -\hat{b}_{t+1} \leq R_{t,t+1}(-\Delta z_t).$$

Moreover, since the eigenprocess is not in the market span (because $\hat{\gamma} < \hat{\rho}$), the last inequality is strict in at least one state s in S. Consider the Markovian default thresholds $\hat{g} = \underline{g} + \hat{b}$ in L^+. At no loss of generality, we shall then show that inequality (7.1) holds true at $t = 0$.

Consider the optimal plan under low borrowing from initial period $t = 0$ when the sovereign has initial debt $\underline{g}_0$ in L_0^+. This satisfies, before default, the recursive condition

$$\hat{V}_t(w_t|\underline{g}) = u(c_t) + \delta \mathbb{E}_t \max\left\{\hat{V}_{t+1}(w_{t+1}|\underline{g}), \underline{V}_{t+1}\right\}$$

subject to

$$\hat{Q}_t(z_t|\underline{g}) + c_t = y_t + w_t,$$

with

$$w_{t+1} = R_{t,t+1}(z_t),$$

where the value of default is given by Assumption 7. Adding the debt roll-over plan, we obtain that

$$R_{t,t+1}(z_t) \geq -\underline{g}_{t+1} \text{ only if } R_{t,t+1}(z_t - \Delta z_t) \geq -\hat{g}_{t+1}.$$

This in turn yields

$$\hat{Q}_t(z_t - \Delta z_t|\hat{g}) + c_t \leq y_t + \left(w_t - \hat{b}_t\right).$$

Therefore, the modified plan is also feasible under default thresholds $\hat{g}$ in L^+, so delivering

$$\hat{V}_t\left(w_t - \hat{b}_t|\hat{g}\right) \geq u(c_t) + \delta\mathbb{E}_t \max\left\{\hat{V}_{t+1}\left(w_{t+1} - \hat{b}_{t+1}|\hat{g}\right), \underline{V}_{t+1}\right\}.$$

Furthermore, at least one inequality is strict by the previous observation. This shows that

$$\hat{V}_0(-\hat{g}_0|\hat{g}) > \hat{V}_0\left(\underline{g}_0|\underline{g}\right).$$

As the argument can be replicated at any other non-initial contingency, this proves the claim. □

Proof of Lemma 7.2. As fundamentals, prices and thresholds are all measurable with respect to the finite state space S, it is sufficient to verify that inequality (7.2) is preserved at the initial period $t = 0$. To this purpose, consider the optimal plan in the benchmark economy when the sovereign begins with maximum liability $\hat{g}_0$ in L_0^+ subject to default thresholds $\hat{g}$ in L^+. Also consider the following change in consumption in the perturbed economy over a sufficiently large finite horizon $\mathbb{T}^n = \{0, 1, \ldots, n\}$:

$$\Delta c_t = \hat{Q}_t(z_t|\hat{g}) - Q_t(z_t|\hat{g}).$$

This basically compensates the change in asset prices by a balanced adjustment in consumption, so that the portfolio plan remains unvaried. Because of discounting, consumption out of the large finite horizon has little impact on the initial utility value and can therefore be ignored. As prices are only slightly perturbed, for a sufficiently small $\epsilon > 0$, $|\Delta c_t| < \epsilon$. So, we only have to verify that, when $\epsilon > 0$ is sufficiently small, the adjusted consumption plan over the given large finite horizon remains positive. This is proved next.

Notice that, as the plan is optimal in the benchmark economy, it satisfies, before default, the recursive condition

$$\hat{V}_t(w_t|\hat{g}) = u(c_t) + \delta\mathbb{E}_t \max\left\{\hat{V}_{t+1}(w_{t+1}|\hat{g}), \underline{V}_{t+1}\right\}.$$

As default has not occurred yet, this implies

$$u(0) + \delta \mathbb{E}_t \bar{V}_{t+1} < \underline{V}_t \leq u(c_t) + \delta \mathbb{E}_t \max \left\{ \hat{V}_{t+1}(w_{t+1}|\hat{g}), \underline{V}_{t+1} \right\} \leq u(c_t) + \delta \mathbb{E}_t \bar{V}_{t+1},$$

where $\bar{V}$ in L is the maximum utility when consumption is unrestricted as in Assumption 4. As both $\underline{V}$ and $\bar{V}$ in L are S-measurable processes, and $u : \mathbb{R}^+ \to \mathbb{R}$ is increasing, this establishes the existence of some $\epsilon > 0$ such that $c_t > \epsilon$ uniformly, thus completing the proof. □

Proof of Proposition 7.1. Preliminary notice that, at no loss of generality, $N \geq \hat{g}$, where process N in L^+ is defined in (3.1) and represents an upper bound on borrowing for the planner. Indeed, by Claim B.3, process N in L^+ grows unboundedly as $\gamma < 1$ converges to the limit $\hat{\gamma} = 1$. We show that $V_t^*(-\hat{g}_t) > \underline{V}_t$, where V^* in $\mathcal{V}$ is the value function for the planner. At no loss of generality, we develop the argument at $t = 0$ for simplicity.

Consider the optimal plan under thresholds $\hat{g}$ in L^+ when the sovereign begins with debt $\hat{g}_0$ in L_0^+. Before default, the budget constraint imposes

$$Q_t(z_t|\hat{g}) + c_t = y_t + w_t,$$

with

$$w_{t+1} = R_{t,t+1}(z_t).$$

Moreover, the consumption plan satisfies $V_t(w_t|\hat{g}) = U_t^*(c)$, with the planner's utility function being defined by (4.1). Observe that

$$\begin{aligned} w_{t+1} \geq -\hat{g}_{t+1} \quad &\text{only if} \quad V_{t+1}(w_{t+1}|\hat{g}) \geq V_{t+1}(-\hat{g}_{t+1}|\hat{g}) \\ &\text{only if} \quad U_{t+1}^*(c) \geq \underline{V}_{t+1}. \end{aligned}$$

As in the previous proofs, this observation reveals that

$$Q_t(z_t|c) \leq Q_t(z_t|\hat{g}),$$

thus showing the same plan is also in the default-adjusted budget set of the planner. □

Proof of Proposition 7.2. The proof is rather convoluted and, to simplify, we only expand the steps that are more unconventional. Fix any sufficiently large n in $\mathbb{T}$, and consider a planner program that is restricted on the time interval $\mathbb{T}^n = \{0, 1, \ldots, n\}$. The planner chooses consumption and portfolio plans over the finite horizon $\mathbb{T}^n$, with continuation utility in the terminal period given as

$$V_{n+1}^{\alpha,n}\left(w_{n+1}\right)=\alpha V_{n+1}\left(w_{n+1}|\bar{g}\right)+\left(1-\alpha\right)V_{n+1}\left(w_{n+1}|\underline{g}\right),$$

where α lies in $[0,1]$. The planner's utility at every t in $\mathbb{T}_n$ is recursively determined by

$$U_t^{\alpha,n}\left(c|w_{n+1}\right)=u\left(c_t\right)+\delta\mathbb{E}_t\max\left\{U_{t+1}^{\alpha,n}\left(c|w_{n+1}\right),\underline{V}_{t+1}\right\},$$

along with the terminal condition $U_{n+1}^{\alpha,n}\left(c|w_{n+1}\right)=V_{n+1}^{\alpha,n}\left(w_{n+1}\right)$. Securities issued by the planner on the finite time horizon $\mathbb{T}^n$ are priced as in section 4. This program is well defined and delivers a value $V_0^{\alpha,n}\left(w_0\right)$ in L_0^*. We now argue that, by construction, the extreme values correspond to the efficient high-debt and inefficient low-debt equilibrium.

It is clear that $V_0^{1,n}\left(w_0\right)=V_0\left(w_0|\bar{g}\right)$ because the planner cannot credibly commit to repay a debt exceeding the efficient default threshold $\bar{g}$ in L^+ and, so, the truncation imposes no actual restriction. In addition, $V_0^{0,n}\left(w_0\right)=V_0\left(w_0|\underline{g}\right)$. This follows from backward induction: if default is expected at any level of debt exceeding the pessimistic thresholds in the following period, the planner optimally defaults on any level of debt exceeding the pessimistic thresholds in the current period, because pessimistic thresholds are an equilibrium. We now argue that the planner's initial value changes continuously with α in $[0,1]$ and w_0 in L_0 on a relevant domain.

To establish our claim, notice that the default-adjusted budget set is closed and, as argued in the proof of Lemma A.1, it is compact in the product topology. Hence, the planner's feasible set is given as an upper hemicontinuous correspondence. We then show that it is also lower hemicontinuous jointly in (w_0,α) in $L_0\times[0,1]$ on an open set around any point satisfying $V_0^{\alpha,n}\left(w_0\right)\geq\underline{V}_0$. Notice that any slight contraction of initial wealth can be accommodated by means of an equal contraction of initial consumption, which is certainly strictly positive under the stated assumptions (because of the boundary conditions in Assumption 4). So, we only have to consider a sequence $\left(\alpha_k\right)_{k\in\mathbb{N}}$ in $[0,1]$ converging to α in $(0,1]$. We show that any default-adjusted budget feasible plan in the limit can be approximated along the sequence, thus establishing lower hemicontinuity.

Let c in L^+ be a limit budget-feasible consumption with corresponding portfolio plan z in Z. Clearly, these plans are only defined before default. Given n in $\mathbb{N}$, let η_k in $\mathbb{R}^{++}$ be such that, conditional on no default in the previous periods,

$$\max\left\{V_{n+1}^{\alpha_k,n}\left(w_{n+1}+\eta_k\right),\underline{V}_{n+1}\right\}\geq\max\left\{V_{n+1}^{\alpha,n}\left(w_{n+1}\right),\underline{V}_{n+1}\right\}.$$

In other terms, we compute the minimal decrease in outstanding terminal debt ensuring that solvency conditions in the limit are also preserved along the sequence for sufficiently large k in $\mathbb{N}$. This adjustment can be constructed so that

$$\lim_{k\to\infty}\eta_k = 0.$$

We now consider an approximated budget-feasible consumption plan c^k in L^+ as follows: the planner sacrifices a fraction of initial consumption, that is, $c_0^k = c_0 - \epsilon_k$ for some sufficiently small ϵ_k in $\mathbb{R}^{++}$; this amount is invested in a fund over time delivering, conditional on no default in the previous periods, at least the amount η_k in $\mathbb{R}^{++}$ in the terminal period; over the finite horizon $\mathbb{T}_n$, with the exception of the initial period, consumption is not modified, that is, $c_t^k = c_t$; the portfolio is modified by the additional saving plan. Notice that this perturbation is constructed in such a way that, except in the initial period,

$$U_t^{\alpha_k,n}\left(c^k|w_{n+1}+\eta_k\right) \geq U_t^{\alpha,n}\left(c|w_{n+1}\right).$$

This ensures that the planner will not default along the sequence when it is not defaulting in the limit, so proving budget-feasibility as required to establish lower hemicontinuity. Our claim then follows from the Maximum Theorem (see Aliprantis and Border [9, Theorem 17.31]).

Fix now any arbitrary g_0 in $\left(\underline{g}_0, \bar{g}_0\right) \subset L_0^+$. Given n in $\mathbb{T}$, consider the following adjustment rule:

$$f(\alpha) = \operatorname{argmin}_{\hat{\alpha}\in[0,1]}\hat{\alpha}\left(V_0^{\alpha,n}\left(-g_0\right) - \underline{V}_0\right).$$

The conventional Kakutani Fixed Point Theorem can be applied (see Aliprantis and Border [9, Theorem 17.55]), delivering $\alpha(n)$ in $[0,1]$ such that $V_0^{\alpha(n),n}\left(-g_0\right) = \underline{V}_0$. This proves that the initial default threshold can be determined in the truncated planning program arbitrarily by means of an appropriate choice of α in $[0,1]$. We now relax the truncation and obtain indeterminacy of equilibrium.

For any sufficiently large n in $\mathbb{T}$, we can determine a process for default thresholds g^n in L^+ consistent with the planning program truncated on the finite horizon $\mathbb{T}_n$, that is, such that

$$V_t^{\alpha(n),n}\left(-g_t^n\right) = \underline{V}_t.$$

This process is such that g_t^n lies in $\left[\underline{g}_t, \bar{g}_t\right] \subset L_t^+$ and its initial value satisfies $g_t^n = g_0$ for a arbitrarily given g_0 in $\left(\underline{g}_0, \bar{g}_0\right) \subset L_0^+$ not varying with the truncation period n in $\mathbb{T}$. Possibly extracting a subsequence, by Tychonoff's Theorem (see Aliprantis and Border [9, Theorem

2.61]), we can assume that the process for default thresholds converges to g in L^+. We then show that the limit process is indeed an equilibrium.

We prove that $V_0(-g_0|g) = \underline{V}_0$ and, so, as the argument can be replicated at any other future date, that $V_t(-g_t|g) = \underline{V}_t$ at every t in $\mathbb{T}$. Supposing that $V_0(-g_0|g) > \underline{V}_0$, there exists a budget feasible consumption plan c in L^+, subject to default thresholds g in L^+, such that, for some sufficiently large k in $\mathbb{T}$,

$$U_0^*\left(c^k\right) > \underline{V}_0,$$

where c^k in L^+ is the truncation of the plan at k in $\mathbb{T}$, that is, $c_t^k = c_t$ for every t in $\mathbb{T}_k$ and $c_t^k = 0$ for every t in $(\mathbb{T}/\mathbb{T}_k)$. By sacrificing a small fraction of the initial consumption, the planner can run a saving policy Δz in Z ensuring that, for some sufficiently small ϵ in $\mathbb{R}^{++}$, at every t in $\mathbb{T}_k$ before default,

$$R_{t,t+1}(z_t) \geq -g_{t+1} \text{ only if } R_{t,t+1}(z_t + \Delta z_t) \geq -g_{t+1} + \epsilon.$$

For a sufficiently large n in $\mathbb{T}$, we thus obtain, at every t in $\mathbb{T}_k$ before default,

$$R_{t,t+1}(z_t) \geq -g_{t+1} \text{ only if } R_{t,t+1}(z_t + \Delta z_t) \geq -g_{t+1}^n.$$

This proves budget feasibility for every sufficiently large n in $\mathbb{T}$, thus implying

$$V_0^{\alpha(n),n}(g_0) > \underline{V}_0,$$

a contradiction establishing our claim. □

Proof of Proposition 8.1. The proof is a simple adaptation of our previous arguments. Suppose the claim is not true. As default thresholds are bounded, we have a sufficiently small ϵ in $\mathbb{R}^{++}$ such that, at all contingencies,

$$g_t\left(z_t^-\right) \leq \tilde{g}_t\left(z_t^-\right) + (1+\epsilon)\, b_t$$

and, at some contingency,

$$g_t\left(z_t^-\right) > \tilde{g}_t\left(z_t^-\right) + (1+\rho\epsilon)\, b_t$$

where ρ in $(0,1) \subset \mathbb{R}^+$ is given in Assumption 9. At no loss of generality, we can assume that the last inequality holds true at $t=0$. Notice that, by definition of the premium on the trade in long-term securities, along with $(1-\rho)\, b_0 \geq \Delta g_0\left(z_0^-\right)$, we have

$$V_0^{\text{short}}\left(z_0^-, (1-\rho)\, b_0 - g_0\left(z_0^-\right)\right) \geq V_0\left(z_0^-, -g_0\left(z_0^-\right)\right) = \underline{V}_0.$$

We can then unfold the replication policy as in the proof of Proposition 5.1, so as to obtain

$$\begin{aligned}\tilde{V}_0\left(z_0^-, -\tilde{g}_0\left(z_0^-\right)\right) &> \tilde{V}_0\left(z_0^-, (1+\rho\epsilon)\, b_0 - g_0\left(z_0^-\right)\right)\\ &= \tilde{V}_0\left(z_0^-, (1-\rho)\, b_0 + \rho\,(1+\epsilon)\, b_0 - g_0\left(z_0^-\right)\right)\\ &\geq V_0^{\text{short}}\left(z_0^-, (1-\rho)\, b_0 - g_0\left(z_0^-\right)\right),\end{aligned}$$

a contradiction. The crucial step is the last inequality: the optimal plan under thresholds $\left(g_t\left(z_t^-\right)\right)_{t\in\mathbb{T}}$ in $L(y)$, beginning with additional resources $(1-\rho)\, b_0$ in L_0^+, when long-term securities are not actively traded, can be replicated under thresholds $\left(\tilde{g}_t\left(z_t^-\right)\right)_{t\in\mathbb{T}}$ in $L(y)$ when initial wealth is increased of an amount $(1+\rho\epsilon)\, b_0$ in L_0^+ so as to compensate for the difference in default thresholds. Long-term securities create no interference because the replication portfolio only involves short-term securities, as initially observed by Aguiar et al. [7]. In other terms, in the replication argument, the only effect on the prices of securities in given by condition (8.1). □

Appendix B. Dominant root

We provide a self-contained basic treatment of the dominant root theory in a Markov environment governed by an irreducible transition $P : S \to \Delta(S)$ on a finite state space S. We illustrate properties of dominant roots and establish existence of their associated eigenprocesses, as required in our paper.

The asset market is represented as finite set of securities J with (linear) payoff functional $R_s : Z \to V$ in state s in S, where $Z = \mathbb{R}^J$ is the portfolio space and $V = \mathbb{R}^S$ is the payoff space. We assume that, for every state s in S, there exists a portfolio z_s^f in Z such that

$$R_s\left(z_s^f\right) \gg_s 0,$$

where the (weak) ordering on space V holds almost surely conditional on the current state s in S, that is,

$$y \geq_s x \text{ if and only if } P_s\left(\{\hat{s} \in S : y_{\hat{s}} \geq x_{\hat{s}}\}\right) = 1.$$

Securities are priced by a (linear) functional $Q_s : Z \to \mathbb{R}$ conditional on current state s in S. We assume that prices of securities are no arbitrage-free, so that

$$R_s\left(z_s\right) >_s 0 \text{ only if } Q_s\left(z_s\right) > 0.$$

No further restrictions are imposed on the asset market. We develop the dominant root theory in this simplified framework.

The upper dominant root $\rho(Q)$ in $\mathbb{R}^{++}$ is the greatest ρ in $\mathbb{R}^{+}$ satisfying, for some non-trivial b in V^{+},

$$\rho b = \Pi(b).$$

Analogously, the lower dominant root $\gamma(Q)$ in $\mathbb{R}^{++}$ is the greatest γ in $\mathbb{R}^{+}$ satisfying, for some non-trivial b in V^{+},

$$\gamma b = -\Pi(-b).$$

Both dominant roots exist, with $\rho(Q) \geq \gamma(Q)$.

Claim B.1 (Upper dominant root)**.** *The upper dominant root $\rho(Q)$ exists. Furthermore, the upper dominant eigenprocess b in V^{+} is strictly positive.*

Proof. For existence, consider the map $\Phi : \Delta \to \Delta$ given by

$$\Phi(b) = \frac{\Pi(b)}{\|\Pi(b)\|_1},$$

where $\Delta = \{v \in V^{+} : \|v\|_1 = 1\}$. This is well-defined because $b > 0$ implies $\Pi(b) > 0$. A fixed point exists and corresponds to an eigenvalue ρ in $\mathbb{R}^{++}$ with eigenprocess b in V^{+}. We now show that eigenprocess b in V^{+} is strictly positive. To this purpose, suppose that $b_s = 0$ for some state s in S. Observe that, by no arbitrage, $b_s = 0$ only if $b =_s 0$, because the claim would otherwise have a strictly positive cost. By irreducibility, we obtain $b = 0$, a contradiction.

Suppose that there exists another eigenvalue ρ' in $\mathbb{R}$ with corresponding eigenprocess b' in V. Notice that

$$|b'| \leq b \text{ implies } |\rho'| \, |b'| = |\Pi(b')| \leq \Pi(|b'|) \leq \Pi(b) = \rho b,$$

This can only be consistent with $|\rho'| \leq \rho$, so proving our claim. □

Claim B.2 (Lower dominant root)**.** *The lower dominant root $\gamma(Q)$ exists, and cannot exceed the upper dominant root $\rho(Q)$. Furthermore, when a single security is traded, the lower dominant eigenprocess b in V^{+} is strictly positive. Finally, if the lower dominant eigenprocess b in V^{+} satisfies*

$$\gamma(Q)\, b = \Pi(b), \tag{B.1}$$

then $\rho(Q) = \gamma(Q)$.

Proof. For any sufficiently small ϵ in $\mathbb{R}^{++}$, consider the map $\Phi^\epsilon : \Delta \to \Delta$ given by

$$\Phi^\epsilon (b) = \frac{-\Pi(-b) + \epsilon \mathbf{1}}{\|-\Pi(-b) + \epsilon \mathbf{1}\|_1}.$$

A fixed point exists, and gives us

$$\gamma^\epsilon b^\epsilon = -\Pi(-b^\epsilon) + \epsilon \mathbf{1}.$$

Notice that

$$\gamma^\epsilon \leq \|\Pi_s(\mathbf{1}) + \epsilon \mathbf{1}\|_1,$$

which imposes an upper bound on γ^ϵ in $\mathbb{R}^{++}$. Extracting a convergent sequence, we obtain existence of a lower eigenvalue γ in $\mathbb{R}^{++}$. Noticing that $\Pi(b) + \Pi(-b) \geq 0$ by sublinearity, arguing as in the proof of the previous claim, we obtain $\gamma \leq \rho(Q)$. Hence, the greatest of such eigenvalues $\gamma(Q)$ exists and satisfies $\gamma(Q) \leq \rho(Q)$.

With a single security, $b_s > 0$ only if $b \gg_s 0$. This happens because, otherwise, $R_s\left(z_s^f\right) \geq -b$ would imply $z_s^f \geq 0$, and so $\Pi_s(-b) = 0$ by no arbitrage, a contradiction. Irreducibility, then, suffices to establish our claim.

To verify our last claim, notice that, under condition (B.1), $b_s = 0$ only if $b =_s 0$, which in turn implies $b = 0$ by irreducibility, thus yielding a contradiction. Therefore, b lies in the interior of V^+. The upper dominant root $\rho(Q)$ satisfies, for some $\hat{b}$ in V^+,

$$\rho(Q)\hat{b} = \Pi\left(\hat{b}\right).$$

Assuming $\hat{b} \leq b$ and $\hat{b} \not\ll b$ at no loss of generality, monotonicity implies

$$\rho(Q)\hat{b} = \Pi\left(\hat{b}\right) \leq \Pi(b) = \gamma(Q)b,$$

which violates our premises. □

We now establish that, given a strictly positive endowment y in V^+, the natural debt limit is finite if and only if $\gamma(Q) < 1$. The natural debt limit is determined by the recursive equation

$$v = y - \Pi(-v). \tag{B.2}$$

This in fact computes the maximum amount of debt that can be repaid in (almost) finite time out of available endowment.

Claim B.3 (Natural debt limit)**.** *Given a strictly positive endowment y in V^+, the natural debt limit is finite if and only if $\gamma(Q) < 1$.*

Proof. Suppose that $\gamma(Q) < 1$, and define a sequence, beginning from $v^0 = 0$ in V^+, by means of

$$v^{n+1} = y - \Pi(-v^n).$$

This sequence increases monotonically, that is, $v^{n+1} \geq v^n$. If $\lim_{n\in\mathbb{N}} \|v^n\|_1 < \infty$, the sequence converges to the natural debt limit, proving our claim. Otherwise, notice that, by sublinearity,

$$\frac{v^n}{\|v^n\|_1} \leq \frac{v^{n+1}}{\|v^n\|_1} = \frac{y}{\|v^n\|_1} - \Pi\left(-\frac{v^n}{\|v^n\|_1}\right).$$

Extracting a converging sequence, we obtain

$$\Delta^* = \{v \in \Delta : v \leq -\Pi(-v)\} \text{ is non-empty.}$$

We now show that $\gamma(Q) \geq 1$, a contradiction.

Consider the map $\Phi : \Delta^* \to \Delta^*$ given by

$$\Phi(v) = \frac{-\Pi(-v)}{\|-\Pi(-v)\|_1}.$$

This map is well-defined, because

$$\begin{aligned} v \leq -\Pi(-v) \quad &\text{only if} \quad v \leq \|-\Pi(-v)\|_1 \Phi(v) \\ &\text{only if} \quad -\Pi(-v) \leq -\|-\Pi(-v)\|_1 \Pi(-\Phi(v)) \\ &\text{only if} \quad \Phi(v) \leq -\Pi(-\Phi(v)). \end{aligned}$$

We notice that Δ^* is also convex, as sublinearity implies

$$\begin{aligned} \lambda v + (1-\lambda) v' &\leq -\lambda\Pi(-v) - (1-\lambda)\Pi(-v') \\ &\leq -\Pi(-(\lambda v + (1-\lambda) v')), \end{aligned}$$

where the first inequality exploits the fact that v and v' are both in Δ^*. Hence, a fixed point exists, giving $\gamma(Q) \geq 1$, a contradiction.

Assuming instead that $\gamma(Q) \geq 1$, consider a solution v in V^+ to recursive equation (B.2). As the lower dominant eigenprocess is given up to a factor of proportionality, at no loss of generality, we can assume that $v \geq b$ and $v \not\gg b$, because v in V^+ is strictly positive.

Monotonicity so implies

$$v = y - \Pi(-v) \geq y - \Pi(-b) \geq y + b \gg b,$$

a contradiction. $\square$