Elections and Strategic Voting: Condorcet and Borda

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• voting rule (social choice function)
  method for choosing social alternative (candidate) on basis of voters’ preferences (rankings, utility functions)

• prominent examples
  – Plurality Rule (MPs in Britain, members of Congress in U.S.)
    choose alternative ranked first by more voters than any other
  – Majority Rule (Condorcet Method)
    choose alternative preferred by majority to each other alternative
- **Run-off Voting (presidential elections in France)**
  - choose alternative ranked first by more voters than any other, unless number of first-place rankings less than majority
    - among top 2 alternatives, choose alternative preferred by majority

- **Rank-Order Voting (Borda Count)**
  - alternative assigned 1 point every time some voter ranks it first, 2 points every time ranked second, etc.
  - choose alternative with lowest point total

- **Utilitarian Principle**
  - choose alternative that maximizes sum of voters’ utilities
• Which voting rule to adopt?
• Answer depends on what one wants in voting rule
  – can specify criteria (axioms) voting rule should satisfy
  – see which rules best satisfy them
• One important criterion: nonmanipulability
  – voters shouldn’t have incentive to misrepresent preferences, i.e., vote strategically
  – otherwise
    not implementing intended voting rule
decision problem for voters may be hard
• But basic negative result
  Gibbard-Satterthwaite (GS) theorem
  – if 3 or more alternatives, no voting rule is always nonmanipulable
    (except for dictatorial rules — where one voter has all the power)

• Still, GS overly pessimistic
  – requires that voting rule never be manipulable
  – but some circumstances where manipulation can occur may be unlikely

• In any case, natural question:
  Which (reasonable) voting rule(s) nonmanipulable most often?

• Paper tries to answer question
• \( X = \) finite set of social alternatives
• society consists of a continuum of voters \([0,1]\)
  – typical voter \(i \in [0,1]\)
  – reason for continuum clear soon
• utility function for voter \(i\) \(U_i : X \to \mathbb{R}\)
  – restrict attention to strict utility functions
    if \(x \neq y\), then \(U_i(x) \neq U_i(y)\)
    \(\mathcal{U}_X\) = set of strict utility functions
• profile \(U\) -- specification of each individual's utility function
• voting rule (generalized social choice function) $F$
  
  for all profiles $U$ and all $Y \subseteq X$
  \[
  F(U, Y) \in Y
  \]

  $F(U, Y)$ = optimal alternative in $Y$ if profile is $U$

• definition isn’t quite right -- ignores ties
  
  – with plurality rule, might be two alternatives that are both ranked first the most
  – with rank-order voting, might be two alternatives that each get lowest number of points

• But exact ties unlikely with many voters
  
  – with continuum, ties are nongeneric

• so, correct definition:

  for generic profile $U$ and all $Y \subseteq X$
  \[
  F(U, Y) \in Y
  \]
plurality rule:
\[ f^P(U_i, Y) = \{ a \mid \mu \{ i \mid U_i(a) \geq U_i(b) \text{ for all } b \} \geq \mu \{ i \mid U_i(a') \geq U_i(b) \text{ for all } b \} \text{ for all } a' \} \]
majority rule:
\[ f^C(U_i, Y) = \{ a \mid \mu \{ i \mid U_i(a) \geq U_i(b) \} \geq \frac{1}{2} \text{ for all } b \} \]
rank-order voting:
\[ f^B(U_i, Y) = \left\{ a \mid \int r_{U_i}(a) d\mu(i) \leq \int r_{U_i}(b) d\mu(i) \text{ for all } b \right\} , \]
where \( r_{U_i}(a) = \# \{ b \mid U_i(b) \geq U_i(a) \} \)
utilitarian principle:
\[ f^U(U_i, Y) = \{ a \mid \int U_i(a) d\mu(i) \geq \int U_i(b) d\mu(i) \text{ for all } b \} \]
What properties should reasonable voting rule satisfy?

• *Pareto Property* (P): if \( U_i(x) > U_i(y) \) for all \( i \) and \( x \in Y \), then \( y \neq F(U_i, Y) \)
  
  – if everybody prefers \( x \) to \( y \), \( y \) should not be chosen

• *Anonymity* (A): suppose \( \pi : [0,1] \rightarrow [0,1] \) measure-preserving permutation. If \( U_i^{\pi} = U_{\pi(i)} \) for all \( i \), then
  
  \[
  F\left(U_i^{\pi}, Y\right) = F\left(U_i, Y\right)
  \]
  
  – alternative chosen depends only on voters’ preferences and not who has those preferences
  – voters treated symmetrically
• **Neutrality (N):** Suppose \( \rho : Y \to Y \) permutation. 
  If \( U_{i}^{\rho,Y}(\rho(x)) > U_{i}^{\rho,Y}(\rho(y)) \iff U_{i}(x) > U_{i}(y) \) for all \( x, y, i \), then 
  \[
  F\left(U_{\rho,Y}^{\rho}, Y\right) = \rho\left(F\left(U_{\rho,Y}, Y\right)\right). 
  \]
  – alternatives treated symmetrically

• All four voting rules – plurality, majority, rank-order, utilitarian – satisfy P, A, N

• Next axiom most controversial
  still
  • has quite compelling justification
  • invoked by both Arrow (1951) and Nash (1950)
• **Independence of Irrelevant Alternatives (I):**

\[
\text{if } x = F(U, Y) \text{ and } x \in Y' \subseteq Y \\
\text{then} \\
x = F(U, Y')
\]

– if \( x \) chosen and some non-chosen alternatives removed, \( x \) still chosen

– Nash formulation (rather than Arrow)

– no “spoilers” (e.g. Nader in 2000 U.S. presidential election, Le Pen in 2002 French presidential election)
• Majority rule and utilitarianism satisfy I, but others don’t:
  – plurality rule

\[
\begin{array}{ccc}
0.35 & 0.33 & 0.32 \\
\underline{x} & \underline{y} & \underline{z} \\
y & z & x \\
z & x & y \\
\end{array}
\]

\[f^P(U_x, \{x, y, z\}) = x\]

\[f^P(U_x, \{x, y\}) = y\]

– rank-order voting

\[
\begin{array}{cc}
0.55 & 0.45 \\
\underline{x} & \underline{y} \\
y & z \\
z & x \\
\end{array}
\]

\[f^B(U_x, \{x, y, z\}) = y\]

\[f^B(U_x, \{x, y\}) = x\]
Final Axiom:

- *Nonmanipulability (NM):*

  \[ x = F(U_i, Y) \text{ and } x' = F(U'_i, Y), \]
  
  where \( U'_j = U_j \) for all \( j \not\in C \subseteq [0,1] \)

  then

  \[ U_i(x) > U_i(x') \text{ for some } i \in C \]

  - the members of coalition \( C \) can’t all gain from misrepresenting utility functions as \( U'_i \)
• NM implies voting rule must be ordinal (no cardinal information used)

• $F$ is ordinal if whenever, for profiles $U_x$ and $U'_x$, 
  $$U_i(x) > U_i(y) \iff U'_i(x) > U'_i(y)$$ 
  for all $i, x, y$

$(*)$ \quad $F(U_x, Y) = F(U'_x, Y)$ for all $Y$

• Lemma: If $F$ satisfies NM and I, $F$ ordinal
  
  – suppose $x = F(U_x, Y) \quad y = F(U'_x, Y)$, where $U_x$ and $U'_x$ same ordinally
  
  – then $x = F(U_x, \{x, y\}) \quad y = F(U'_x, \{x, y\})$, from I

  – suppose $\begin{bmatrix} C & -C \\ y & x \end{bmatrix}$

  – if $F(U'_c, U_{-c}, \{x, y\}) = y$, then $C$ will manipulate
  
  – if $F(U'_c, U_{-c}, \{x, y\}) = x$, then $-C$ will manipulate

• NM rules out utilitarianism
But majority rule also violates NM

- \( F^C \) not even always defined

\[
\begin{array}{ccc}
.35 & .33 & .32 \\
.35 & .33 & .32 \\
\hline
x & y & z \\
y & z & x \\
z & x & y \\
\end{array}
\]

\( F^C \left(U \cap \{x, y, z\}\right) = \emptyset \)

- example of Condorcet cycle
- \( F^C \) must be extended to Condorcet cycles
- one possibility

\[
F^{C/B} \left(U \cap Y\right) = \begin{cases} 
F^C \left(U \cap Y\right), & \text{if nonempty} \\
F^B \left(U \cap Y\right), & \text{otherwise} 
\end{cases}
\]

(Black's method)

- extensions make \( F^C \) vulnerable to manipulation

\[
\begin{array}{ccc}
.35 & .33 & .32 \\
.35 & .33 & .32 \\
\hline
x & y & z \\
y & z & x \\
z & x & y \\
\end{array}
\]

\( F^{C/B} \left(U \cap \{x, y, z\}\right) = x \)

\[
\begin{array}{ccc}
z & y & x \\
\hline
z & y & x \\
\end{array}
\]

\( F^{C/B} \left(U \cap \{x, y, z\}\right) = z \)
**Theorem**: There exists no voting rule satisfying P, A, N, I and NM

**Proof**: similar to that of GS

overly pessimistic --- many cases in which some rankings unlikely
**Lemma:** Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election

![Graph showing Nader, Gore, Bush]

unlikely that many had ranking Bush or Nader

Nader or Bush

Gore or Gore

- strongly-felt candidate
  - in 2002 French election, 3 main candidates: Chirac, Jospin, Le Pen
  - voters didn’t feel strongly about Chirac and Jospin
  - felt strongly about Le Pen (ranked him first or last)
• Voting rule $F$ works well on domain $\mathcal{U}$ if satisfies P,A,N,I,NM when utility functions restricted to $\mathcal{U}$

  - e.g., $F^c$ works well when preferences single-peaked
• **Theorem 1**: Suppose \( F \) works well on domain \( \mathcal{U} \), then \( F^C \) works well on \( \mathcal{U} \) too.

• Conversely, suppose that \( F^C \) works well on \( \mathcal{U}^C \).

Then if there exists profile \( U^\circ \) on \( \mathcal{U}^C \) such that
\[
F\left(U^\circ, Y\right) \neq F^C\left(U^\circ, Y\right)
\]
for some \( Y \),

there exists domain \( \mathcal{U}' \) on which \( F^C \) works well but \( F \) does not.

**Proof**: From NM and I, if \( F \) works well on \( \mathcal{U} \), \( F \) must be ordinal.

• Hence result follows from
  
  Dasgupta-Maskin (2008), *JEEA*
  
  shows that Theorem 1 holds when NM replaced by ordinality
To show this D-M uses

Lemma: $F^C$ works well on $\mathcal{U}$ if and only if $\mathcal{U}$ has no Condorcet cycles

- Suppose $F$ works well on $\mathcal{U}$

- If $F^C$ doesn't work well on $\mathcal{U}$, Lemma implies $\mathcal{U}$ must contain

  Condorcet cycle $x \ y \ z$
  
  $y \ z \ x$
  
  $z \ x \ y$
• Consider
\[
U^1_\square = \begin{array}{cccc}
1 & 2 & \ldots & n \\
 x & z & z & \\
 z & x & x & \\
\end{array}
\]

\[(*) \text{ Suppose } F\left(U^1_\square, \{x, z\}\right) = z\]

• \[
U^2_\square = \begin{array}{cccc}
1 & 2 & 3 & n \\
x & y & z & z \\
y & z & x & x \\
z & x & y & y \\
\end{array}
\]

\[
F\left(U^2_\square, \{x, y, z\}\right) = x \implies \text{ (from I) } F\left(U^2_\square, \{x, z\}\right) = x, \text{ contradicts (*)}
\]

\[
F\left(U^2_\square, \{x, y, z\}\right) = y \implies \text{ (from I) } F\left(U^2_\square, \{x, y\}\right) = y, \text{ contradicts (*) (A,N)}
\]

so

\[
F\left(U^2_\square, \{x, y, z\}\right) = z
\]

• so \[F\left(U^2_\square, \{y, z\}\right) = z \quad \text{(I)}\]

• so for
\[
U^3_\square = \begin{array}{cccc}
1 & 2 & 3 & \ldots & n \\
x & x & z & z & \\
z & z & x & x & \\
\end{array}
\]

\[
F\left(U^3_\square, \{x, z\}\right) = z \quad \text{(N)}
\]

• Continuing in the same way, let \[
U^4_\square = \begin{array}{cccc}
1 & \ldots & n-1 & n \\
x & x & z & \\
z & z & x & \\
\end{array}
\]

\[
F\left(U^4_\square, \{x, z\}\right) = z, \text{ contradicts (*)}
\]
• So $F$ can’t work well on $\not\succeq$ with Condorcet cycle

• Conversely, suppose that $F^C$ works well on $\not\succeq^C$ and

$$F\left(U_\circ, Y\right) \neq F^C\left(U_\circ, Y\right)$$

for some $U_\circ$ and $Y$

• Then there exist $\alpha$ with $1 - \alpha > \alpha$ and

$$U_\circ = \frac{1 - \alpha}{x} \frac{\alpha}{y}$$

such that

$$x = F^C\left(U_\circ, \{x, y\}\right) \text{ and } y = F\left(U_\circ, \{x, y\}\right)$$

• But not hard to show that $F^C$ unique voting rule satisfying P, A, N, and NM when $|X| = 2$ - - contradiction
• Let’s drop I
  – most controversial

• *no* voting rule satisfies P,A,N,NM on $\mathcal{X}$
  – GS again

• *$F$ works nicely* on $\mathcal{Y}$ if satisfies P,A,N,NM on $\mathcal{X}$
Theorem 2: $|X| = 3$

- Suppose $F$ works nicely on $\mathbb{U}$, then $F^C$ or $F^B$ works nicely on $\mathbb{U}$ too.

- Conversely suppose $F^*$ works nicely on $\mathbb{U}^*$, where $F^* = F^C$ or $F^B$.

Then, if there exists profile $\mathbb{U}^\infty$ on $\mathbb{U}^*$ such that

$$F\left(\mathbb{U}^\infty, Y\right) \neq F^*\left(\mathbb{U}^\infty, Y\right)$$

for some $Y$,

there exists domain $\mathbb{U}'$ on which $F^*$ works nicely but $F$ does not

**Proof:**

- $F^C$ works nicely on any Condorcet-cycle-free domain

- $F^B$ works nicely only when $\mathbb{U}$ is subset of Condorcet cycle

- so $F^C$ and $F^B$ complement each other

  - if $F$ works nicely on $\mathbb{U}$ and $\mathbb{U}$ does not contain Condorcet cycle, $F^C$ works nicely too
  
  - if $F$ works nicely on $\mathbb{U}$ and $\mathbb{U}$ contains Condorcet cycle, then $\mathbb{U}$ can’t contain any other ranking (otherwise *no* voting rule works nicely)
  
  - so $F^B$ works nicely on $\mathbb{U}$.
Striking that the 2 longest-studied voting rules (Condorcet and Borda) are also

- *only two* that work nicely on maximal domains