Technical Details for In-Model Interpolations

1 Variable Names

- CG_i : annual capital gain of the year i.
- X_t : log of quarterly capital gains.
- Y_t : quarterly variables consisting of macro variables and S&P index.

2 Information Structure

- Y_t is observed at the end of each quarter in the current year.
- X_t is not observed.
- CG_i, the observed annual capital gain of the year i, is the average of unobserved quarterly capital gains. More specifically,

$$CG_i = \frac{1}{4}\left[\exp(X_{i:Q1}) + \exp(X_{i:Q2}) + \exp(X_{i:Q3}) + \exp(X_{i:Q4})\right] \tag{1}$$

- CG_i is observed at the end of the forth quarter of the year i, right after $Y_{i:Q4}$ is observed.

3 Labelling alternative capital gains series

- $\{X^F_{i:Q1}, X^F_{i:Q2}, X^F_{i:Q3}, X^F_{i:Q4}\}$
  - quarterly capital gains constructed using all information in

$$\begin{aligned}\Omega_{i1} &= \{CG_{i-1,}CG_{i-2,}...\} \cup \\ &\{Y_{i-1:Q4,}Y_{i-1:Q3,}Y_{i-1:Q2,}Y_{i-1:Q1,}...\}\end{aligned}$$

  - called "**forecasts**" of capital gains for the year i.
- $\{X^*_{i:Q1}, X^*_{i:Q2}, X^*_{i:Q3}, X^*_{i:Q4}\}$
  - capital gains constructed using all information contained in

$$\begin{aligned}&\Omega_{i2} = \{CG_{i-1,}CG_{i-2,}...\} \cup \\ &\{Y_{i:Q4,}Y_{i:Q3,}Y_{i:Q2,}Y_{i:Q1,}Y_{i-1:Q4,}Y_{i-1:Q3,}Y_{i-1:Q2,}Y_{i-1:Q1,}...\}\end{aligned}$$

  - called "**estimates**" of capital gains for the year i.
- $\{X^{**}_{i:Q1}, X^{**}_{i:Q2}, X^{**}_{i:Q3}, X^{**}_{i:Q4}\}$

– capital gains constructed using all information contained in

$$\begin{aligned}\Omega_{i3} = \quad & \{CG_i, CG_{i-1}, CG_{i-2}, ...\} \cup \\ & \{Y_{i:Q4}, Y_{i:Q3}, Y_{i:Q2}, Y_{i:Q1}, Y_{i-1:Q4}, Y_{i-1:Q3}, Y_{i-1:Q2}, Y_{i-1:Q1}, \cdots\}\end{aligned}$$

– called "**reconciled"** capital gains for the year i.

Remark 1 *As is evident from the information sets, the informational advantage of the "estimates" over the "forecasts" comes from the current year quarterly observations of Y, while the informational advantage of the "reconciled" gains over the "estimates" is due to the year-end observations of CG.*

4 Overview of Methodology

- STEP I: Initialization in a base year
  - Assume that, for some base year i_0, a "seed" series $\{X_t^{**}, t \leq i0 : Q4\}$ is available.
  - A seed is assumed to be a reconciled, satisfying the condition (1) for each year up to $i0$.
- STEP II : Constructing a BVAR.
  - Estimate a BAVR(5) of the form

$$\begin{aligned} Z_t &= \mu + B_1 Z_{t-1} + B_2 Z_{t-2} + B_3 Z_{t-3} + B_4 Z_{t-4} + B_5 Z_{t-5} + \varepsilon_t, \\ \varepsilon_t &\mid \mu, B_i \backsim N(0, \Sigma) \end{aligned} \tag{2}$$

where

$$Z_t = \begin{bmatrix} X_t \\ Y_t \end{bmatrix}, \quad t \leq i0 : Q4$$

and the seed is used as if it were actual quarterly capital gain series.

- – Using the estimated version of (2), generate 4-quarter ahead forecasts of capital gains, $\{X^F_{i_0+1:Q1}, X^F_{i_0+1:Q2}, X^F_{i_0+1:Q3}, X^F_{i_0+1:Q4}\}$.
- STEP III : Getting Estimated Capital Gains.
  - Assume the estimates $(\widehat{\mu}, B_i, \widehat{\Sigma})$ obtained in STEP II are invariant for the following year $i_0 + 1$.
  - Then, the evolution of the system (2) in the year $i_0 + 1$ can be represented by the following state space model:

State Equation	$Z_t = \widehat{\mu} + \widehat{B}_1 Z_{t-1} + \widehat{B}_2 Z_{t-2} + \widehat{B}_3 Z_{t-3} + \widehat{B}_4 Z_{t-4} + \widehat{B}_5 Z_{t-5} + \varepsilon_t, \quad \varepsilon_t \mid \widehat{\mu}, \widehat{B}_i \backsim N(0, \widehat{\Sigma})$
Measurement Equation	$Y_t = H Z_t$

  where$H$ is the matrix that singles out the observables form the whole system vector Z_t.
  - It is now evident that the estimated quarterly capital gains for the year $i_0 + 1$ are $E[X_t \mid \Omega_{i1}]$, *i.e.*, the smoothed X_t for $t = i0 + 1 : Q1, ..., i0 + 1 : Q4$.

- STEP IV: Reconciliation
  - The resulting "estimates" $\{X^*_{i_0+1:Q1}, X^*_{i_0+1:Q2}, X^*_{i_0+1:Q3}, X^*_{i_0+1:Q4}\}$ do not generally match the actual annual capital gain $CG_{i_0+1.}$
  - For reconciliation, the following perturbation method is used:
    - From the estimated version of (2), take the first equation governing the evolution of X_t.
    - The estimated capital gains, $\{X^*_{i_0+1:Q1}, X^*_{i_0+1:Q2}, X^*_{i_0+1:Q3}, X^*_{i_0+1:Q4}\}$, are subjected to a common estimation error δ each period.
    - The error to the estimate for a quarter is carried over to the next quarter, according to the (estimated) first equation of (2).
    - Therefore, the perturbed estimates$\{X^*(\delta)_{i_0+1:Q1}, X^*(\delta)_{i_0+1:Q2}X^*(\delta)_{i_0+1:Q3}X^*(\delta)_{i_0+1:Q4}\}$ are complicated functions of the common error δ.
    - Find the δ^{**} that satisfies

$$CG_{i_0} = \frac{1}{4}\left[\begin{array}{c}\exp(X^*(\delta)_{i_0+1:Q1}) + \exp(X^*(\delta)_{i_0+1:Q2}) \\ + \exp(X^*(\delta)_{i_0+1:Q3}) + \exp(X^*(\delta)_{i_0+1:Q4})\end{array}\right]$$

    - The resulting perturbed values are $\{X^{**}_{i_0+1:Q1}, X^{**}_{i_0+1:Q2}, X^{**}_{i_0+1:Q3}, X^{**}_{i_0+1:Q4}\}$, the reconciled quarterly capital gains for the year $i0+1$.

- STEP V: Iterate the process.
  - Append the reconciled capital gains to the existing seed.
  - Follow the next steps to get the forecasts, estimates, and reconcilates of quarterly capital gains for the year $i0+2$.