Equilibrium in a Pure Currency Economy

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1. Introduction

This paper studies the determination of the equilibrium price level in a stationary economy in which all exchange involves the trade of fiat money for goods. The use of money in exchange is guaranteed by the imposition of a constraint, as suggested by Clower (1967),[1] which requires that purchases of goods must necessarily be paid for by currency held over from the preceding period. The models examined also resemble closely that studied by Friedman (1969, first part). Individual behavior resembles that captured in inventory-theoretic models of money demand, as studied by Baumol (1952) and Tobin (1956), so that another way to think of the paper is as an attempt to study the transaction demand for money in as simple as possible a general equilibrium setting.

In section 2, an example with perfect certainty is analyzed, with a digression to motivate the cash-in-advance constraint. In this example, which is a special case of the much more general setup treated by Grandmont and Younes (1972, 1973), equilibrium velocity is determined in an entirely mechanical way by the assumed payments period. In section 3, individual uncertainty is introduced, giving rise to a precautionary motive for holding currency and a nontrivial problem of equilibrium determination, in which velocity depends on the kinds of economic factors long thought to be important in reality. The analysis of this latter case is continued in sections 4–6.

I think of this exercise not so much as an end in itself, but as an analytical step toward models which capture more and more features which monetary economists believe to be important in understanding actual monetary systems. In the concluding section, then, I will go well beyond the results developed in the paper to venture some opinions on some of these other issues.

2. An Economy With Certainty

Throughout the paper, I will study an economy with a continuum of identical

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[1]Author names and years refer to the works listed at the end of this book.

traders. Each trader is endowed with one unit of labor each period, to which no disutility is attached, which yields y units of a nonstorable consumption good. In the present section, preferences over consumption sequences $\{c_t\}$, $c_t \geq 0$, are

$$(1) \qquad \sum_{t=0}^{\infty} \beta^t U(c_t)$$

where $0 < \beta < 1$, and $U: R^+ \rightarrow R$ is bounded, twice differentiable, with $U'(\cdot) > 0$ and $U''(\cdot) < 0$.

Considering only allocations in which identical traders are identically treated, it is clear that an optimal allocation is $c_t = y$ for all t. Nothing more will be done in this section than to propose a monetary arrangement which will bring this allocation about and to determine the money price of goods under this arrangement.

In order to motivate the need for any monetary arrangement (indeed for any arrangement other than autarchy, in which agents consume their own produce y), I will first reinterpret this model economy as one involving many goods, as follows. Let the good come in n colors, where items of each color are produced under the technology assumed above: one unit of labor yields y units of any color. Consumption is now a vector $(c_{1t}, \ldots, c_{nt})$, where c_{it} is consumption of color i in period t. Let current period utility be

$$V(c_1, \ldots, c_n) = U\left[\prod_{i=1}^{n} \left(\frac{c_i}{\alpha_i}\right)^{\alpha_i} \right]$$

where U is as above and $\Sigma_i \alpha_i = 1$, $\alpha_i > 0$, all i. Let $c = \Sigma_i c_i$. Now given the assumed constant returns to scale technology, equilibrium requires relative prices of unity among all goods.[2] With these prices, consumers will select color proportions c_i/c equal to α_i for all i, and given this mix, $V(c_1, \ldots, c_n) = U(c)$. Without altering the example, one can think of all agents having the same α-weights, of agents distributed by a fixed c.d.f. $F(\alpha)$ of weights, or of each agent drawing a period-t weight α from F in a way which is unpredictable even to that agent. In each of these cases, the equilibrium output mix (per capita) is $(\bar{\alpha}_1 y_1, \ldots, \bar{\alpha}_n y_n)$ each period, where $\bar{\alpha}_i = \int \alpha_i \, dF(\alpha)$. I imagine this sort of elaboration is what we always have in mind when we work with aggregative models.

Next, imagine each agent as consisting of a two-person household, one of whom spends each day shopping (call that person the *shopper*) and the other of whom works at the production of a single color (call that person the *worker*). Production and sale occur at spatially distinct stores. Each day, the worker goes to work at the same store, while the shopper moves from store to store purchasing the mix dictated by the current drawing of α. Equilibrium dictates that the value of the worker's labor y should equal the total expenditures by the shopper over all n (at least) stores.

[2]This remark is technically premature (since equilibrium has yet to be defined) and perhaps substantively questionable as well. This scenario depends on prices being set in a spatially decentralized manner, as opposed to in a single, centralized auction, so that it may not be clear how a constant-returns technology is manifested in the structure of equilibrium prices. In the present paper, the discussion will be confined to stationary examples in which one can easily imagine a constant relative price structure arising from custom. In a situation in which market equilibrium were subject to shocks, it would, I think, be necessary to treat this issue with more care.

What will assure that this equilibrium is, in fact, executed? What, for example, prevents a shopper from collecting $2y$ in various goods in the course of a day? To get an idea of the importance of this question, let us suppose that each store keeps an exact record of each shopper's purchases and continuously informs all other stores throughout the day as to how many credits (it is almost impossible even to discuss this matter without using language suggesting securities) have been used up. Then for each shopper, each of the first $n-1$ stops necessitates $n-1$ messages, or $(n-1)^2$ per household per day. Let the workday be 8 hours, and let each message require 6 seconds of a worker's time to send. Then with 101 stores, this information transmission activity utilizes $(100)^2 \cdot 6/(60)^2 = 16\frac{2}{3}$ hours, or more than twice national product!

This issue could be pursued further by spelling out in more detail a technology for information storage, transmission, and processing and the available methods for enforcing against fraud, after the fact. An easier route is suggested by the observation that the adoption of paper currency can reduce these costs essentially to zero. Let each shopper, at the beginning of a period, be issued claims to y units of consumption. Proceeding from store to store, these claims are exhausted and redistributed to workers at the end of each day. This system (except for resources used to print currency and prevent counterfeiting) economizes perfectly on informational costs. Note that nothing has been said as to how this monetary solution to the information problem might come into being, nor is it at all clear how an individual agent, or a collection of agents, could act so as to bring this system into existence. The monetary solution involves a social convention, with the property that if (for some reason) everyone else adopts it, then it is in one's own interest to adopt it as well.

A formal definition of a monetary equilibrium with a constant money supply M which embodies this convention is developed by means of the optimal value function $v(m)$, interpreted as the value of the objective function (1) for a consumer who begins the current period with nominal balances m and behaves optimally. This function v must satisfy

$$v(m) = \max_{c, m' \geq 0} \{U(c) + \beta v(m')\} \tag{2}$$

subject to

$$m' = m - pc + py \tag{3}$$

$$m \geq pc. \tag{4}$$

Here p is the constant equilibrium price level, c is current goods consumption, and m' is end-of-period balances. Equation (3) is the standard budget constraint, and (4) is the cash-in-advance constraint discussed above. Then in terms of v, equilibrium is defined as follows:[3]

[3]An alternative to this definition would be to define an equilibrium as an element of a space of infinite sequences $\{c_t, p_t, m_t\}$ of consumptions, prices, and money demands, satisfying feasibility, utility maximization, and market clearing. In this alternative setup, the equilibrium specified below (a constant sequence) is the only one, but this must (and can) be argued using a transversality condition. The stationarity built into the definition used here will prove convenient in the section following. Whether it rules out any behavior of economic interest is not known, though my own opinion is that, in the present context, it does not.

DEFINITION. *An equilibrium in the certainty economy is a number $p \geqslant 0$ and a continuous bounded function $v: R^+ \to R$ such that*

1. *Given p, v satisfies (2)–(4).*
2. *$(c, m') = (y, M)$ attains $v(M)$.*

That is, consumers behave optimally (condition 1) and money demand equals money supply (condition 2).

Enough has been said already to make it clear that the unique equilibrium on this definition involves $p = M/y$ and $v(M) = u(y)/1-\beta$. That is, each household spends all of its current money balances M on goods each period, replenishing these holdings with the worker's end-of-period pay. Since this example is a special case of the one analyzed in the next section, a formal substantiation of this claim is omitted.

3. An Economy With Individual Uncertainty

In this section, the technology and trading arrangements will be assumed the same as in section 2, but individual preferences will be taken to be subject to uncertainty, unpredictable even to the household itself. (Think of an unanticipated medical need or the unexpected discovery of an item of a particularly attractive color.) Formally, let the shock to preferences be a drawing, independent over time and over persons at a point in time, of a random variable θ from the fixed c.d.f. $F(\theta)$. Take F to be strictly increasing on the interval $I = [\underline{\theta}, \overline{\theta}] \subset R$ with $F(\underline{\theta}) = 0$ and $F(\overline{\theta}) = 1$. Then with a continuum of agents, there is no aggregative uncertainty: the state of the economy will not change from period to period.

Let preferences be given by

$$E\left\{\sum_{t=0}^{\infty} \beta^t \, U(c_t, \theta_t)\right\} \tag{5}$$

where $0 < \beta < 1$, $U: R^+ \times I \to R$ is bounded, twice differentiable, with $U_c > 0$, $U_\theta > 0$, $U_{cc} < 0$, and $U_{c\theta} > 0$. Require also that for all $c \geqslant y$

$$\lim_{\theta \to \overline{\theta}} U_c(c, \theta) = \infty \tag{6}$$

or that consumption may be arbitrarily urgent.[4] At the time the t^{th} period decision is taken, regard θ_t as known, so that the expectation in (5) is taken with respect to the distribution of $(\theta_1, \theta_2, \ldots)$, with θ_0 given.

As in the preceding section, I will study an economy with the constant money supply M and seek an equilibrium in which the price level is constant at p. The situation of any *individual*, however, cannot be expected to settle down, since individuals are continually shocked by new drawings of θ. I will first develop the problem faced by a representative trader, then discuss what is meant by market clearing in this context, and then summarize these in a formal definition of equilibrium. With this accomplished, I will turn to the analytical issues involved in constructing and characterizing the equilibrium.

The budget constraints facing an agent are as in the preceding section, but

[4]Condition (6) is used only in the proof of Lemma 1, section 4, where it is clear from the context that it could be replaced, with appropriate modification in the argument, with a much weaker condition.

in this case it is convenient to let m denote an individual's *real* balances (nominal balances divided by the constant price level p). Let $v(m, \theta)$ be the optimum value function for a consumer who begins the current period with real balances m, draws an urgency to consume θ, and behaves optimally. Then that person's current period decision problem is

$$v(m, \theta) = \max_{c, m' \geq 0} \left\{ U(c, \theta) + \beta \int_I v(m', \theta') \, dF(\theta') \right\} \tag{7}$$

subject to

$$c + m' \leq m + y \tag{8}$$

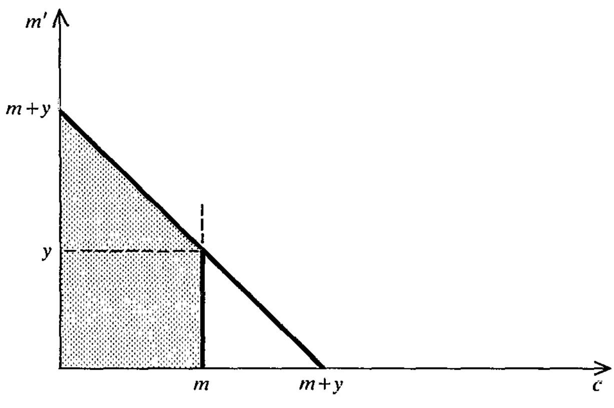
$$c \leq m. \tag{9}$$

The opportunity set defined by (8) and (9) is as drawn in Figure 1 (for given m and y). If the derived preference function for c and m', $U(c, \theta) + \int v(m', \theta') dF(\theta')$, has indifference curves of the usual shape, then the household will either locate at a tangency point to the line given when (8) holds with equality, spending less than its initial balances, or it will spend all it has, choosing $c = m$ and $m' = y$. Assuming this problem has a solution, denote the individual demand functions for goods and end-of-period balances by $c = c(m, \theta)$ and $m' = g(m, \theta)$. This individual decision problem will receive more detailed attention in the next section.

In *market* equilibrium in this economy, it must be true that the per capita demand for real balances (averaged over agents) equals per capita balances supplied M/p at the given nominal quantity supplied M and the assumed constant equilibrium price level p. To calculate per capita end-of-period demand one needs to know the distribution of agents by beginning-of-period balances, $\Psi(m)$, say. Given Ψ, per capita demand is

$$\iint g(m, \theta) \, d\Psi(m) \, dF(\theta)$$

Figure 1



so the equilibrium condition is

(10) $$\iint g(m, \theta)\, d\Psi(m)\, dF(\theta) = \frac{M}{p}\,.$$

In general, the p-value satisfying (10) will depend on the distribution $\Psi(m)$ of real balances over persons. In order, then, for consumers' expectations that p be constant over time to be rational (or correct) we require also that $\Psi(m)$ replicate itself over time, or that it be a stationary distribution for the stochastic difference equation

$$m_{t+1} = g(m_t, \theta_t).$$

This requirement is just that Ψ solve[5]

(11) $$\Psi(m') = \iint_{A(m')} d\Psi(m)\, dF(\theta)$$

where $A(m')$ is the region of the (m, θ) plane defined by

(12) $$A(m') = \{(m, \theta): m \geq 0, \theta \in I, g(m, \theta) \leq m'\}.$$

The foregoing considerations can be summarized in the following.

DEFINITION. *An equilibrium in the economy with individual uncertainty is a number $p > 0$, a continuous bounded function $v: R^+ \times I \rightarrow R$, a pair of continuous functions $c, g: R^+ \times I \rightarrow R^+$ and a c.d.f. $\Psi: R^+ \rightarrow [0, 1]$ such that*

1. *$v(m, \theta)$ solves (7).*
2. *(c, g) solves the maximum problem (7) for each (m, θ).*
3. *g, Ψ, and p satisfy (10).*
4. *g and Ψ satisfy (11).*

4. Construction of the Equilibrium

Equilibrium was defined as four unknown functions together with a positive number. This simultaneous system may be solved sequentially, first by finding a function v which satisfies condition 1, then finding the policy functions c and g satisfying condition 2, then finding the c.d.f. Ψ satisfying condition 4, and finally finding the price p which satisfies condition 3.

The relevant facts about the value function $v(m, \theta)$ are given in

PROPOSITION 1. *There is exactly one continuous bounded function $v(m, \theta)$ satisfying (7). This solution v is strictly increasing and strictly concave with respect to m.*

Proof. The proof is standard (see, for example, Lucas 1978b), involving the following formulation and facts. Let L be the Banach space of continuous bounded functions $u: R^+ \times I \rightarrow R$, normed by

$$\| u \| = \sup_{m, \theta} |u(m, \theta)|.$$

[5]Notice that if (11) holds, (10) is equivalent to

$$\iint m\, d\Psi(m)\, dF(\theta) = \frac{M}{p}\,.$$

Define T as the operator on L such that (7) reads $v = Tv$. Using Berge 1963, p. 116, $T: L \to L$. Using Blackwell 1965, thm. 5, T is a contraction, so that $Tv = v$ has a unique solution $v^* \in L$ and $\| T^n u - v^* \| \to 0$ as $n \to \infty$ for all $u \in L$.

It is easy to verify that T takes nondecreasing, concave functions of m into strictly increasing, strictly concave functions of m. It follows that $v^* = \lim_{n\to\infty} T^n \cdot 0$ is nondecreasing and concave, and then, since $v^* = Tv^*$, that these properties hold strictly.

PROPOSITION 2. *There exist unique, continuous functions $c(m, \theta)$, $g(m, \theta)$: $R^+ \times I \to R^+$ such that $c = c(m, \theta)$ and $m' = g(m, \theta)$ attain the right-hand side of (7) for each (m, θ).*

Proof. The maximum problem (7) involves maximizing a continuous strictly concave function over a compact convex set. Hence $c(m, \theta)$, $g(m, \theta)$ are uniquely defined. Their continuity follows from Berge 1963, p. 116.

PROPOSITION 3. *The solution v to (7) is continuously differentiable with respect to m, for each fixed θ, and, if $c(m, \theta) > 0$,*

$$v_m(m, \theta) = U_c[c(m, \theta), \theta]. \tag{13}$$

Proof. In the interior of the region of the (m, θ) plane on which (9) is not binding, the proof follows that in Lucas 1978b, prop. 2. In the interior of the region on which (9) is binding,

$$v(m, \theta) = U(m, \theta) + \beta \int v(y, \theta')\, dF(\theta')$$

and (13) follows since $c(m, \theta) \equiv m$ in this region. Since the one-sided derivatives agree on the boundary of these two regions, the result follows.

Now the function $g(m, \theta)$ and the c.d.f. F of θ together define a Markov process

$$m_{t+1} = g(m_t, \theta_t) \tag{14}$$

with state space R^+. That is, given an initial distribution of persons by cash balances $\Psi_0(m)$, say, where $\Psi_0(m)$ is the fraction of consumers beginning period 0 with initial balances less than or equal to m, the distribution $F(\theta)$ and the difference equation (14) together determine the sequence of distributions $\Psi_1(m)$, $\Psi_2(m)$, ... which prevail at times 1, 2, ... Our interest will be in the limiting behavior of this sequence.

The behavior of this sequence of distributions can be studied by examining the characteristics of the transition probabilities of the process defined by (14) and F. For $m \geq 0$ and any measurable $A \subseteq R^+$, these are given by

$$P(m, A) = \int_{B(m)} dF(\theta)$$

where

$$B(m) = \{\theta \in I : g(m, \theta) \in A\}.$$

Then if $\mathcal{m}$ is the set of probability measures μ on R^+, define $S: \mathcal{m} \to \mathcal{m}$ by

$$(S\mu)(A) = \int_0^\infty P(m, A)\, \mu(dm).$$

Then if $\Psi_0(m) = \int_0^m \mu_o\,(du)$ is the initial distribution mentioned above, the t^{th} term in the sequence is

$$\Psi_t(m) = \int_0^m (S^t\mu_o)\,(du).$$

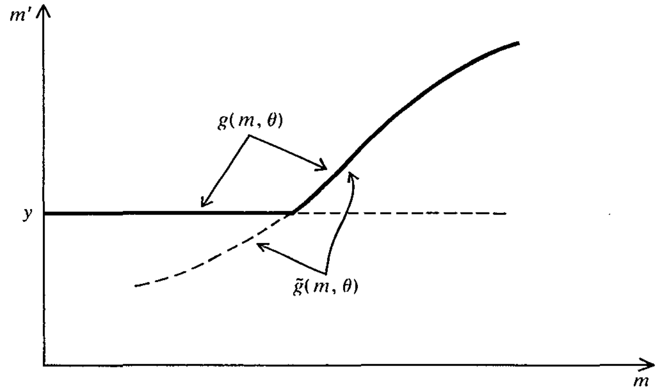
A solution μ^* to $S\mu = \mu$ corresponds to a solution $\Psi^*(m) = \int_0^m \mu^*(du)$ to (11): a stationary distribution of agents by real balances. The process (14) will be studied here via S using results from Doob 1953, pp. 190–218[6] and the implications of the maximum problem (7).

The first-order condition for the maximum problem (7) when (9) is ignored and (8) is used to eliminate the variable c is, in view of Proposition 3,

(15) $$U_c(m + y - m', \theta) = \beta\int v_m(m', \theta')\,dF(\theta').$$

It then follows from the strict concavity of U and v in their first arguments that the m' value, call it $\tilde{g}(m, \theta)$, satisfying (15) is increasing in m and decreasing in θ. Then, clearly, the money demand function $g(m, \theta) = \max[y, \tilde{g}(m, \theta)]$ so that $g(m, \theta)$ is as drawn in Figure 2, for θ fixed. The ergodic set for the

Figure 2



process (14) is included in $[y, \infty)$, since $g(m, \theta) \geq y$ for all (m, θ). An upper bound on m can be obtained from examination of

(16) $$U_c(y, \underline{\theta}) = \beta\int v_m(m, \theta')\,dF(\theta')$$

which is the form (15) takes if $(c, m') = (y, m)$ is optimal at $(m, \underline{\theta})$, where $\underline{\theta}$ is the lower bound on θ. The right-hand side of (16) is a decreasing function of m. Since $v(m, \theta)$ is, for θ fixed, an increasing, concave, bounded, and differentiable function of m, we have

$$v(0, \theta) \leq v(m, \theta) + v_m(m, \theta)\,(-m)$$

[6]A very useful recent treatment of the same issues is given in Futia undated.

or

$$0 \leqslant m\, v_m(m, \theta) \leqslant v(m, \theta) - v(0, \theta) \leqslant B - v(0, \theta)$$

where B is a bound for v. Hence $v_m(m, \theta) \rightarrow 0$ as $m \rightarrow \infty$. It follows that (16) is solved for a unique $m = \overline{m} \geqslant y$ if

$$\beta \int v_m(y, \theta')\, dF(\theta) \geqslant U_c(y, \underline{\theta})$$

and has no solution otherwise. In the latter case, the ergodic set for the process (14) is just $E = \{y\}$. In the case where $\overline{m} > y$ satisfies (16), $g(\overline{m}, \underline{\theta}) = \overline{m}$, so that initial balances are just maintained. For $\theta > \underline{\theta}$, $g(\overline{m}, \theta) < \overline{m}$, and also for $m > \overline{m}$, $g(m, \theta) < m$, for all θ. Thus the ergodic set of the process (14) is $E = [y, \overline{m}]$. There are no cyclically moving subsets.

The next result verifies that the Doeblin condition (Doob 1953, cond. D, p. 192) holds on $[y, \overline{m}] = E$.

LEMMA 1. *There is a finite measure* λ *on* E *and an* $\epsilon > 0$ *such that* $\lambda(A) \leqslant \epsilon$ *implies* $P(m, A) \leqslant 1 - \epsilon$, *for all* $m \in E$.

Proof. For the case $\overline{m} = y$ the result is trivial. For the case $\overline{m} > y$, assign measure $\lambda_o \in (0, 1)$ to the point y and let $\lambda([m_1, m_2]) = (1-\lambda_o)\,(m_2-m_1)/(\overline{m}-y)$ for $y < m_1 \leqslant m_2 \leqslant \overline{m}$, so that $\lambda(E) = 1$. Now using (15), $g(m, \theta) = y$ whenever

$$U_c(m, \theta) > \beta \int v_m(y, \theta')\, dF(\theta')$$

so that

$$P(m, \{y\}) = Pr\{U_c(m, \theta) > \beta \int v_m(y, \theta')\, dF(\theta')\}.$$

Then for $m \in E$,

$$P(m, \{y\}) \geqslant Pr\{U_c(\overline{m}, \theta) > \beta \int v_m(y, \theta')\, dF(\theta')\}.$$

By condition (6) $\theta^o < \overline{\theta}$ can be chosen such that $\theta \geqslant \theta^o$ implies $U_c(\overline{m}, \theta) > \beta \int U_c(y, \theta')\, dF(\theta')$, so that if $Pr\{\theta^o \leqslant \theta \leqslant \overline{\theta}\} = b$,

$$P(m, \{y\}) \geqslant b.$$

Choose $\epsilon > 0$ with $\epsilon < \lambda_o$ and $\epsilon < b$. Then $\lambda(A) \leqslant \epsilon$ implies $y \notin A$ so that for all m,

$$P(m, A) \leqslant 1 - P(m, \{y\}) \leqslant 1 - b \leqslant 1 - \epsilon.$$

This proves Lemma 1.

It then follows from Doob 1953, p. 214, that

PROPOSITION 4. *Given* g *as in Proposition 2, there is exactly one solution* Ψ *to (11) (or solution* μ *to* $S\mu = \mu$*), and* $\Psi(m) = 0$ *for* $m < y$, $\Psi(y) > 0$, *and* $\Psi(\overline{m}) = 1$.

The final step in establishing the existence of a unique equilibrium is taken by observing that (10) can be solved, given M, for a unique, positive price p.

5. Discussion of the Equilibrium

In constructing the equilibrium distribution of persons by real balance holdings, $\Psi(m)$, we began with an arbitrary distribution $\Psi_0(m)$ and then studied the limit of the sequence of distributions $\Psi_t(m)$ (that is, measures $S^t\mu_0$). This was merely a technical device for arriving at a solution Ψ to (11), but the sequence $\Psi_t(m)$ has an economic interpretation. It is the sequence of distributions which would prevail, in an economy starting at Ψ_0, if all agents believed that the current price level will prevail into the next period (that the nominal yield on money is always zero). In fact, if $\Psi_0 \neq \Psi$, prices will not be constant, so that these consumer beliefs will be confirmed only in the limit. In the vocabulary of growth theory, this equilibrium is a stationary point of an economy with static expectations, where the distributions Ψ_t play the role of capital. This equilibrium is not a golden rule (a stationary state with discount factor $\beta = 1$). In contrast to optimal growth paths, however, only the stationary state can be interpreted as an equilibrium: along any approach path, agents taking prices as given can increase their utility. This seems to me to mirror exactly Friedman's statement, in a very similar context, that while "it is easy to see what the final position [following a change in M] will be... it is much harder to say anything about the transition" (Friedman 1969, p. 6).

The shape of the equilibrium distribution of real balances is shown in Figure 3. There is a mass point at the institutional minimum holding y, and then a smooth distribution on $(y, \bar{m}]$. The existence of a mass point clearly follows from the economics of the situation: if individuals did not occasionally spend all available cash (return to y), they would be holding too much money. Money is an inventory, held against a particular contingency, and one never has an optimal inventory bounded away from zero. There is, however, no presumption that the lower bound y is visited frequently or, which comes to the same thing, that a large fraction of consumers will be at $m = y$ at any point in time.

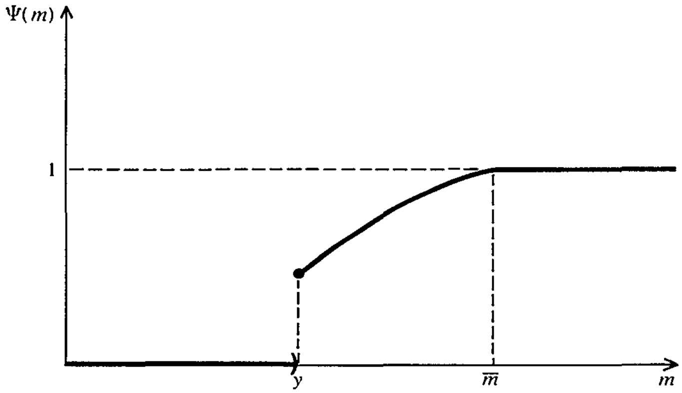
The determinants of the demand for money, or of velocity, in this model are a mix of institutional and economic factors. Clearly, the length of a "day" will affect the equilibrium; indeed, there are economists to whom a constraint of the form $pc \leq M$ (in units, $\$/t \leq \$$) must appear unthinkable. As long as one remembers not to vary the length of a "day" in midargument, this raises no problems, however. Moreover, the rate at which the earth rotates does have important economic implications, and there is nothing to be gained in insisting on an economic explanation for this phenomenon.

The economic factors affecting money demand are preferences U, the discount factor β, the volatility F of the shocks θ and income y. Thus the amount of risk and people's attitude toward it (U and F) will affect money demand, as is appropriate in a model stressing the precautionary motive; so too will the rate of time preference. One's intuition as to the direction of effect of changes in these forces is fairly strong, but rigorous verification is somewhat complicated. The next section, a treatment of the income effect on real balance demand, illustrates a useful method for answering questions of this type and also addresses a question of substantive interest.

6. Engel Curves for Real Balances

The relationship of the demand for real balances to the level of real income has received a great deal of attention, both theoretically and empirically.

Figure 3



Early inventory-theoretic treatments suggested an income elasticity less than unity, a prediction which has never found any empirical confirmation. Friedman's early empirical work (1959) led to estimated income elasticities of around 1.8, which he rationalized in terms of conventional consumer demand theory by concluding that real balances, as a consumer durable, are a "luxury" good. I think it is now recognized that any empirically estimated Engel curve can be rationalized theoretically as well as any other, so that the issue is purely an empirical one. The model studied in this paper does not suggest modifications to this open conclusion, but it can be utilized to isolate the contributions of the several determinants of the income elasticity of money demand somewhat more satisfactorily than can be done with theories at the level of individual behavior only.

Real output (per capita productivity) was taken as a constant in sections 2–4. This assumption will be maintained here for each individual agent, but it will be assumed that each agent's constant income y is taken from a distribution $\Lambda(y \mid \bar{y})$ where $\bar{y}$ is mean income:

$$\bar{y} \equiv \int y \, \Lambda \left(dy \mid \bar{y}\right).$$

One may then consider the *individual* Engel curve, describing the way average real balances vary with y for given $\bar{y}$, and the *market* Engel curve, describing the way average balances vary across economies with different average income levels $\bar{y}$.

In these seemingly more complex economies, individuals continue to solve (7). Denote the resulting value and policy functions, constructed exactly as in section 4, by $v(m, \theta, y)$, $c(m, \theta, y)$, $g(m, \theta, y)$. Similarly, (11) and (12) continue to define the stationary distribution of real balances, conditioned on y, as constructed in section 4. Call this c.d.f. $\Psi\left(m \mid y\right)$. This is the fraction of time an agent with income y will hold balances less than or equal to m, independent of the average income $\bar{y}$ in society. The individual Engel curve is then

$$h(y) = \int_{y}^{\bar{m}(y)} m \, d\Psi\left(m \mid y\right). \tag{17}$$

The market Engel curve requires averaging over $\Lambda(y \mid \bar{y})$. It is

$$(18) \qquad k(\bar{y}) = \int_0^\infty \int_y^{\overline{m}(y)} m \, d\Psi(m \mid y) d\Lambda(y \mid \bar{y}) = \int_0^\infty h(y) \, d\Lambda(y \mid \bar{y}).$$

Market equilibrium (price level determination) is obtained by replacing (10) with

$$(19) \qquad k(\bar{y}) = \frac{M}{P}.$$

I shall turn, then, to methods for learning about the function $h(y)$, with the reader forewarned by the introduction to this section that sharp predictions are not likely to be forthcoming.

The function $h(y)$, evaluated at a particular y-value, is the mean value of the function $f(m) \equiv m$ with respect to the stationary distribution $\Psi(m \mid y)$. This function is continuous and therefore bounded on the interval $[y, \overline{m}(y)]$. I shall utilize well-known facts about mean values of continuous bounded functions with respect to stationary distributions. First,

LEMMA 2. *If $\mu^* = S\mu^*$ and for all measurable $A \subseteq R^+$*

$$\lim_{t\to\infty} (S^t \mu_o)(A) = \mu^*(A)$$

independent of μ_0, then for all continuous bounded f_0

$$(20) \qquad \lim_{t\to\infty} \int_0^\infty f_0(m)(S^t\mu_o)(dm) = \int_0^\infty f_0(m)\mu^*(dm).$$

Proof. See Feller 1966, p. 243.

Second, using (14), one notices that

$$(21) \qquad \int_0^\infty f_0(m)(S^{t+1}\mu_o)(dm) = \int_0^\infty \int_I f_0[g(m,\theta)]dF(\theta)(S^t\mu_o)(dm)$$

since both sides of (21) express the mean value of $f_0(m_{t+1})$ given the initial distribution μ_o. Then if the sequence $\{f_t\}$ is defined recursively from f_0 by

$$(22) \qquad f_{t+1}(m) = \int_I f_t[g(m,\theta)] \, dF(\theta)$$

repeated application of (21) gives (compare Feller 1966, p. 266)

$$(23) \qquad \int_0^\infty f_t(m)\mu_o(dm) = \int_0^\infty f_0(m)(S^t\mu_o)(dm), t = 0, 1, 2, \ldots .$$

Thus (20) may be replaced by

$$(24) \qquad \lim_{t\to\infty} \int_0^\infty f_t(m)\mu_o(dm) = \int_0^\infty f_0(m)\mu^*(dm).$$

Moreover, since the choice of μ_o was arbitrary, $\{f_t\}$ must converge (almost everywhere) to a constant function, so that (24) [or (20)] can be replaced by

$$\lim_{t\to\infty} f_t(m) = \int_0^\infty f_0(m)\mu^*(dm) \text{ for all } m \geq 0. \tag{25}$$

We know from Proposition 4, section 4, that the hypotheses of Lemma 2 are satisfied for each fixed y. Then Lemma 2 with (20) replaced by (25) provides an inductive method for verifying statements about mean values of functions of m with respect to the stationary distribution.

Returning to the particular function $f_0(m) \equiv m$ of interest here, we have

LEMMA 3. *Suppose $g(m, \theta, y)$ is a nondecreasing function of m and y. Then $h(y)$ as defined in (17) is a nondecreasing function of y.*

Proof. The proof is an induction on the sequence $\{f_t\}$ defined by (22) and $f_0(m) \equiv m$. Clearly $f_0(m) = m$ is nondecreasing in m and y. Then if f_t has these properties, so does f_{t+1}, from (22) and the hypotheses on $g(m, \theta, y)$. The result then follows from Lemma 2, the fact that (20) implies (25), and (25).

To verify the hypotheses of Lemma 3, we need to go back to the maximum problem (7). In section 4 (compare Figure 2) we found that $g(m, \theta, y)$ is nondecreasing in m. In the (m, θ) region on which $g(m, \theta, y) = y$, g is clearly increasing in y. From (15), one can see that this is also true when $g(m, \theta, y) > y$. This proves, applying Lemma 3,

PROPOSITION 5. *$h(y)$ is a nondecreasing function of y. From (18), it also follows that if increases in mean income $\bar{y}$ shift the entire distribution $\Lambda(y|\bar{y})$ to the right, then $k(\bar{y})$ is also an increasing function.*

It has been established, then, that both the individual and market Engel curves for real balances are upward-sloping (really, only that they are never downward-sloping) or that real balances are a normal good. The methods used to establish this fact make it fairly clear, I think, that no sharper predictions on the magnitude of the slope of this curve will be obtained without much stronger restrictions on preferences (on U and F). Put backwards, any empirically found slope would be consistent with the theory.

Since the model of this paper is inventory-theoretic, one might wonder why the scale economies which played such a prominent role in earlier theory do not seem to arise here. One way to answer this is by suggesting a modification of the model which would, or might, reintroduce them. In section 2, I suggested that the cash-in-advance constraint facing households be motivated as imposed on a household in which one member spends a day spending the cash earned by the other member on the preceding day. No provision was made for the shopper to make visits during a day to the store of the worker, picking up currency earned there in the first hour, the second hour, and so on. That is, I have taken the payments period to be institutionally rather than economically determined. Were this convention relaxed, it might be the case that increases in y would induce the number of intraday currency reorders to rise, so that real balances demanded would rise less than proportionally with income y. This modification would introduce no new possibilities for the shape of $h(y)$ into the theory. It is possible, though not a conjecture I would expend much effort to verify, that it would rule out some $h(y)$ possibilities. The cross-section results obtained by Meltzer (1963) suggest that this role of scale economies may safely be abstracted from.

7. Concluding Comments

One of many issues not touched upon above is that of the economic *efficiency* of the monetary equilibrium found in sections 3 and 4. Clearly, the equilibrium in the economy with certainty (section 2) is efficient.[7] With individual uncertainty introduced, even to define efficiency in a satisfactory way is a problem of some complexity. If one thinks of each individual's current θ as observable to all, a marginal condition expressing the idea of from each according to ability, to each according to need can be derived. Presumably, however, one is interested in the case in which agents observe their own θ, but not anyone else's, in which case issues of incentive compatibility of allocative arrangements become central.

Without exploring this difficult matter further one can, I think, see that on any efficiency criterion which takes these issues into account, the monetary equilibrium of sections 3 and 4 will not be efficient. In any period, there will be some households in a run of low θ's, with large real balances accumulated but no particular urgency to spend them. There will be others in a run of high θ's, with balances of y and a high marginal utility of current consumption. Here, then, are two sides to a nonexistent credit market on which some would gladly lend at positive interest rather than the zero yield provided by currency and others would gladly pay this premium to consume today at the expense of future consumption.

Can this gap be filled by a government-engineered deflation, in which currency is withdrawn from the system via lump-sum taxes and a positive real yield thereby created? Clearly not, though by some efficiency criteria this policy may be utility-increasing. The problem here is not one involving the attractiveness of currency on average, but one of permitting the benefits of gains from trade between differently situated agents.

The introduction of a credit market into this economy would, with impatient agents ($\beta < 1$), be associated with a positive interest rate and hence with real balance holdings at the institutionally fixed minimum level (as in section 2). (With arbitrarily short periods, this would imply arbitrarily high, or infinite, velocity.) With the introduction of some real cost associated with dealing in a credit market (say, the time involved for one's credit worthiness to be established), one can imagine a model in which currency demand is governed by a mechanism such as that studied above coexisting with a credit mechanism for larger transactions. The analysis of such a hybrid system must be left for future research.

In the present model, as in more complex elaborations which one may imagine, there is a clear sense in which money is a second-rate asset. It serves a role and commands resources only insofar as it enables the economy to economize on some sort of record keeping or other transaction cost. At best, then, money is viewed as a means of approximating some idealized real resource allocation. This feature may be contrasted with the role of money in the intergenerational models introduced by Samuelson (1958). There, money converts an economy which is allocating resources inefficiently into an efficient one. It does not provide a cheap approximation to an idealized and

[7]The assumption in section 2 that no disutility is attached to labor supply is crucial to this conclusion. See Locay and Palmon 1978, where in a context very similar to that of this paper, but with disutility attached to labor, it is shown that a Friedmanlike deflationary policy is required for the monetary allocation to be efficient. An earlier, more general treatment of this efficiency question is given in Grandmont and Younes 1972.

efficient real allocation which one can at least imagine being achieved in a decentralized, nonmonetary way; it is the only device short of centralized planning by which an efficient real allocation may be attained.

This theoretical second-rateness of money seems to me a virtue of models in which its use is motivated by a cash-in-advance constraint and therefore a reason for attempting to pursue the analysis of models of this type more deeply. In the first place, money (or currency, certainly) really is a second-rate asset: if any of us were to have free overnight access to Federal funds, we would take advantage of it. In the second place, this view of money as an aid in approximately attaining real general equilibrium is consistent with the way economists use real general equilibrium or relative price theory. When we apply theories of barter economies to problems in, say, public finance or labor economics, it is not our intent to obtain results applicable only to primitive or prehistoric societies. We apply this body of theory to money-using economies such as our own because we believe that for many problems the fact that money is used in attaining equilibrium can be abstracted from, or that the theoretical barter economy is a tractable, idealized model which approximates well (is well-approximated by) the actual, monetary economy. If this practice is sound, then we want monetary theories which rationalize it or at least which do not radically conflict with it.