Some Remarks on Monetary Policy in an Overlapping Generations Model

W. A. Brock and J. A. Scheinkman*

1. Introduction

It is the purpose of this paper to show that certain results (derived from rational expectations monetary models where real balance services enter the utility function directly)—such as that an increase in the *mean* of the rate of growth of the money supply induces a welfare loss and an increase in the *variance* of the rate of growth of the money supply may cause an increase in welfare—are not dependent upon the Friedman-Patinkin-Samuelson device of inserting real balances into the utility function.

More specifically, we set up an overlapping generations model in which money balances are the only way of carrying wealth from periods of high endowment to periods of low endowment. Our setup is like that of Samuelson (1958),[1] Cass and Yaari (1966b), Gale (1973), Lucas (1972), and Cass, Okuno, and Zilcha (this volume). In particular, we use the Cass-Okuno-Zilcha discussion of rational expectations equilibria with a positive price of money and the Gale analysis of Gale's "Samuelson case." However, no one except Lucas (1972) has addressed the question of whether an increase in the mean rate of growth in the money supply would cause a welfare loss in an overlapping generations model. In Lucas, though, monetary transfers during the period were proportional to the individual's beginning-of-period balances, so Lucas' agents suffered no capital losses on money due to inflation. Furthermore, although Eden (1975) studied the case of random money supply and obtained the result that an increase in the variance of the rate of growth of money could increase equilibrium welfare, no one has studied such a question in a model without real balances in the utility function. We take up both issues here.

Section 2 sets up the model and studies rational expectations equilibria

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[1]Author names and years refer to the works listed at the end of this book.

when the money supply grows at a constant nonrandom rate σ. It is shown that an increase in the rate of growth of the money supply leads to welfare loss. Other properties of equilibria are explored as well.

In section 3 uncertainty in the rate of growth of the money supply is introduced. It is shown that if there is no uncertainty in tastes and endowments then there is an equivalent deterministic monetary policy in the sense that the same stationary real balance equilibrium is generated by that policy as was generated by the random policy.

Section 4 contains a study of equilibria when real balances converge to 0 as $t \to \infty$ and a discussion of the economic meaning of such equilibria.

Section 5 contains a summary and conclusion.

Before we begin we want to acknowledge that we have borrowed ideas from Cass and Shell's notes (undated), Okuno and Zilcha 1977, and Wallace 1977.

Finally, we would like to acknowledge the intellectual influence of Black (1974) upon us in stressing the importance of the nonuniqueness problem discussed in section 4 for monetary policy. Calvo (1978, 1979) has written two interesting papers on related topics.

2. The Model

The model presented here is a simple overlapping generations model. Each consumer lives two periods. There is no population growth, and there is only one consumption good. Consumers born in period t are called *young* at t and *old* at $t + 1$. They each receive endowments of w_y in period t and $w_o < w_y$ in period $t + 1$. Furthermore, in period $t + 1$ they receive a monetary transfer h_{t+1}. The consumption good is perishable, but consumers may carry money from t to $t + 1$. Their utility function is given by $u[c_y(t)] + \delta u[c_o(t)]$. Formally, each consumer solves

$$\text{maximize } u[c_y(t)] + \delta u[c_o(t+1)] \tag{1}$$

subject to

$$p_t c_y(t) + m_y(t) = p_t w_y$$

$$p_{t+1} c_o(t+1) = m_y(t) + h_{t+1} + p_{t+1} w_o$$

where $c_y(t)$ is consumption of a young person at t, $c_o(t + 1)$ is consumption of an old person at $t + 1$, and $m_y(t)$ is money demand by young people at t. If u is strongly concave [that is, $u''(x) < 0$] and $u'(0) = +\infty$, a sufficient condition for a solution to (1) is given by

$$\frac{u'[c_y(t)]}{u'[c_o(t+1)]} = \frac{p_t}{p_{t+1}} \delta \tag{2}$$

and

$$c_y < w_y.$$

Now we are ready to define a monetary equilibrium for a money supply sequence

$$\{m_t^s\}_{t=0}^{\infty}.$$

DEFINITION. *A sequence* $\{p_t\}_{t=0}^{\infty}$, $p_t > 0$, $t = 0,1,\ldots$ *is a monetary equilibrium if*

(a) *For all* $t, c_y(t) + c_o(t) = w_o + w_y = \overline{w}$.
(b) $m_t^s = m_y(t)$, $t=0,1,2,\ldots$.

Note that (b) is just the requirement that money supply be equal to money demand in every period since old people demand no money. We are now ready to prove an easy but basic result for the case of certainty.

PROPOSITION 1. *If* $h_t \equiv m_t^s - m_{t-1}^s \equiv \sigma m_{t-1}^s, \sigma > -1$, *and if* c_y *and* c_o *are positive numbers such that* $u'(c_y) = \delta u'(c_o)/(1+\sigma)$, $c_y + c_o = \overline{w}$, *and if* $0 < c_o - w_o$, *then* $p_{t+1} \equiv (1+\sigma)p_t$, $p_0 \equiv m_0^s/(c_o - w_o)$ *forms an equilibrium sequence.*

Proof. We will show that $\{p_t\}_{t=1}^{\infty}$, and $c_y(t) \equiv c_y$, $t = 0, 1, \ldots, c_o(t) \equiv c_o$, $t = 0, 1, \ldots$ satisfy (2) and the definition of a monetary equilibrium. On the one hand, it is obvious from the definition of $\{p_t\}$ that (2) is satisfied. On the other hand, we must show that market clearing obtains. It is enough to show that the money market clears. Now

$$m_y(0) = p_0(w_y - c_y) = p_0(c_o - w_o) = m_0^s.$$

Hence (b) holds for $t = 0$. Suppose now that (b) holds for $t \leqq \tau - 1$. Then

$$p_0 = \frac{m_0^s}{(c_o - w_o)}$$

and therefore

$$p_\tau = \frac{m_0^s(1+\sigma)^\tau}{(c_o - w_o)} \equiv \frac{m_\tau^s}{(c_o - w_o)}.$$

From the budget constraint of the young consumer and the construction of $\{p_t\}$ we have

$$m_y(\tau) = p_\tau(w_y - c_y) = p_\tau(c_o - w_o) = m_\tau^s.$$

This ends the proof.

Before going on we warn the reader that has not looked at Gale 1973 or the conference papers of Cass-Okuno-Zilcha and Wallace that there is a continuum of perfect foresight equilibria. The stationary equilibrium of Proposition 1 is just one of these. We defer study of the others until section 4.

It may be helpful to develop a graphical depiction of equilibria before doing any more analysis. For the case $h_t = 0$, $t = 1,2,\ldots$, follow the Cass-Okuno-Zilcha (COZ) paper and recognize that the solution $[c_y(t), c_o(t+1)]$ of the consumer's problem lies on the offer curve. Then depict the dynamics of a perfect foresight equilibrium as do COZ.[2] Unfortunately this neat depic-

[2]David Cass pointed out to us that if we work with real transfers y^{th} directly then to each sequence of real transfers $y_1, y_2, \ldots$, in particular the stationary sequence $\bar{y}$, we may draw an offer curve through the point $(w_y, w_o + y_t)$. If we do this for the stationary case $(w_y, w_o + \bar{y})$, we may trivially adopt COZ's technique to depict the equilibrium dynamics graphically and hence apply their analysis directly to our stationary σ-equilibria. This is so because the young's demand

tion of the equilibrium dynamics is not as useful when $h_t \neq 0$ since the offer curve at each date depends upon $\{h_t\}$.

We can, however, depict the stationary σ-equilibria graphically. Given a utility function $U(c_y,c_o)$ consider the income consumption curve—call it ICC(σ)—generated by the problem

$$\text{maximize } U(c_y,c_o)$$

subject to

$$c_y + (1+\sigma)c_o = I$$

as I varies. Stationary σ-equilibria are contained in the set of points $E(\sigma)$ where the ICC(σ) intersects the economy's production possibility frontier

$$\text{PPF} = \{(c_y,c_o) \mid c_y + c_o = w_y + w_o \equiv \overline{w}\}.$$

A point $[\overline{c}_y(\sigma),\overline{c}_o(\sigma)] \in E(\sigma)$ is indeed a σ-equilibrium if positive real balances are carried at $[\overline{c}_y(\sigma),\overline{c}_o(\sigma)]$. In order to check for positive real balances, just check that the elderly are spending more than w_o, or what is the same thing, that the young are spending less than w_y. Graphically speaking, points in $E(\sigma)$ that lie to the left of (w_y,w_o) on the PPF are σ-equilibria.

Such a graphical apparatus may be used with profit in constructing examples. We shall rely on COZ to aid us in briefly expositing some examples.

The first example is one where the barter equilibrium is Pareto optimal, but $\sigma < 0$ may be chosen so that the resulting stationary monetary equilibrium is Pareto optimal. This shows that the "existence proposition" (as COZ call it) in Samuelson's basic model does not obtain when $\sigma \neq 0$. To construct the example, set $(w_y,w_o) \equiv (\hat{c}_y,\hat{c}_o)$ and follow Proposition 2(b) below. In other words, choose $\sigma < 0$ and check that the ICC(σ) cuts the PPF left of $(\hat{c}_y,\hat{c}_o)$ in the normal case. Show this by noting that the ICC(0) cuts the PPF exactly at $(\hat{c}_y,\hat{c}_o)$ and by noticing that c_o is cheaper than c_y when $\sigma < 0$.

The first example is what COZ call a "coexistence example." We did not need heterogeneity of tastes, as did COZ, but we did need $\sigma \neq 0$ in order to create such an example.

Turn now to a second example.

In order to create what COZ call a "nonoptimality example," choose (w_y,w_o) so that it is not Pareto optimal. Anywhere to the right of $(\hat{c}_y,\hat{c}_o)$ on the PPF will do. Now choose $\sigma > 0$ so that the ICC(σ) cuts the PPF to the right of $(\hat{c}_y,\hat{c}_o)$ but to the left of (w_y,w_o). By Proposition 2(a) such a σ-equilibrium is not Pareto optimal.

It is important to recognize that in the COZ examples $\sigma = 0$. The counterexamples that they present are more fundamental in the sense that gov-

$[c_y(t),c_o(t+1)]$ lies on the offer curve through $(w_y,w_o+\overline{y})$. Equilibrium $[c_y(t),c_o(t)]$ must satisfy $c_y(t) + c_o(t) = w_y + w_o$.

In order to depict an equilibrium path graphically, start at a point $[c_y(0),c_o(0)]$ on the

$$\text{PPF} \equiv \{(c_y,c_o) \mid c_y + c_o \equiv w_y + w_o \equiv \overline{w}\}.$$

Find a point $[c_y(0),c_o(1)]$ on the offer curve through $(w_y,w_o+\overline{y})$. Now find a point $[c_y(1),c_o(1)]$ on the PPF. Continue in this manner. Therefore the minor adaptation of the COZ analysis that was suggested above applies directly to the stationary σ-equilibria.

ernment is not doing anything to distort the terms of trade between present and future in their examples.

We feel, however, that the case $\sigma \neq 0$ drives home the point that the mere act itself of introducing a social mechanism such as money that allows finitely lived people in a world of perishable goods to store value has little to do with efficiency per se even when preferences are very well behaved. Take note that we are not saying that there is no way that government can manage the money supply that will improve efficiency. COZ do not tell us that there is no way to steer their nonoptimality example to a Pareto optimum by a well chosen monetary policy. It would be interesting to ask what conditions on preferences and endowments guarantee the existence of some monetary policy that will lead to a Pareto optimum equilibrium.

Let us get back to analysis. For fixed δ, what happens when σ varies?

Since

$$U_y(c_y,c_o) = \frac{1}{(1+\sigma)} U_o(c_y,c_o)$$

$$c_y + c_o = \overline{w}$$

$$c_y + (1+\sigma)c_o = I,$$

for the case $U(c_y,c_o) \equiv u(c_y) + \delta u(c_o)$ we have

$$\frac{\partial c_y}{\partial \sigma} = -\left[u''(c_y) + \frac{\delta u''(\overline{w} - c_y)}{(1+\sigma)}\right]^{-1} \delta u'(\overline{w} - c_y) \frac{1}{(1+\sigma)^2} > 0$$

and therefore

$$\frac{\partial c_y}{\partial \sigma} > 0.$$

But notice that this is valid only for $c_y < w_y$. (By Proposition 1 the requirement that $c_y < w_y$ is needed for equilibrium, namely, $c_o > w_o$, obtains.) Thus a higher rate of monetary expansion induces a higher consumption when young.

What can be said about Pareto optimality? We can prove this:

PROPOSITION 2.

(a) *For $\sigma > 0$, if a time stationary equilibrium exists, then it is Pareto inefficient.*

(b) *For $\sigma \leqq 0$, if a time stationary equilibrium exists, it is Pareto optimum.*

Proof.

(a) Let $\hat{c}_y$, $\hat{c}_o$ be defined by

$$\hat{c}_y + \hat{c}_o = \overline{w}$$

and

$$u'(\hat{c}_y) = \delta u'(\hat{c}_o).$$

Call $(\hat{c}_y,\hat{c}_o)$ the *global Pareto optimum*. Then, if $\bar{c}_o(\sigma)$ and $\bar{c}_y(\sigma)$ denote the consumption bundles in the competitive equilibrium associated with $\sigma \neq 0$ we know that $\bar{c}_y(\sigma) > \hat{c}_y$ and $\bar{c}_o(\sigma) < \hat{c}_o$. We show that the allocation $(\hat{c}_o,\hat{c}_y)$ Pareto dominates $[\bar{c}_o(\sigma),\bar{c}_y(\sigma)]$. Clearly the old consumer at time zero is better off. Also

$$u(\hat{c}_y) + \delta u(\hat{c}_o) > u[\bar{c}_y(\sigma)] + \delta u[\bar{c}_o(\sigma)]$$

since $\hat{c}_y,\hat{c}_o$ solve the problem

$$\text{maximize } u(c_y) + \delta u(c_o)$$

subject to

$$c_y + c_o = \overline{w}$$

and

$$\bar{c}_y(\sigma),\bar{c}_o(\sigma) \text{ satisfy } \bar{c}_y(\sigma) + \bar{c}_o(\sigma) = \overline{w}.$$

This ends the proof of part (a).

(b) Now consider the case $\sigma \leqq 0$. The proof is by contradiction. Suppose we may make some individual better off without making anyone else worse off. Then we must give that person more in old age. This is so because if we give her or him more in youth we must take it away from the old one who is still alive. Hence we may suppose without loss of generality that it is the first person who is made better off by the proposed reallocation. Hence there exists a feasible allocation $\{c_y(t), c_o(t)\}_{t=0}^{\infty}$ with

$$u[c_y(t)] + \delta u[c_o(t+1)] \geqq u[\bar{c}_y(\sigma)] + \delta u[\bar{c}_o(\sigma)] \text{ for } t = 0,1,2,\ldots$$

and

$$c_o(0) > \bar{c}_o(\sigma).$$

Now since u is strictly concave,

$$\begin{aligned} u[c_y(0)] - u[\bar{c}_y(\sigma)] &< u'[\bar{c}_y(\sigma)]\,[c_y(0) - \bar{c}_y(\sigma)] \\ &= u'[\bar{c}_y(\sigma)][\bar{c}_o(\sigma) - c_o(0)]. \end{aligned}$$

Since the proposed reallocation Pareto dominates $[\bar{c}_y(\sigma),\bar{c}_o(\sigma)]$, u is concave, and $c_y(0) + c_o(0) \leqq \overline{w}$, we must have

$$\begin{aligned} \delta u[c_o(1)] - \delta u[\bar{c}_o(\sigma)] &\geqq u[\bar{c}_y(\sigma)] - u[c_y(0)] \\ &\geqq u'[\bar{c}_y(\sigma)][\bar{c}_y(\sigma) - c_y(0)] \\ &\geqq u'[\bar{c}_y(\sigma)][c_o(0) - \bar{c}_o(\sigma)]. \end{aligned}$$

The chain of inequalities derived above gives us

$$\delta u[c_o(1)] - \delta u[\bar{c}_o(\sigma)] > u'[\bar{c}_y(\sigma)][c_o(0) - \bar{c}_o(\sigma)] > 0. \tag{3}$$

Again, the concavity of u gives us

(4) $$\delta u[c_o(1)] - \delta u[\bar{c}_o(\sigma)] \leqq \delta u'[\bar{c}_o(\sigma)][c_o(1) - \bar{c}_o(\sigma)].$$

From (3) and (4) together we get

$$\frac{c_o(1) - \bar{c}_o(\sigma)}{c_o(0) - \bar{c}_o(\sigma)} > \frac{u'[\bar{c}_y(\sigma)]}{\delta u'[\bar{c}_o(\sigma)]} = \frac{1}{(1+\sigma)}.$$

If $\sigma < 0$, then replacing 0 by 1 and 1 by 2 in the above reasoning, continuing in the above manner we eventually obtain

$$c_o(t) - \bar{c}_o(\sigma) > \left(\frac{1}{1+\sigma}\right)^t [c_o(0) - \bar{c}_o(\sigma)] > \overline{w} - \bar{c}_o(\sigma)$$

for t large enough which is a contradiction to $c_o(t) \leqq \overline{w}$.

For the case $\sigma = 0$, it may be possible that

$$\lim_{t\to\infty} [c_o(t) - \bar{c}_o(0)] \equiv \gamma < \overline{w} - \bar{c}_o(\sigma).$$

In such a case we must have

$$\lim_{t\to\infty}[\bar{c}_y(0) - c_y(t)] = \gamma.$$

But note that $\bar{c}_y(0) \equiv \hat{c}_y, \bar{c}_o(0) \equiv \hat{c}_o$. The strong concavity of u guarantees us that for a t sufficiently large

$$u[c_y(t)] + \delta u[c_o(t)] < u(\hat{c}_y) + \delta u(\hat{c}_o)$$

since $\hat{c}_y, \hat{c}_o$ solve

$$\text{maximize } u(c_y) + \delta u(c_o)$$

subject to

$$c_y + c_o = \overline{w}.$$

This is a contradiction to the hypothesis that $\{c_y(t), c_o(t)\}$ Pareto dominates $(\hat{c}_y, \hat{c}_o)$.

Q.E.D.

Notice that our results concerning Pareto optimality differ from those of Lucas (1972). The reason is that Lucas assumed that transfers were proportional to the money holdings of the consumer. In our model transfers are given exogenously; hence, for higher rates of expansion of the money supply the individual tries to economize in real money balances and so consumes more when young. More to the point, $\sigma \neq 0$ imposes a wedge between the marginal rate of substitution (MRS) and the marginal rate of transformation which is unity along the PPF.

Existence of a stationary equilibrium with a positive price for money requires that the MRS evaluated at the initial endowment point be less than $1/(1+\sigma)$. When $w_o > 0$, if the factor of monetary expansion, $1 + \sigma$, becomes too large, there will exist no pair c_y, c_o with $u'(c_y) = \delta u'(c_o)/(1+\sigma)$, $c_y + c_o = \overline{w}$, *and* $c_o > w_o$. Hence in such a case no stationary monetary equilibrium will exist. We can, however, prove the following result:

PROPOSITION 3. *For a given* $\overline{w}$ *suppose* $w_y > \hat{c}_y$ *and consequently* $w_o < \hat{c}_o$. *Then there exists* $\epsilon > 0$ *such that for each* $\sigma \leqq \epsilon$, *there exists* $p_0(\sigma)$ *such that the sequence* $p_t(\sigma)$ *with* $p_{t+1}(\sigma) = (1 + \sigma)p_t(\sigma)$, $t = 0,1,\ldots$ *form an equilibrium.*

Proof. By assumption $u'(w_y) < \delta u'(w_o)$ and hence $u'(w_y) < \delta u'(w_o)/(1+\sigma)$ for any $\sigma < \epsilon$ for some $\epsilon > 0$. Consequently we may find $c_y(\sigma)$ and $c_o(\sigma)$ such that $c_y(\sigma) + c_o(\sigma) = \overline{w}$ and $u'[c_y(\sigma)] = \delta u'[c_o(\sigma)]/(1+\sigma)$ with $c_y(\sigma) > w_y$ and hence $c_o(\sigma) < w_o$. The result follows from Proposition 1.

The reader should notice that when the rate of deflation is high enough *every generation but the* 0^{th} *one is made worse off than at its initial endowment.* It may seem strange that the introduction of money makes one enter into transactions that lower utility. One should not forget, however, that in a deflation $h_t < 0$. Hence the deflation is equivalent to a tax on old people (except, of course, the first generation of old people).

If one would consider a model as the one considered here but with t ranging from $-\infty$ to ∞, then it is obvious that $\sigma = 0$ would yield the unique Pareto optimum stationary equilibrium. This is because in such a case there will be no 0^{th} generation being made better off by the increase in the real value of their money holdings caused by the expected deflation.

Turn now to the case of uncertainty.

3. The Money Supply Is Random

The purpose of this section is to formulate a notion of rational expectations equilibrium in the overlapping generations setup introduced in section 2 for the case when the money supply is *random*. After doing this we will demonstrate that for a given variance of the rate of growth of the money supply there is a deterministic growth rate that gives the same real allocation in equilibrium. This follows from observing that the price level is proportional to the quantity of money in a stationary rational expectations equilibrium. Hence it turns out that when the variance of the money supply increases the variance of the price level increases, and this in turn may increase the average real rate of return on cash over any interval of time due to the convexity of the function $f(p) \equiv 1/p$. Given that money provides the only means of carrying wealth forward from youth to old age in this model, it seems plausible that people will be made better off if the average real rate of return on this asset is increased.

Let us get into the details.

The real side of the model is the same as that in section 2. The only difference is that given $\tilde{m}_t^s$ at each $t + 1$ the transfer is given by

$$\tilde{h}_{t+1} \equiv (\tilde{A} - 1)\tilde{m}_t^s \equiv \tilde{\sigma}\tilde{m}_t^s \tag{5}$$

where $\tilde{A}$ is a random variable. For example, we will put

$$(6) \qquad \tilde{A} = \begin{Bmatrix} A + \lambda, \text{probability } 1/2 \\ A - \lambda, \text{probability } 1/2 \end{Bmatrix}$$

where A is constant and study what happens to equilibrium values when λ increases.

At each time t the young solve

$$(7) \qquad \text{maximize } u[\tilde{c}_y(t)] + \delta E_t u[\tilde{c}_o(t+1)]$$

subject to

$$(8) \qquad \tilde{p}_t \tilde{c}_y(t) + \tilde{m}_y(t) = \tilde{p}_t w_y$$

$$(9) \qquad \tilde{p}_{t+1}\tilde{c}_o(t+1) = \tilde{m}_y(t) + \tilde{h}_{t+1} + \tilde{p}_{t+1}w_o$$

$$= \tilde{m}_y(t) + (\tilde{A} - 1)\tilde{m}_t^s + \tilde{p}_{t+1}w_o.$$

Here E_t denotes the mathematical expectation conditional on information available at t. Also, variables with tildes are random.

The old consume at t according to

$$(10) \qquad \tilde{p}_t \tilde{c}_o(t) = \tilde{m}_y(t-1) + \tilde{h}_t + \tilde{p}_t w_o.$$

We may now define a rational expectations equilibrium.

DEFINITION. *Given the random process $\{\tilde{m}_t^s\}_{t=0}^{\infty}$, a sequence of random variables $\{p_0, \tilde{p}_1, \tilde{p}_2, \ldots\}$ is a rational expectations equilibrium (R.E.) if money demand equals money supply and goods demand equals goods supply for almost every realization of $\{\tilde{m}_t^s\}_{t=0}^{\infty}$. That is, for almost all realizations of $\{\tilde{m}_t\}_{t=0}^{\infty}$ we have*

$$(11) \qquad \tilde{c}_y(t) + \tilde{c}_o(t) = w_o + w_y = \overline{w}, \quad t = 0,1,2,\ldots$$

$$(12) \qquad \tilde{m}_t^s = \tilde{m}_y(t), \quad t = 0,1,2,\ldots.$$

The definition just says that people know the probability distribution of the money supply process and the price level process and that their expectations are confirmed.

Time 0 is special since we must specify the initial stock of money as the holdings of the old that are living at time 0. That is,

$$(13) \qquad m_0^s = m_o(0)$$

and

$$(14) \qquad p_0 c_o(0) = m_o(0) + p_0 w_o = m_0^s + p_0 w_o.$$

Hence, as in the certainty case, we are assuming that

$$(15) \qquad p_0 = m_0^s / [c_o(0) - w_o].$$

As in the certainty case we will choose $c_o(0)$ and define p_0 via (14) in order to create an equilibrium candidate that is stationary in the real variables.

In order to solve for an R.E., let us derive the first-order necessary conditions for an optimum of $(\tilde{P})$ given $\{\tilde{p}_t\}$. From (8) and (9) we get

(16) $$u[c_y(t)] + \delta E_t u[c_o(t+1)]$$
$$= u[w_y - \tilde{m}_y(t)/\tilde{p}_t] + \delta E_t u\left\{[\tilde{m}_y(t) + (\tilde{A} - 1)m_t^s]/\tilde{p}_{t+1} + w_o\right\}.$$

Differentiate both sides of (16) with respect to $\tilde{m}_y(t)$ to get

(17) $$-u'[c_y(t)](1/\tilde{p}_t) + \delta E_t\{u'[c_o(t+1)](1/\tilde{p}_{t+1})\} \leqq 0 \quad [=0, \text{ if } \tilde{m}_y(t) > 0].$$

Now in an R.E. it is necessary that

(18) $$\tilde{m}_t^s = \tilde{m}_y(t), \quad t = 0, 1, 2, \ldots$$
$$m_0^s \equiv m_o(0).$$

Put (17) in real balance form, and use (18) to get

(19) $$-u'[c_y(t)]\tilde{x}_t + \delta E_t\{u'[c_o(t+1)][\tilde{m}_y(t)/\tilde{p}_{t+1}]\}$$
$$= -u'[c_y(t)]\tilde{x}_t$$
$$+ \delta E_t\{u'[c_o(t+1)][\tilde{m}_y(t+1)/\tilde{p}_{t+1}][\tilde{m}_y(t)/\tilde{m}_y(t+1)]\}$$
$$= -u'[c_y(t)]\tilde{x}_t + \delta E_t\{u'[c_o(t+1)](\tilde{x}_{t+1})[\tilde{m}_s(t)/\tilde{m}_s(t+1)]\} \leqq 0.$$

Here

(20) $$\tilde{x}_t \equiv \tilde{m}_y(t)/\tilde{p}_t.$$

In order to find a solution to the R.E. equations we will try for an R.E. of the form

(21) $$\tilde{c}_o(t) = \bar{c}_o, \quad \tilde{c}_y(t) = \bar{c}_y, \quad \tilde{x}_t = \bar{x} > 0, \quad \tilde{p}_t = \tilde{m}_s(t)/\bar{x}$$

where all barred quantities are constant and nonrandom.

From (19) and (21), if $\bar{x} > 0$, we must have

(22) $$-u'(\bar{c}_y) + \delta u'(\bar{c}_o) E_t\{\tilde{m}_s(t)/\tilde{m}_s(t+1)\} = 0.$$

But

(23) $$E_t\{\tilde{m}_s(t)/\tilde{m}_s(t+1)\} = E_t\{1/\tilde{A}\} = 1/2\{[1/(A+\lambda)] + [1/(A-\lambda)]\}.$$

Notice that the right-hand side of (23) increases as λ increases for $0 \leqq \lambda < A$. Let

(24) $$\bar{R} \equiv E_t\{1/\tilde{A}\}.$$

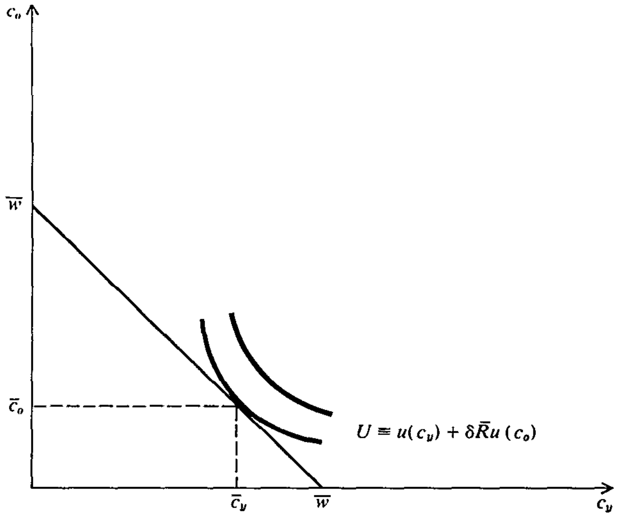
Can we find $(\bar{c}_y, \bar{c}_o)$ such that

(25) $u'(\bar{c}_y) = \delta\bar{R}\, u'(\bar{c}_o)$

(26) $\bar{c}_y + \bar{c}_o = \overline{w} \equiv w_y + w_o$?

Examine Figure 1.

Figure 1



Obviously the point of tangency in Figure 1 solves (25) and (26). Now define p_0 by

(27) $p_0\bar{c}_o = m_0^s + p_0 w_o$

that is,

(28) $p_0 = m_0^s/(\bar{c}_o - w_o)$.

Define $\bar{x}$ by

(29) $m_0^s/p_0 = \bar{x} = \bar{c}_o - w_o$.

Put

(30) $\tilde{p}_1 = \tilde{m}_1^s/\bar{x},\ \tilde{p}_2 = \tilde{m}_2^s/\bar{x}, \ldots.$

We may sum up the above into

THEOREM 1. *If (25) and (26) have a solution* $\bar{c}_y > 0$, $\bar{c}_o > 0$, *and* $\bar{c}_o - w_o > 0$, *then* $\bar{x} = \bar{c}_o - w_o$, $p_0 = m_0^s/(\bar{c}_o - w_o)$, $\tilde{p}_t = \tilde{m}_t^s/\bar{x}$, $t = 1, 2, \ldots$ *is an R.E.*

Proof. This follows immediately by the construction of the R.E. welfare economics.

It is interesting to explore what values of (A, δ_o, λ) lead to a Pareto improvement over $(A_o, \delta_o, \lambda_o)$. To examine this question notice that the young achieve the highest utility when $(\bar{c}_y, \bar{c}_o)$ solves

$$\text{(31)} \qquad \text{maximize } u(c_y) + \delta u(c_o)$$
$$\text{subject to } c_y + c_o = \overline{w} \equiv w_y + w_o.$$

This is so because steady state utility for people born at 0, 1, 2, . . . is highest for solutions of (31).

However, the utility of old people living at $t = 0$ may not be the highest possible at $c_o = \bar{c}_o$.

We will show that there exist initial conditions $(A_o, \delta_o, \lambda_o)$ such that increasing the variance of the money supply will make everyone better off in R.E. For let $A_o > 1$, $\lambda_o = 0$. Then

$$\delta_o \bar{R} = \delta_o / A_o < \delta_o.$$

Let

$$c_y\,(A_o, \delta_o, \lambda_o), \quad c_o\,(A_o, \delta_o, \lambda_o)$$

denote the solution of (25) and (26). Now let λ_o increase to a value $\bar{\lambda}_o$ such that

$$1/2\delta \left\{ \frac{1}{A_o - \lambda_o} + \frac{1}{A_o + \bar{\lambda}_o} \right\} = \delta_o.$$

Obviously the utility of the people born at 0, 1, 2, ... is increased by this move since

$$\delta_o \bar{R} = \delta_o$$

after the increase in λ_o to $\bar{\lambda}_o$.

What about the utility of the old people at time 0? Their consumption rises since

$$c_o(A_o, \delta_o, \lambda_o)$$

increases in λ for $A_o > 1$, $\lambda_o = 0$. Hence the elderly living at 0 are better off as well.

What are the economics behind this apparently paradoxical result? Since $A_o > 1$, inflation is going on at the rate $A_o - 1$ when $\lambda = 0$. Money is the only way for the young to carry wealth from the period of high endowment to the period of low endowment. Recall that $w_y > w_o$. But when $A_o > 1$, money is depreciating at the rate $A_o - 1$. So the young consume more in their youth than is optimal, for they would like to consume to maximize $u(c_y) + \delta u(c_o)$ subject to $c_y + c_o = \overline{w}$.

But now let $\lambda > 0$. The real rate[3] of return on cash balances increases due to the convexity of $f(p) = 1/p$. Hence money depreciates less, and the young are made better off in lifetime utility terms since they save more for their old age even though on the average inflation is proceeding at the rate $A_o - 1$.

Turn to the old at time 0. Only the elderly receive fiat money transfers in this model. The money held by the elderly at time 0 is worth more to the young living at time 0 after λ increases. Hence the young are willing to pay more in terms of the consumption goods to the old at time 0 in order to obtain their money. Hence the old at 0 are better off.

For what positions $Z_o \equiv (A_o, \delta_o, \lambda_o)$ is it possible to find $Z = (A, \delta_o, \lambda)$ such that everyone is better off in the R.E. $[c_y(Z), c_o(Z)]$? The lifetime utility of young people living at $t = 0,1,2,\ldots$ is increased whenever $\bar{R}$ is made closer to unity. This is so because lifetime utility is maximized at the choice of (c_y, c_o) that solves

$$\text{maximize } u(c_y) + \delta u(c_o)$$

subject to

$$c_y + c_o = \overline{w}$$

that is (provided that $c_y > 0$, $c_o > 0$),

$$u'(c_y) = \delta u'(c_o), \quad c_y + c_o = \overline{w}. \tag{32}$$

But in R.E. we have

$$u'(c_y) = \delta \bar{R}\, u'(c_o), \quad c_y + c_o = \overline{w}. \tag{33}$$

Hence driving $\bar{R}$ nearer to unity leads to an increase in lifetime utility for young people living at $t = 0, 1, 2, \ldots$.

[3]This result is similar to the result that the real rate of return on bonds may increase when the variance of the rate of inflation increases. In order to see the latter, let

(a) $Q_{t+1} = (1 + r)Q_t$

(b) $P_{t+1} = (1 + \pi)P_t$

describe the nominal rate of return on bonds and the evolution of the price level, respectively. Here r and π are the nominal rate of return on bonds and the (random) rate of inflation, respectively. Then the expected real rate of return on bonds in $t+1$ given t is given by

(c) $E_t\{[Q_{t+1}/P_{t+1} - Q_t/P_t]/(Q_t/P_t)\} = E_t\{(1 + r)/(1 + \pi) - 1\}$.

Now for simplicity consider the random variable

$$\pi \begin{cases} = \bar{\pi} + \lambda, \text{ probability } ½ \\ = \bar{\pi} - \lambda, \text{ probability } ½. \end{cases}$$

An increase in λ corresponds to an increase in the variance of π about its mean $\bar{\pi}$. It is trivial to check that an increase in π will increase the right-hand side of (c). Hence, in this case, an increase in the variance of the rate of inflation will increase the real rate of return on bonds. This type of result is general for symmetric distribution functions.

What about the old living at $t = 0$? It is easy to see that an increase in $\bar{R}$ leads to an increase in c_0. Totally differentiate both sides of (33) with respect to $\bar{R}$ in order to prove this result. Hence any increase in R_o when $R_o < 1$ is Pareto superior. If $R_o \geqq 1$, further increases benefit the aged at $t = 0$ but harm everyone else.[4]

R_o may be increased by decreasing A or by increasing λ. Hence when $R_o < 1$ we see that increasing the variance of the rate of growth of the money supply or decreasing the mean of the rate of growth of the money supply leads to a welfare improvement for everyone.

It is worth pointing out how the graphical apparatus used to illuminate Propositions 1 and 2 may be used to study the case of uncertainty. Notice that

$$\bar{R} \equiv E_t\{\tilde{m}_s(t) / \tilde{m}_s(t+1)\}$$

plays the role of $1/(1 + \sigma)$ in Propositions 1 and 2. Hence, define $\bar{\sigma}$ by

$$\frac{1}{1+\bar{\sigma}} \equiv \bar{R}.$$

Call $\bar{\sigma}$ the equivalent deterministic growth rate. Then look at the intersection of the ICC($\bar{\sigma}$) and the PPF as in section 2. This intersection is the R.E. studied earlier in section 3. Increasing the variance of $\tilde{A}$ amounts to decreasing $\bar{\sigma}$, so it is not surprising that welfare is increased if $\bar{\sigma} > 0$. In fact, as pointed out to us by Don Roper of the University of Utah, it might be more natural to view $\bar{\sigma}$ as a measure of the mean rate of growth of the money supply.

It is natural to look for the analogue of Friedman's optimum quantity of money in this model for the case $\lambda = 0$. A Pareto optimum is located at $R_o = 1$:

$$R_o \equiv 1/A_o \equiv 1. \tag{34}$$

Hence constant money supply is Pareto optimum. This is the same as Friedman's case, since he manages the money supply so that the real rate of return on it is zero. His case corresponds to $R_o \equiv 1$ in our model. Because people discount the utility of their progeny, Friedman needs to contract the money supply at the rate of time preference in order to force the real rate of return on it to be zero. However, all policies $A_o \leqq 1$ are Pareto optimal in our model. The reader is referred to Brock 1974 for an explicitly formulated version of Friedman's model.

[4]It is fruitful to think of $R - 1$ as a measure of distortion introduced by monetary policy. This is so because $R = 1$ is optimal in the sense that steady-state utility and hence equilibrium utility is maximal for all generations except the elderly living at time 0.

If one asks the question as has been suggested by Hurwicz — What monetary policy R will optimize equilibrium utility for all generations beginning at $T = -\infty$ and ending at $t = +\infty$? — then the answer is to put $R = 1$.

The $R = 1$ policy is attractive since a move to it from any policy $R \neq 1$ will help an infinitude of generations and harm at most the elderly living at time 0. Out of all the Pareto optima $R \geqq 1$, the $R = 1$ policy appears especially attractive since it maximizes capitalized steady-state utility.

One should not make too much of the result that an increase in variance of the money supply may increase welfare. There is no real uncertainty in our model. That is, real balances are deterministic. The result merely says that for a given random rate of growth of the money supply there is an equivalent deterministic monetary policy that yields the same value of $\bar{R}$. We are not advocating that the Fed start flipping coins in order to figure out what to do next.

4. Other Equilibria

It was Gale (1973) who first showed in the nonstochastic case with constant money stock that besides the stationary equilibrium studied above there exists in general a continuum of equilibria. This result still persists in our model. The treatment that follows borrows from Gale's treatment of what he calls the "Samuelson case," and we wish to thank Charles Wilson of the University of Wisconsin for pointing out to us the relevance of Gale's work to our problem.

Turn back to the beginning of section 3. Let us write equation (19) in real balance form in order to study expeditiously all the R.E. equilibria with $\bar{p}_t > 0$, $t = 0,1,2,\ldots$ that are *not random* in real balances, that is,

$$-u'(w_y - x_t)x_t + \delta E_t\{u'(w_o + x_{t+1})(x_{t+1})[m_s(t)/m_s(t+1)]\} \leqq 0 \quad (= 0 \text{ if } x_t > 0). \tag{35}$$

Notice that we did not put a tilde on x_t, x_{t+1} since we are restricting ourselves to the study of the selections of (35) that are nonrandom. Now

$$E_t\{\tilde{m}_s(t)/\tilde{m}_s(t+1)\} = E_t\{1/\tilde{A}\} \equiv \bar{R}. \tag{36}$$

Hence (35) becomes

$$-u'(w_y - x_t)x_t + \delta\bar{R}u'(w_o + x_{t+1})x_{t+1} \leqq 0 \quad (= 0 \text{ if } x_t > 0). \tag{37}$$

Let us restrict attention to those solutions of (37) where $x_t > 0$, $t = 0,1,2,\ldots$. A sufficient condition for $x_t > 0$, $t = 0,1,2,\ldots$ will be given shortly. Hence

$$u'(w_y - x_t)x_t \equiv A(x_t) = \delta\bar{R}u'(w_o + x_{t+1})x_{t+1} \equiv B(x_{t+1}). \tag{38}$$

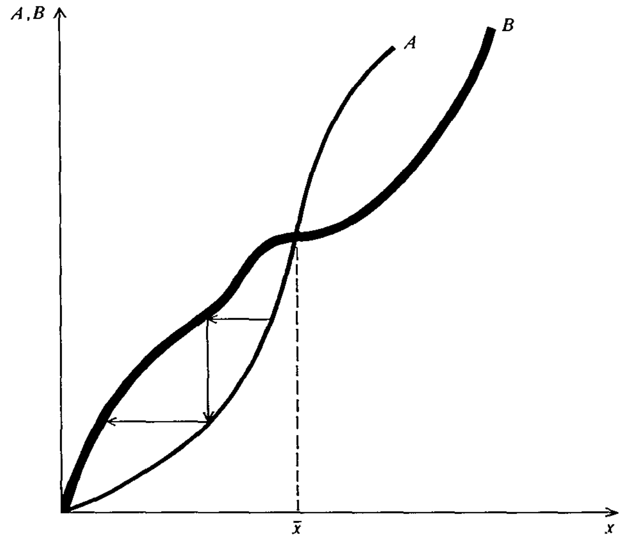
We will argue below that under reasonable assumptions on u Figure 2 tells the story. Figure 2 is intended to capture the following basic properties:

(a) $A(0) = B(0) = 0$.
(b) B may not decrease for $x > \bar{x}$.
(c) A is always increasing.
(d) $A'(0) < B'(0)$.
(e) There is just one $\bar{x} > 0$ such that $A(\bar{x}) = B(\bar{x})$; that is, $A(x) < B(x)$ for $x \in (0,\bar{x})$.

If $w_y > 0$ and $w_o > 0$, then (a) obviously holds. Property (b) will be assumed throughout. To prove (c), calculate

$$A'(x) = -u''(w_y - x)x + xu'(w_y - x) > 0, \quad \text{for } x > 0 \tag{39}$$

Figure 2



since

$$u'' \leqq 0, \quad u' > 0, \quad x > 0.$$

Turn now to (d). From (39) we obtain

(40) $$A'(0) = 0, \text{ provided } w_y > 0.$$

Calculate $B'(0)$. We have

(41) $$B'(x) = \delta R u'(w_o + x) + \delta R u''(w_o + x)x, \quad B'(0) = \delta R u'(w_o) > 0.$$

Hence (d) holds.

Let us examine (e). Here for $x > 0$

(42) $$A(x) = B(x)$$

if

(43) $$u'(w_y - x) = \delta R u'(w_o + x).$$

The left-hand side of this is increasing while the right is decreasing. Hence the assumption that $u'(w_y) < \bar{R}u'(w_o)$ is sufficient for (e). This assumption will be maintained throughout.

The following theorem obtains.

THEOREM 2. *Let the difference equation $A(x_t) = B(x_{t+1})$ satisfy properties (a)–(e). Then (i) $x_t \searrow 0$, $t \to \infty$ or (ii) $x_t = \bar{x}$, $t = 0,1,2,\ldots$ or (iii) $x_t \nearrow \infty$, $t \to \infty$, for any solution $\{x_t\}$.*

Proof. This is obvious from Figure 2. Here $x_t \searrow 0$ means that the sequence x_t decreases to 0, $t \to \infty$. Analogously $x_t \nearrow \infty$, $t \to \infty$.

Remark. Notice that the function $B(\cdot)$ is not required to be nondecreasing on $[0, \bar{x}]$ and that $A(x_t) = B(x_{t+1})$ may be satisfied for more than one x_{t+1} for each x_t.

We have proven above that (c), (d), and (e) must hold in the class of models we are considering and that (a) holds when $w_y > 0$ and $w_o > 0$. The following lemma will be useful in verifying condition (b).

LEMMA. *If for $y > 0$, $[-u'(y)] / [u''(y)\, y] \geqslant 1$ (that is, the coefficient of relative risk aversion is $\geqslant 1$), then $B'(x) > 0$ for $x > 0$.*

Proof. $[-u'(y)] / [u''(y)\, y] \geqslant 1$ implies $u''(w_o + x)(w_o + x) + u'(w_o + x) > 0$ for any $x > 0$. Now since $u''(w_o + x)\, w_o < 0$, we must have $u''(w_o + x)\, x + u'(w_o + x) > 0$. The result follows from (41).

Q.E.D.

Solutions $\{x_t\}$ such that $x_t \to \infty$ as $t \to \infty$ cannot be R.E.'s, since from the budget constraint $x_t \leqslant w_y$ must hold. All other solutions are equilibria, however.

THEOREM 3. *Let $\{x_t\}_{t=0}^{\infty}$ be any solution of $A(x_t) = B(x_{t+1})$ with $x_t \leqslant w_y$ for $t \geqslant 0$. Let $\tilde{p}_t = \tilde{m}_t^s / x_t$, $t = 0,1,2,\ldots$.* Then the stochastic process $\{\tilde{p}_t\}_{t=0}^{\infty}$ *is an R.E. price level sequence.*

Proof. Just check that $\{\tilde{p}_t\}$ satisfies the definition of an R.E.

Notice that $x_t \to 0$, $t \to \infty$ implies that people are ultimately driven to the consumption of their endowments. Such an equilibrium is not Pareto optimal because there exists a reallocation that makes some generations better off while making no other generation worse off. To construct such an allocation, simply wait until a time T_o when $x(T_o)$ is near zero and give some of the young person's endowment to the old living at T_o. Do this for all $t \geqq T_o$. It is trivial to see that this proposed reallocation is a Pareto improvement because at T_o, for example, the young who are harmed at T_o are given enough more (in their old age) by the young at $T_o + 1$ so that on the whole they are better off.

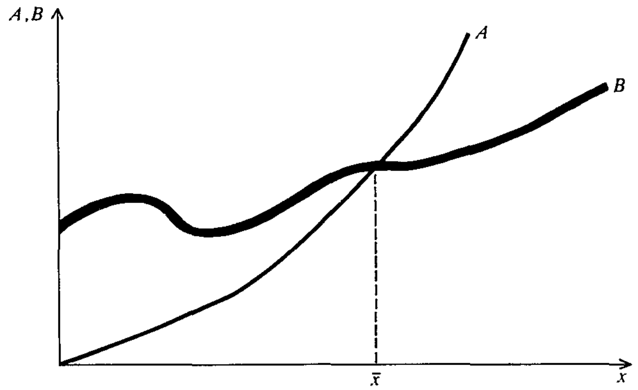
Notice that in such a case the welfare implications of a change in the rate of expansion of the money supply is no longer obvious, since one has to know which equilibria the economy will go to. Two solutions to this problem are possible. One is to introduce a weaker ordering requiring only the existence of better equilibria, and in this case the results of section 4 will hold. Another is to try to get rid of the nonstationary equilibria. This can be accomplished either by making an alternative assumption to (a) or by changing the specification of the model.

We first study the case in which the following condition holds besides (b)–(e).

(a') $B(0) > 0$, $A(0) = 0$.

The picture is now as in Figure 3. As before, if $x_0 < \bar{x}$ then $x_{t+1} < x_t$. But now $B(x) > \epsilon$ for all $x \geqslant 0$ for some $\epsilon > 0$. And $\lim_{x \to 0} A(x) = 0$. Hence eventually

Figure 3



solutions with $x_0 < \bar{x}$ cannot be continued. Hence we have proven

THEOREM 4. *Under (a′) and (b)–(e), the stationary equilibrium is the unique R.E. which is nonrandom in real balances.*

An obvious example in which (a′) holds is given by $u(y) = \log y$ and $w_y > 0$, $w_o = 0$. In this case (b) also holds, as can be verified by using the lemma. Since (c), (d), and (e) are general properties in our model, Theorem 4 will be true in such a case.

We now return to the case where (a) holds instead of (a′). What is the economic meaning of equilibria when $x_t \to 0$, $t \to \infty$, and should we take such equilibria seriously or do they just reflect a specification of the model that is too loose for sound economics? As the model stands, it is silent about which of the multitude of equilibria the economy will follow for a given money supply process $\{\tilde{m}_t^s\}_{t=0}^{\infty}$. Notice also that the price level grows faster than the money supply along equilibria such that $x_t \to 0$, $t \to \infty$. We shall henceforth call such equilibria *hyperinflationary*.

The economic story is as follows. Suppose people expect a hyperinflationary path of the price level and act on such expectations. They will be fulfilled! Hence people by the very act of forming expectations on the nominal price level can have an impact on the evolution of real balances and drive themselves into a hyperinflation. Furthermore, only one of the infinitude of equilibria is not hyperinflationary.[5]

[5]It is perhaps appropriate at this point to quote from an Australian journalist, P. P. McGuinness (1977), who is describing to his readers what he calls a "recent innovation in monetarist thinking," namely, "rational expectations":

> For example, it has been suggested that attitudes to government deficit financing will have a major influence on whether it is inflationary or not. If people believe that the financing of the deficit must be inflationary, then they will act in such a way that that belief will be validated.

McGuinness comes very close here to describing a situation where beliefs themselves, independent of the underlying money supply process, determine the equilibrium path of the price level, and that is exactly the situation described by Theorem 3.

We are not sure whether to attach much economic significance to such equilibria for reasons that are outside the model. For example, with an additively separable utility function, if any change is made in the specification of the model that requires the holding of at least $\epsilon > 0$ real balances no matter how small ϵ is, then only one of the equilibria remains: $x_t = \bar{x}$, $t = 0,1,2,\ldots$. This is so because looking at Figure 2 we see that there will be a first time where the demand for real balances by the young which is greater than or equal to ϵ will be smaller than the supply by the old.

What changes in the specification of the model will lead to the holding of at least $\epsilon > 0$ worth of real balances for all t? A time-honored one is to require each generation while young to set aside ϵ units of purchasing power which is paid out in taxes to the government during that youth. This money is then returned to the young in a lump sum at the beginning of their old age. This social security system in effect requires the young to carry ϵ units of real balances. Notice that ϵ may be chosen arbitrarily provided it is less than the difference between w_y and the consumption of the young at the stationary equilibrium. Hence, for example, in our model an *arbitrarily small amount of social security coupled with a monetary policy that makes* $\bar{R} = 1$ *leads to a Pareto optimum.* Generating a demand for fiat money by requiring it to be used for payment of taxes is a common device in the theoretical money literature. But Edward Sieper of the Australian National University reminded us of Gale's "business cycle" which is depicted in Figure 3b of COZ. So a device that rules out equilibria where real balances go to zero is not enough to rule out equilibria that cycle, for instance. Furthermore, after looking at COZ we are convinced that one can construct just about any type of equilibrium.

Another device is to introduce a transaction technology into the model that leads to an indirect utility function that would lead to the demand for at least $\epsilon > 0$ units of real balances in order to facilitate transactions. It seems reasonable to conjecture that there exists a Becker-Lancaster activity analysis model of the household that would lead to the retention of some positive number $\epsilon > 0$ units of real balances for a given future bounded interval of rates of change of the price level.[6]

[6]The requirement that there exist a bounded interval that contains the rate of price level changes against which the economy is posited to retain real cash balances of at least $\epsilon > 0$ is important. Obviously no such $\epsilon > 0$ will exist in any reasonable economic model of the demand for money if the rate of change of the price level is unbounded.

Let us examine the nonrandom case.

Is p_{t+1}/p_t associated with $x_t \to 0$, $t \to \infty$, $A(x_t) = B(x_{t+1})$ bounded? Let $m_t^s = (1 + \sigma)^t m_0^s$, $m_0^s > 0$. Then

$$x_{t+1}/x_t = (1 + \sigma)(p_t/p_{t+1})$$

so that

$$p_{t+1}/p_t = (1 + \sigma)(x_t/x_{t+1}).$$

Now

$$u'(w_y - x_t)x_t = \delta R u'(w_o + x_{t+1})x_{t+1}$$

so that

$$x_t/x_{t+1} = \delta R u'(w_o + x_{t+1})/u'(w_y - x_t) \simeq \delta R u'(w_o)/u'(w_y)$$

More devices of introducing a minimal demand for cash balances for rates of change of the price level contained in a given bounded interval may be generated by the imaginative economist. Notice that the $x_t \searrow 0$ equilibria depicted in Figure 2 have bounded rates of change of the associated price level. This line of thought is pursued in Scheinkman's discussion in this volume.

5. Some Remarks on the Meaning of All This

In this paper we set up an overlapping generations model of a monetary economy that followed Samuelson 1958 and used it to study, for example, the robustness of some results that had been previously derived from models with real balances in the utility function. The device of putting real balances into the utility function has been controversial since the services of money are not explicitly modeled in such a formulation.

We found that, roughly speaking, perfectly anticipated inflations lead to welfare losses in such a model. In particular, let

$$R = E(1/\tilde{A})$$

where $A = 1 + \tilde{\sigma}$, $\tilde{\sigma}$ equals the random rate of growth of the money supply, and E equals the mathematical expectation of the random variable $1/\tilde{A}$. Then if $R_0 < 1$, any change that leads to an increase in R_0 leads to an increase in welfare for all generations. If $R_0 \geqq 1$, an increase in R_0 leads to an increase in the welfare of the elderly living at $t = 0$ but a fall in the welfare of all other generations.

These results follow from the assumptions that the indifference curve of the young at the endowment point cuts the PPF from below (that is, the marginal rate of substitution is less than one) and that money is the only means of transferring wealth from youth to old age.

It is natural to ask if these results are themselves robust to minor alterations of the model. Clearly if a capital good (which may be just inventories of consumption goods) is introduced that pays a return greater than that on cash balances (which is negative in an inflation), then the demand for money may disappear; with an alternative, more efficient store of value, no equilibrium with a positive price of money may exist. But if one introduces transaction costs for getting in and out of capital, then the demand for money with associated positive equilibrium price for money is likely to reappear if such transaction costs are large enough, and our results on the inflation tax, for example, are likely to obtain.

If the indifference curve of the young cuts the PPF from above at the endowment point, then there will be difficulties with existence of equilibrium with a positive price of money, since the young will want to borrow but the old will want to consume rather than lend. This observation leads us naturally to the question of whether overlapping generations models are a good foundation for monetary theory.

Erecting a monetary theory on the formulation of an overlapping generations model with the time span of a generation covering some twenty-five

for x_t, x_{t+1} near zero. Hence the growth factor p_{t+1}/p_t converges to the finite number

$$(1 + \sigma)\delta R u'(w_o)/u'(w_y) = \delta u'(w_o)/u'(w_y)$$

as $t \to \infty$. And hence the rate of growth of the price level is bounded along equilibria $x_t \to 0$, $t \to \infty$.

The random case is similar.

years will strike the reader as bizarre indeed. We defend studying such a model because it allows us to gain insights into what is likely to happen in a more complicated model where a "generation" is one or two weeks, an individual lives for from five to six thousand "generations," and people carry money in order to consume in weeks they don't receive a paycheck and because it is too costly to move in and out of higher yielding capital goods every few weeks.

Now in such a model, for every individual who receives a large money endowment and a small real endowment at the beginning of "generation" t there must be another individual whose net money endowment is low and whose whole real endowment is high at the beginning of "generation" t, so that gains from trade are possible, and thus the aggregate stock of money has a chance of being held, in equilibrium, by the economy as a whole. Furthermore, the endowment (or wage) pattern (for each individual) of real goods and money must alternate in order to give individuals an incentive to carry money forward. Are such conditions satisfied well enough in the real world so that Gale's "Samuelson case" is a good abstraction? We don't know.

It is to capture the salient features of this more complicated economy that the fictions of generation, small endowment at old age, large endowment at youth, and each old generation having an offsetting young generation were introduced. Obviously in the overlapping generations case it may be more realistic to look at the case of low endowment at youth and high endowment in the second period of life, which Gale (1973) calls the "classical case." But there is no equilibrium with positive price of money in this case unless the indifference curve of the young through (w_y, w_o) cuts the PPF from below, and this is unlikely if $w_y < w_o$. There seems to be no compelling reason to reject Gale's "classical case" just because it is inconvenient to those who want to erect monetary theory on the foundations of the overlapping generations model (OLG).

Wallace's paper in this volume makes a spirited argument that overlapping generations models are likely to be the best models of money. He argues that fiat money is intrinsically useless and inconvertible. Hence, he argues, the continuum of equilibria in the OLG reflects just the "tenuousness" of equilibrium value of an asset that is intrinsically useless.

But two papers by Charles Wilson (1978a,b) explore a model that is more in the spirit of Clower in the sense that the equilibrium value of money ultimately derives from the "Clower budget constraint" that the value of an agent's current consumption cannot exceed the value of the agent's money holdings at the beginning of the trading period. (The Lucas paper in this volume also studies a model of this type.) Wilson's papers show that a multitude of equilibria exist when agents are finite lived. This seems to capture the tenuousness that Wallace is looking for without having to go through such contortions as ruling out Gale's "classical case" or having to rule out capital goods that can be costlessly traded.

How are we as economists to view the multitude of equilibria? The OLG gives no clue as to which one the economy will converge to, if any. Are we to believe that all are equally likely? Are we to make even money bets that a constitutional amendment to hold the money supply constant in a constant population deterministic economy with zero technological change will lead to rampant inflation? Hardly.

Yet to rule out such equilibria because they do not accord well with empirical work that supports the quantity theory as a long-run proposition (see Lucas 1978a) leaves unexplained the mechanism by which the economy converges to an equilibrium where the price level is proportional to the quantity of money.

We can follow Scheinkman's reasoning in this volume that if money is essential (in a sense that he makes precise) then all candidate equilibria with real balances converging to zero are untenable.

But this argument won't take care of periodic equilibria. In long-horizon models with a small rate of time preference and near linear utility, intertemporal arbitrage will crush periodic equilibria. This suggests that in the real world the correct time metric for an OLG model to be a good model of money would allow it to admit long and/or damped cycles only. After all, if the cycle had a period of two weeks and a high amplitude it would be arbitraged down in the real world. If it had a period of ten years then matters are more problematic. Utility should display more curvature and time preference should increase as the time metric (relative to which such utility and time preference are measured) is elongated.

What about equilibria such that real balances remain bounded away from zero and infinity but display no recognizable growth pattern? Again, for reasonable preferences relative to a sensible time metric, the price level would not move much weekly, monthly, or possibly even yearly on such equilibria. What about movements over a decade? Here again the case is not so clear.

About all we can say for sure is that the complications of multiple equilibria for monetarist doctrine and monetary policy need more research.