Liquidity Traps and Capital Flows*

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Abstract

Do free capital flows lead to an efficient re-balancing of global demand in a world economy where some - but not all - countries experience a liquidity trap and monetary policy is conducted non-cooperatively? We study this question in a highly tractable non-linear continuous-time version of a standard New Keynesian open-economy model of the world economy featuring two country blocks: North and South. We consider a demand shock that pushes the North, but not the South, to the zero lower bound (ZLB) on interest rates. In the absence of capital account interventions, capital naturally flows downstream (i.e. from North to South) during the liquidity trap scenario. These flows reduce the adverse effects of the liquidity trap. But due to aggregate demand externalities, free capital flows are inefficient in the sense that they are Pareto-dominated by a regime of capital account management that subsidizes downstream flows during the trap and subsidizes upstreams flows thereafter. Exploiting the global efficiency gains from capital flow management nonetheless requires policy coordination, due to these policies' inherent redistributive implications.

*The views expressed in this paper are entirely those of the authors. They do not necessarily represent the views of the Federal Reserve Bank of New York or the Federal Reserve System.

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1 Introduction

Following the 2007-2008 financial crises, a large number of advanced economies (including the U.S., the U.K. and the Eurozone) have entered a period of very low or zero interest rates that resembles a liquidity trap. This period has also featured a surge in capital flows from advanced to emerging economies, which has prompted some emerging market governments to adopt policies restricting capital inflows. These policy responses have led observers to warn against the adverse spillover effects of such actions and to recommend that policies be better coordinated in a world where deficient aggregate demand is perceived as a first-order concern. In this paper, we question the efficiency property of an international monetary system of free capital flows in the context of a liquidity trap episode. To this end, we study the (i) direction, (ii) intensity, and (iii) timing of capital flows associated with regional liquidity traps, both when capital flows freely across countries *and* under capital flow management regimes where these flows are optimally taxed or subsidized.

We investigate the role played by capital flows and their management in a model of the world economy where a large demand shock in one region causes a liquidity trap, and induces capital to flow towards the rest of the world. The model features a continuum of small open economies, with some belonging to a *North* block and others belonging to a *South* block. The North is meant to represent a region of the world economy that suffers from temporarily deficient demand.[1] When the North is hit by the shock, it enters a regional liquidity trap. Under free capital flows, capital flows downstream (i.e. from the North to the South) during the liquidity trap and then comes to a halt. This pattern, we argue, is *inefficient* in the sense that there exists alternative capital flow management regimes that make some countries better off without making any country worse off throughout and past the liquidity trap episode. In particular, we find that the most desirable regime features larger downstream flows during the liquidity trap and positive upstream flows thereafter. This capital flow pattern contributes to re-direct demand to the region where it is temporarily relatively stronger, above and beyond what would occur under free capital flows.

Existing studies of liquidity traps in general equilibrium models of the world economy assume that monetary policy is set either cooperativey or naively.[2] In constrast, we assume

[1] Through the lens of the model, the demand-deficient North corresponds broadly to the US and Europe while the South can be thought of as emerging markets like Brazil, India etc. Under such an interpretation it is more natural to think of the southern economies as the ones who use capital controls to manage their currencies.

[2] See for example, Fujiwara et al. (2013) and Devereux and Yetman (2014)

that monetary policy is set non-cooperatively in each country.[3] Our analysis is conducted in a continuous time fully non-linear version of the Gali and Monacelli (2005) model with two key features. First, the world economy is split into two homogenous blocks of countries: North and South. Second, the ZLB on nominal interest rates is explicitly taken into account. In line with the closed economy literature on liquidity traps, we model a liquidity trap scenario using a temporary negative shock to the discount rate that generates a desire to temporarily reduce consumption and increase savings. In the basic New Keynesian closed economy without capital, this desire puts downward pressure on the natural real interest rate, and when demand stabilization commands a nominal interest rate below zero, the ZLB sets in motion an adverse feedback loop between plummeting demand, deflation and an excessively high real interest rate. In contrast, in an open economy context, a desire to save in one region can be accommodated by capital flows that result in an accumulation of claims vis-à-vis the rest of the world. But the extent to which this margin operates depends on the capital flow regime. When capital accounts are closed, this margin is absent and output dynamics mirrors consumption dynamics, as in the closed economy. Under free capital flows, in contrast, a negative demand shock in the North causes downstream flows during the liquidity trap scenario, and leads to a build up of claims of the North vis-à-vis the South that survives the temporary shock. These flows allow output dynamics to decouple from consumption dynamics, which results in a smoother output gap (or labor wedge) and an earlier ZLB exit in the North than under closed capital accounts.

Our main objective is to shed light on the normative properties of a free capital flow regime during a liquidity trap episode. To this end, we formulate a planning problem that consists of choosing paths for taxes or subsidies on downstream capital flows to maximize the North countries' welfare subject to making South countries as well off as under free (i.e. untaxed) capital flows. We show that a path of zero taxes is not a solution to the planning problem, i.e. that *free capital flows are inefficient*. Importantly, this qualitative result does not hinge on the planner's ability to use compensating transfers across countries. We then show that the solution to this planning problem features subsidies on downstream flows during the liquidity trap and taxes on downstream flows (or subsidies on upstream flows) for a brief period after the trap. Under the efficient capital flow management regime, downstream flows are therefore larger than under free capital flows during the trap, and upstream flows are positive after the trap. This tax pattern achieves an optimal decoupling of output from consumption in the North: when the marginal propensity to consume (MPC) in the North is low (during the

[3] Given that most central banks have explicit mandates to promote domestic interests, we view this assumption as the most plausible one.

trap), South consumption is subsidized (North consumption is taxed), but once MPC in the North recovers (immediately after the trap), South consumption is taxed (North consumption is subsidized).

Our results go against the conventional wisdom in international macroeconomics that free capital flows lead to efficient outcomes. Perhaps paradoxically, they also stand in sharp contrast to a recent literature on capital flow management that argues that free capital flows might be excessively volatile.[4] The reason behind this contrast is simple. This literature studies capital flow management from the perspective of individual capital flow recipient countries, while we take a global efficiency standpoint. In fact, our model features excessively volatile capital flows from the perspective of capital flow recipient countries, even though it features insufficiently volatile flows from a global perspective. To illustrate this point, we study a case where individual South countries optimally manage their capital account. We find that these countries optimally tax inflows during the liquidity trap. Such taxes on inflows reduce downstream flows, and as a result they increase output gap fluctuations and delay the optimal ZLB exit time in the North. Thus, capital flow management measures that are optimal from an inflow recipient perspective might hamper and delay macroeconomic stabilization in countries with deficient aggregate demand during a liquidity trap. These findings validate concerns about adverse spillover effects of capital controls in a liquidity trap.

Throughout our analysis, we adopt the assumption that prices are entirely rigid in producers' currency. This assumption makes us abstract from producer price index (PPI) inflation, but it allows us to streamline aspects of macroeconomic adjustment tied to capital flows and exchange rate dynamics. Despite assuming away PPI inflation, our model captures the feedback mechanism between consumer price index (CPI) deflation and the output gap that is believed to be a key characteristic of liquidity trap episodes. The main virtue of assuming fully rigid prices is to allow for an analytical characterization of optimal policy in the non-linear model as well as an exact numerical solution of the path of all relevant variables across all capital flow regimes we consider. Given the notorious non-stationarity of countries' wealth positions in open-economy models and the centrality of the wealth distribution dynamics for our question, we view this tractability as undeniable advantage.

The paper is organized as follows. The next subsection contrasts our contribution to the

[4]See Caballero and Krishnamurthy (2001), Korinek (2010), Bianchi and Mendoza (2010), Bianchi (2011), Jeanne and Korinek (2010), Benigno et al. (2013) and Bengui and Bianchi (2014) among others for theories based on financial market frictions. Alternatively, see Farhi and Werning (2012), Schmitt-Grohé and Uribe (2013) and Farhi and Werning (forthcoming) for theories based on goods market frictions.

existing literature. Section 2 describes the basic model setup and describes the equilibrium of a world economy divided into two blocks. Section 3 characterizes the optimal allocations from the point of view of the planner of an individual country. Section 4 describes optimal monetary policy in the North in response to any arbitrary path of capital controls in the South. Section 5 analyzes efficient capital flows. Section 6 analyzes non-cooperative capital flow management, and Section 7 concludes. All proofs can be found in the Appendix.

Related Literature

This paper can be seen broadly at the intersection of three strands of literature. The first studies optimal monetary policy in response to liquidity traps in small open economies. Similar to our findings, this literature finds that in a liquidity trap situation, the economy suffers from a negative output gap and an appreciated real exchange rate. Our findings resonate with Svensson (2001, 2003, 2004) who shows that an intentional currency depreciation and a crawling peg can induce private-sector expectations of a higher future price level and escape from the liquidity trap. Further he shows that this "*Foolproof Way*" implements the optimal policy. Similar to previous closed economy studies (for instance, Krugman (1998), Eggertsson and Woodford (2003) and most closely Werning (2012)), we find that optimal monetary policy calls for an extension of the time spent at the ZLB past the liquidity trap episode. This helps in implementing the "Foolproof Way". However, this paper also shows how the use of capital controls in the South may make such a policy infeasible.

The second strand of literature finds its roots in the Mundellian trilemma and is motivated by the fact that in recent decades, capital flows into emerging economies have largely been driven by global factors like cyclical interest rates rather than domestic fundamentals (Calvo et al., 1993, 1996). This literature looks at the rationale for the use of capital controls in emerging economies so as to enable independent monetary policy.[5] A large literature looks at the optimality of a managed capital account to prevent over-borrowing in response to sudden stops in non-monetary models controls (See Bianchi and Mendoza (2010), Bianchi (2011) and Jeanne and Korinek (2010) among others). However, these are real models and do not look at the rationale for controls in the presence of Keynesian Aggregate Demand externalities which arise in models with nominal rigidities. Farhi and Werning (forthcoming), much like this paper, look at the rationale for controls in an economy with nominal rigidities but unlike this paper assume that there is no feedback from/ on the rest of the world. Furthermore, they do not

[5] See for example Rey (2013).

look at the interaction of capital controls in response to a ZLB episode.

The third strand of literature looks at international policy spillovers at the ZLB and in this context look at the role for policy cooperation. Fujiwara et al. (2013) using a two-country model show that the optimal rate of inflation in one country is affected by whether or not the other country is in a liquidity trap. Haberis and Lipinska (2012) find that more stimulatory foreign policy worsens the home policymakers trade-off between stabilizing inflation and the output gap when home and foreign goods are close substitutes. Jeanne (2009) studies international policy spillovers and shows that a temporary increase in both countries' inflation targets may restore the first-best level of employment and welfare. Devereux and Yu (2014) show that international financial integration helps to diversify risk but also may increase the transmission of crises across countries.

A paper similar in spirit to ours is Devereux and Yetman (2014). As in this paper, they find that monetary policy effectiveness may be restored by the imposition of capital controls, which inhibit the transmission of these shocks across countries. However, the key difference from this paper is that they focus on a setting where monetary policy is set cooperatively across countries and conclude that even though capital controls may facilitate effective monetary policy, they are not desirable in welfare terms. This paper instead focuses on a setting where monetary policy is set non-cooperatively across countries and find that in some cases, capital controls can in fact raise welfare of all countries. Importantly, we jointly characterize the choice of optimal capital controls in the South and monetary policy in the North as the outcome of a cooperative and non-cooperative equilibria. This allows us to study the cases under which the use of capital controls in response to a liquidity trap can improve welfare in both the North and the South.

2 Model

The world economy consists of a continuum of countries of unit measure. These countries are separated into two blocks. For the purpose of this paper, we will refer to the block of *demand deficient* economies as the *North* and the rest of the economies as the *South*. North economies consists of the countries for which $k \in [0, x]$ and South economies consists of the countries for which $k \in (x, 1]$.[6] For ease of notation, we denote the set of North countries by the set $\mathcal{N}$. Thus, if a country $k \in \mathcal{N}$ refers to a North country while $k \notin \mathcal{N}$ denotes a South country.

[6]This labeling roughly corresponds to the global economic climate following the Great Recession, where the US and European economies were characterized by deficient demand.

2.1 Households

In each country k (we will refer to country k as the 'home' country for ease of exposition), there is a representative household with preferences represented by the utility functional

$$\int_0^\infty e^{-\int_0^t(\rho+\zeta_{k,h})dh}\left\{\frac{(\mathbb{C}_{k,t})^{1-\sigma}}{1-\sigma}-\frac{(N_{k,t})^{1+\phi}}{1+\phi}\right\}dt, \tag{1}$$

where $\mathbb{C}_{k,t}$ is consumption, $N_{k,t}$ is labor supply, σ is the inverse intertemporal elasticity of substitution, ϕ is the inverse Frish elasticity of labor supply, ρ is the discount rate and $\zeta_{k,t}$ is a time-varying and country specific preference shifter. The consumption index $\mathbb{C}_{k,t}$ is defined as

$$\mathbb{C}_{k,t}\equiv\left[(1-\alpha)^{\frac{1}{\eta}}\left(C_{k,t}^H\right)^{\frac{\eta-1}{\eta}}+\alpha^{\frac{1}{\eta}}\left(C_{k,t}^F\right)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{2}$$

where C_k^H denotes an index of all domestically produced varieties in country k, C_k^F is an index of all varieties imported into country k, α is a home bias parameter representing the degree of openess, and η is the elasticity of substitution between domestic and imported goods. Letting $l\in[0,1]$ denote the possible varieties of goods, we can define:

$$C_k^H\equiv\left[\int_0^1 C_k^H(l)^{\frac{\epsilon-1}{\epsilon}}dl\right]^{\frac{\epsilon}{\epsilon-1}} \tag{3}$$

where $C_k^H(l)$ denotes country k's consumption of variety l produced domestically, and ϵ is the elasticity of substitution between varieties produced withing a given country. Similarly, we define:

$$C_k^F\equiv\left[\int_0^1 {C_k^j}^{\frac{\gamma-1}{\gamma}}dj\right]^{\frac{\gamma}{\gamma-1}} \tag{4}$$

and

$$C_k^j\equiv\left[\int_0^1 C_k^j(l)^{\frac{\epsilon-1}{\epsilon}}dl\right]^{\frac{\epsilon}{\epsilon-1}} \tag{5}$$

where C_k^j denotes country k's consumption of the final good produced in country j, and γ is the elasticity of substitution between goods produced in different foreign countries.

The household's budget constraint is given by

$$\begin{aligned}\dot{a}_{k,t} &= i_{k,t}a_{k,t}+W_{k,t}N_{k,t}+T_{k,t}-\int_0^1 P_{k,t}^k(l)C_k^H(l)dl-\int_0^1\int_0^1 P_k^j(l)C_k^j(l)dldj\\ &\quad+\int_0^1\left[i_{j,t}-i_{k,t}+\tau_{k,t}-\tau_{j,t}+\frac{\dot{\mathcal{E}}_{k,t}^j}{\mathcal{E}_{k,t}^j}\right]\mathcal{E}_{k,t}^jD_{k,t}^jdj\end{aligned} \tag{6}$$

where $a_{k,t} \equiv \int_0^1 \mathcal{E}^j_{k,t} D^j_{k,t} dj$ are assets expressed in country k's own currency, $\mathcal{E}^j_{k,t}$ is the nominal exchange rate between country j and k, $D^j_{k,t+1}$ are the bonds issued by country j and held by country k at time t, $W_{k,t}$ is the nominal wage and $T_{k,t}$ denotes lump-sum transfers including the payout of domestic firms. We explicitly allow for taxes and subsidies on capital flows. $\tau_{k,t}$ is a tax on capital inflows (or a subsidy on capital outflows) in country k, and similarly $\tau_{j,t}$ is a tax on capital inflows (or a subsidy on capital outflows) in country j. The proceeds of these taxes are rebated lump sum to the households of country k and j, respectively.

The lump-sum rebate $T_{k,t}$ is given by

$$T_{k,t} = -\tau_{k,t} \int_0^1 \mathcal{E}^j_{k,t} D^j_{k,t} dj + \tau^L_k W_{k,t} N_{k,t} + \Pi_{k,t} \tag{7}$$

where τ^k_L is a constant labor tax levied on firms in country k, and $\Pi_{k,t}$ is firm profits.

2.2 Firms

Technology A firm in each economy k produces a differentiated good $l \in [0, 1]$ with a linear technology:

$$Y_{k,t}(l) = AN_{k,t}(l)$$

For simplicity, we assume that labor productivity A is identical across countries.

Price setting We assume that the price of each variety produced in country k is fully rigid, and normalize this price to 1. An implication of this assumtion is that the PPI of country k output in its own currency is fixed at 1.

The assumption of fully rigid prices can be regarded as an extreme one, but it has the virtue of allowing for an exact non-linear solution of the path of all relevant variables during and after a liquidity trap scenario. Such a non-linear solution is attractive, given the focus on large shocks and the well known unit-root property of the net foreign asset position of countries in open-economy settings.

This assumption rules out PPI inflation or delfation, so our analysis abstracts from the distortion caused by price dispersion typically present in sticky price environments à la Calvo. Instead, it solely focusses on the labor wedge (or output gap) and terms-of-trade management objectives of optimal policy making. At the same time, our price setting assumption does not eliminate the deflation-recession feedback loop that is a key characteristic of liquidity trap episodes, since the relevant measure for that mechanism is CPI inflation rather than PPI inflation, and CPI inflation does respond to nominal exchange rate fluctuations.

2.3 Terms of Trade, Exchange Rates and UIP

Expenditure minimization leads to the country k CPI definition

$$\mathbb{P}_k \equiv \left[(1-\alpha)(P_k^H)^{1-\eta} + \alpha(P_k^F)^{1-\eta}\right]^{\frac{1}{1-\eta}}, \tag{8}$$

where P_k^H is the price of the bundle produced within the home country in terms of the home currency,

$$P_k^H \equiv \left[\int_0^1 P_k^H(l)^{1-\epsilon} dl\right]^{\frac{1}{1-\epsilon}}, \tag{9}$$

and P_k^F is the price index for imported goods in terms of country k's currency,

$$P_k^F \equiv \left[\int_0^1 P_k^{j\,1-\gamma} dj\right]^{\frac{1}{1-\gamma}}, \tag{10}$$

where P_k^j is country j's PPI expressed in country k's currency,

$$P_k^j \equiv \left[\int_0^1 P_k^j(l)^{1-\epsilon} dl\right]^{\frac{1}{1-\epsilon}} \tag{11}$$

and $P_k^j(l)$ denotes the price of variety l produced in country j in terms of country k's currency. Note that $P_k^H \equiv P_k^k$.

$\mathcal{E}_{k,t}^j$ is the nominal exchange rate between country k and country j.[7] The law of one price (LOP) implies $P_k^j(l) = \mathcal{E}_{k,t}^j P_{j,t}^j(l)$. As a result, a similar relationship exists for the price of country j's final good: $P_k^j = \mathcal{E}_k^j P_j^j$. Therefore, we have:

$$P_k^F = \left[\int_0^1 \left(\mathcal{E}_k^j P_j^j\right)^{1-\gamma} dj\right]^{\frac{1}{1-\gamma}} \equiv \mathcal{E}_k P^* \tag{12}$$

where

$$P^* \equiv \left[\int_0^1 P_j^{j\,1-\gamma} dj\right]^{\frac{1}{1-\gamma}} \tag{13}$$

is the world price index and $\mathcal{E}_k$ is country k's effective nominal exchange rate:

$$\mathcal{E}_k \equiv \left[\frac{\int_0^1 \left(\mathcal{E}_k^j P_j^j\right)^{1-\gamma} dj}{\int_0^1 P_j^{j\,1-\gamma} dj}\right]^{\frac{1}{1-\gamma}} \tag{14}$$

[7] An increase in $\mathcal{E}_{k,t}^j$ is a depreciation of country k's currency and an appreciation of country j's currency, since $\mathcal{E}_{k,t}^j = 1/\mathcal{E}_{j,t}^k$

Using the above definitions, the household's budget constraint (6) can be expressed as

$$\dot{a}_{k,t} = i_{k,t}a_{k,t} + W_{k,t}N_{k,t} + T_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t} + \int_0^1 \left[i_{j,t} - i_{k,t} + \tau_{k,t} - \tau_{j,t} + \frac{\dot{\mathcal{E}}^j_{k,t}}{\mathcal{E}^j_{k,t}} \right] \mathcal{E}^j_{k,t} D^j_{k,t} dj \tag{15}$$

The bilateral terms of trade between country k and country j are defined as the relative price of country j's good in terms of country k's good

$$\mathcal{S}^j_k \equiv \frac{P^j_k}{P^k_k}. \tag{16}$$

The effective terms of trade of country k are defined as

$$\mathcal{S}_k \equiv \frac{P^F_k}{P^k_k} = \left[\int_0^1 \mathcal{S}^{j\,1-\gamma}_k dj \right]^{\frac{1}{1-\gamma}}. \tag{17}$$

The bilateral real exchange rate between country k and country j is further defined as the ratio of the two countries' CPIs

$$\mathcal{Q}^j_k \equiv \frac{\mathcal{E}^j_k \mathbb{P}_j}{\mathbb{P}_k} \tag{18}$$

and the effective real exchange rate of country k is defined as

$$\mathcal{Q}_k \equiv \frac{P^F_k}{\mathbb{P}_k} = \frac{\mathcal{E}_k P^*}{\mathbb{P}_k} \tag{19}$$

2.4 Equilibrium Conditions with Symmetric North and South Blocks

We now present equilibrium conditions from the perspective of a home country k in a case where all foreign North countries are identical and all foreign South countries are identical. Equilibrium conditions comprise a set of bilateral Backus-Smith conditions, a goods market clearing condition, an equation relating real exchange rate to the terms of trade, a labor market clearing condition, a domestic bond Euler equation, a set of UIP conditions, and the country's resource constraint.

The Backus-Smith condition between country k and a North country is given by

$$\mathbb{C}_{k,t} = \Theta^n_{k,t}\mathbb{C}_{n,t} \left(\mathcal{Q}^n_k\right)^{\frac{1}{\sigma}} \tag{20}$$

where $\Theta^n_{k,t}$ is the relative Pareto weight of country k with respect to the North country, given by

$$\Theta^n_{k,t} \equiv \Theta^n_{k,0} \exp\left\{\frac{1}{\sigma}\int_0^t \left[\tau_{k,h} - \tau_{n,h} - \zeta_{k,h} + \zeta_{n,h}\right] dh\right\} \tag{21}$$

A similar bilateral condition holds between country k and a South country, but this condition can alternatively be derived from (20) and the Backus-Smith condition between a North country and a South country.

The market clearing condition for output of country k, defined as $Y_{k,t} \equiv \left[\int_0^1 Y_{k,t}(l)^{\frac{\epsilon-1}{\epsilon}} dl\right]^{\frac{\epsilon}{\epsilon-1}}$ is given by

$$Y_{k,t} = (1-\alpha)\left(\frac{\mathcal{Q}_{k,t}}{\mathcal{S}_{k,t}}\right)^{-\eta} \mathbb{C}_{k,t} + \alpha x \left(\mathcal{S}_{n,t}\mathcal{S}^n_{k,t}\right)^{\gamma} \mathcal{Q}^{-\eta}_{n,t}\mathbb{C}_{n,t} + \alpha(1-x)\left(\mathcal{S}_{s,t}\mathcal{S}^s_{k,t}\right)^{\gamma} \mathcal{Q}^{-\eta}_{s,t}\mathbb{C}_{s,t}, \tag{22}$$

where the three terms making up demand for the country k good represent domestic demand, foreign demand from North countries, and foreign demand for South countries, respectively.

The effective and bilateral terms of trade are related through

$$\mathcal{S}_{k,t} = \left[x\left(\mathcal{S}^n_{k,t}\right)^{1-\gamma} + (1-x)\left(\mathcal{S}^s_{k,t}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}, \tag{23}$$

the bilateral real exchange rate is related to the effective and bilateral terms of trade through

$$\mathcal{Q}^j_{k,t} = \frac{\left[(1-\alpha) + \alpha\left(\mathcal{S}_{j,t}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}}{\left[(1-\alpha)\left(\mathcal{S}^j_{k,t}\right)^{-(1-\eta)} + \alpha\left(\mathcal{S}_{j,t}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}} \tag{24}$$

for $j = \{n, s\}$, and the effective real exchange rates is related to the effective terms of trade through

$$\mathcal{Q}_{k,t} = \left[(1-\alpha)\left(\mathcal{S}_{k,t}\right)^{\eta-1} + \alpha\right]^{\frac{1}{\eta-1}}. \tag{25}$$

The labor market clearing condition is given by[8]

$$N_{k,t} = \frac{Y_{k,t}}{A}. \tag{26}$$

The Euler equation for the domestic bond is given by

$$\frac{\dot{\mathbb{C}}_k}{\mathbb{C}_k} = \frac{1}{\sigma}\left[i_k - \pi_k - (\rho + \zeta_k)\right] \tag{27}$$

[8] The price dispersion term $\Delta_{k,t} \equiv \int_0^1 \left(\frac{P^H_{k,t}(l)}{P^H_{k,t}}\right)^{-\epsilon} dl$ is zero due to our rigid prices assumption.

where $\pi_k \equiv \frac{\dot{\mathbb{P}}_k}{\mathbb{P}_k} = \pi_k^H + \frac{\dot{\mathcal{S}}_k}{\mathcal{S}_k} - \frac{\dot{\mathcal{Q}}_k}{\mathcal{Q}_k}$ is CPI inflation.

The interest parity condition between the home bond and the North country bond is given by

$$i_{k,t} - \tau_{k,t} = i_{n,t} - \tau_{n,t} + \frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n}. \tag{28}$$

A similar bilateral condition holds between country k and a South country, but this condition can alternatively be derived from (28) and the interest parity condition between a North country and a South country.

Finally, country k's budget constraint is

$$\dot{B}_{k,t} = (\rho + \zeta_{n,t} - \tau_{n,t})\, B_{k,t} + \mathbb{C}_{n,t}^{-\sigma} \left(\mathcal{Q}_{k,t}^n\right)^{-1} \left[(\mathcal{S}_{k,t})^{-\alpha} Y_{k,t} - \mathbb{C}_{k,t}\right] \tag{29}$$

where $B_{k,t} \equiv \frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{J}_{k,t}^n \mathbb{P}_{n,t}} = \frac{\mathbb{C}_{n,t}^{-\sigma} \int_0^1 \mathcal{J}_{k,t}^j D_{k,t}^j dj}{\mathcal{J}_{k,t}^n \mathbb{P}_{n,t}}$ is a country's net foreign assets at t measured in terms of a North country's CPI $\mathbb{P}_{n,t}$ and normalized by a North country's marginal utility of consumption $\mathbb{C}_{n,t}^{-\sigma}$. Imposing a No-Ponzi game condition, this budget constraint can be written in present value form as

$$B_{k,0} = -\int_0^\infty e^{-\int_0^t [\rho + \zeta_{n,s} - \tau_{n,t}]ds} \mathbb{C}_{n,t}^{-\sigma} \left(\mathcal{Q}_{k,t}^n\right)^{-1} \left[(\mathcal{S}_{k,t})^{-\alpha} Y_{k,t} - \mathbb{C}_{k,t}\right] dt \tag{30}$$

With fully rigid prices, producer prices are fixed at their at their $t = 0$ values in own currency terms.[9]

Cole-Obstfeld parametrization From this point onwards, we impose the Cole-Obstfeld parametrization (Cole and Obstfeld, 1991), which corresponds to $\sigma = \gamma = \eta = 1$. This parametrization is far from unreasonable, and has the advantage of decisively improving the tractability of the non-linear model. To clarify the exposition, we here summarize some key equilibrium conditions that take significantly simpler forms under the adopted parametrization. The Backus-Smith condition is given by

$$\mathbb{C}_{k,t} = \Theta_{k,t}^n \mathbb{C}_{n,t} \left(\mathcal{S}_{k,t}^n\right)^{1-\alpha}, \tag{31}$$

the goods market clearing condition becomes

$$Y_{k,t} = \left[(1-\alpha)\,\Theta_{k,t}^n + \alpha x + \alpha\,(1-x)\,\Theta_{s,t}^n\right] \left(\mathcal{S}_{n,t}^s\right)^{\alpha(1-x)} \mathcal{S}_{k,t}^n \mathbb{C}_{n,t}, \tag{32}$$

and the country lifetime budget constraint reduces to

$$B_{k,0} = \alpha \int_0^\infty e^{-\int_0^t (\rho + \zeta_{n,s} - \tau_{n,t})ds} \left[\Theta_{k,t}^n - x - (1-x)\Theta_{s,t}^n\right] dt. \tag{33}$$

[9]The price of all varieties produced in country k in their own currency is normalized to 1 at $t = 0$.

3 Optimal Allocations

In this section, we expose an efficient benchmark to which we will compare the outcomes of our optimal policy experiments.

We characterize optimal allocations from the point of view of a social planner who only values the welfare of country k. The problem of a country k planner can be described as choosing allocations to maximize the discounted value of lifetime utility of the representative household in economy k (1) subject to (i) a technological constraint given by the production function (26), (ii) the Backus-Smith condition (31), (iii) the market clearing condition (32) and (iv) the country's lifetime budget constraint (33). Given the absence of a dynamic choice variable, the problem simply amounts to statically choosing $Y_{k,t}$, $N_{k,t}$, $\mathbb{C}_{k,t}$ and $\mathcal{S}^n_{k,t}$ to maximize $U(\mathbb{C}_{k,t}, N_{k,t})$ subject to (26), (31) and (32).

Proposition 1 (Optimal allocation for country k)**.** *The optimal allocation from the perspective of country k features constant employment and output:*

$$N_{k,t} \equiv \overline{N} = (1-\alpha)^{\frac{1}{1+\phi}} \qquad \textit{and} \qquad Y_{k,t} \equiv \overline{Y} = A\,(1-\alpha)^{\frac{1}{1+\phi}}$$

Thus, a social planner in country k seeks to keep employment – and therefore output – constant.[10] This allows us to interpret any deviation from this constant level as causing a welfare loss from the perspective of country k's social planner, corresponding to the opening of an output gap.

4 Optimal Monetary Policy and Arbitrary Capital Account Regimes

This section builds intuition regarding how monetary policy deals with a liquidity trap in a region of the world economy. From this point onwards we focus on a scenario in which the

[10]Such an allocation can be decentralized in a flexible price economy by susidizing employment at a rate which balances the incentive of the monetary authority to improve its terms-of-trade and offset the market power distortion. However, unlike in Gali and Monacelli (2005), the optimal allocation cannot be achieved by a time invariant employment subsidy in the flexible price equilibrium as the incentive to manipulate terms-of-trade vary over time in the presence of asymmetry shocks to the discount rate. A second order approximation to the welfare function would therefore not solely depend on squared output gap and inflation terms. However, this is not a concern for our purposes. Since we assume fully rigid prices, rather than comparing the allocations resulting from our optimal policy problems to flexible price outcomes, we can compare them to the optimal allocations of proposition 1 directly. Besides, we do not resort to local approximation and solve for optimal policy using the exact welfare function in the fully non-linear model.

North is subject to an unexpected negative demand shock modeled as a shock to the discount rate. Assumption 1 formalizes this notion.

Assumption 1. *At $t = 0$, agents learn about the path of discount rate shocks for $t \geq 0$. This path is given by $\zeta_{s,t} = 0$ $\forall t$ and*

$$\zeta_{n,t} = \begin{cases} -\bar{\zeta} & \text{for } t \in [0, T) \\ 0 & \text{for } t \geq T \end{cases}$$

with $\bar{\zeta} > 0$.

Hence, agents in the North are temporarily more patient from 0 to T. This path for the discount rate can be thought of as a temporary negative demand shock originating in the North. Depending on the size of the shock, the equilibrium under optimal policy might feature no liquidity traps, regional liquidity traps or global liquidity traps. We now make an assumption on the size of the shock that ensures that under free capital flows, the equilibrium of the non-cooperative optimal monetary policy game results in a regional liquidity trap, where the ZLB binds in the North but not in the South.

Assumption 2. *The size of the demand shock in the North satisfies:*

$$\rho + \frac{\alpha\,(1-x)\,\rho}{[1-\alpha(1-x)]\,(\rho-\bar{\zeta})}\left[\rho - \bar{\zeta}e^{-(\rho-\bar{\zeta})T}\right] < \bar{\zeta} < \rho + \frac{(1-\alpha x)\,\rho}{\alpha x\,(\rho-\bar{\zeta})}e^{-\bar{\zeta}T}\left[\rho - \bar{\zeta}e^{-(\rho-\bar{\zeta})T}\right].$$

This condition on $\bar{\zeta}$ ensures that the demand shock is large enough to push the North to the ZLB, yet small enough so as to not push the South to the ZLB. In the closed economy limit $\alpha \to 0$, this condition trivially reduces to $\rho < \bar{\zeta}$. But more generally, the condition on $\bar{\zeta}$ depends on the degree of openness α and on the relative size of the North block x.[11]

Next, we analyze the optimal response of a monetary policy authority to this demand shock, for an exogenously set path of capital flow taxes at home and abroad. A distinctive feature of our modeling choice is the assumption that countries do not set monetary policy cooperatively A singularity of our approach within the context of the open-economy ZLB literature is the assumption that monetary policy is set non-cooperatively by individual countries.

[11] The parameter set satisfying condition of assumption 2 is non-empty if and only if the following condition holds:

$$T\bar{\zeta} < \ln\left[1 + \frac{1-\alpha}{\alpha\,(1-x)\,\alpha x}\right].$$

Broadly speaking, this condition requires that for a given trap duration T the shock size $\bar{\zeta}$ is not too large, or equivalently that for a given shock size $\bar{\zeta}$, the trap duration T is not too long.

Formally, the optimal policy problem in country k is to choose paths for $\mathbb{C}_{k,t}$, $Y_{k,t}$, $N_{k,t}$, $\mathcal{Q}^n_{k,t}$, $\mathcal{Q}^s_{k,t}$, $\mathcal{Q}_{k,t}$, $\mathcal{S}^n_{k,t}$, $\mathcal{S}^n_{k,t}$, $\mathcal{S}_{k,t}$, $i_{k,t}$ to maximize (1) subject to (20), (22), (23), (24), (25), (26), (27), (28) and a ZLB constraint, for given paths of prices, capital controls, and aggregate variables.[12] Using constraints to substitute out variables, the problem can be written as

$$\max \int_0^\infty e^{-\left(\rho+\zeta_{k,h}\right)dh} \left\{ \log \mathbb{C}_{k,t} - \frac{1}{1+\phi} \left(\frac{Y_{k,t}}{A} \right)^{1+\phi} \right\}$$

subject to:

$$\begin{aligned} Y_{k,t} &= \left[(1-\alpha)\,\Theta^n_{k,t} + \alpha x + \alpha\,(1-x)\,\Theta^n_{s,t} \right] \left(\mathcal{S}^s_{n,t} \right)^{\alpha(1-x)} \left(\mathbb{C}_{k,t} \right)^{\frac{1}{1-\alpha}} \left(\Theta^n_{k,t} \right)^{\frac{-1}{1-\alpha}} \left(\mathbb{C}_{n,t} \right)^{\frac{-\alpha}{1-\alpha}} && (34) \\ \frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} &= (1-\alpha) i_{k,t} + \alpha \left[x \left(i_{n,t} + \tau_{k,t} - \tau_{n,t} \right) + (1-x) \left(i_{s,t} + \tau_{k,t} - \tau_{s,t} \right) \right] - \left(\rho + \zeta_{k,t} \right), && (35) \\ i_{k,t} &\geq 0. && (36) \end{aligned}$$

This is an optimal control problem with state $\mathbb{C}_k$ and control i_k. Two key open economy elements are apparent in the monetary authority' constraints: one pertaining to domestic consumption in level and the other to domestic consumption growth. First, the demand constraint (34) reflects that the intra- and intertemporal market arrangements to which the country subscribes means that for given foreign prices and aggregate variables, the elasticity of domestic consumption with respect to domestic output is equal to the degree of home bias $(1-\alpha)$. An increase in domestic output leads to a terms-of-trade deterioration, and brings about a less than one-for-one increase in domestic consumption. Second, the Euler equation constraint (35) shows that the relevant nominal interest rate for consumption growth is an average of the domestic rate (weighted by the degree of home bias, $1-\alpha$) and of the foreign rates net of capital flow taxes (weighted by the degree of openness α and the relative importances of the North and South blocks, x and $1-x$). This Euler equation embeds the key elements at the source of spillover effects of monetary and capital account management policy across countries.

We shall see that optimal monetary policy in country k seeks to achieve the optimal allocation featuring a constant level of output and employment (defined in Proposition 1), and is able to achieve this goal as long as the ZLB does not bind.

Lemma 1 (Optimal shadow interest rate)**.** *In the absence of a zero-bound on interest rates, optimal interest rate policy is given by*

$$i^*_{k,t} = \mathcal{I}_k \left(\Theta^n_{k,t}, \Theta^n_{s,t}, \zeta_{k,t}, \zeta_{n,t}, \tau_{k,t}, \tau_{n,t}, \tau_{s,t} \right) \tag{37}$$

[12] Our rigid price assumption implies that bilateral terms-of-trade are simply equal to nominal exchange rates: $\mathcal{S}^j_k = \mathcal{E}^j_k$.

where

$$\mathcal{I}_k(\cdot) \equiv \rho + \frac{(1-\alpha)\Theta^n_{k,t}\zeta_{k,t} + \alpha x \zeta_{n,t} + \alpha x\left(\tau_{k,t} - \tau_{n,t}\right) + \alpha(1-x)\Theta^n_{s,t}\left(\tau_{k,t} - \tau_{s,t}\right)}{(1-\alpha)\,\Theta^n_{k,t} + \alpha x + \alpha\,(1-x)\,\Theta^n_{s,t}}. \tag{38}$$

Furthermore, the first best levels of employment and output defined in Proposition 1, $N_{k,t} = \overline{N}$ *and* $Y_{k,t} = \overline{Y}$*, are achieved under that policy.*

In the absence of a ZLB on interest rates, optimal monetary policy achieves the optimal allocation described in section 3. Negative demand shocks at home (ζ_k) or in the foreign North economies (ζ_n) require a stimulation of demand through a lowering of the nominal interest rate. Similarly, domestic taxes on capital inflows (or subsidies to outflows) increase the domestic optimal interest rate, while taxes on capital inflows (or subsidies to capital outflows) by foreign countries (whether North or South) lower it. Intuitively, domestic taxes on capital inflows or foreign taxes on outflows require an expected appreciation of the home currency. This expected appreciation brings about a depreciation of the domestic currency on impact, which stimulates (home and foreign) demand for the home good and reduces the need for the nominal interest rate to do the job.

From the optimal shadow interest rate expression (38), it is apparent that if a country could use capital flow taxes as a stabilization policy tool, it would always have the ability to set a high enough tax on inflows τ_k to push the shadow rate above zero and thereby fully relax the ZLB constraint.[13] Such a policy may not necessarily be optimal, though, since it could entail large intertemporal consumption distortions. Rather than focusing on how capital controls can be used by an individual country to relax a ZLB constraint, we take a broader efficiency standpoint and ask whether there exists a global policy regime entailing non-zero capital flow taxes that Pareto-dominates the free capital flow regime benchmark.[14]

We now establish that under the retained assumptions on parameters, the ZLB binds in the North but not in the South, as long as capital flow taxes are not too large.

[13] Similar arguments are presented in Farhi and Werning (2013) and Korinek (2014).

[14] This global efficiency question is addressed in Section 5. There, we assume that taxes are zero in the North ($\tau_{n,t} = 0$), and consider a planner who sets taxes in the South aiming for constrained Pareto-optimal outcomes. For the Pareto problem, the assumption that $\tau_{n,t} = 0$ is without loss of generality since in a symmetric constellation the only relevant quantity is the wedge in the interest parity condition $\tau_{s,t} - \tau_{n,t}$. To understand the distributional implications of capital flow taxes, we also consider non-cooperative optimal capital flow management in Section 6. There, we again assume that taxes are zero in the North, and consider planners in each individual South country who set domestic taxes to maximize the country's welfare. Given our intepretation of North and South countries, the case where only South countries use capital flow management seems the most empirically relevant.

Lemma 2. *Suppose that Assumptions 1 and 2 hold, and that capital flow taxes are sufficiently small (in absolute value) for $t \geq 0$. Then the optimal interest rate policy specified in Lemma 1 is not feasible for a North economy, because $\mathcal{I}_n(\cdot) < 0$ for $0 \leq t < T$, but it is feasible for a South economy ($\mathcal{I}_s(\cdot) > 0 \; \forall t$).*

The above Lemma states that under the retained assumption for the size of the demand shock, monetary policy is unable to perform a full stabilization of output and employment in the North, but is able to do so in the South. The failure to lower the interest rate in the North to the level prescribed by Lemma 1 results in a real interest rate that is "too high." This excessively high interest rate requires consumption, and therefore output, to grow, and results in a temporary recession. Optimal monetary policy at the ZLB, it turns out, commands to keep the interest rate at zero beyond the liquidity trap in order to center the employment (or output) path around the target attainable under the unconstrained policy. The next proposition summarizes optimal policy in such a scenario.

Proposition 2 (Optimal Monetary Policy with the ZLB)**.** *Suppose that Assumptions 1 and 2 hold, and that capital flow taxes are sufficiently small (in absolute value) for $t \geq 0$. Then optimal policy in the South is unconstrained by the ZLB and achieves the optimal employment at all time:*

$$i_{s,t} = \mathcal{I}_s\left(\cdot\right) \qquad \forall t \geq 0,$$

but optimal policy in the North hits the ZLB and fails to achieve the optimal employment at all time. In particular, the ZLB exit in the North is delayed to $\widehat{T}_n > T$, so as to center employment around its optimal level:

$$i_{k,n} = \widetilde{\mathcal{I}}_n\left(\cdot\right) = \begin{cases} 0 & \text{if } t \in [0, \widehat{T}_n], \\ \\ \mathcal{I}_n\left(\cdot\right) & \text{if } t > \widehat{T}_n, \end{cases}$$

where the exit time $\widehat{T}_n$ and the output path satisfy

$$0 = \int_0^{\widehat{T}_n} e^{-\int_0^t \left(\rho + \zeta_{n,h}\right) dh} \left[1 - \left(\frac{Y_{n,t}}{\overline{Y}}\right)^{1+\phi}\right] dt.$$

Proposition 2 underscores the fact that optimal policy is *not* given by $i^*_{k,t} = \max\left\{0, \mathcal{I}_k(\cdot)\right\}$. That path would imply that the policy maker raise nominal rates as soon as the liquidity trap episode ends. Instead, optimal policy demands that nominal rates be kept at zero even after the liquidity trap scenario has ended.

Consider the case in which the South does not use capital controls ($\tau_{s,t} = 0$), $\forall t \geq 0$.[15] In this case, under the optimal interest rate policy, the currency of country any Northern country k undergoes a large depreciation on impact of the demand shock at time 0. This exchange rate then appreciates gradually and overshoots its final level. The initial depreciation makes the output of northern economy k cheaper for the souther block of countries and allows it to export some of its output since it's own consumption demand is low. Capital flows from the North to the South and consequently, the Northern countries build claims against the Southern economies. In the long run, this results in a permanently lower consumption for the southern economies. However, the consumption of Southern economies goes up during the liquidity trap period.

Note that the optimal policy prescription is similar to that in the closed economy liquidity trap models.[16] By keeping nominal rates at zero for longer than the duration of the liquidity trap the policy makers commits to a consumption boom after the end of the liquidity trap which helps him arrest part of the fall in output on impact. The policy is also similar to that in Svensson (2001, 2003, 2004) who shows that the *Foolproof Way* to exit a liquidity trap is through an intentional currency depreciation and a crawling peg can induce private-sector expectations of a higher future price level and escape from the liquidity trap.

Earlier literature seemed to suggest that keeping nominal rates at zero beyond the trap was useful as it allowed the policymaker to commit to future inflation which helps in lowering the real rate (Krugman, 1998). However, Werning (2012) shows that the extension of zero rates, in fact, does not require a commitment to future inflation. It is in fact the commitment to the consumption boom in the future which is the key. Following this logic, it is clear why in this model economy, even though producer prices are fully sticky in their own currency, a similar policy is optimal. The result here can be seen as an open economy counterpart of the result. Furthermore, as far as we know, this is the first analytic characterization of the optimality of keeping nominal rates at zero past the end of the liquidity trap in an open economy setting.

[15] We contrast this with the case with positive controls in subsequent sections.

[16] See for example, Eggertsson and Woodford (2003), Jung et al. (2005), Adam and Billi (2006) and Werning (2012).

4.1 Free capital flows

The openness of the economy brings with it both stabilizing and destabilizing forces. The stabilizing aspect of an open economy is that capital flows towards economies where demand is relatively stronger. The terms of trade adjust (through the depreciation on impact) to counter the lack of demand of northern economies output by making it cheaper for emerging economies to consume the output of northern economy goods. As a result, the fall in output in Northern economies is not as large as in the corresponding closed economy case where a fall in demand causes an equally large fall in output. However, there is also a destabilizing effect of openness of the economy. After the initial depreciation of the currency in the north economy, there is persistent appreciation of its currency. This results in CPI deflation which raises the real rate and worsens the liquidity trap in the advanced economies. This is the destabilizing effect.

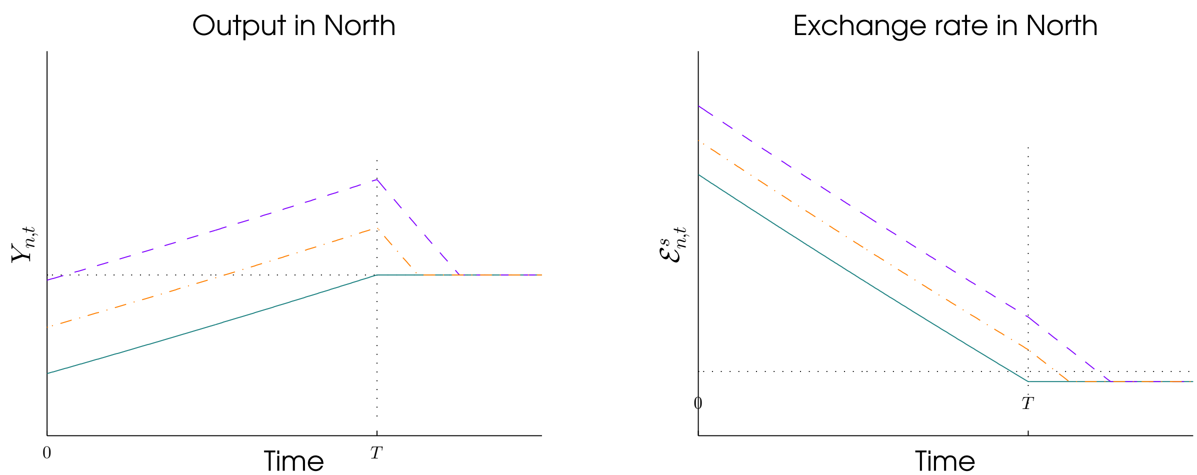

Figure 1: Variable paths under arbitrary and optimal ZLB exit times (and free capital flows).

The policy maker in Northern economies, by keeping nominal rates at zero for longer than the duration of the liquidity trap, tries to balance these stabilizing and destabilizing forces. Committing to keep nominal rates at zero for too long cause the initial depreciation of the advanced economy to be *too large*. Output falls less on impact but the strong and is followed by a vigorous appreciation of its currency. This results in CPI deflation which implies an even higher real interest rate which further depresses demand. To counter this, the policy maker in the North must then stimulate a much larger consumption boom at home after the trap has ended. This requires employment to be above the social optimum for longer and hence, lowers welfare. On the other hand, raising rates *too early* helps the planner control the extent of CPI deflation following the depreciation on impact but this reduces the stabilizing effect by limiting the terms of trade adjustment on impact. In other words, it impedes the reallocation of demand on impact to the relatively high demand economies resulting in a much larger drop in output on impact in advanced economies. Figure 1 contrasts the path of output in the advanced economy if the planner renormalizes *too early* or *too late*. By picking the exit-time optimally, the planner smooths the deviations in employment before and after the liquidity trap. The *too early* exit scenario displayed depicts the a case where the monetary policy maker in the North raises rates as soon as the liquidity trap ends. This happens to correspond to the optimal policy in the absense of commitment. Output on impact of the shock falls by more in this case. The exchange rate in the North depreciates by less which limits the demand that the South is willing to take up.

4.2 Closed capital account

The case of the closed capital account is the the polar opposite of the free capital flows. The closed capital account case corresponds to a sequence of capital controls $\tau_{e,t} = \bar{\zeta}$ for $t \in [0, T]$ imposed by emerging countries. We will refer to this scenario also as the case with "naive" controls. Such a sequence of controls imply that the terms of trade cannot adjust on realization of the demand shock. The fall in domestic demand for Northern country output causes it's output to fall relative to southern output. This raises the relative price of northern output and further reduces the demand for northern output both domestically and from the southern countries making demand and output fall more. Thus, contrary to the case with free capital flows, here, the terms of trade move adversely and end up harming the northern countries. On impact, rather than a depreciation of its currency, there is an appreciation and then gradual depreciation. Thus, capital controls imposed by the south can impede the ability of the north to implement the *Foolproof Way* to exit a liquidity trap.

Notice that this case is very similar to if the economy was in fact closed. The following proposition contrasts the the path of interest rates in a setting with perfect capital mobility versus a setting with a closed capital account.

Proposition 3. *In an open economy with perfect capital mobility the planner optimally renormalizes faster than in the corresponding closed economy. Also, the speed of renormalization is faster than if the capital account was closed.*

The intuition underlying the above proposition is straightforward. The inability of capital

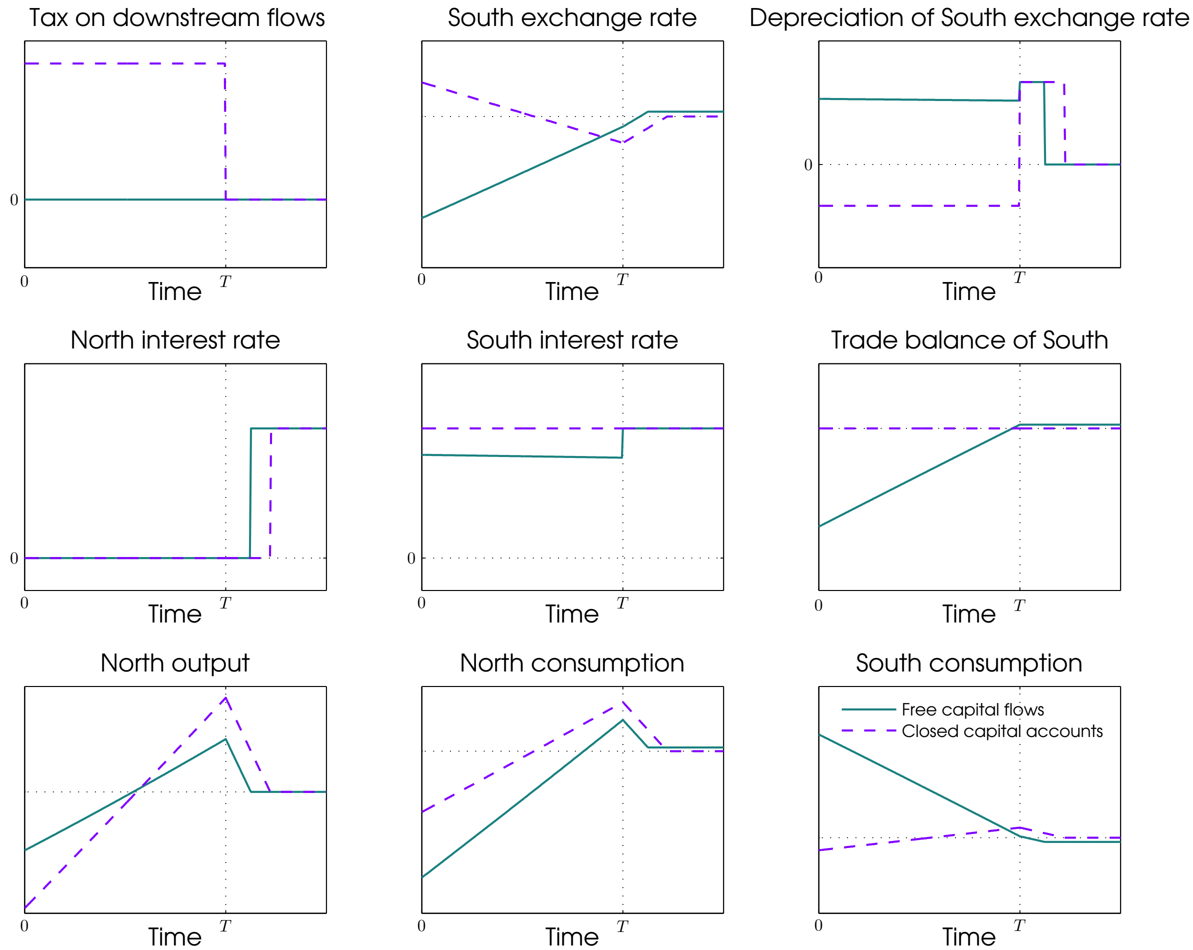

Figure 2: Variable paths under optimal monetary policy in all countries: free capital flows (solid) vs. closed capital accounts (dashed).

to flow to relatively high demand economies hampers the favorable adjustment of the terms of trade. This limits the re-allocation of demand to the South from the North during the liquidity trap. Consequently, there is a larger initial drop in output in the North which in turn requires a larger consumption boom in the future to compensate. Figure 2 shows that under naive controls, on impact the exchange rate in a Northern country instead of depreciating, actually appreciates. This makes it more expensive for the South to consume Northern output. In contrast to the free capital flows case, the South does not absorb the excess output in the North. This implies that output in the North must fall by more. FThe figure also show that the reallocation of demand through a terms of trade adjustment does not occur in the naive controls case.

5 Efficient Capital Account Regime

The previous sections highlighted how optimal monetary policy in the North has to adapt to capital controls imposed in the South. The free-capital flow scenario highlighted how the openness of the economy may allow the north to ameliorate the effects of the negative demand shock. At the other end of the spectrum, the closed capital account scenario highlighted how capital controls might make it a lot more painful for the north to adjust to the negative demand shock. This raises the question about how capital controls affect welfare in the two sets of countries.

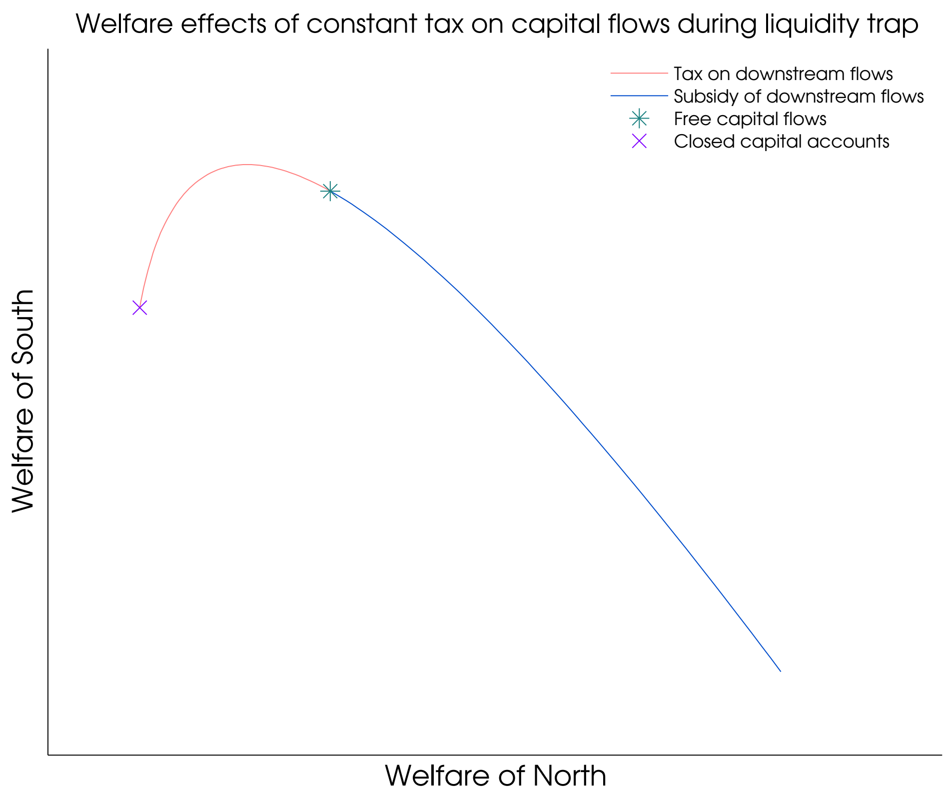

Figure 3: Welfare in the North and in the South as a function of constant taxes on downstream flows during the liquidity trap (i.e. from 0 to T).

Figure 3 depicts the distribution of welfare in the North and the South depending on the capital controls imposed. A point higher up relative to the y-axis implies higher utility for each Southern country k while a higher x-coordinate implies higher utility in country $k \in \mathcal{N}$. The solid curve displays the distribution of welfare across different levels of constant capital controls[17] Notably, the black dot denotes the scenario with no capital controls. Points to the left of the black dot represent scenarios with a tax on capital inflows in to the South while points to the right denoted welfare under subsidies to capital inflows from the North in to the South. The blue diamond represents the "naive controls" scenario while the beginning of the curve in the south-east is a case when the south subsidizes capital inflows from the north during the liquidity trap period. The figure clearly shows that the relationship between total welfare (and even welfare in the South) is not monotonic in the average level of capital controls. Starting from the right of the no controls case and moving towards it (raising capital controls/ reducing the subsidy) decreases the welfare in the North while increasing welfare in the South. If the South subsidizes capital inflows, it allows the North to reduce its consumption during the liquidity trap without having to reduce employment. In addition it builds up claims against the South which allows it a higher consumption after the trap is over. The flip-side of this is that the terms-of-trade move unfavorably for the South. Cheaper consumption today comes at the cost of lower consumption in the future which results in an overall lower welfare compared to the scenario if the South had used incrementally higher controls to counter the unfavorable movement in the terms-of-trade.

However, higher average capital controls do not unambiguously help the South. As can be seen from the figure, very high levels of capital controls actually end up reducing the welfare of both the North and the South relative to the free capital flows case. The clearest illustration of this is the naive controls case represented by the blue-diamond. Very high controls do not allow the terms-of trade to adjust. Consequently, the North is unable to export as much and output (and employment) fall much more in the North compared to a case with lower taxes on inflows. Also, the high taxes on capital inflows does not allow the South to avail cheaper consumption during the liquidity trap.

The previous figure highlights the tension between the North and the South in setting capital controls. With a regime of constant controls during the liquidity trap, it is not possible to raise the welfare of one party without hurting the other. This zero-sum redistributive aspect of imposing controls suggests why common perception about the desirability of the use of capital controls and might be seen as one of the main reasons why the use of capital controls

[17] This corresponds to a scenario in which capital controls are kept constant between $t = 0$ and $t = T$.

has been frowned upon by institutions such as the IMF in the past which have promoted the idea of free capital flows.

This raises the question whether there is any path of capital controls which would allow one to increase welfare in both the North and the South compared to the zero capital control or free capital flow case. Additionally, is there such a path for capital controls even when the local authorities ignore possible spillovers when setting monetary policy?

We frame the answer to the question above as the solution to a constrained Pareto problem in which a Global-Planner chooses a path of capital controls to maximize welfare in the North while ensuring that the South enjoys at least the welfare that it would enjoy in the case with no capital controls. Furthermore, the planner must also respect the constraint that local authorities in the North and the South set monetary policy independently and will also respond to the capital controls that she chooses. The problem is formalized below:

Definition 1 (Constrained Pareto Problem)**.** *The problem of a Global-Planner maximizing welfare in the North while ensuring that the South enjoys at least the welfare that it would enjoy in the case with no capital controls. Monetary policy is set by the local authorities and respond to the choice of capital controls chosen by the global-planner.*

$$\max_{\{\tau_{s,t}\}_{t\in[0,\infty)}} \int_0^\infty e^{-\int_0^h(\rho+\zeta_{n,h})dh} \left\{ \log \mathbb{C}_{n,t} - \frac{1}{1+\phi}\left(\frac{Y_{n,t}}{A}\right)^{1+\phi} \right\} dt$$

subject to:

$$\int_0^\infty e^{-\rho t} \left\{ \log \mathbb{C}_{s,t} - \frac{1}{1+\phi}\left(\frac{Y_{s,t}}{A}\right)^{1+\phi} \right\} dt \geq \overline{\mathbb{W}}_s \tag{39}$$

$$
\begin{aligned}
Y_{n,t} &= \left[1-\alpha(1-x)+\alpha\left(1-x\right)\Theta^{n}_{s,t}\right]\left(\Theta^{n}_{s,t}\right)^{\frac{\alpha(1-x)}{1-\alpha}}\left(\mathbb{C}_{s,t}\right)^{-\frac{\alpha(1-x)}{1-\alpha}}\left(\mathbb{C}_{n,t}\right)^{\frac{1-\alpha x}{1-\alpha}} && (40)\\
Y_{s,t} &= \left[(1-\alpha x)\Theta^{n}_{s,t}+\alpha x\right]\left(\mathbb{C}_{s,t}\right)^{\frac{1-\alpha(1-x)}{1-\alpha}}\left(\Theta^{n}_{s,t}\right)^{-\frac{1-\alpha(1-x)}{1-\alpha}}\left(\mathbb{C}_{n,t}\right)^{-\frac{\alpha x}{1-\alpha}} && (41)\\
\frac{\dot{\Theta}^{n}_{s,t}}{\Theta^{n}_{s,t}} &= \tau_{s,t}+\zeta_{n,t} && (42)\\
\frac{\dot{\mathbb{C}}_{n,t}}{\mathbb{C}_{n,t}} &= \left[1-\alpha(1-x)\right]i_{n,t}+\alpha(1-x)\left[i_{s,t}-\tau_{s,t}\right]-\left(\rho+\zeta_{n,t}\right) && (43)\\
\frac{\dot{\mathbb{C}}_{s,t}}{\mathbb{C}_{s,t}} &= (1-\alpha x)i_{s,t}+\alpha x\tau_{s,t}+\alpha x i_{n,t}-\rho && (44)\\
i_{k,t} &= \begin{cases}\mathcal{I}_k\left(\cdot\right) & \textit{if } \mathcal{I}_k(\cdot)>0 \textit{ at } t=0\\ \\ \widetilde{\mathcal{I}}_k\left(\cdot\right) & \textit{else}\end{cases} && (45)
\end{aligned}
$$

Equation (39) constrains the global planner to provide each Southern country k at least $\overline{W}_s$ units of welfare. $\overline{W}_s$ corresponds to the welfare a country $k \notin \mathcal{N}$ would enjoy under the free capital flows scenario. Constraints (40)-(41) represent the market clearing constraints for the two blocks of economies. Equation (42) restricts the response of $\Theta^{n}_{s,t}$ to a choice of $\tau_{s,t}$ while equations (43)-(44) describe the path of consumption in the North and the South in response to a choice of $\tau_{s,t}$. Equation (45) implies that the global planner must respect that fact that monetary policy is set (optimally in response to the choice of controls) by each country.

Proposition 4 (Pareto Inefficiency of Free Capital Flows)**.** *A regime of free capital flows is Pareto-Inefficient, in the sense that a path of zero taxes on capital flows between the North and the South is not a solution to the Constrained Pareto Problem 1.*

Proposition 4 implies that a policy of active management of the capital account can actually raise welfare in both the North and the South. Conventional belief has suggested that zero capital controls are preferred since the use of controls is thought of as raising the welfare of one party at the expense of the other. The above Proposition suggests that this is not accurate in situations with liquidity traps.

Proposition 5. *Capital flows "too slowly" during and after the liquidity trap in the free capital flow case relative to the optimal policy of capital account management.*

Figures 4 graphically illustrates the claims of the two propositions above. The solid line represents the free capital flows case, while the dashed line represents the Pareto optimal

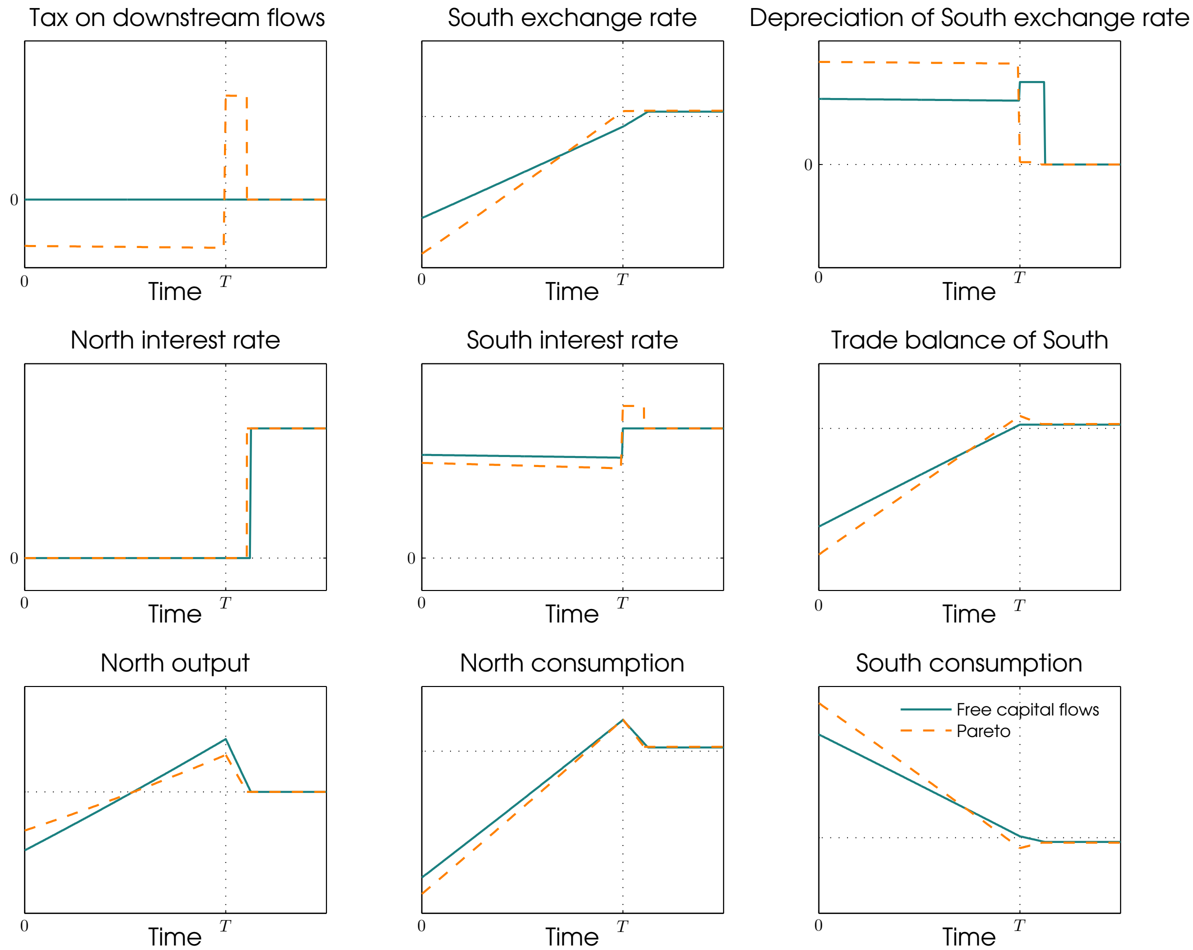

Figure 4: Variable paths under optimal monetary policy in all countries: free capital flows (solid) vs. efficient capital account management (dashed).

capital account management regime. Under the optimal regime, the planner subsidizes downstream flows during the liquidity trap (from 0 to T) and then subsidizes upstream flows after the liquidity trap and until renormalization of monetary policy in the North (from T to $\widehat{T}$). This capital flows management policy makes the South currency appreciate (resp. the North currency depreciate) more on impact than under free capital flows. By engineering a larger depreciation of the North currency and a larger appreciation of the South currency, the planner induces expenditure switching away from the South good toward the North good in the initial stage of the liquidity trap. Stabilizing output in the South requires monetary policy to be more accomodating than under free capital flows. As a consequence of expenditure switching, North output falls by less on impact than under free capital flows. The reallocation of demand achieved under the optimal capital account regime results in a larger trade and current account deficit for the South (resp. larger trade and current account surpluses for the North) in the initial stage of the liquidity trap. However, these larger initial deficits (resp. surpluses) are partially compensated by smaller deficits (resp. smaller surpluses) towards the end of and past the liquidity trap. A notable feature of the optimal capital account regime is the tax on downstream flows (or subsidy to upstream flows) between the end of the liquidity trap and the optimal ZLB exit time in the North. At that stage, the North is in the midst of a (promised) consumption boom, and due to home bias, this high consumption has to be met partly by a high level of output in the North, resulting in a negative output gap. A tax on downstream flows contributes to depreciating the South currency (resp. appreciating the North currency) and thereby tilt North consumption away from the North good and toward the South good. This inverse expenditure switching dampens output gap fluctuations.

The discussion above highlighted that the optimal capital flow management regime calls for subsidization of capital flows from the North to the South during the liquidity trap episode followed by a tax on capital till monetary policy renormalizes in the North. Importantly, this policy raises the overall welfare in the world economy without leaving any country worse off. Given these gains from capital account management, a pertinent question is whether these gains can be realized if each Southern country is allowed to independently choose capital controls. We answer this question in section 6.

6 Optimal Non-Cooperative Capital Flow Management

The previous section indicated that managing the capital account in the South in response to the liquidiity trap in the North Pareto-dominates the zero capital control scenario. Following this, we explore whether allowing economies in the south to independently choose their capital account management policy can generate outcomes which achieve the welfare distributions on the Pareto Frontier described in section 5.

Recall that section 4 highlighted how capital controls imposed by the south could influence monetary policy in the northern economies. In particular, sections 4.1 and 4.2 showed how the use of high controls in the South may reduce welfare in the North by requiring monetary policy to keep nominal rates at zero for a much longer duration. This section shows that in the absence of coordination the South choose capital controls that are “too high” during the liquidity trap duration and "too low" in the interim period after the end of the liquidity trap but before the nominal rates are raised in the North.

To understand the motives of a southern country k behind setting controls, recall that by using the nominal interest rate, northern countries can control the exchange rate appreciation in southern countries. For example, by keeping rates at zero for an extended period after the liquidity trap episode ends, the advanced country can causes the exchange rate to appreciate appreciably in the south. As a consequence of the cheap capital flowing in from the north, countries in the south are induced to consume more today. This flow of capital and increased consumption in the south results in the northern countries building up claims against the south. Once the liquidity trap in the north ends, they enjoy higher consumption than the south permanently as the south needs to repay all that it borrowed.

From the point of view of southern country k, cheap credit today allows increased consumption today but it is at the cost of lower consumption in the future. The high trade deficits in the South today have to be compensated by trade surpluses in the future. In such a scenario, the planner in a Southern country K has incentives to distort the dynamic terms of trade. During the liquidity trap when the South is running a trade deficit, the planner in southern country k would like to distort the prices downwards by lowering domestic consumption. This decrease in consumption today results in higher consumption in the future. This force is what Costinot et al. (2014) refer to as dynamic terms-of-trade manipulation. Economy k can accomplish this goal by taxing capital inflows. Thus, the use of capital controls allows the planner to spread the benefit from cheap credit today over time. Another way to see this is that by using controls, the planner can limit the claims that the northern economies can build up against it over time and hence enjoy on average a higher level of consumption over time.

Since each southern economy has an incentive to manipulate the dynamic terms of trade, each of them taxes the inflow of capital. This itself generates another reason for a single southern country to use controls. To see this consider southern economy k which does not impose capital controls while other southern countries do. Since the other southern countries tax capital inflows, cheap credit will flow even more so into country k which will cause too large an appreciation on impact. Not only does the currency of country k appreciate in relation tho the north but also to other countries in the south. Thus, the choice of controls by southern economies can be seen as an outcome of this non-cooperative game. Consequently, in the uncoordinated capital control game, the average level of controls are "too high". These excessively high controls also prevent the terms of trade from adjusting enough on impact of the demand shock in the North . This curtails the ability of the northern economies to exit the liquidity trap in the "Fool Proof" way. Formally, the problem of the country $k \notin \mathcal{N}$ policy maker in choosing capital controls can be expressed as:

$$\max_{\{\mathbb{C}_{k,t},\tau_{k,t}\}} \int_0^\infty e^{-\int_0^t (\rho+\zeta_{k,h})dh} \mathbb{W}_t^k \left(\mathbb{C}_{k,t}; \Theta_{k,t}^n, \mathcal{E}_{n,t}^s, \mathbb{C}_{n,t}\right) dt$$

subject to

$$\begin{aligned} B_{k,0} &= \alpha \int_0^\infty e^{-\int_0^t [\rho+\zeta_{n,h}]dh} \left\{\Theta_{k,t}^n - x - (1-x)\,\Theta_{s,t}^n\right\} dt && (46)\\ i_{k,t} &= \mathcal{I}_k(\cdot), && (47)\\ \frac{\dot{\Theta}_{k,t}^n}{\Theta_{k,t}^n} &= \tau_{k,t} - \tau_{n,t} + \zeta_{n,t} - \zeta_{k,t} && (48)\\ \frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} &= (1-\alpha)i_{k,t} + \alpha\tau_{k,t} + \alpha x i_{n,t} + \alpha(1-x)\left[i_{s,t} - \tau_{s,t}\right] - (\rho + \zeta_{k,t}) && (49) \end{aligned}$$

for given sequences of prices $\left\{i_{n,t}, i_{s,t}, \mathcal{E}_{n,t}^s\right\}$, controls $\{\tau_{s,t}\}$ and aggregate variables $\left\{\mathbb{C}_{n,t}, \Theta_{s,t}^n\right\}$ and shocks $\{\zeta_{n,t}, \zeta_{k,t}\}$ and

$$\mathbb{W}_{k,t} = \log \mathbb{C}_{k,t} - \left[\frac{\left[(1-\alpha)\,\Theta_{k,t}^n + \alpha x + \alpha\,(1-x)\,\Theta_{s,t}^n\right]\left(\mathcal{E}_{n,t}^s\right)^{\alpha(1-x)}}{A(1+\phi)^{\frac{1}{1+\phi}}\left(\mathbb{C}_{n,t}\right)^{\frac{\alpha}{1-\alpha}}\left(\Theta_{k,t}^n\right)^{\frac{1}{1-\alpha}}}\right]^{1+\phi} \mathbb{C}_{k,t}^{\frac{1+\phi}{1-\alpha}}$$

$\mathbb{W}_{k,t}$ is the instantaneous utility function of the representative household in country.

Proposition 6 (Symmetric Nash Equilibrium of the Optimal Capital Control Game)**.** *The symmetric non-cooperative Nash equilibrium of the capital control game in the southern economies features positive controls in the interval* $t \in [0, T]$ *and zero controls from then onwards. Furthermore, the level of controls in the non-cooperative Nash-equilibrium is never larger than naive controls at any instant. Additionally, for* $t \in [0, T]$*, the optimal non-cooperative controls are strictly smaller than the naive controls.*

Under the optimal non-cooperative controls, the currency in the Northern countries is not allowed to depreciate as much on impact as under the free capital control. This curtails the adjustment in the terms of trade. The controls arrest the decrease in the relative price of Northern (compared to the free capital flows case). Consequently, compared to the free capital flow case, the South runs a smaller current account deficit during the liquidity trap as a result of which it needs to run smaller trade surpluses in the future. Hence, it can enjoy higher consumption in the long run. Non-cooperatively set controls make it harder and more painful for the North to exit a liquidity trap. A Northern economy monetary authority must commit to a much longer consumption boom following the liquidity trap so as to be able to arrest the fall in output on impact. This delay in renormalization of interest rates results in a welfare loss in the North.

Proposition 7 (Optimal Monetary and Capital Control Policy in the Non-Cooperative Equilibrium)**.** *The optimal duration for which the Northern monetary authority keeps nominal rates at zero in response to a liquidity trap is longer with naive controls than under no capital controls.*

Additionally, under the optimal non-cooperative capital controls, the optimal exit time from zero rates for the Northern monetary authority is after the optimal exit time under no controls but earlier than the optimal exit under "naive controls". Moreover, under the Pareto-optimal regime of capital account management, the exit from zero nominal rates is faster than under free capital flows.

Figure 5 highlights the differences in the response of the economies in the North and the South depending on the capital-flow management regime. One important difference in the Pareto capital flow regime and the no-coordination regime is that capital flows are taxed rather than subsidized in the non-coordination regime during the liquidity trap. Further, with no coordination, capital controls go to zero as soon as the liquidity trap ends while the Pareto regime required capital inflows to be taxed between T and $\widehat{T}$. Thus, capital flows "too slowly" in the uncoordinated regime compared to the Pareto regime (it flows "too slowly" even compared to the free capital flows case).

Figure 5 also shows that the initial depreciation of the exchange rate is the largest under the Pareto regime while it is the smallest in the no-coordination regime. Consequently, output falls the least on impact in the Pareto regime and the most in the no-coordination regime. As a result, the monetary authority needs to commit to a much shorter consumption boom following the liquidity trap. Consequently, monetary policy renormalizes faster relative to the free capital flows or non-coordinated capital flows regime contributing to welfare gains.

The above discussion underlines that fact that the welfare gains possible by managing the capital account in the Pareto Regime are not achievable if countries in the South are allowed to choose controls in the absence of coordination. This highlights the need for policy coordination.

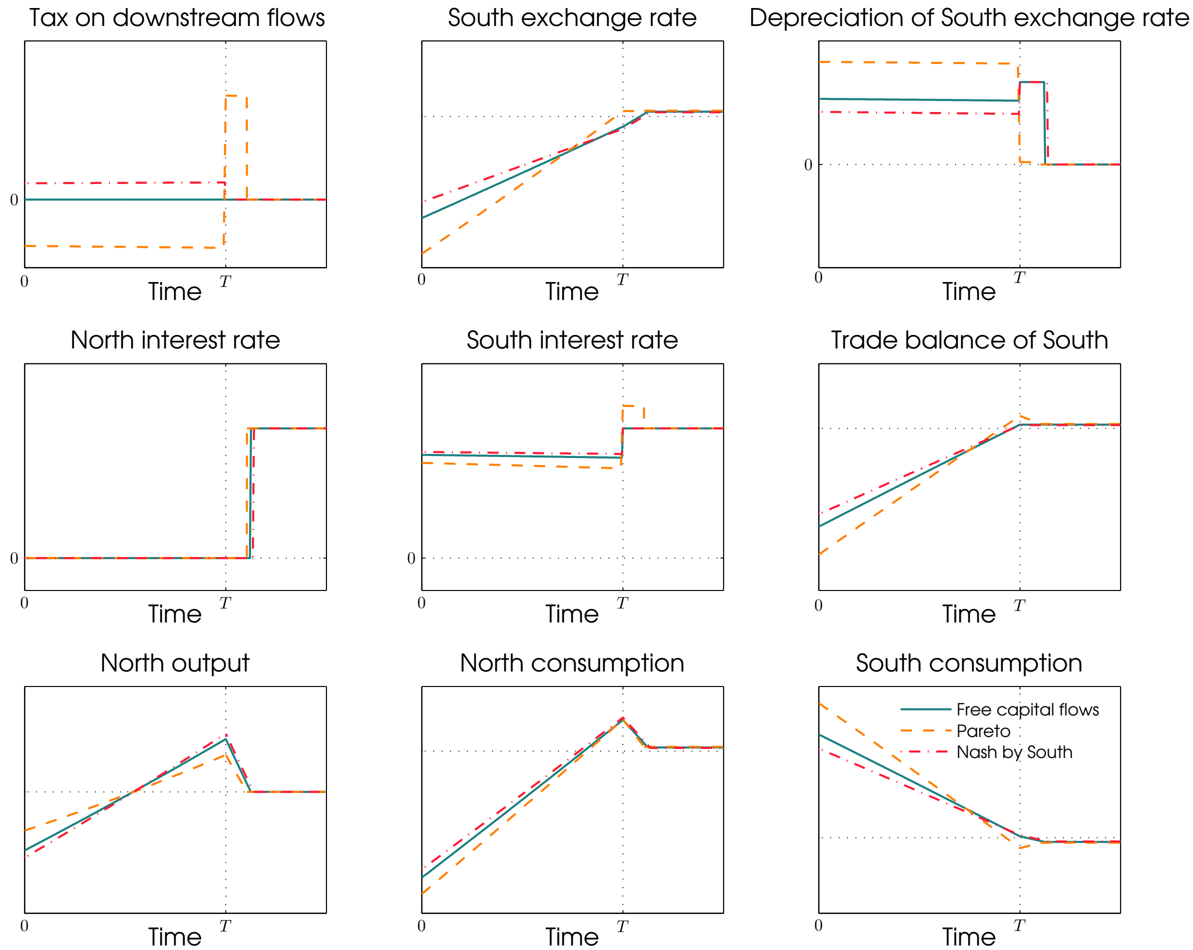

Figure 5: Variable paths under optimal monetary policy in all countries: free capital flows (solid) vs. efficient capital account management (dashed) vs. optimal non-coordinated capital account management (dot-dashed).

7 Conclusion

The recent liquidity trap in the US and the Eurozone caused these advanced economies to cut nominal interest rates to zero in a bid to restore deficient demand and bolster growth prospects. One outcome of such a policy was large flows of capital into the emerging markets like Brazil and India. Faced with imminent appreciation and overheating in their own economies, these countries imposed taxes on capital inflows.

The international spillovers associated with such policies have raised concerns by both policymakers in the emerging economies as well as in advanced economies. None resonate more than the claims by Guido Mantega of being in a "currency war" or Rajan (2014), who calls for advanced economies to be cognizant of the effects of their quantitative easing policies. On the other hand, Olivier Blanchard and others have spoken of how a re-balancing of consumption demand towards the south has to take place for the global economic environment to improve. In other words, the call for international policy cooperation and the need to understand these spillovers has never been stronger.

This paper makes an attempt to study these international policy spillovers in a monetary model with nominal rigidities and isolates an important role for cooperation and internalization of spillovers in setting monetary policy and capital controls internationally. In particular, the paper finds that the use of capital controls in the south may delay the time at which the advanced economies are able to exit the zero lower bound, thus, delaying a global recovery. This points to a novel spillover channel operating from southern to northern economies which needs to be kept in mind as the use of controls prevents the efficient reallocation of demand causing lower global output.

The paper also analyses the equally important spillover from the north to the south. The large inflow of capital into the South because of low rates in the north results in vigorous appreciation of the southern economies which affects their current accounts and growth prospects adversely. The paper finds that under cooperation, the optimal level of capital flows balances these two forces allowing for a more efficient demand reallocation during the liquidity trap but also prescribes capital flows from the South to the North after the trap is over.

Thus, this paper makes a contribution in the growing literature which tries to understand the role of international spillovers in economies where output is partially demand driven. The findings of this paper are very important in the current global economic climate as we need to better understand spillovers associated with policy in this increasingly interconnected world.

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Appendix to "Liquidity Traps and Capital Flows"

Sushant Acharya and Julien Bengui

A Some Derivations

A.1 Budget Constraint of Representative household in country k

$$\int_0^1 \mathcal{E}^j_{k,t}\left[D^j_{k,t+\Delta t} - D^j_{k,t}\right] dj = \left[W_{k,t}N_{k,t} + T_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t}\right]\Delta t + \int_0^1 \left[\frac{\left(1+\tau_{k,t-\Delta t}\right)^{\Delta t}\left(1+i_{j,t-\Delta t}\right)^{\Delta t}}{\left(1+\tau_{j,t-\Delta_t}\right)^{\Delta t}} - 1\right]\mathcal{E}^j_{k,t}D^j_{k,t}dj \tag{A.1.1}$$

Define $(\Delta t) = n \times \overline{\delta t}$ to get:

$$\int_0^1 \mathcal{E}^j_{k,t}\left[D^j_{k,t+\Delta t} - D^j_{k,t}\right] dj = \left[W_{k,t}N_{k,t} + T_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t}\right]\Delta t + \int_0^1 \left[\frac{\left(1+\tau_{k,t-\Delta t}/n\right)^{n\Delta t}\left(1+i_{j,t-\Delta t}/n\right)^{n\Delta t}}{\left(1+\tau_{j,t-\Delta_t}/n\right)^{n\Delta t}} - 1\right]\mathcal{E}^j_{k,t}D^j_{k,t}dj \tag{A.1.2}$$

Taking the limit $\overline{\delta t} \to 0$:

$$\int_0^1 \mathcal{E}^j_{k,t}\left[D^j_{k,t+\Delta t} - D^j_{k,t}\right] dj = \left[W_{k,t}N_{k,t} + T_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t}\right]\Delta t + \int_0^1 \left[e^{(\tau_{k,t-\Delta t}+i_{j,t-\Delta t}-\tau_{j,t-\Delta t})\Delta t} - 1\right]\mathcal{E}^j_{k,t}D^j_{k,t}dj \tag{A.1.3}$$

Using the expansion of e^x, yields:

$$\int_0^1 \mathcal{E}^j_{k,t}\left[D^j_{k,t+\Delta t} - D^j_{k,t}\right] dj = \left[W_{k,t}N_{k,t} + T_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t}\right]\Delta t + \int_0^1 \left[\sum_{h=1}^{\infty}\frac{(\tau_{k,t-\Delta t}+i_{j,t-\Delta t}-\tau_{j,t-\Delta t})^h(\Delta t)^h}{h!}\right]\mathcal{E}^j_{k,t}D^j_{k,t}dj \tag{A.1.4}$$

Divide by Δt:

$$\int_0^1 \mathcal{E}_{k,t}^j \left[\frac{D_{k,t+\Delta t}^j - D_{k,t}^j}{\Delta t}\right] dj = W_{k,t}N_{k,t} + T_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t} + \int_0^1 \left[\sum_{h=1}^{\infty} \frac{(\tau_{k,t-\Delta t} + i_{j,t-\Delta t} - \tau_{j,t-\Delta t})^h (\Delta t)^{h-1}}{h!}\right] \mathcal{E}_{k,t}^j D_{k,t}^j dj \tag{A.1.5}$$

Take limit $(\Delta t) \to 0$:

$$\int_0^1 \mathcal{E}_{k,t}^j \dot{D}_{k,t}^j dj = W_{k,t}N_{k,t} + T_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t} + \int_0^1 [\tau_{k,t} + i_{j,t} - \tau_{j,t}]\, \mathcal{E}_{k,t}^j D_{k,t}^j dj \tag{A.1.6}$$

Add $\int_0^1 \dot{\mathcal{E}}_{k,t}^j D_{k,t}^j dj$ to both sides and use the fact that $a_{k,t} = \int_0^1 \mathcal{E}_{k,t}^j D_{k,t}^j dj$:

$$\dot{a}_{k,t} = i_{k,t}a_{k,t} + W_{k,t}N_{k,t} + T_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t} + \int_0^1 \left[\tau_{k,t} + i_{j,t} - \tau_{j,t} - i_{k,t} + \frac{\dot{\mathcal{E}}_{k,t}^j}{\mathcal{E}_{k,t}^j}\right] \mathcal{E}_{k,t}^j D_{k,t}^j dj \tag{A.1.7}$$

A.2 Optimal Decisions of Representative Household in country k

The problem of the household can be written in form of the following Hamiltonian:

$$\begin{aligned}\mathcal{H} &= \frac{\mathbb{C}_k^{1-\sigma}}{1-\sigma} - \frac{N_k^{1+\phi}}{1+\phi} + \lambda_k \left\{ i_k a_k + W_k N_k + T_k - \mathbb{P}_{k,t}\mathbb{C}_{k,t} + \int_0^1 \left[i_j - \tau_k + \tau_j - i_k + \frac{\dot{\mathcal{E}}_k^j}{\mathcal{E}_k^j}\right] \mathcal{E}_k^j D_k^j dj \right\} \\ &\quad + \mu_{k,t}\left[a_k - \int_0^1 \mathcal{E}_k^j D_k^j dj\right]\end{aligned}$$

The optimal hoice of $\mathbb{C}_k, N_k, D_k^j (j \neq k)$ and D_k^k are given by the following equations (in the same order) :

$$\frac{\partial \mathcal{H}}{\partial \mathbb{C}_k} = \mathbb{C}_k^{-\sigma} - \lambda_k \mathbb{P}_k = 0 \tag{A.2.1}$$

$$\frac{\partial \mathcal{H}}{\partial N_k} = -\mathbb{N}_k^{\phi} + \lambda_k W_k = 0 \tag{A.2.2}$$

$$\frac{\partial \mathcal{H}}{\partial D_k^j} = \left\{ \lambda_k \left[i_j + \tau_k - \tau_j - i_k + \frac{\dot{\mathcal{E}}_k^j}{\mathcal{E}_k^j}\right] - \mu_k \right\} \mathcal{E}_k^j = 0 \ , \ \forall j \neq k \tag{A.2.3}$$

$$\frac{\partial \mathcal{H}}{\partial D_k^k} = -\mu_k \mathcal{E}_k^k = 0 \tag{A.2.4}$$

Equation (A.2.4) implies that $\mu_k = 0$. Along with equation (A.2.3), this implies:

$$i_k = i_j + \tau_k - \tau_j + \frac{\dot{\mathcal{E}}_k^j}{\mathcal{E}_k^j} \tag{A.2.5}$$

$$i_j - \tau_j + \frac{\dot{\mathcal{E}}_k^j}{\mathcal{E}_k^j} = i_{j'} - \tau_{j'} + \frac{\dot{\mathcal{E}}_k^{j'}}{\mathcal{E}_k^{j'}}, \forall j, j' \neq k \tag{A.2.6}$$

The evolution of the costate λ_k can be expressed as:

$$\frac{\dot{\lambda}_k}{\lambda_k} = \rho + \zeta_k - i_k \tag{A.2.7}$$

Equations (A.2.1) and (A.2.7) imply:

$$\frac{\dot{\mathbb{C}}_k}{\mathbb{C}_k} = \frac{1}{\sigma}\left[i_k - \pi_k - (\rho + \zeta_k)\right] \tag{A.2.8}$$

where $\pi_k = \frac{\dot{\mathbb{P}}_k}{\mathbb{P}_k}$.

A.3 Backus-Smith condition

Consider the Euler equation of country k agents and of country j agents for the country j nominal bond:

$$\frac{\dot{\mathbb{C}}_k}{\mathbb{C}_k} = \frac{1}{\sigma}\left[i_j + \tau_k - \tau_j + \frac{\dot{\mathcal{E}}_k^j}{\mathcal{E}_k^j} - \pi_k - (\rho + \zeta_k)\right] \tag{A.3.1}$$

$$\frac{\dot{\mathbb{C}}_j}{\mathbb{C}_j} = \frac{1}{\sigma}\left[i_j - \pi_j - (\rho + \zeta_j)\right] \tag{A.3.2}$$

Equations (A.3.1) and (A.3.2) can be expressed as:

$$\frac{\dot{\mathbb{C}}_k}{\mathbb{C}_k} = \frac{1}{\sigma}\left[i_j - \pi_j + \tau_k - \tau_j + \frac{\dot{\mathcal{Q}}_k^j}{\mathcal{Q}_k^j} - (\rho + \zeta_k)\right] \tag{A.3.3}$$

$$\frac{\dot{\mathbb{C}}_j}{\mathbb{C}_j} = \frac{1}{\sigma}\left[i_j - \pi_j - (\rho + \zeta_j)\right] \tag{A.3.4}$$

Subtract equation (A.3.4) from (A.3.3):

$$\frac{\dot{\mathbb{C}}_k}{\mathbb{C}_k} - \frac{\dot{\mathbb{C}}_j}{\mathbb{C}_j} = \frac{1}{\sigma}\left[\tau_k - \tau_j + \frac{\dot{\mathcal{Q}}_k^j}{\mathcal{Q}_k^j} + \zeta_j - \zeta_k\right] \tag{A.3.5}$$

This can be re-written as:

$$\frac{d}{dt}\left[\ln\left(\frac{\mathbb{C}_{k,t}}{\mathbb{C}_{j,t}}\left(\mathcal{Q}_k^j\right)^{-\frac{1}{\sigma}}\right)\right] = \frac{1}{\sigma}\left[\tau_k - \tau_j + \zeta_j - \zeta_k\right] \tag{A.3.6}$$

which implies that:

$$\frac{\mathbb{C}_{k,t}}{\mathbb{C}_{j,t}}\left(\mathcal{Q}^{j}_{k,t}\right)^{-\frac{1}{\sigma}} = \Theta^{j}_{k,t} \tag{A.3.7}$$

where

$$\Theta^{j}_{k,t} \equiv \Theta^{j}_{k,0}\exp\left\{\frac{1}{\sigma}\int_0^t \left[\tau_{k,h}-\tau_{j,h}\right]dh\right\}\times\exp\left\{\frac{1}{\sigma}\int_0^t \left[\zeta_{j,h}-\zeta_{k,h}\right]dh\right\} \tag{A.3.8}$$

is a bilateral Pareto weight, so we have the bilateral Backus-Smith condition between country k and country j given by

$$\mathbb{C}_{k,t} = \Theta^{j}_{k,t}\mathbb{C}_{j,t}\left(\mathcal{Q}^{j}_{k,t}\right)^{\frac{1}{\sigma}}. \tag{A.3.9}$$

A.4 Goods market clearing condition

The goods market clearing condition for variety l in country k (date subscript is omitted) is

$$\begin{aligned}
Y_k(l) &= C^H_k(l) + \int_0^1 C^k_j(l)\,dj \\
&= \left(\frac{P^H_k(l)}{P^H_k}\right)^{-\epsilon}\left[(1-\alpha)\left(\frac{P^H_k}{\mathbb{P}_k}\right)^{-\eta}\mathbb{C}_k + \alpha\int_0^1\left(\frac{P^k_j}{P^F_j}\right)^{-\gamma}\left(\frac{P^F_j}{\mathbb{P}_j}\right)^{-\eta}\mathbb{C}_j dj\right]
\end{aligned} \tag{A.4.1}$$

Plugging tequation (A.4.1) into the definition of aggregate country k output $Y_k = \left[\int_0^1 Y_k(l)^{\frac{\epsilon-1}{\epsilon}}dl\right]^{\frac{\epsilon}{\epsilon-1}}$, we get:

$$\begin{aligned}
Y_k &= (1-\alpha)\left(\frac{P^H_k}{\mathbb{P}_k}\right)^{-\eta}\mathbb{C}_k + \alpha\int_0^1\left(\frac{P^k_j}{P^F_j}\right)^{-\gamma}\left(\frac{P^F_j}{\mathbb{P}_j}\right)^{-\eta}\mathbb{C}_j dj \\
&= (1-\alpha)\left(\frac{P^F_k}{\mathbb{P}_k}\frac{P^H_k}{P^F_k}\right)^{-\eta}\mathbb{C}_k + \alpha\int_0^1\left(\frac{P^H_j}{P^F_j}\frac{P^k_j}{P^H_j}\right)^{-\gamma}\left(\frac{P^F_j}{\mathbb{P}_j}\right)^{-\eta}\mathbb{C}_j dj \\
&= (1-\alpha)\left(\frac{\mathcal{Q}_k}{\mathcal{S}_k}\right)^{-\eta}\mathbb{C}_k + \alpha\int_0^1\left(\mathcal{S}_j\mathcal{S}^j_k\right)^{\gamma}\mathcal{Q}_j^{-\eta}\mathbb{C}_j dj \\
&= (1-\alpha)\left(\frac{\mathcal{Q}_k}{\mathcal{S}_k}\right)^{-\eta}\mathbb{C}_k + \alpha\int_0^x\left(\mathcal{S}_n\mathcal{S}^n_k\right)^{\gamma}\mathcal{Q}_n^{-\eta}\mathbb{C}_n dj + \alpha\int_x^1\left(\mathcal{S}_s\mathcal{S}^s_k\right)^{\gamma}\mathcal{Q}_s^{-\eta}\mathbb{C}_s dj \\
&= (1-\alpha)\left(\frac{\mathcal{Q}_k}{\mathcal{S}_k}\right)^{-\eta}\mathbb{C}_k + \alpha x\left(\mathcal{S}_n\mathcal{S}^n_k\right)^{\gamma}\mathcal{Q}_n^{-\eta}\mathbb{C}_n + \alpha(1-x)\left(\mathcal{S}_s\mathcal{S}^s_k\right)^{\gamma}\mathcal{Q}_s^{-\eta}\mathbb{C}_s
\end{aligned} \tag{A.4.2}$$

Under the Cole-Obstfeld parametrization this condition becomes

$$\begin{aligned}
Y_k &= (1-\alpha)\left(\frac{\mathcal{Q}_k}{\mathcal{S}_k}\right)^{-1}\mathbb{C}_k + \alpha x\left(\mathcal{S}_n\mathcal{S}_k^n\right)\mathcal{Q}_n^{-1}\mathbb{C}_n + \alpha(1-x)\left(\mathcal{S}_s\mathcal{S}_k^s\right)\mathcal{Q}_s^{-1}\mathbb{C}_s \\
&= (1-\alpha)\left(\mathcal{S}_k\right)^{\alpha}\mathbb{C}_k + \alpha x\left(\mathcal{S}_n\right)^{\alpha}\mathcal{S}_k^n\mathbb{C}_n + \alpha(1-x)\left(\mathcal{S}_s\right)^{\alpha}\mathcal{S}_k^s\mathbb{C}_s \\
&= (1-\alpha)\left(\mathcal{S}_k\right)^{\alpha}\Theta_k^n\mathbb{C}_n\mathcal{Q}_k^n + \alpha x\left(\mathcal{S}_n\right)^{\alpha}\mathcal{S}_k^n\mathbb{C}_n + \alpha\left(1-x\right)\left(\mathcal{S}_s\right)^{\alpha}\mathcal{S}_k^s\Theta_s^n\mathbb{C}_n\mathcal{Q}_s^n \\
&= (1-\alpha)\left(\mathcal{S}_k^n\right)^{\alpha}\left(\mathcal{S}_n^s\right)^{\alpha(1-x)}\Theta_k^n\mathbb{C}_n\left(\mathcal{S}_k^n\right)^{1-\alpha} + \alpha x\left(\mathcal{S}_n\right)^{\alpha}\mathcal{S}_k^n\mathbb{C}_n + \alpha\left(1-x\right)\left(\mathcal{S}_s\right)^{\alpha}\mathcal{S}_k^n\mathcal{S}_n^s\Theta_s^n\mathbb{C}_n\left(\mathcal{S}_s^n\right)^{1-\alpha} \\
&= (1-\alpha)\left(\mathcal{S}_n^s\right)^{\alpha(1-x)}\Theta_k^n\mathbb{C}_n\mathcal{S}_k^n + \alpha x\left(\mathcal{S}_n^s\right)^{\alpha(1-x)}\mathcal{S}_k^n\mathbb{C}_n + \alpha\left(1-x\right)\left(\mathcal{S}_n^s\right)^{-\alpha x}\mathcal{S}_k^n\mathcal{S}_n^s\Theta_s^n\mathbb{C}_n\left(\mathcal{S}_n^s\right)^{-1+\alpha} \\
&= \left[(1-\alpha)\Theta_k^n + \alpha x + \alpha\left(1-x\right)\Theta_s^n\right]\left(\mathcal{S}_n^s\right)^{\alpha(1-x)}\mathcal{S}_k^n\mathbb{C}_n
\end{aligned} \tag{A.4.3}$$

The 2nd line used the facts that $\mathcal{Q}_k = \mathcal{S}_k^{1-\alpha}$, $\mathcal{Q}_n = \mathcal{S}_n^{1-\alpha}$ and $\mathcal{Q}_s = \mathcal{S}_s^{1-\alpha}$. The 3rd line used the Backus-Smith conditions between country k and a representative advanced country, and between a representative emerging and a representative advanced to substitute out $\mathbb{C}_k$ and $\mathbb{C}_k$, respectively. The 4th line used the fact that $\mathcal{Q}_k^n = \left(\mathcal{S}_k^n\right)^{1-\alpha}$, $\mathcal{Q}_s^n = \left(\mathcal{S}_s^n\right)^{1-\alpha}$, $\mathcal{S}_k = \mathcal{S}_k^n\left(\mathcal{S}_n^s\right)^{1-x}$, $\mathcal{S}_k^s = \mathcal{S}_k^n\mathcal{S}_n^s$. The 5th line used the fact that $\mathcal{S}_n = \left(\mathcal{S}_n^s\right)^{1-x}$, $\mathcal{S}_s = \left(\mathcal{S}_n^s\right)^{-x}$ and $\mathcal{S}_n^s = 1/\mathcal{S}_s^n$.

For $k = n$:

$$Y_n = \left[1-\alpha\left(1-x\right)+\alpha\left(1-x\right)\Theta_s^n\right]\left(\mathcal{S}_n^s\right)^{\alpha(1-x)}\mathbb{C}_n$$

For $k = s$:

$$Y_s = \left[\left(1-\alpha x\right)\Theta_s^n + \alpha x\right]\left(\mathcal{S}_n^s\right)^{\alpha(1-x)-1}\mathbb{C}_n$$

The Backus-Smith condition implies:

$$\begin{aligned}
\mathbb{C}_{s,t} &= \Theta_{s,t}^n\mathbb{C}_{n,t}\mathcal{Q}_{s,t}^n = \Theta_{s,t}^n\mathbb{C}_{n,t}\left(\mathcal{S}_{s,t}^n\right)^{1-\alpha} = \Theta_{s,t}^n\mathbb{C}_{n,t}\left(\mathcal{S}_{n,t}^s\right)^{\alpha-1} \\
\Rightarrow \mathcal{S}_{n,t}^s &= \left[\frac{\Theta_{s,t}^n\mathbb{C}_{n,t}}{\mathbb{C}_{s,t}}\right]^{\frac{1}{1-\alpha}}
\end{aligned}$$

which implies that:

$$\begin{aligned}
Y_{n,t} &= \left[1-\alpha\left(1-x\right)+\alpha\left(1-x\right)\Theta_{s,t}^n\right]\left[\frac{\Theta_{s,t}^n}{\mathbb{C}_{s,t}}\right]^{\frac{\alpha(1-x)}{1-\alpha}}\left(\mathbb{C}_{n,t}\right)^{\frac{1-\alpha x}{1-\alpha}} \\
Y_{s,t} &= \left[\left(1-\alpha x\right)\Theta_{s,t}^n + \alpha x\right]\left[\frac{\Theta_{s,t}^n}{\mathbb{C}_{s,t}}\right]^{\frac{\alpha(1-x)-1}{1-\alpha}}\left(\mathbb{C}_{n,t}\right)^{\frac{-\alpha x}{1-\alpha}}
\end{aligned}$$

A.5 Labor market clearing condition

Aggregate employment is defined as $N \equiv \int_0^1 N\left(l\right)dl$. Therefore, labor market clearing requires (omitting t subscripts)

$$N_k = \int_0^1 N_k\left(l\right)dl = \int_0^1 \frac{Y_k\left(l\right)}{A}dl = \frac{Y_k}{A}\int_0^1\left(\frac{P_k^H\left(l\right)}{P_k^H}\right)^{-\epsilon}dl \equiv \frac{Y_k}{A}\Delta_k \tag{A.5.1}$$

with $\Delta_k \equiv \int_0^1\left(\frac{P_k^H(l)}{P_k^H}\right)^{-\epsilon}dl$.

A.6 Relation Between CPI and PPI inflation in a country

Note that

$$
\begin{aligned}
\pi_k &\equiv \frac{\dot{\mathbb{P}}_k}{\mathbb{P}_k} \\
&= \frac{\dot{P}_k^H}{P_k^H} + \frac{\dot{P}_k^F}{P_k^F} - \frac{\dot{P}_k^k}{P_k^k} + \frac{\dot{\mathbb{P}}_k}{\mathbb{P}_k} - \frac{\dot{P}_k^F}{P_k^F}\ , \text{ (since } P_k^H = P_k^k) \\
&= \pi_k^H + \frac{\dot{\mathcal{S}}_k}{\mathcal{S}_k} - \frac{\dot{\mathcal{Q}}_k}{\mathcal{Q}_k} \qquad \text{(A.6.1)}
\end{aligned}
$$

A.7 UIP condition

$$
\tau_k - \tau_j = i_k - i_j - \frac{\dot{\mathcal{E}}_k^j}{\mathcal{E}_k^j}
$$

into the definition of the Pareto-weight (A.3.8)

$$
\Theta_{k,t}^j \equiv \frac{\mathbb{C}_{k,0}}{\mathbb{C}_{j,0}} \left(\mathcal{Q}_{k,0}^j\right)^{-\frac{1}{\sigma}} \exp\left\{\frac{1}{\sigma}\int_0^t \left[\tau_{k,h} - \tau_{j,h} + \zeta_{j,h} - \zeta_{k,h}\right] dh\right\}
$$

we get

$$
\Theta_{k,t}^j \equiv \frac{\mathbb{C}_{k,0}}{\mathbb{C}_{j,0}} \left(\mathcal{Q}_{k,0}^j\right)^{-\frac{1}{\sigma}} \exp\left\{\frac{1}{\sigma}\int_0^t \left[i_{k,h} - i_{j,h} - \frac{\dot{\mathcal{E}}_{k,h}^j}{\mathcal{E}_{k,h}^j} + \zeta_{j,h} - \zeta_{k,h}\right] dh\right\} \qquad \text{(A.7.1)}
$$

A.8 Resource constraint

$$
\begin{aligned}
T_{k,t} &= -\tau_{k,t}\int_{j\neq k} \mathcal{E}_{k,t}^j D_{k,t}^j dj + \tau_{k,t}\int_{j\neq k} D_{j,t}^k dj + \tau_k^L W_{k,t}N_{k,t} + \Pi_{k,t} \\
&= -\tau_{k,t}\int_0^1 \mathcal{E}_{k,t}^j D_{k,t}^j dj + \tau_{k,t}D_{k,t} + \tau_{k,t}\int_{j\neq k} D_{j,t}^k dj + \tau_k^L W_{k,t}N_{k,t} + \Pi_{k,t} \\
&= -\tau_{k,t}\int_0^1 \mathcal{E}_{k,t}^j D_{k,t}^j dj + \tau_{k,t}\left(D_{k,t} + \int_{j\neq k} D_{j,t}^k dj\right) + \tau_k^L W_{k,t}N_{k,t} + \Pi_{k,t} \\
&= -\tau_{k,t}\int_0^1 \mathcal{E}_{k,t}^j D_{k,t}^j dj + \tau_k^L W_{k,t}N_{k,t} + \Pi_{k,t} \qquad \text{(A.8.1)}
\end{aligned}
$$

Substituting (A.8.1) into the household's budget constraint (A.1.7) yields the economy's resource constraint

$$
\begin{aligned}
\dot{a}_{k,t} &= i_{k,t}a_{k,t} + W_{k,t}N_{k,t} + \tau_k^L W_{k,t}N_{k,t} + \Pi_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t} - \tau_{k,t}\int_0^1 \mathcal{E}_{k,t}^j D_{k,t}^j dj \\
&= \left(i_{k,t} - \tau_{k,t}\right)a_{k,t} + P_{k,t}^H Y_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t} \\
&= \left(i_{k,t} - \tau_{k,t}\right)a_{k,t} + P_{k,t}^H Y_{k,t} - \mathbb{P}_{k,t}\mathbb{C}_{k,t}
\end{aligned}
\qquad \text{(A.8.2)}
$$

Multiplying by $\frac{\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}}$ yields

$$\begin{aligned}
&\frac{\dot{a}_{k,t}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}\mathbb{C}_{n,t}^{-\sigma}} = \left(i_{k,t}-\tau_{k,t}\right)\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} + \frac{P_{k,t}^{H}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}}Y_{k,t}\mathbb{C}_{n,t}^{-\sigma} - \frac{\mathbb{P}_{k,t}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}}\mathbb{C}_{k,t}\mathbb{C}_{n,t}^{-\sigma}\\
\Rightarrow\quad &\frac{\dot{n}_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} = \left(i_{k,t}-\tau_{k,t}\right)\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} + \left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\left(\mathcal{S}_{k,t}\right)^{-\alpha}Y_{k,t}\mathbb{C}_{n,t}^{-\sigma} - \left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\mathbb{C}_{k,t}\mathbb{C}_{n,t}^{-\sigma}
\end{aligned}$$

Subtracting $\left(\pi_{n,t}+\frac{\dot{\mathcal{E}}_{k,t}^{n}}{\mathcal{E}_{k,t}^{n}}\right)\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}}$ on both sides

$$\frac{\dot{a}_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} - \left(\pi_{n,t}+\frac{\dot{\mathcal{E}}_{k,t}^{n}}{\mathcal{E}_{k,t}^{n}}\right)\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} = \left(i_{k,t}-\tau_{k,t}-\pi_{n,t}-\frac{\dot{\mathcal{E}}_{k,t}^{n}}{\mathcal{E}_{k,t}^{n}}\right)\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} + \mathbb{C}_{n,t}^{-\sigma}\left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\left[\left(\mathcal{S}_{k,t}\right)^{-\alpha}Y_{k,t}-\mathbb{C}_{k,t}\right]$$

Using the advanced country's Euler equation for country k bonds, we have

$$\frac{\dot{a}_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} - \left(\pi_{n,t}+\frac{\dot{\mathcal{E}}_{k,t}^{n}}{\mathcal{E}_{k,t}^{n}}\right)\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} = \left(\sigma\frac{\dot{\mathbb{C}}_{n,t}}{\mathbb{C}_{n,t}}+\rho+\zeta_{n,t}-\tau_{n,t}\right)\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} + \mathbb{C}_{n,t}^{-\sigma}\left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\left[\left(\mathcal{S}_{k,t}\right)^{-\alpha}Y_{k,t}-\mathbb{C}_{k,t}\right]$$

and therefore

$$\frac{d\left(\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}}\right)}{dt} = \left(\rho+\zeta_{n,t}-\tau_{n,t}\right)\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} + \mathbb{C}_{n,t}^{-\sigma}\left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\left[\left(\mathcal{S}_{k,t}\right)^{-\alpha}Y_{k,t}-\mathbb{C}_{k,t}\right]$$

or

$$\frac{dB_{k,t}}{dt} - \left(\rho+\zeta_{n,t}-\tau_{n,t}\right)B_{k,t} = \mathbb{C}_{n,t}^{-\sigma}\left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\left[\left(\mathcal{S}_{k,t}\right)^{-\alpha}Y_{k,t}-\mathbb{C}_{k,t}\right]$$

with $B_{k,t}\equiv\frac{a_{k,t}\mathbb{C}_{n,t}^{-\sigma}}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}} = \frac{\mathbb{C}_{n,t}^{-\sigma}\int_0^1\mathcal{E}_{k,t}^{j}D_{k,t}^{j}dj}{\mathcal{E}_{k,t}^{n}\mathbb{P}_{n,t}}$. The above equation can be written as:

$$\frac{d}{dt}\left[B_{k,t}e^{-\int_0^t[\rho+\zeta_{n,h}-\tau_{n,t}]ds}\right] \quad = \quad e^{-\int_0^t[\rho+\zeta_{n,h}-\tau_{n,t}]ds}\mathbb{C}_{n,t}^{-\sigma}\left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\left[\left(\mathcal{S}_{k,t}\right)^{-\alpha}Y_{k,t}-\mathbb{C}_{k,t}\right]$$

This can be written as:

$$B_{k,T}e^{-\int_0^T[\rho+\zeta_{n,h}-\tau_{n,h}]dh} \quad = \quad B_{k,0}+\int_0^T\left\{e^{-\int_0^t[\rho+\zeta_{a,h}-\tau_{n,h}]dh}\mathbb{C}_{n,t}^{-\sigma}\left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\left[\left(\mathcal{S}_{k,t}\right)^{-\alpha}Y_{k,t}-\mathbb{C}_{k,t}\right]\right\}dt$$

Taking limits $T\to\infty$ and using the NPG condition yields:

$$B_{k,0} = -\int_0^\infty e^{-\int_0^t[\rho+\zeta_{n,h}-\tau_{n,h}]dh}\mathbb{C}_{n,t}^{-\sigma}\left(\mathcal{Q}_{k,t}^{n}\right)^{-1}\left[\left(\mathcal{S}_{k,t}\right)^{-\alpha}Y_{k,t}-\mathbb{C}_{k,t}\right]dt \tag{A.8.3}$$

A.9 PVIC / Discounted Lifetime Resource Constraint

Substituting out the Backus-Smith condition between country k and a representative northern country (A.3.9) and the goods market clearing condition (A.4.3) into the present value resource constraint

(A.8.3), noting that $\mathcal{Q}_{k,t}^n = \left(\mathcal{S}_{k,t}^n\right)^{1-\alpha}$ and $\mathcal{S}_{k,t} = \mathcal{S}_{k,t}^n \left(\mathcal{S}_{n,t}^s\right)^{1-x}$, we get

$$\begin{aligned} B_{k,0} &= -\int_0^\infty e^{-\int_0^t [\rho+\zeta_{n,h}]dh} \mathbb{C}_{n,t}^{-\sigma} \left(\mathcal{Q}_{k,t}^n\right)^{-1} \left[\left(\mathcal{S}_{k,t}\right)^{-\alpha} Y_{k,t} - \mathbb{C}_{k,t}\right] dt \\ &= \alpha \int_0^\infty e^{-\int_0^t [\rho+\zeta_{n,h}]dh} \left[\Theta_{k,t}^n - x - (1-x)\,\Theta_{s,t}^n\right] dt \qquad \text{(A.9.1)} \end{aligned}$$

which is also known as the present value implementability constraint (PVIC).
For $k = n$:

$$B_{n,0} = \alpha(1-x) \int_0^\infty e^{-\int_0^t [\rho+\zeta_{n,h}]dh} \left[1 - \Theta_{s,t}^n\right] dt$$

For $k = s$:

$$B_{s,0} = -\alpha x \int_0^\infty e^{-\int_0^t [\rho+\zeta_{n,h}]dh} \left[1 - \Theta_{s,t}^n\right] dt$$

A.10 Relationship between nominal exchange rates using UIP's

The UIP for country k implies:

$$i_{k,t} = i_{s,t} + \tau_{k,t} - \tau_{s,t} + \frac{\dot{\mathcal{E}}_{k,t}^s}{\mathcal{E}_{k,t}^s} \qquad \text{(A.10.1)}$$

$$i_{k,t} = i_{n,t} + \tau_{k,t} - \tau_{n,t} + \frac{\dot{\mathcal{E}}_{k,t}^n}{\mathcal{E}_{k,t}^n} \qquad \text{(A.10.2)}$$

The UIP between advanced and emerging economies imply:

$$i_{n,t} = i_{s,t} + \tau_{n,t} - \tau_{s,t} + \frac{\dot{\mathcal{E}}_{n,t}^s}{\mathcal{E}_{n,t}^s} \qquad \text{(A.10.3)}$$

Note that from the definitons of the nominal exchange rates:

$$\mathcal{E}_k^n = \mathcal{E}_s^n \mathcal{E}_k^s \Leftrightarrow \frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} = \frac{\dot{\mathcal{E}}_s^n}{\mathcal{E}_s^n} + \frac{\dot{\mathcal{E}}_k^s}{\mathcal{E}_k^s} \qquad \text{(A.10.4)}$$

This can then be used to re-write the second equations as:

$$i_{k,t} = i_{n,t} + \tau_{k,t} - \tau_{n,t} + \frac{\dot{\mathcal{E}}_s^n}{\mathcal{E}_s^n} + \frac{\dot{\mathcal{E}}_k^s}{\mathcal{E}_k^s} \qquad \text{(A.10.5)}$$

Combining the UIP between a and e with this equation yields:

$$i_{k,t} = i_{s,t} + \tau_{k,t} - \tau_{s,t} + \frac{\dot{\mathcal{E}}_{k,t}^s}{\mathcal{E}_{k,t}^s} + \frac{\dot{\mathcal{E}}_s^n}{\mathcal{E}_s^n} + \frac{\dot{\mathcal{E}}_n^s}{\mathcal{E}_n^s} \qquad \text{(A.10.6)}$$

Since $\frac{\dot{\mathcal{E}}_s^n}{\mathcal{E}_s^n} = -\frac{\dot{\mathcal{E}}_n^s}{\mathcal{E}_n^s}$, the above equation is the same as the UIP condition between k and e. Thus, one out the three constraints is redundant.

A.11 Relationship $\mathcal{S}_k, \mathcal{Q}_k, \mathcal{E}_k$ etc. in the rigid price case

Note that under the Cole-Obstfeld parameterization, $\mathcal{S}_k^n = \frac{P_k^n}{P_k^k} = \frac{\mathcal{E}_k^n P_n^n}{P_k^k} = \mathcal{E}_k^n$ and $\mathcal{S}_k^s = \mathcal{E}_k^s$.[18] Hence,

$$\frac{\dot{\mathcal{S}}_k^n}{\mathcal{S}_k^n} = \frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} \text{ and } \frac{\dot{\mathcal{S}}_k^s}{\mathcal{S}_k^s} = \frac{\dot{\mathcal{E}}_k^s}{\mathcal{E}_k^s}$$

Also,

$$\mathcal{S}_k = \left(\mathcal{S}_k^n\right)^x \left(\mathcal{S}_k^s\right)^{1-x} = \left(\mathcal{E}_k^n\right)^x \left(\mathcal{E}_k^s\right)^{1-x} = \left(\mathcal{E}_k^n\right)^x \left(\mathcal{E}_k^n \mathcal{E}_n^s\right)^{1-x} = \left(\mathcal{E}_k^n\right)^x \left(\mathcal{E}_k^n\right)^{1-x} \left(\mathcal{E}_n^s\right)^{1-x} = \mathcal{E}_k^n \left(\mathcal{E}_n^s\right)^{1-x}$$

and consequently:

$$\frac{\dot{\mathcal{S}}_k}{\mathcal{S}_k} = x\frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} + (1-x)\frac{\dot{\mathcal{E}}_k^s}{\mathcal{E}_k^s}$$

Note that from the definitons of the nominal exchange rates:

$$\mathcal{E}_k^n = \mathcal{E}_s^n \mathcal{E}_k^s \Leftrightarrow \frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} = \frac{\dot{\mathcal{E}}_s^n}{\mathcal{E}_s^n} + \frac{\dot{\mathcal{E}}_k^s}{\mathcal{E}_k^s}$$

As a result:

$$\frac{\dot{\mathcal{S}}_k}{\mathcal{S}_k} = x\frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} + (1-x)\left[\frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} - \frac{\dot{\mathcal{E}}_s^n}{\mathcal{E}_s^n}\right] = \frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} - (1-x)\frac{\dot{\mathcal{E}}_s^n}{\mathcal{E}_s^n} = \frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} + (1-x)\frac{\dot{\mathcal{E}}_n^s}{\mathcal{E}_n^s}$$

Also, we have

$$\mathcal{Q}_k = \left(\mathcal{S}_k\right)^{1-\alpha}$$

so that

$$\frac{\dot{\mathcal{Q}}_k}{\mathcal{Q}_k} = (1-\alpha)\,\frac{\dot{\mathcal{S}}_k}{\mathcal{S}_k} = (1-\alpha)\left[\frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} + (1-x)\frac{\dot{\mathcal{E}}_n^s}{\mathcal{E}_n^s}\right]$$

A.12 Euler equation in the rigid price case

$$\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} = i_{k,t} - \left(\frac{\dot{\mathcal{S}}_{k,t}}{\mathcal{S}_{k,t}} - \frac{\dot{\mathcal{Q}}_{k,t}}{\mathcal{Q}_{k,t}}\right) - (\rho + \zeta_{k,t})$$

$$\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} = i_{k,t} - \left(\frac{\dot{\mathcal{S}}_{k,t}}{\mathcal{S}_{k,t}} - (1-\alpha)\,\frac{\dot{\mathcal{S}}_{k,t}}{\mathcal{S}_{k,t}}\right) - (\rho + \zeta_{k,t})$$

$$\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} = i_{k,t} - \alpha\frac{\dot{\mathcal{S}}_{k,t}}{\mathcal{S}_{k,t}} - (\rho + \zeta_{k,t})$$

$$\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} = i_{k,t} - \alpha\left[\frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} + (1-x)\frac{\dot{\mathcal{E}}_n^s}{\mathcal{E}_n^s}\right] - (\rho + \zeta_{k,t})$$

[18] In the rigid price case, $P_k^k = P_n^n = 1$ etc.

$$\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} = i_{k,t} - \alpha\left[\frac{\dot{\mathcal{E}}_k^n}{\mathcal{E}_k^n} + (1-x)\frac{\dot{\mathcal{E}}_n^s}{\mathcal{E}_n^s}\right] - (\rho + \zeta_{k,t})$$

$$\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} = i_{k,t} - \alpha\left[(i_{k,t} - i_{n,t} - \tau_{k,t} + \tau_{n,t}) + (1-x)\left(i_{n,t} - i_{s,t} - \tau_{n,t} + \tau_{s,t}\right)\right] - (\rho + \zeta_{k,t})$$

$$\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} = i_{k,t} - \alpha\left[i_{k,t} - \tau_{k,t} - x\left(i_{n,t} - \tau_{n,t}\right) - (1-x)\left(i_{s,t} - \tau_{s,t}\right)\right] - (\rho + \zeta_{k,t})$$

$$\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} = (1-\alpha)\, i_{k,t} + \alpha\tau_{k,t} + \alpha x\left(i_{n,t} - \tau_{n,t}\right) + \alpha(1-x)\left(i_{s,t} - \tau_{s,t}\right) - (\rho + \zeta_{k,t})$$

For $k = n$

$$\frac{\dot{\mathbb{C}}_{n,t}}{\mathbb{C}_{n,t}} = \left[1 - \alpha\left(1-x\right)\right] i_{n,t} + \alpha(1-x)\left(i_{s,t} - \tau_{s,t}\right) - (\rho + \zeta_{n,t})$$

For $k = s$:

$$\frac{\dot{\mathbb{C}}_{s,t}}{\mathbb{C}_{s,t}} = (1-\alpha x)\, i_{s,t} + \alpha x\tau_{s,t} + \alpha x i_{n,t} - \rho$$

A.13 The path of $\Theta_{s,t}^n$ after the shock

We have an expression for $\Theta_{s,t}^n$

$$\Theta_{s,t}^n = \Theta_{e,0}^n e^{\int_0^t [\tau_{e,h} + \zeta_{n,h}]dh}$$

while PVIC gives

$$B_{k,0} = \alpha x \int_0^\infty e^{-\int_0^t [\rho + \zeta_{n,h}]dh}\left\{\Theta_{s,t}^n - 1\right\} dt$$

Substituting $\Theta_{s,t}^n$ into the PVIC we get

$$\begin{aligned}
B_{e,0} &= \alpha x \int_0^\infty e^{-\int_0^t [\rho + \zeta_{n,h}]dh}\left\{\Theta_{e,0}^n e^{\int_0^t [\tau_{e,h} + \zeta_{n,h}]dh} - 1\right\} dt \\
B_{e,0} &= \alpha x \int_0^\infty \left\{\Theta_{e,0}^n e^{-\int_0^t [\rho + \zeta_{n,h}]dh} e^{\int_0^t [\tau_{e,h} + \zeta_{n,h}]dh} - e^{-\int_0^t [\rho + \zeta_{n,h}]dh}\right\} dt \\
B_{e,0} &= \alpha x \Theta_{e,0}^n \int_0^\infty \left\{e^{-\int_0^t [\rho + \zeta_{n,h}]dh} e^{\int_0^t [\tau_{s,h} + \zeta_{n,h}]dh}\right\} dt + \alpha x \int_0^\infty \left\{-e^{-\int_0^t [\rho + \zeta_{n,h}]dh}\right\} dt
\end{aligned}$$

This implies:

$$
\begin{aligned}
\Theta_{e,0}^{n} &= \frac{B_{e,0}+\alpha x\int_0^\infty \left\{e^{-\int_0^t[\rho+\zeta_{n,h}]dh}\right\}dt}{\alpha x\int_0^\infty \left\{e^{-\int_0^t[\rho+\zeta_{n,h}]dh}e^{\int_0^t[\tau_{s,h}+\zeta_{n,h}]dh}\right\}dt} \\
&= \frac{B_{e,0}+\alpha x\int_0^\infty \left\{e^{-\int_0^t[\rho+\zeta_{n,h}]dh}\right\}dt}{\alpha x\int_0^\infty \left\{e^{-\int_0^t[\rho-\tau_{s,h}]dh}\right\}dt} \\
&= \frac{1}{\int_0^\infty \left\{e^{-\int_0^t[\rho-\tau_{s,h}]dh}\right\}dt}\left\{\frac{B_{e,0}}{\alpha x}+\int_0^\infty \left\{e^{-\int_0^t[\rho+\zeta_{n,h}]dh}\right\}dt\right\} \\
&= \frac{1}{\left[\int_0^T \left\{e^{-\rho t+\int_0^t \tau_{s,h}dh}\right\}dt+\int_T^\infty \left\{e^{-\rho t+\int_0^T \tau_{s,h}dh}\right\}\right]dt}\left\{\frac{B_{e,0}}{\alpha x}+\left[\int_0^T e^{-[\rho-\bar{\zeta}]t}dt+e^{\bar{\zeta}T}\int_T^\infty e^{-\rho t}dt\right]\right\} \\
&= \frac{1}{\left[\int_0^T \left\{e^{-\rho t+\int_0^t \tau_{s,h}dh}\right\}dt+\frac{1}{\rho}e^{-\rho T+\int_0^T \tau_{s,h}dh}\right]}\left\{\frac{B_{e,0}}{\alpha x}+\frac{e^{[\bar{\zeta}-\rho]T}-1}{\bar{\zeta}-\rho}+\frac{e^{[\bar{\zeta}-\rho]T}}{\rho}\right\} \\
&= \frac{1}{\left[\int_0^T \left\{e^{-\rho t+\int_0^t \tau_{s,h}dh}\right\}dt+\frac{1}{\rho}e^{-\rho T+\int_0^T \tau_{s,h}dh}\right]}\left\{\frac{B_{e,0}}{\alpha x}+\frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)}\right\}
\end{aligned}
$$

Since $\frac{\dot{\Theta}_{s,t}^{n}}{\Theta_{s,t}^{n}}=\tau_{s,t}+\zeta_{n,t}$, we can express the evolution of $\Theta_{s,t}^{n}$ as follows (as long as capital controls are zero after date T:

$$
\Theta_{s,t}^{n}=\begin{cases}
\frac{e^{\int_0^t \tau_{s,h}dh-\bar{\zeta}t}}{\left[\int_0^T \left\{e^{-\rho t+\int_0^t \tau_{s,h}dh}\right\}dt+\frac{1}{\rho}e^{-\rho T+\int_0^T \tau_{s,h}dh}\right]}\left\{\frac{B_{e,0}}{\alpha x}+\frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)}\right\} & \text{if } 0\le t\le T \\
\frac{e^{\int_0^T \tau_{s,h}dh-\bar{\zeta}T}}{\left[\int_0^T \left\{e^{-\rho t+\int_0^t \tau_{s,h}dh}\right\}dt+\frac{1}{\rho}e^{-\rho T+\int_0^T \tau_{s,h}dh}\right]}\left\{\frac{B_{e,0}}{\alpha x}+\frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)}\right\} & \text{if } t>T
\end{cases}
$$

with zero initial NFA, we have

$$
\Theta_{s,t}^{n}=\begin{cases}
\frac{e^{\int_0^t \tau_{s,h}dh-\bar{\zeta}t}}{\left[\int_0^T \left\{e^{-\rho t+\int_0^t \tau_{s,h}dh}\right\}dt+\frac{1}{\rho}e^{-\rho T+\int_0^T \tau_{s,h}dh}\right]}\times\frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)} & \text{if } 0\le t\le T \\
\frac{e^{\int_0^T \tau_{s,h}dh-\bar{\zeta}T}}{\left[\int_0^T \left\{e^{-\rho t+\int_0^t \tau_{s,h}dh}\right\}dt+\frac{1}{\rho}e^{-\rho T+\int_0^T \tau_{s,h}dh}\right]}\times\frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)} & \text{if } t>T
\end{cases}
$$

so that

$$
\Theta_{e,0}^{n}=\frac{1}{\left[\int_0^T \left\{e^{-\rho t+\int_0^t \tau_{s,h}dh}\right\}dt+\frac{1}{\rho}e^{-\rho T+\int_0^T \tau_{s,h}dh}\right]}\times\frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)}
$$

fro which it is apparent that higher controls lower $\Theta_{e,0}^{n}$.

$$
\Theta_{e,0}^{n}=\frac{1}{\frac{\rho-\tau_e e^{(\tau_e-\rho)T}}{\rho(\rho-\tau_e)}}\times\frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)}
$$

For the particular case wher $\tau_{s,h} = \rho$, we have

$$\Theta^n_{s,t} = \begin{cases} \frac{e^{(\rho-\bar{\zeta})t}}{[\rho T+1]} \times \frac{\bar{\zeta}e^{[\bar{\zeta}-\rho]T}-\rho}{(\bar{\zeta}-\rho)} & \text{if } 0 \le t \le T \\ \\ \frac{e^{(\rho-\bar{\zeta})T}}{[\rho T+1]} \times \frac{\bar{\zeta}e^{[\bar{\zeta}-\rho]T}-\rho}{(\bar{\zeta}-\rho)} & \text{if } t > T \end{cases}$$

In case of zero controls (and $B_{e,0} = 0$):

$$\Theta^n_{s,t} = \begin{cases} e^{-\bar{\zeta}t} \times \frac{\bar{\zeta}e^{[\bar{\zeta}-\rho]T}-\rho}{(\bar{\zeta}-\rho)} & \text{if } 0 \le t \le T \\ \\ e^{-\bar{\zeta}T} \times \frac{\bar{\zeta}e^{[\bar{\zeta}-\rho]T}-\rho}{(\bar{\zeta}-\rho)} & \text{if } t > T \end{cases}$$

In case of naive controls (and $B_{e,0} = 0$):

$$\Theta^n_{s,t} = \begin{cases} \frac{1}{\frac{\frac{\bar{\zeta}}{\rho}e^{(\bar{\zeta}-\rho)T}-1}{(\bar{\zeta}-\rho)}} \times \frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)} = 1 & \text{if } 0 \le t \le T \\ \\ \frac{1}{\frac{\frac{\bar{\zeta}}{\rho}e^{(\bar{\zeta}-\rho)T}-1}{(\bar{\zeta}-\rho)}} \times \frac{\frac{\bar{\zeta}}{\rho}e^{[\bar{\zeta}-\rho]T}-1}{(\bar{\zeta}-\rho)} = 1 & \text{if } t > T \end{cases}$$

In case of constant controls τ_e (and $B_{e,0} = 0$):

$$\Theta^n_{s,t} = \begin{cases} \frac{(\rho-\tau_e)e^{(\tau_e-\bar{\zeta})t}}{\rho-\tau_e e^{-(\rho-\tau_e)T}} \times \frac{\bar{\zeta}e^{[\bar{\zeta}-\rho]T}-\rho}{(\bar{\zeta}-\rho)} & \text{if } 0 \le t \le T \\ \\ \frac{(\rho-\tau_e)e^{(\tau_e-\bar{\zeta})T}}{\rho-\tau_e e^{-(\rho-\tau_e)T}} \times \frac{\bar{\zeta}e^{[\bar{\zeta}-\rho]T}-\rho}{(\bar{\zeta}-\rho)} & \text{if } t > T \end{cases}$$

wrt the caveat of the special case where $\tau_{s,h} = \rho$.

B Optimal Policy Problems

B.1 Optimal Allocations

The resource constraint of economy k, the resource constraint of a country in the north and the Backus-Smith condition implies:

$$\mathbb{C}_{k,t} = \left[\frac{1-\alpha(1-x)+\alpha(1-x)\Theta^n_{s,t}}{(1-\alpha)\Theta^n_{k,t}+\alpha x+\alpha(1-x)\Theta^n_{s,t}}\right]^{1-\alpha}\left(\frac{AN_{k,t}}{Y_{n,t}}\right)^{1-\alpha}\Theta^n_{k,t}\mathbb{C}_{n,t}$$

The economy k planner's problem is given below:

$$\max_{N_{k,t}} \int_0^\infty e^{-\int_0^t [\rho+\zeta_{k,h}]dh} \left\{ \log\left(\left[\frac{1-\alpha(1-x)+\alpha\,(1-x)\,\Theta^n_{s,t}}{(1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}} \right]^{1-\alpha} \left(\frac{AN_{k,t}}{Y_{n,t}} \right)^{1-\alpha} \Theta^n_{k,t}\mathbb{C}_{n,t} \right) - \frac{N_{k,t}^{1+\phi}}{1+\phi} \right\} dt$$

subject to the country budget constraint:

$$B_{k,0} = \alpha \int_0^\infty e^{-\int_0^t (\rho+\zeta_{a,s})ds} \left[\Theta^n_{k,t} - x - (1-x)\Theta^n_{s,t} \right]$$

The corresponding Lagrangian can be written as:

$$\begin{aligned} \mathcal{L} &= \int_0^\infty e^{-\int_0^t [\rho+\zeta_{k,h}]dh} \Bigg\{ (1-\alpha)\log\left[\frac{1-\alpha(1-x)+\alpha\,(1-x)\,\Theta^n_{s,t}}{(1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}} \right] + \log\Theta^n_{k,t} \\ &\quad -\alpha\Gamma_k\Theta^n_{k,t}e^{-\int_0^t [\zeta_{a,h}-\zeta_{k,h}]dh} \Bigg\} dt + \int_0^\infty e^{-\int_0^t [\rho+\zeta_{k,h}]dh} \left((1-\alpha)\log N_{k,t} - \frac{N_{k,t}^{1+\phi}}{1+\phi} \right) dt + \cdots \end{aligned}$$

where Γ_k is the multiplier on the PVIC.

So again the optimal allocation for the advanced economy planner features:

$$N_{k,t} = (1-\alpha)^{\frac{1}{1+\phi}}$$

$$e^{\int_0^t [\zeta_{a,s}-\zeta_{k,s}]ds} \left[\frac{1}{\Theta^n_{k,t}} - \frac{(1-\alpha)^2}{(1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}} \right] = \alpha\Gamma_k$$

Taking the time derivitive:

$$[\zeta_{n,t}-\zeta_{k,t}] \left[\frac{1}{\Theta^n_{k,t}} - \frac{(1-\alpha)^2}{g\left(\Theta^n_{k,t},\Theta^n_{s,t}\right)} \right] + \frac{\alpha\,(1-x)\,(1-\alpha)^2\,\Theta^n_{s,t}}{\left[g\left(\Theta^n_{k,t},\Theta^n_{s,t}\right)\right]^2} \frac{\dot{\Theta}^n_{s,t}}{\Theta^n_{s,t}} = \left[\frac{1}{\Theta^n_{k,t}} - \frac{(1-\alpha)^3\,\Theta^n_{k,t}}{\left[g\left(\Theta^n_{k,t},\Theta^n_{s,t}\right)\right]^2} \right] \frac{\dot{\Theta}^n_{k,t}}{\Theta^n_{k,t}}$$

where

$$g\left(\Theta^n_{k,t},\Theta^n_{s,t}\right) = (1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}$$

and $\mathbb{C}_{k,t}$ moves around:

$$\mathbb{C}_{k,t} = \left[\frac{1-\alpha(1-x)+\alpha\,(1-x)\,\Theta^n_{s,t}}{(1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}} \right]^{1-\alpha} \left(\frac{A(1-\alpha)^{\frac{1}{1+\phi}}}{Y_{n,t}} \right)^{1-\alpha} \Theta^n_{k,t}\mathbb{C}_{n,t}$$

B.2 Optimal Monetary Policy for Given Controls

The choice of optimal monetary policy for an arbitary path of capital controls can be given by:

$$\max \int_0^\infty e^{-(\rho+\zeta_{k,h})dh} \left\{ \log \mathbb{C}_{k,t} - \frac{1}{1+\phi} \left(\frac{Y_{k,t}}{A} \right)^{1+\phi} \right\}$$

subject to:

$$\begin{aligned}
i_{k,t} &\geq 0 \\
\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} &= (1-\alpha)i_{k,t} + \alpha x i_{a,t} + \alpha \tau_{k,t} + \alpha(1-x)\left[i_{e,t} - \tau_{e,t}\right] - \rho - \zeta_{k,t}, \\
Y_{k,t} &= \left[(1-\alpha)\,\Theta^n_{k,t} + \alpha x + \alpha\,(1-x)\,\Theta^n_{s,t}\right] \left(\mathcal{E}^s_{n,t}\right)^{\alpha(1-x)} \left(\mathbb{C}_{k,t}\right)^{\frac{1}{1-\alpha}} \left(\Theta^n_{k,t}\right)^{\frac{-1}{1-\alpha}} \left(\mathbb{C}_{n,t}\right)^{\frac{-\alpha}{1-\alpha}}
\end{aligned}$$

B.3 Optimal Capital Account Management Policy in the South

This section lays out the problem facing a Southern economy k is choosing the optimal level of tax/subsidy on the capital inflows from the North. Appendix B.2 provides the solution to the problem of such a country in choosing monetary policy optimally in response to any arbitary path of controls chosen by all other Southern countries. Since we concentrate on the case where the South does not get pushed to the zero lower bound, Appendix B.2 implies that the optimal interest rate policy of a southern country k is given by $i_{k,t} = \mathcal{I}_k(\cdot)$ (See Propositions 1 and 2 for details). The problem of choosing capital controls can be written as:

$$\max_{\{\mathbb{C}_{k,t},\tau_{k,t}\}} \int_0^\infty e^{-\rho t} \left\{ \log \mathbb{C}_{k,t} - \frac{1}{1+\phi} \left(\frac{Y_{k,t}}{A} \right)^{1+\phi} \right\}$$

subject to

$$\begin{aligned}
Y_{k,t} &= \left[(1-\alpha)\,\Theta^n_{k,t} + \alpha x + \alpha\,(1-x)\,\Theta^n_{s,t}\right] \left(\mathcal{E}^s_{n,t}\right)^{\alpha(1-x)} \left(\mathbb{C}_{k,t}\right)^{\frac{1}{1-\alpha}} \left(\Theta^n_{k,t}\right)^{\frac{-1}{1-\alpha}} \left(\mathbb{C}_{n,t}\right)^{\frac{-\alpha}{1-\alpha}} \\
B_{k,0} &= \alpha \int_0^\infty e^{-\int_0^t [\rho+\zeta_{a,s}]ds} \left\{ \Theta^n_{k,t} - x - (1-x)\,\Theta^n_{s,t} \right\} dt \\
i_{k,t} &= \mathcal{I}_k\left(\Theta^n_{k,t}, \Theta^n_{s,t}, \zeta_{k,t}, \zeta_{n,t}, \tau_{s,t}, \tau_{k,t}\right) \\
\frac{\dot{\Theta}^n_{k,t}}{\Theta^n_{k,t}} &= \tau_{k,t} + \zeta_{n,t} \\
\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} &= (1-\alpha)i_{k,t} + \alpha\tau_{k,t} + \alpha x i_{n,t} + \alpha(1-x)\left[i_{s,t} - \tau_{s,t}\right] - (\rho + \zeta_{k,t})
\end{aligned}$$

for given sequences of prices $\left\{i_{n,t}, i_{s,t}, \mathcal{E}^s_{n,t}\right\}$, controls $\{\tau_{s,t}\}$ and aggregate variables $\left\{\mathbb{C}_{n,t}, \Theta^n_{s,t}\right\}$ and shocks $\{\zeta_{n,t}, \zeta_{k,t}\}$.

C Proofs

Proof of Lemma 1

The problem of the monetary policy authority is an optimal control problem with state $\mathbb{C}_{k,t}$ and control $i_{k,t}$. The Hamiltonian is given by

$$\mathcal{H}_{k,t} = e^{-\rho t+\zeta_{k,t}}\mathbb{W}_{k,t}+\mu^c_{k,t}\mathbb{C}_{k,t}\left\{(1-\alpha)i_{k,t}+\alpha\left[x\left(i_{a,t}+\tau_{k,t}-\tau_{a,t}\right)+(1-x)\left(i_{e,t}+\tau_{k,t}-\tau_{e,t}\right)\right]-(\rho+\zeta_{k,t})\right\}$$

with

$$\begin{aligned}
\mathbb{W}_{k,t} &\equiv \ln\mathbb{C}_{k,t}-\frac{1}{1+\phi}\left(\frac{Y_{k,t}}{A}\right)^{1+\phi}\\
Y_{k,t} &= \left[(1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}\right]\left(\mathcal{S}^s_{n,t}\right)^{\alpha(1-x)}\left(\mathbb{C}_{k,t}\right)^{\frac{1}{1-\alpha}}\left(\Theta^n_{k,t}\right)^{\frac{-1}{1-\alpha}}\left(\mathbb{C}_{n,t}\right)^{\frac{-\alpha}{1-\alpha}}.
\end{aligned}$$

The optimality conditions are given by

$$\begin{aligned}
\frac{\partial\mathcal{H}_{k,t}}{\partial i_{k,t}} &= (1-\alpha)\mu^c_{k,t}\mathbb{C}_{k,t}\le 0 \qquad (\text{with } = \text{ if } \ i_{k,t}>0) &\text{(B.3.1)}\\
-\dot{\mu}^c_{k,t} &= \frac{\partial\mathcal{H}_{k,t}}{\partial\mathbb{C}_{k,t}}=\frac{e^{-\rho t+\zeta_{k,t}}}{\mathbb{C}_{k,t}}\left\{1-\frac{1}{1-\alpha}\left(\frac{Y_{k,t}}{A}\right)^{1+\phi}\right\}+\mu^c_{k,t}\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} &\text{(B.3.2)}
\end{aligned}$$

In the absence of the ZLB (or when it never binds), (B.3.1) implies that $\mu^c_{k,t}=0\ \forall t$. (B.3.2) then directly implies that $Y_{k,t}=A(1-\alpha)^{\frac{1}{1+\phi}}\ \forall t$. This in term implies an employment level of $N_{k,t}=(1-\alpha)^{\frac{1}{1+\phi}}\ \forall t$.

Using the Backus-Smith condition between a North and a South country, $\mathbb{C}_{s,t}=\Theta^n_{s,t}\mathbb{C}_{n,t}\left(\mathcal{S}^n_{s,t}\right)^{1-\alpha}$, to subsitute out $\mathcal{S}^s_{n,t}\left(=\mathcal{S}^n_{s,t}\right)$ from (B.3.1), taking natural logarithm and differentiating with respect to time yields

$$\begin{aligned}
\frac{\dot{Y}_{k,t}}{Y_{k,t}} &= \left[\frac{(1-\alpha)\,\Theta^n_{k,t}}{(1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}}-\frac{1}{1-\alpha}\right]\frac{\dot{\Theta}^n_{k,t}}{\Theta^n_{k,t}}+\frac{1}{1-\alpha}\frac{\dot{\mathbb{C}}_{k,t}}{\mathbb{C}_{k,t}} &\text{(B.3.3)}\\
&\quad+\alpha\,(1-x)\left[\frac{\Theta^n_{s,t}}{(1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}}+\frac{1}{1-\alpha}\right]\frac{\dot{\Theta}^n_{s,t}}{\Theta^n_{s,t}}-\frac{\alpha\,(1-x)}{1-\alpha}\frac{\dot{\mathbb{C}}_{s,t}}{\mathbb{C}_{s,t}}-\frac{\alpha x}{1-\alpha}\frac{\dot{\mathbb{C}}_{n,t}}{\mathbb{C}_{n,t}}.
\end{aligned}$$

The Euler equation (35) for representative North and South countries are given by

$$\begin{aligned}
\frac{\dot{\mathbb{C}}_{n,t}}{\mathbb{C}_{n,t}} &= [1-\alpha(1-x)]i_{n,t}+\alpha(1-x)\left(i_{s,t}+\tau_{n,t}-\tau_{s,t}\right)-(\rho+\zeta_{n,t}) &\text{(B.3.4)}\\
\frac{\dot{\mathbb{C}}_{s,t}}{\mathbb{C}_{s,t}} &= (1-\alpha x)i_{s,t}+\alpha x\left(i_{n,t}+\tau_{s,t}-\tau_{n,t}\right)-\rho. &\text{(B.3.5)}
\end{aligned}$$

Substituting (35), (B.3.4), (B.3.5), $\frac{\dot{\Theta}^n_{k,t}}{\Theta^n_{k,t}}=\tau_{k,t}-\tau_{n,t}-\zeta_{k,t}+\zeta_{n,t}$ and $\frac{\dot{\Theta}^n_{s,t}}{\Theta^n_{s,t}}=\tau_{s,t}-\tau_{n,t}+\zeta_{n,t}$ into (B.3.3), while imposing $Y_{k,t}=A(1-\alpha)^{\frac{1}{1+\phi}}\ \forall t$ yields the optimal interest rate rule in (38).

Proof of Lemma 2

We proceed by proving the result for zero taxes, and then arguing that it must hold as well for small taxes do to the continuity of the interest rate policy function $\mathcal{I}_k(\cdot)$ in $\tau_{k,t}$, $\tau_{n,t}$ and $\tau_{s,t}$. In a symmetric equilibrium with zero taxes, the interest rate policy function specified in Lemma 1, for $0 \le t \le T$, reduce to

$$\begin{aligned}
\mathcal{I}_n(1, \Theta^n_{s,t}, -\bar{\zeta}, -\bar{\zeta}, 0, 0, 0) &= \rho - \frac{1-\alpha(1-x)}{1-\alpha(1-x)+\alpha(1-x)\Theta^n_{s,t}}\bar{\zeta}, && \text{(B.3.6)}\\
\mathcal{I}_s(\Theta^n_{s,t}, \Theta^n_{s,t}, 0, -\bar{\zeta}, 0, 0, 0) &= \rho - \frac{\alpha x}{(1-\alpha x)\Theta^n_{s,t}+\alpha x}\bar{\zeta}. && \text{(B.3.7)}
\end{aligned}$$

for a North and for a South country, respectively. Now, noting that $\Theta^n_{s,0} = \frac{\rho}{\rho-\zeta} - \frac{\bar{\zeta}}{\rho-\zeta}e^{-(\rho-\bar{\zeta})T}$ and $\Theta^n_{s,T} = \Theta^n_{s,0}e^{-\bar{\zeta}T}$, the condition of assumption 2 can be written as

$$\rho + \frac{\alpha(1-x)\rho}{[1-\alpha(1-x)]}\Theta^n_{s,0} < \bar{\zeta} < \rho + \frac{(1-\alpha x)\rho}{\alpha x}\Theta^n_{s,T}.$$

or

$$\frac{1-\alpha(1-x)+\alpha(1-x)\Theta^n_{s,0}}{1-\alpha(1-x)}\rho < \bar{\zeta} < \frac{\alpha x + (1-\alpha x)\Theta^n_{s,T}}{\alpha x}\rho. \qquad \text{(B.3.8)}$$

Given that $\Theta^n_{s,T} < \Theta^n_{s,t} < \Theta^n_{s,0}$ for $0 < t < T$, the left inequality in (B.3.8), together with (B.3.6), implies that $\mathcal{I}_n(\cdot) < 0$ for $0 \le t < T$. Similarly, the right inequality in (B.3.8), together with (B.3.7), implies that $\mathcal{I}_s(\cdot) > 0$ for $0 \le t < T$. For $t > T$, we have $\mathcal{I}_n(\cdot) = \mathcal{I}_s(\cdot) = \rho > 0$.

Proof of Lemma 3

The planning problem is given by[19]

$$\max \int_0^\infty e^{-\rho t + \bar{\zeta}\min[t,T]}\mathbb{W}_{n,t}dt$$

[19] With compensating transfers, the planner is not tied to the countries's date 0 budget constraints since he can freely transfer resources across countries at date 0. We can thus abstract from these constraints in the problem.

subject to:

$$
\begin{aligned}
\overline{\mathbb{W}}_s &\leq \int_0^\infty e^{-\rho t}\mathbb{W}_{s,t}dt \\
\mathbb{W}_{n,t} &= \log \mathbb{C}_{n,t} - \frac{1}{1+\phi}\left(\frac{Y_{n,t}}{A}\right)^{1+\phi} \\
\mathbb{W}_{s,t} &= \log \mathbb{C}_{s,t} - \frac{1}{1+\phi}\left(\frac{Y_{s,t}}{A}\right)^{1+\phi} \\
Y_{n,t} &= \left[1-\alpha(1-x)+\alpha\left(1-x\right)\Theta^n_{s,t}\right]\left(\Theta^n_{s,t}\right)^{\frac{\alpha(1-x)}{1-\alpha}}\left(\mathbb{C}_{s,t}\right)^{-\frac{\alpha(1-x)}{1-\alpha}}\left(\mathbb{C}_{n,t}\right)^{\frac{1-\alpha x}{1-\alpha}} \\
Y_{s,t} &= \left[(1-\alpha x)\Theta^n_{s,t}+\alpha x\right]\left(\mathbb{C}_{s,t}\right)^{\frac{1-\alpha(1-x)}{1-\alpha}}\left(\Theta^n_{s,t}\right)^{-\frac{1-\alpha(1-x)}{1-\alpha}}\left(\mathbb{C}_{n,t}\right)^{-\frac{\alpha x}{1-\alpha}} \\
\frac{\dot{\Theta}^n_{s,t}}{\Theta^n_{s,t}} &= \tau_{s,t}+\zeta_{n,t} \\
\frac{\dot{\mathbb{C}}_{n,t}}{\mathbb{C}_{n,t}} &= \left[1-\alpha(1-x)\right]i_{n,t}+\alpha(1-x)\left[i_{s,t}-\tau_{s,t}\right]-(\rho+\zeta_{n,t}) \\
\frac{\dot{\mathbb{C}}_{s,t}}{\mathbb{C}_{s,t}} &= (1-\alpha x)i_{s,t}+\alpha x\left(i_{n,t}+\tau_{s,t}\right)-\rho \\
\mu^c_{k,t}i_{k,t} &= 0\ ,\ k=n,s \\
\dot{\mu}^c_{n,t} &= -\frac{e^{-\rho t+\bar{\zeta}\min[t,T]}}{\mathbb{C}_{n,t}}\left\{1-\frac{1}{1-\alpha}\left[\frac{Y_{n,t}}{A}\right]^{1+\phi}\right\}-\mu^c_{n,t}\frac{\dot{\mathbb{C}}_{n,t}}{\mathbb{C}_{n,t}} \\
\dot{\mu}^c_{s,t} &= -\frac{e^{-\rho t}}{\mathbb{C}_{s,t}}\left\{1-\frac{1}{1-\alpha}\left[\frac{Y_{s,t}}{A}\right]^{1+\phi}\right\}-\mu^c_{s,t}\frac{\dot{\mathbb{C}}_{s,t}}{\mathbb{C}_{s,t}} \\
\dot{\mu}^c_{n,t} &= \mu^c_{n,t}=0\ \forall t\geq \hat{T} \\
\mu^c_{s,0} &= \mu^c_{s,0}=0
\end{aligned}
$$

The present value Hamiltonian — with controls variables $i_{n,t}, i_{s,t}, \tau_{s,t}$, state variables $\Theta^n_{s,t}, \mathbb{C}_{n,t}, \mathbb{C}_{s,t}, \mu^c_{n,t}, \mu^c_{s,t}$, and co-state variables $\mu^\theta_t, \lambda^c_{n,t}, \lambda^c_{s,t}, \varphi^\mu_{n,t}, \varphi^\mu_{s,t}$ — is given by

$$
\begin{aligned}
\mathcal{H} \;=\;& e^{-\rho t\bar{\zeta}\min[t,T]}\mathbb{W}_{n,t}+\Lambda e^{-\rho t}\mathbb{W}_{s,t}+\mu^\theta_t\Theta^n_{s,t}\left[\tau_{s,t}+\zeta_{n,t}\right] \\
&+\lambda^c_{n,t}\mathbb{C}_{n,t}\left[\left[1-\alpha(1-x)\right]i_{n,t}+\alpha(1-x)\left[i_{s,t}-\tau_{s,t}\right]-(\rho+\zeta_{n,t})\right] \\
&+\lambda^c_{s,t}\mathbb{C}_{s,t}\left[(1-\alpha x)i_{s,t}+\alpha x\tau_{s,t}+\alpha x i_{n,t}-\rho\right] \\
&+\varphi^\mu_{n,t}\left[-\frac{e^{-\rho t+\bar{\zeta}\min[t,T]}}{\mathbb{C}_{n,t}}\left\{1-\frac{1}{1-\alpha}\left[\frac{Y_{n,t}}{A}\right]^{1+\phi}\right\}-\mu^c_{n,t}\left\{\left[1-\alpha(1-x)\right]i_{n,t}+\alpha(1-x)\left[i_{s,t}-\tau_{s,t}\right]-(\rho+\zeta_{n,t})\right\}\right] \\
&+\varphi^\mu_{s,t}\left[-\frac{e^{-\rho t}}{\mathbb{C}_{s,t}}\left\{1-\frac{1}{1-\alpha}\left[\frac{Y_{s,t}}{A}\right]^{1+\phi}\right\}-\mu^c_{s,t}\left\{(1-\alpha x)i_{s,t}+\alpha x\tau_{s,t}+\alpha x i_{n,t}-\rho\right\}\right]
\end{aligned}
$$

The optimality conditions are given by:

$$\frac{\partial \mathcal{H}}{\partial i_{n,t}} = [1-\alpha(1-x)]\left\{\lambda^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\mu^c_{n,t}\right\} + \alpha x\left\{\lambda^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\mu^c_{s,t}\right\} \leq 0 \tag{B.3.9}$$

$$0 = i_{n,t}\left\{[1-\alpha(1-x)]\left[\lambda^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\mu^c_{n,t}\right] + \alpha x\left[\lambda^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\mu^c_{s,t}\right]\right\} \tag{B.3.10}$$

$$\frac{\partial \mathcal{H}}{\partial i_{s,t}} = \alpha(1-x)\left[\lambda^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\mu^c_{n,t}\right] + (1-\alpha x)\left[\lambda^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\mu^c_{s,t}\right] \leq 0 \tag{B.3.11}$$

$$0 = i_{s,t}\left\{\alpha(1-x)\left[\lambda^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\mu^c_{n,t}\right] + (1-\alpha x)\left[\lambda^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\mu^c_{s,t}\right]\right\} \tag{B.3.12}$$

$$\frac{\partial \mathcal{H}}{\partial \tau_{s,t}} = \mu^\theta_t\Theta^n_{s,t} + \alpha x\left[\lambda^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\mu^c_{s,t}\right] - \alpha(1-x)\left[\lambda^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\mu^c_{n,t}\right] = 0 \tag{B.3.13}$$

$$\begin{aligned}-\dot{\mu}^\theta_t &= e^{-\rho t}\left\{e^{\bar{\zeta}\min[t,T]}\frac{\partial \mathbb{W}_{n,t}}{\partial \Theta^n_{s,t}} + \Lambda\frac{\partial \mathbb{W}_{s,t}}{\partial \Theta^n_{s,t}}\right\} + \mu^\theta_t\left(\tau_{s,t}+\zeta_{n,t}\right) + \left\{\frac{\varphi^\mu_{n,t}}{\mathbb{C}_{n,t}}e^{-\rho t+\bar{\zeta}\min[t,T]}\frac{1+\phi}{1-\alpha}\left[\frac{Y_{n,t}}{A}\right]^\phi\frac{1}{A}\frac{\partial Y_{n,t}}{\partial \Theta^n_{s,t}}\right\} \\ &\quad + \left\{\frac{\varphi^\mu_{s,t}}{\mathbb{C}_{s,t}}e^{-\rho t}\frac{1+\phi}{1-\alpha}\left[\frac{Y_{s,t}}{A}\right]^\phi\frac{1}{A}\frac{\partial Y_{s,t}}{\partial \Theta^n_{s,t}}\right\}\end{aligned} \tag{B.3.14}$$

$$\begin{aligned}-\dot{\lambda}^c_{n,t} &= e^{-\rho t}\left\{e^{\bar{\zeta}\min[t,T]}\frac{\partial \mathbb{W}_{n,t}}{\partial \mathbb{C}_{n,t}} + \Lambda\frac{\partial \mathbb{W}_{s,t}}{\partial \mathbb{C}_{n,t}}\right\} + \lambda^c_{n,t}\frac{\dot{\mathbb{C}}_{a,t}}{\mathbb{C}_{n,t}} + \left\{\frac{\varphi^\mu_{n,t}e^{-\rho t+\bar{\zeta}\min[t,T]}}{\left(\mathbb{C}_{n,t}\right)^2}\left[1-\frac{1}{1-\alpha}\left(\frac{Y_{n,t}}{A}\right)^{1+\phi}\right]\right\} \\ &\quad + \left\{\frac{\varphi^\mu_{n,t}e^{-\rho t+\bar{\zeta}\min[t,T]}}{\mathbb{C}_{n,t}}\frac{1+\phi}{1-\alpha}\left(\frac{Y_{n,t}}{A}\right)^\phi\frac{1}{A}\frac{\partial Y_{n,t}}{\partial \mathbb{C}_{n,t}}\right\} + \left\{\frac{\varphi^\mu_{s,t}e^{-\rho t}}{\mathbb{C}_{s,t}}\frac{1+\phi}{1-\alpha}\left(\frac{Y_{s,t}}{A}\right)^\phi\frac{1}{A}\frac{\partial Y_{s,t}}{\partial \mathbb{C}_{n,t}}\right\}\end{aligned} \tag{B.3.15}$$

$$\begin{aligned}-\dot{\lambda}^c_{s,t} &= e^{-\rho t}\left\{e^{\bar{\zeta}\min[t,T]}\frac{\partial \mathbb{W}_{n,t}}{\partial \mathbb{C}_{s,t}} + \Lambda\frac{\partial \mathbb{W}_{s,t}}{\partial \mathbb{C}_{s,t}}\right\} + \lambda^c_{s,t}\frac{\dot{\mathbb{C}}_{e,t}}{\mathbb{C}_{s,t}} + \left\{\frac{\varphi^\mu_{s,t}e^{-\rho t}}{\left(\mathbb{C}_{s,t}\right)^2}\left[1-\frac{1}{1-\alpha}\left(\frac{Y_{s,t}}{A}\right)^{1+\phi}\right]\right\} \\ &\quad + \left\{\frac{\varphi^\mu_{n,t}e^{-\rho t+\bar{\zeta}\min[t,T]}}{\mathbb{C}_{n,t}}\frac{1+\phi}{1-\alpha}\left(\frac{Y_{n,t}}{A}\right)^\phi\frac{1}{A}\frac{\partial Y_{n,t}}{\partial \mathbb{C}_{s,t}}\right\} + \left\{\frac{\varphi^\mu_{s,t}e^{-\rho t}}{\mathbb{C}_{s,t}}\frac{1+\phi}{1-\alpha}\left(\frac{Y_{s,t}}{A}\right)^\phi\frac{1}{A}\frac{\partial Y_{s,t}}{\partial \mathbb{C}_{s,t}}\right\}\end{aligned} \tag{B.3.16}$$

$$-\dot{\varphi}^\mu_{n,t} = -\varphi^\mu_{n,t}\frac{\dot{\mathbb{C}}_{n,t}}{\mathbb{C}_{n,t}} \tag{B.3.17}$$

$$-\dot{\varphi}^\mu_{s,t} = -\varphi^\mu_{s,t}\frac{\dot{\mathbb{C}}_{s,t}}{\mathbb{C}_{s,t}} \tag{B.3.18}$$

where

$$\frac{\partial Y_{n,t}}{\partial \Theta^n_{s,t}} = Y_{n,t}\frac{\alpha(1-x)}{1-\alpha}\left\{\frac{1-\alpha(1-x)+(1-\alpha x)\Theta^n_{s,t}}{\left[1-\alpha(1-x)+\alpha(1-x)\Theta^n_{s,t}\right]\Theta^n_{s,t}}\right\}, \quad \frac{\partial Y_{n,t}}{\partial \mathbb{C}_{n,t}} = \frac{1-\alpha x}{1-\alpha}\frac{Y_{n,t}}{\mathbb{C}_{n,t}}, \quad \frac{\partial Y_{n,t}}{\partial \mathbb{C}_{s,t}} = -\frac{\alpha(1-x)}{1-\alpha}\frac{Y_{n,t}}{\mathbb{C}_{s,t}},$$

$$\frac{\partial Y_{s,t}}{\partial \Theta^n_{s,t}} = -Y_{s,t}\frac{\alpha x}{(1-\alpha)}\left\{\frac{(1-\alpha x)\Theta^n_{s,t}+[1-\alpha(1-x)]}{\left[(1-\alpha x)\Theta^n_{s,t}+\alpha x\right]\Theta^n_{s,t}}\right\}, \quad \frac{\partial Y_{s,t}}{\partial \mathbb{C}_{n,t}} = -\frac{\alpha x}{1-\alpha}\frac{Y_{s,t}}{\mathbb{C}_{n,t}}, \quad \frac{\partial Y_{s,t}}{\partial \mathbb{C}_{s,t}} = \frac{1-\alpha(1-x)}{1-\alpha}\frac{Y_{s,t}}{\mathbb{C}_{s,t}},$$

$$\frac{\partial \mathbb{W}_{n,t}}{\partial \mathbb{C}_{n,t}} = \frac{1}{\mathbb{C}_{n,t}}\left[1-\left(\frac{Y_{n,t}}{A}\right)^{1+\phi}\frac{1-\alpha x}{1-\alpha}\right], \quad \frac{\partial \mathbb{W}_{n,t}}{\partial \mathbb{C}_{s,t}} = \frac{1}{\mathbb{C}_{s,t}}\left(\frac{Y_{n,t}}{A}\right)^{1+\phi}\frac{\alpha(1-x)}{1-\alpha}$$

$$\frac{\partial \mathbb{W}_{n,t}}{\partial \Theta^n_{s,t}} = -\left(\frac{Y_{n,t}}{A}\right)^{1+\phi} \frac{\alpha\,(1-x)}{(1-\alpha)} \left\{ \frac{1-\alpha(1-x)+(1-\alpha x)\,\Theta^n_{s,t}}{\left[1-\alpha(1-x)+\alpha\,(1-x)\,\Theta^n_{s,t}\right]\Theta^n_{s,t}} \right\},$$

$$\frac{\partial \mathbb{W}_{s,t}}{\partial \mathbb{C}_{n,t}} = \frac{1}{\mathbb{C}_{n,t}} \left(\frac{Y_{s,t}}{A}\right)^{1+\phi} \frac{\alpha x}{1-\alpha}, \quad \frac{\partial \mathbb{W}_{s,t}}{\partial \mathbb{C}_{s,t}} = \frac{1}{\mathbb{C}_{s,t}} \left[1 - \left(\frac{Y_{s,t}}{A}\right)^{1+\phi} \frac{1-\alpha(1-x)}{1-\alpha}\right],$$

$$\frac{\partial \mathbb{W}_{s,t}}{\partial \Theta^n_{s,t}} = \left(\frac{Y_{s,t}}{A}\right)^{1+\phi} \frac{\alpha x}{(1-\alpha)} \left\{ \frac{(1-\alpha x)\,\Theta^n_{s,t} + [1-\alpha(1-x)]}{\left[(1-\alpha x)\Theta^n_{s,t}+\alpha x\right]\Theta^n_{s,t}} \right\}.$$

Define $\hat{T}$ as the time when both countries have exited the ZLB. Then for $t \geq \hat{T}$, from the complementary slackness conditions (B.3.10) and (B.3.12), we must have

$$\left\{[1-\alpha(1-x)]\left[\lambda^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\mu^c_{n,t}\right] + \alpha x \left[\lambda^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\mu^c_{s,t}\right]\right\} = 0$$

and

$$\left\{\alpha(1-x)\left[\lambda^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\mu^c_{n,t}\right] + (1-\alpha x)\left[\lambda^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\mu^c_{s,t}\right]\right\} = 0$$

Further, for $t \geq \hat{T} > T$ we must have $\mu^c_{n,t} = \mu^c_{s,t} = 0$. The above two equations thus imply $\lambda^c_{n,t} = \lambda^c_{s,t} = 0$ for $t \geq \hat{T} > T$. Equations (B.3.17) and (B.3.18) hence imply

$$0 = \left(e^{\bar{\zeta}T} + \Lambda\right)\alpha x + \left[\delta_n \frac{1+\phi}{1-\alpha}\right] e^{\bar{\zeta}T}\,(1-\alpha x) - \left[\delta_s \frac{1+\phi}{1-\alpha}\right]\alpha x$$

$$0 = \left(e^{\bar{\zeta}T} + \Lambda\right)\alpha\,(1-x) - \left[\delta_n \frac{1+\phi}{1-\alpha}\right] e^{\bar{\zeta}T}\alpha\,(1-x) + \left[\delta_s \frac{1+\phi}{1-\alpha}\right][1-\alpha\,(1-x)]$$

This is a linear system δ_n and δ_s, whose solution can be expressed as

$$\left[\delta_n \frac{1+\phi}{1-\alpha}\right] = -\left(1 + e^{-\bar{\zeta}T}\Lambda\right)\left\{\psi x + (1-\psi)(1-x)e^{-\bar{\zeta}T}\right\}\frac{\alpha x}{(1-\alpha)} \quad \text{and} \quad \left[\delta_s \frac{1+\phi}{1-\alpha}\right] = -\left(e^{\bar{\zeta}T} + \Lambda\right)\frac{\alpha\,(1-x)}{(1-\alpha)}. \tag{B.3.19}$$

Next, equation (B.3.13) implies

$$\mu^\theta_t \Theta^n_{s,t} + \alpha x\left[\lambda^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\mu^c_{s,t}\right] - \alpha(1-x)\left[\lambda^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\mu^c_{n,t}\right] = 0. \tag{B.3.20}$$

Differentiating this equation with respect to time, we obtain

$$\mu^\theta_t\dot{\Theta}^n_{s,t} + \dot{\mu}^\theta_t\Theta^n_{s,t} + \alpha\left[x\left(\lambda^c_{s,t}\dot{\mathbb{C}}_{e,t} + \dot{\lambda}^c_{s,t}\mathbb{C}_{s,t} - \varphi^\mu_{s,t}\dot{\mu}_{s,t} - \dot{\varphi}_{s,t}\mu^c_{s,t}\right) - (1-x)\left(\lambda^c_{n,t}\dot{\mathbb{C}}_{a,t} + \dot{\lambda}^c_{n,t}\mathbb{C}_{n,t} - \varphi^\mu_{n,t}\dot{\mu}_{n,t} - \dot{\varphi}_{n,t}\mu^c_{n,t}\right)\right] = 0 \tag{B.3.21}$$

Using equation (B.3.14) to substitute for $\dot{\mu}^\theta_t\Theta^n_{s,t} + \mu^\theta_t\dot{\Theta}^n_{s,t}$, equation (B.3.16) to substitute for $\lambda^c_{s,t}\dot{\mathbb{C}}_{e,t} + \dot{\lambda}^c_{s,t}\mathbb{C}_{s,t}$, equations (law of motion for $\mu^c_{s,t}$) and (B.3.18) to substitute for $\varphi^\mu_{s,t}\dot{\mu}_{s,t} + \dot{\varphi}_{s,t}\mu^c_{s,t}$, equation (B.3.15) to substitute for $\lambda^c_{n,t}\dot{\mathbb{C}}_{b,t} + \dot{\lambda}^c_{n,t}\mathbb{C}_{n,t}$, equations (law of motion for $\mu^c_{n,t}$) and (B.3.17) to substitute for $\varphi^\mu_{n,t}\dot{\mu}_{n,t} + \dot{\varphi}_{n,t}\mu^c_{n,t}$, equation (B.3.21) can be simplified to

$$\begin{aligned} 0 \quad = \quad & \left(1 - \delta_n\frac{1+\phi}{1-\alpha}\right)\frac{\alpha\,(1-x)\,\Theta^n_{s,t}}{\left[1-\alpha(1-x)+\alpha\,(1-x)\,\Theta^n_{s,t}\right]}\left(\frac{Y_{n,t}}{A}\right)^{1+\phi} \\ & - e^{-\bar{\zeta}\min[t,T]}\left[\Lambda - \delta_s\frac{1+\phi}{1-\alpha}\right]\frac{\alpha x}{\left[(1-\alpha x)\Theta^n_{s,t}+\alpha x\right]}\left(\frac{Y_{s,t}}{A}\right)^{1+\phi} \\ & + \alpha\left[(1-x) - x\Lambda e^{-\bar{\zeta}\min[t,T]}\right] \end{aligned}$$

Using (B.3.19), and assuming that $0 < \alpha < 1$, one can express this relationship as

$$\begin{aligned} 0 &= \frac{\left[1-\alpha\left(1-x\right)+\alpha x\Lambda e^{-\bar{\zeta}T}\right]\Theta_{s,t}^{n}\frac{(1-x)}{1-\alpha}}{\left[1-\alpha(1-x)+\alpha\left(1-x\right)\Theta_{s,t}^{n}\right]}\left(\frac{Y_{n,t}}{A}\right)^{1+\phi}-\frac{\left[\Lambda\left(1-\alpha x\right)+\alpha\left(1-x\right)e^{\bar{\zeta}T}\right]\frac{x}{1-\alpha}}{\left[(1-\alpha x)\Theta_{s,t}^{n}+\alpha x\right]}e^{-\bar{\zeta}\min[t,T]}\left(\frac{Y_{s,t}}{A}\right)^{1+\phi} \\ &\quad +\left[(1-x)-x\Lambda e^{-\bar{\zeta}\min[t,T]}\right] \end{aligned} \tag{B.3.22}$$

The equation needs to hold for any t. In particular, it holds for $t = \hat{T} > T$. Then, since $Y_{n,t} = Y_{s,t} = A\left(1-\alpha\right)^{\frac{1}{1+\phi}}$, it implies $\Lambda = \frac{(1-x)}{x}e^{\bar{\zeta}T}\Theta_{s,\hat{T}}^{n}$. Defining

$$\nu_{1,t} \equiv \frac{\left[1-\alpha\left(1-x\right)+\alpha\left(1-x\right)\Theta_{s,\hat{T}}^{n}\right]\frac{\Theta_{s,t}^{n}}{1-\alpha}}{\left[1-\alpha(1-x)+\alpha\left(1-x\right)\Theta_{s,t}^{n}\right]}\left(\frac{Y_{n,t}}{A}\right)^{1+\phi} \tag{B.3.23}$$

$$\nu_{2,t} \equiv \frac{\left[\alpha x+\Theta_{s,\hat{T}}^{n}\left(1-\alpha x\right)\right]\frac{1}{1-\alpha}}{\left[\alpha x+(1-\alpha x)\Theta_{s,t}^{n}\right]}e^{\bar{\zeta}\max\{0,T-t\}}\left(\frac{Y_{s,t}}{A}\right)^{1+\phi} \tag{B.3.24}$$

(B.3.22) can be compactly written as

$$0 = \nu_{1,t} - \nu_{2,t} + \left[1-\Theta_{s,\hat{T}}^{n}e^{\bar{\zeta}\max\{0,T-t\}}\right] \tag{B.3.25}$$

Differentiating this expression wrt to time gives

$$0 = \dot{\nu}_{1,t} - \dot{\nu}_{2,t} - \Theta_{s,\hat{T}}^{n}e^{\bar{\zeta}\max\{0,T-t\}}\zeta_{n,t} \tag{B.3.26}$$

where

$$\dot{\nu}_{1,t} = \nu_{1,t}\left(\frac{\left[1-\alpha(1-x)\right]}{\left[1-\alpha(1-x)+\alpha\left(1-x\right)\Theta_{s,t}^{n}\right]}\frac{\dot{\Theta}_{s,t}^{n}}{\Theta_{s,t}^{n}}+(1+\phi)\frac{\dot{Y}_{nt}}{Y_{n,t}}\right)$$

$$\dot{\nu}_{2,t} = \nu_{2,t}\left(-\frac{\left(1-\alpha x\right)\Theta_{s,t}^{n}}{\left[\alpha x+\left(1-\alpha x\right)\Theta_{s,t}^{n}\right]}\frac{\dot{\Theta}_{s,t}^{n}}{\Theta_{s,t}^{n}}+\zeta_{n,t}+(1+\phi)\frac{\dot{Y}_{s,t}}{Y_{s,t}}\right)$$

Using (B.3.25), (B.3.26) can be written as

$$0 = \nu_{1,t}\left(\frac{\left[1-\alpha(1-x)\right]}{\left[1-\alpha(1-x)+\alpha\left(1-x\right)\Theta_{s,t}^{n}\right]}\frac{\dot{\Theta}_{s,t}^{n}}{\Theta_{s,t}^{n}}+(1+\phi)\frac{\dot{Y}_{n,t}}{Y_{n,t}}\right) \tag{B.3.27}$$

$$-\nu_{2,t}\left(-\frac{\left(1-\alpha x\right)\Theta_{s,t}^{n}}{\left[\alpha x+\left(1-\alpha x\right)\Theta_{s,t}^{n}\right]}\frac{\dot{\Theta}_{s,t}^{n}}{\Theta_{s,t}^{n}}+(1+\phi)\frac{\dot{Y}_{s,t}}{Y_{s,t}}\right)-\left(\nu_{1,t}+1\right)\zeta_{n,t} \tag{B.3.28}$$

When the ZLB does not bind

Then $\frac{\dot{Y}_{n,t}}{Y_{n,t}} = \frac{\dot{Y}_{s,t}}{Y_{s,t}} = 0$ and the above equation reduces to

$$\frac{\dot{\Theta}_{s,t}^{n}}{\Theta_{s,t}^{n}} = \frac{1+\nu_{1,t}}{\nu_{1,t}\frac{\left[1-\alpha(1-x)\right]}{\left[1-\alpha(1-x)+\alpha(1-x)\Theta_{s,t}^{n}\right]}+\nu_{2,t}\frac{(1-\alpha x)\Theta_{s,t}^{n}}{\left[\alpha x+(1-\alpha x)\Theta_{s,t}^{n}\right]}}\zeta_{n,t}$$

with

$$\nu_{1,t} \equiv \frac{\left[1-\alpha\left(1-x\right)+\alpha\left(1-x\right)\Theta^n_{s,\hat{T}}\right]\Theta^n_{s,t}}{\left[1-\alpha(1-x)+\alpha\left(1-x\right)\Theta^n_{s,t}\right]} \qquad \text{and} \qquad \nu_{2,t} \equiv \frac{\left[\alpha x+\Theta^n_{s,\hat{T}}\left(1-\alpha x\right)\right]}{\left[\alpha x+(1-\alpha x)\Theta^n_{s,t}\right]}e^{\bar{\zeta}\max\{0,T-t\}}$$

ZLB binds only in the North In this case, we need to make use of the growth rate of output in the North:

$$\frac{\dot{Y}_{n,t}}{Y_{n,t}} = -\left(\rho+\zeta_{n,t}\right)+\frac{\alpha\left(1-x\right)\Theta^n_{s,t}}{1-\alpha(1-x)+\alpha\left(1-x\right)\Theta^n_{s,t}}\frac{\dot{\Theta}^n_{s,t}}{\Theta^n_{s,t}}$$

and we find the ODE

$$\frac{\dot{\Theta}^n_{s,t}}{\Theta^n_{s,t}} = \frac{\nu_{1,t}\left(1+\phi\right)\left(\rho+\zeta_{n,t}\right)+\left(\nu_{1,t}+1\right)\zeta_{n,t}}{\nu_{1,t}\left[1+\phi\frac{\alpha(1-x)\Theta^n_{s,t}}{1-\alpha(1-x)+\alpha(1-x)\Theta^n_{s,t}}\right]+\nu_{2,t}\frac{(1-\alpha x)\Theta^n_{s,t}}{\left[\alpha x+(1-\alpha x)\Theta^n_{s,t}\right]}}$$

In order to have an ODE in $\Theta^n_{s,t}$ only, we use the fact that $Y_{n,t}\exp\left\{\int_t^{\hat{T}}\frac{\dot{Y}_{n,h}}{Y_{n,h}}dh\right\} = A\left(1-\alpha\right)^{\frac{1}{1+\phi}}$ and get

$$Y_{n,t} = A\left(1-\alpha\right)^{\frac{1}{1+\phi}}e^{\rho\left(\hat{T}-t\right)-\bar{\zeta}\max\{0,T-t\}}\left(\frac{1-\alpha(1-x)+\alpha\left(1-x\right)\Theta^n_{s,t}}{1-\alpha(1-x)+\alpha\left(1-x\right)\Theta^n_{s,\hat{T}}}\right)$$

so that we have

$$\nu_{1,t} \equiv e^{(1+\phi)\left[\rho\left(\hat{T}-t\right)-\bar{\zeta}\max\{0,T-t\}\right]}\Theta^n_{s,t}\left(\frac{1-\alpha(1-x)+\alpha\left(1-x\right)\Theta^n_{s,t}}{1-\alpha(1-x)+\alpha\left(1-x\right)\Theta^n_{s,\hat{T}}}\right)^{\phi}$$

$$\nu_{2,t} \equiv \frac{\left[\alpha x+\Theta^n_{s,\hat{T}}\left(1-\alpha x\right)\right]}{\left[\alpha x+(1-\alpha x)\Theta^n_{s,t}\right]}e^{\bar{\zeta}\max\{0,T-t\}}$$

Proof of Proposition 1

The planner's problem is

$$\max_{Y_{k,t},N_{k,t},\mathbb{C}_{k,t},\mathcal{S}^n_{k,t}} \ln\left(\mathbb{C}_{k,t}\right)-\frac{\left(N_{k,t}\right)^{1+\phi}}{1+\phi}$$

subject to

$$Y_{k,t} = AN_{k,t} \tag{C.1.1}$$

$$\mathbb{C}_{k,t} = \Theta^n_{k,t}\mathbb{C}_{n,t}\left(\mathcal{S}^n_{k,t}\right)^{1-\alpha} \tag{C.1.2}$$

$$Y_{k,t} = \left[\left(1-\alpha\right)\Theta^n_{k,t}+\alpha x+\alpha\left(1-x\right)\Theta^n_{s,t}\right]\left(\mathcal{S}^s_{n,t}\right)^{\alpha(1-x)}\mathcal{S}^n_{k,t}\mathbb{C}_{n,t} \tag{C.1.3}$$

Using (C.1.1)-(C.1.3) to substitute out for $Y_{k,t}, \mathbb{C}_{k,t}, \mathcal{S}^n_{k,t}$ yields an unconstrained problem in $N_{k,t}$

$$\max_{N_{k,t}} \quad \ln\left(\Theta^n_{k,t}\mathbb{C}_{n,t}\left[\frac{AN_{k,t}}{\mathbb{C}_{n,t}\left[(1-\alpha)\,\Theta^n_{k,t}+\alpha x+\alpha\,(1-x)\,\Theta^n_{s,t}\right]\left(\mathcal{S}^s_{n,t}\right)^{\alpha(1-x)}}\right]^{1-\alpha}\right)-\frac{(N_{k,t})^{1+\phi}}{1+\phi},$$

whose first-order condition simply yields a constant employment $N_{k,t}=(1-\alpha)^{\frac{1}{1+\phi}}$.

Proof of Proposition 2

[To be added]

Proof of Proposition 3

[To be added]

Proof of Proposition 4

[To be added]

Proof of Proposition 5

By contradiction. Suppose that $\tau_{s,t}=0\ \forall t\geq 0$. In this case, we know that the ZLB binds in the North only, so that $\frac{\dot{\Theta}^n_{s,t}}{\Theta^n_{s,t}}=\zeta_{n,t}$ implies

$$\zeta_{n,t}=\frac{\nu_{1,t}\,(1+\phi)\,(\rho+\zeta_{n,t})+(\nu_{1,t}+1)\,\zeta_{n,t}}{\nu_{1,t}\left[1+\phi\frac{\alpha(1-x)\Theta^n_{s,t}}{1-\alpha(1-x)+\alpha(1-x)\Theta^n_{s,t}}\right]+\nu_{2,t}\frac{(1-\alpha x)\Theta^n_{s,t}}{\left[\alpha x+(1-\alpha x)\Theta^n_{s,t}\right]}}$$

This equation must be valid for $t\geq 0$, and therefore also for any t such that $T\leq t<\hat{T}$ (under free capital flows we know for a fact that $T<\hat{T}$). Then, since $\zeta_{n,t}=0$, the equation implies

$$0=\frac{\nu_{1,t}\,(1+\phi)\,\rho}{\nu_{1,t}\left[1+\phi\frac{\alpha(1-x)\Theta^n_{s,t}}{1-\alpha(1-x)+\alpha(1-x)\Theta^n_{s,t}}\right]+\nu_{2,t}\frac{(1-\alpha x)\Theta^n_{s,t}}{\left[\alpha x+(1-\alpha x)\Theta^n_{s,t}\right]}}$$

which is a contradiction since denominator and numerator of the fraction are both stricly positive.