DISCUSSION PAPER NO. 451

POST-WAR U.S. BUSINESS CYCLES: AN EMPIRICAL INVESTIGATION */

by

Robert J. Hodrick 1/

and

Edward C. Prescott 2/

Revised November 1980

*Support of the National Science Foundation is acknowledged. We also acknowledge helpful comments by the participants at 1979 Summer Warwick Workshop on Expectation and the money workshops at the Universities of Chicago and Virginia and at Carnegie-Mellon University. In particular, we would like to thank Robert Avery, V. V. Chari, Charles R. Nelson, Thomas J. Sargent and John H. Wood for comments. We also thank the Wharton Economic Forecasting Associates for providing the data.

1/ Carnegie-Mellon University

2/ University of Minnesota and Northwestern University

1. Introduction

In this article aggregate economic fluctuations in the post-war U.S. economy are investigated using quarterly data. The time series are decomposed into a growth and a cyclical component. The growth component includes those variations that are sufficiently smooth to be consistent with slowly changing demographic and technological factors and accumulation of capital stocks. The cyclical component is defined to be those variations that appear to be too rapid to be explained by such considerations. This decomposition reflects the division within economics between studies of growth and studies of business cycles.

There are many ways in which the series might be decomposed into these components, but, without strong prior knowledge concerning the processes governing the growth and the cyclical components, standard statistical theory is of limited use in choosing among procedures. We therefore proceed in a more cautious manner that requires weaker prior knowledge. The maintained hypothesis, based upon growth theory considerations, is that the growth component varies smoothly over time. The sense in which it varies smoothly is made explicit in section 2.

Our approach is very much in the tradition of Wesley Mitchell [1913] who documented some important systematic fluctuations that could not readily be accounted for by standard growth considerations.[1] Our approach differs in two respects from his. The first and not so important respect is that our method involves a minimum of judgment and is easily reproduced at small costs. It is computationally more intensive, but, this is hardly a consideration now that high speed computers can compute in seconds that which would have taken Mitchell decades. At a substantive level our primary objective is not to identify specific cycles. Instead it is to examine the magnitudes and stability of covariances between various economic time series and real output and the autocovariances of real output.

Several researches using alternative methods have and are adding to our understanding of business cycle regularities.[2] Our view is not that our methods dominate theirs. Rather our view is that many of the methods, including our own, all contribute to a better description of the empirical regularities.

2. Decomposition Procedure

The observed time series are viewed as the sum of a cyclical and growth component. Actually, there is also a seasonal component, but as the data are seasonally adjusted, this component has already been removed by those preparing the data series. If growth theory provided estimates of the growth component with errors that were small relative to the cyclical component, computing the cyclical component would be just a matter of calculating the difference between the observed value and the growth component. Growth theory accounting (cf. Denison [1974]), in spite of its considerable success, is far from adequate for providing such numbers. If our prior knowledge were sufficiently strong that we could model the growth component as a deterministic component, possibly conditional on exogenous data, plus a stochastic process and the cyclical component as some other stochastic process, estimating the cyclical component would be an exercise in modern time series analysis. Our prior knowledge is not of this variety, so these powerful methods are not applicable. Our prior knowledge is that the growth component varies "smmothly" over time.

Our conceptual framework is that a given time series y_t is the sum of a growth component g_t and a cyclical component c_t:

$$y_t = g_t + c_t \quad \text{for } t = 1,\ldots,T . \tag{1}$$

Our measure of the smoothness of the $\{g_t\}$ path is the sum of the squares of its second difference. The c_t are deviations from g_t and our conceptual framework is that over long time periods, their average is near zero. These considerations lead to the following programming problem for determining the growth component:

$$\text{(2)} \qquad \underset{\{g_t\}_{t=-1}^{T}}{\text{Min}} \left\{ \sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=1}^{T} [(g_t - g_{t-1}) - (g_{t-1} - g_{t-2})]^2 \right\}$$

where $c_t = y_t - g_t$. The parameter λ is a positive number which penalizes variability in the growth component series. The larger the value of λ, the smoother is the solution series. For sufficiently large λ, at the optimum $g_{t+1} - g_t$ must be near some constant β and g_t near $g_0 + \beta t$. This implies that the limit of solutions as λ approaches infinity is the least squares fit with a linear time trend.

Out method has a long history of use, particularly in the actuarial sciences. There it is called the Whittaker-Henderson Type A method (Whittaker [1923]) of graduating or smoothing mortality experiences in constructing mortality tables. The method is still in use.[3] As pointed out in Stigler's [1978] historical review paper, closely related methods were developed by the Italian astronomer Schiaparelli in 1867 and in the ballistic literature by, among others, von Neuman in the early forties.

Value of the Smoothness Parameter

The data analyzed, with the exception of the interest rates, are in logs so the change in the growth component, $g_t - g_{t-1}$, corresponds to a growth rate.

The growth rate of labor's productivity has varied considerably over this period (see McCarthy [1978]). In the 1947-53 period, the annual growth rate was 4.20 percent, in the 1953-68 period 2.61 percent, in the 1968-73 period only 1.41 percent and in the subsequent period it was even smaller. Part of these changes can be accounted for by a changing capital/labor ratio and changing composition of the labor force. But, as shown by McCarthy, a sizable and variable unexplained component remains, even after correcting for cyclical factors. The assumption that the growth rate has been constant over our thirty-year sample period, 1950-79, is not tenable. To proceed as if it were would result in errors in modeling the growth component and these errors are likely to be non-trivial relative to the cyclical component. For this reason, an infinite value for the smoothness parameter was not selected.

If the cyclical components and the second differences of the growth components were identically and independently distributed normal variates with means zero and variances σ_1^2 and σ_2^2 respectively (which they are not), one would solve the programming problem

$$\text{(3)} \qquad \min_{\{g_t\}_{t=-1}^{T}} \left\{ \sigma_1^{-2} \sum_{t=1}^{T} c_t^2 + \sigma_2^{-2} \sum_{t=1}^{T} (\Delta^2 g_t)^2 \right\}$$

to determine the conditional expectations of the g_t as a function of the observations.[4] This minimization has the same solution as does program (2) if $\sqrt{\lambda} = \sigma_1/\sigma_2$. Our prior view is that a five percent cyclical component is moderately large as is a one-eighth of one percent change in the growth rate in a quarter. This led us to select $\sqrt{\lambda} = 5/(1/8) = 40$ or $\lambda = 1600$ as a value for the smoothing parameter.

One important issue is how sensitive are the results to the value

λ selected. To explore this issue various other values of λ were tried. Table 1 contains the (sample) standard deviations and autocorrelations of cyclical real GNP for the selected values of the smoothing parameter. These numbers change little if λ is reduced by a factor of four to 400 or increased by a factor of four to 6400. As λ increases, the standard deviation increases and there is greater persistence, with the results being very different for $\lambda = \infty$.

As a second test, the sixth order autoregressive least square fit of the cyclical series for this same set of values of the smoothing parameter were computed; that is, compute the least squares fit to the equation

$$c_t = \alpha + \sum_{i=1}^{6} \beta_i c_{t-i} . \tag{4}$$

The unit input response function are the values of c_t obtained if $c_t = 0$ for $t < 0$ and $c_0 = 1$. These response functions are plotted in Figure 1 for the selected λ's.[5] As can be seen, the response pattern is very different for the linear time trend model ($\lambda = \infty$) than for the other values of λ. For the linear time trend model there is much greater persistence and the response function never dips below zero.

With our procedure for identifying the growth component, the annual rate of change of the growth component varied between 2.3 and 4.9 percent over the sample period with the minima occurring in 1957 and in 1974. The maximum growth rate occurred in 1964 with another peak of 4.4 percent in 1950. The average growth rate over the period was 3.4 percent.

TABLE 1

STANDARD DEVIATIONS AND SERIAL CORRELATIONS OF
CYCLICAL GNP FOR DIFFERENT VALUES OF THE SMOOTHING PARAMETER

	λ=400	λ=1600	λ=6400	λ=infinity
Standard Deviations	1.56%	1.80%	2.03%	3.12%
Auto Correlations				
Order 1	.80	.84	.87	.94
Order 2	.48	.57	.65	.84
Order 3	.15	.27	.41	.73
Order 4	-.14	-.01	.17	.61
Order 5	-.32	-.20	.00	.52
Order 6	-.39	-.30	-.11	.44
Order 7	-.42	-.38	-.20	.38
Order 8	-.44	-.44	-.27	.31
Order 9	-.41	-.44	-.31	.25
Order 10	-.36	-.41	-.32	.20

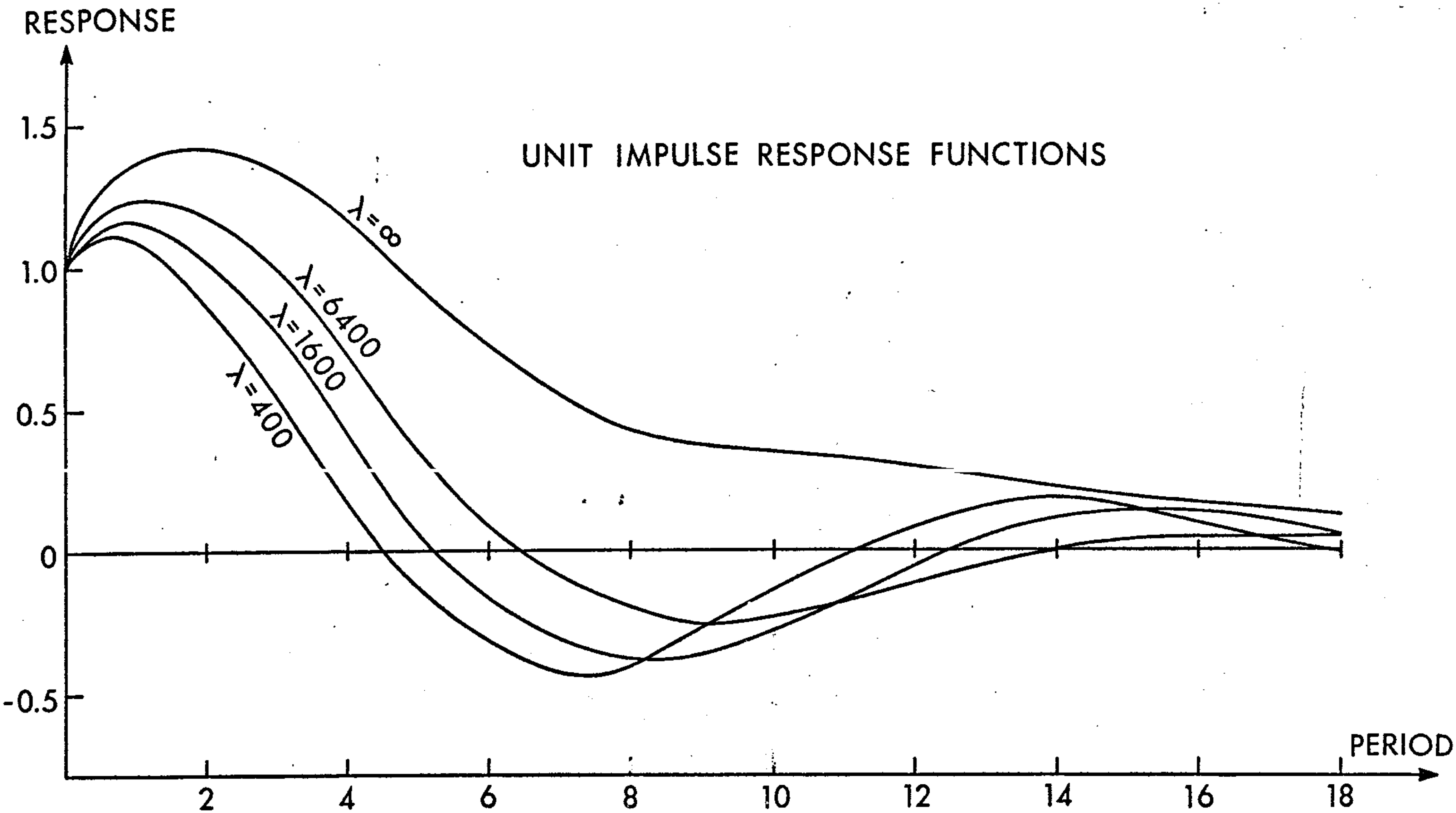

RESPONSE
1.5
1.0
0.5
0
-0.5
UNIT IMPULSE RESPONSE FUNCTIONS
λ=∞
λ=6400
λ=1600
λ=400
PERIOD
2
4
6
8
10
12
14
16
18

FIG. 1

The differences between our cyclical components and those obtained with perfect smoothing ($\lambda = \infty$) are depicted in Figure 2 along with the cyclical component. The smoothness of the variation in this difference and its small size relative to the cyclical component indicates that the smoothing parameter chosen is reasonable. Of greater importance, it also provides confirming evidence for viewing the series as the sum of a slowly and a rapidly varying component.

No decomposition method is perfect. If the thirties were part of the sample, the method would be inappropriate. In that period, there is another component which swamps the others--namely, the Great Depression, an important topic of study in itself. The analysis is not suited for identifying cycles of long durations. If, for example, Schumpeter's (1939) view is correct and there are Kondratieff cycles with durations measured in decades and Juglar cycles with average duration of nine or ten years, our method would assign virtually all these variations to the growth component. With our method, only the rapid or high frequency fluctuations are included in the cyclical component.

The same transformation was used for all series; that is, for each series j

$$g_{jt} = \sum_{i=1}^{T} w_{it}^{T} y_{ji} , \tag{5}$$

where T is the length of the sample period. The coefficients w_{it}^{T} used were the same for all series j. If the sample size were infinite, it would not be necessary to index these coefficients by t and

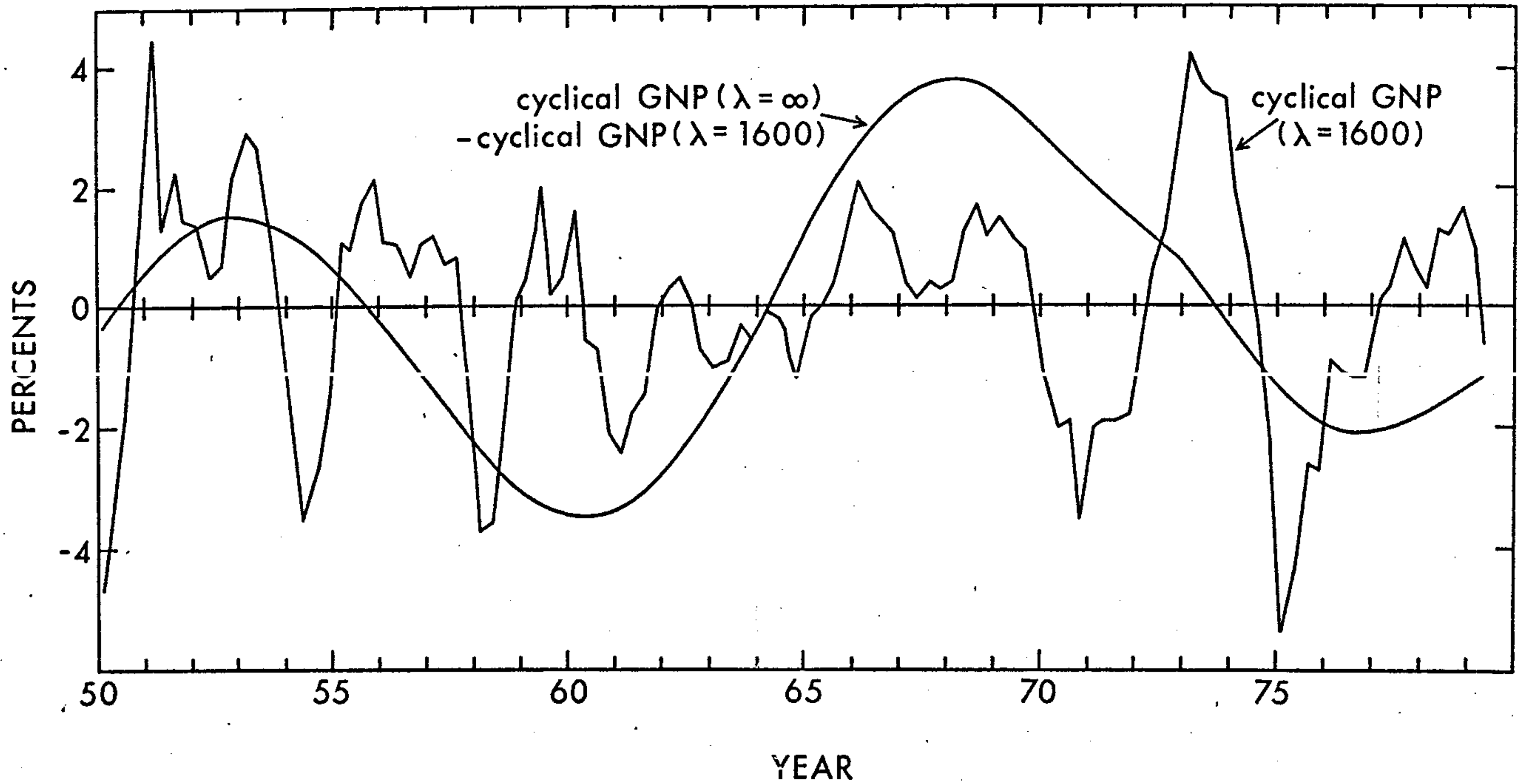

cyclical GNP (λ = ∞)
- cyclical GNP (λ = 1600)
cyclical GNP
(λ = 1600)
PERCENTS
4
2
0
-2
-4
50
55
60
65
70
75
YEAR

FIG. 2

$$g_{jt} = \sum_{i=-\infty}^{\infty} w_i^{\infty} y_{j,t+i} \tag{6}$$

where

$$w_i^{\infty} = .8941^i \, [.056168 \cos(.11168\, i) + .055833 \sin(.11168\, i) \tag{7}$$

for $i \geqq 0$ and $w_i = w_{-i}$ for $i < 0$.[6] For t far from either the end or the beginning of the sample, the w_{it}^T are near w_{t-i}^{∞} so our method is approximately a two-way moving average with weights subject to a damped harmonic. The advantage of using the exact solution is that observations near the beginning and the end of the sample period are not lost.

3. Variability and Covariability of the Series

The components being studied are the cyclical component and subsequently all references to a series relate to its cyclical component. The sample standard deviations of a series is our measure of a series's variability and the correlation with real GNP our measure of a series's covariability. These measures are computed for the first and the second half of the sample as well as for the entire sample. This is a check for the stability of the measures over time.

A variable might be strongly associated with real output but lead or lag real output. Therefore, as a second measure of the strength of association with real output, the R-squared for the regression

$$c_{jt} = \alpha_j + \sum_{i=-2}^{2} \beta_{ji} \, GNP_{t+i} \tag{8}$$

for each series j was computed. As a measure of the instability of the relationship, the following statistic was computed:

$$\frac{(SS_\omega - SS_\Omega)/6}{SS_\Omega/102} ,$$

where SS_ω is the sum of squared residuals when the coefficients of the regression were constrained to be equal in the first and the second half of the sample and SS_Ω when they were not. As the sample size was 114, the degrees of freedom under Ω are 102. Further ω is obtained by imposing six linear constraints upon Ω. Consequently, if the errors were identically and independently distributed normal variates, this statistic would have an F distribution with degrees of freedom 6 and 102.

TABLE 2

AGGREGATE DEMAND COMPONENTS: STANDARD DEVIATIONS WITH GNP

Sample Period: 1950.1 - 1979.2

	Standard Deviations in Percents			Correlations with Real Output			Average Percent of Real GNP
	Whole	First Half	Second Half	Whole	First Half	Second Half	
Real GNP	1.8	1.7	1.9	-	-	-	-
Total Consumption	1.3	1.2	1.4	.739	.503	.917	61.7
Services	.7	.7	.6	.615	.441	.781	26.8
Non-Durables	1.2	1.0	1.3	.714	.575	.808	26.5
Durables	5.6	6.1	5.0	.574	.298	.884	8.4
Total Invest. Fixed	5.1	4.2	5.9	.714	.454	.884	14.2
Residential	10.7	8.5	12.4	.436	.123	.637	4.4
Nonresidential	4.9	4.4	5.3	.684	.554	.777	9.7
Equipment	5.8	5.6	5.9	.707	.642	.760	6.0
Structures	4.5	3.8	5.1	.512	.225	.698	3.7
Total Government	4.8	6.5	2.2	.258	.353	.152	22.6
Federal	8.7	11.6	4.2	.266	.377	.125	10.8
State & Local	1.3	1.6	1.0	-.170	-.408	.131	11.8

TABLE 3

AGGREGATE DEMAND COMPONENTS
STRENGTH OF ASSOCIATION WITH GNP AND
MEASURE OF STABILITY

Sample Period: 1950.1 - 1979.2

	Correlation with Real Output Squared	R^2 for Regressions $c_{jt}=\alpha_j+\sum_{i=-2}^{2} \beta_{ji}GNP_{t+i}$	Measure of Instability
Total Consumption	.546	.620	2.16
Services	.378	.424	1.38
Non-Durables	.510	.589	.48
Durables	.329	.415	2.34
Total Investment Fixed	.509	.552	4.00
Residential	.190	.441	3.26
Non-residential	.468	.602	2.73
Equipment	.500	.631	2.92
Structures	.262	.367	2.05
Total Government	.067	.119	4.43
Federal	.071	.129	2.82
State & Local	.029	.095	2.27

Aggregate Demand Components

The first set of variables studied are the real aggregate demand components. The results are summarized in Tables 2 and 3. The series that are most stable are consumption of services, consumption of non-durables and state and local government purchases of goods and services. Each of these has standard deviation less than the 1.8 percent value for real output. The investment components including consumer durable expenditures are about three times as variable as output. Covariabilities of consumption and investment with output are much stronger than the covariability of government expenditures with output.

Factors of Production

The second set of variables considered are the factors of production and productivity which is output per hour. These results are summarized in Tables 4 and 5. There is a strong and stable positive relationship between hours and output. In addition, the variability in hours is comparable to the variability in output. The contemporaneous association between productivity and output is weak and unstable with the standard deviation of productivity being much smaller than the standard deviation of output. It is interesting to note that when lead and lag GNP's are included, the association between GNP and productivity increases dramatically with the R-squared increasing from .010 to .453.

Capital stocks both in durable goods and non-durable goods industries are less variable than real output and negatively associated with output. Inventory stocks, on the other hand, have a variability comparable to output and the correlation with output is positive. Further,

TABLE 4

FACTORS OF PRODUCTION: STANDARD DEVIATIONS AND CORRELATIONS WITH GNP

Sample Period: 1950.1 - 1979.2

	Standard Deviations in Percents			Correlations with Real Output		
	Whole	First Half	Second Half	Whole	First Half	Second Half
Real GNP	1.8	1.7	1.9	–	–	–
Capital Stocks						
Inventory	1.7	2.0	1.4	.507	.686	.309
Capital Stock Durables	1.2	1.4	1.0	-.210	-.178	-.274
Capital Stock Non-Durables	.7	.7	.7	-.236	-.185	-.297
Hours	2.0	2.1	1.8	.853	.896	.824
Work Week	.5	.6	.5	.820	.854	.800
Employees	1.4	1.6	1.2	.773	.831	.732
Productivity	1.0	1.0	1.1	.100	-.231	.361

TABLE 5

FACTORS OF PRODUCTION
STRENGTH OF ASSOCIATION WITH GNP
AND MEASURE OF STABILITY

Sample Period: 1950.1 - 1979.2

	Correlation with Real Output Squared	R^2 for Regressions $c_{jt}=\alpha_j+\sum_{i=-2}^{2}\beta_{ji}GNP_{t+i}$	Measure of Instability
Capital Stocks			
Inventory	.257	.622	7.86
Capital Stock Durable	.044	.235	3.07
Capital Stock Non-Durable	.056	.129	2.89
Hours	.728	.838	4.85
Work Week	.672	.700	4.62
Employees	.600	.801	4.92
Average Product of Labor	.010	.453	4.80

the strength of association of inventories with GNP increases when lag and lead GNP's are included in the regression. This is indicated by the increase in the R-squared from .257 to .622.

Monetary Variables

Results for the final set of variables are presented in Tables 6 and 7. Correlations between nominal money, velocity, and real money with GNP are all positive. The differences in the correlations in the first and second half of the sample with the exception of nominal M1 suggests considerable instability over time in these relationships. A similar conclusion holds for the short term interest rate. The correlations of GNP with the price variables are positive in the first half of the sample and negative in the second half with the correlation for the entire period being small and negative.

TABLE 6

MONETARY AND PRICE VARIABLES
STANDARD DEVIATIONS AND CORRELATIONS
WITH GNP

Sample Period: 1950.1 - 1979.2

	Standard Deviations in Percents			Correlations with Real Output		
	Whole Period	First Half	Second Half	Whole Period	First Half	Second Half
Real GNP	1.8	1.7	1.9	-	-	-
M1						
Nominal Value	.9	.8	1.0	.661	.675	.649
Velocity	1.6	2.0	1.0	.614	.801	.415
Real Value	1.5	1.2	1.7	.565	.079	.865
M2						
Nominal	1.1	.9	1.3	.480	.175	.665
Velocity	1.9	2.4	1.2	.529	.818	.131
Real Value	1.8	1.4	2.1	.432	-.221	.828
Interest Rate						
Short	.24	.27	.19	.510	.738	.255
Long	.06	.06	.06	.193	.640	-.175
Price Indexes						
GNP Deflator	1.0	1.0	1.1	-.239	.490	-.814
CPI	1.3	1.3	1.3	-.316	.223	-.799

TABLE 7

MONEY AND PRICE VARIABLES
STRENGTH OF ASSOCIATION WITH GNP AND
MEASURE OF STABILITY

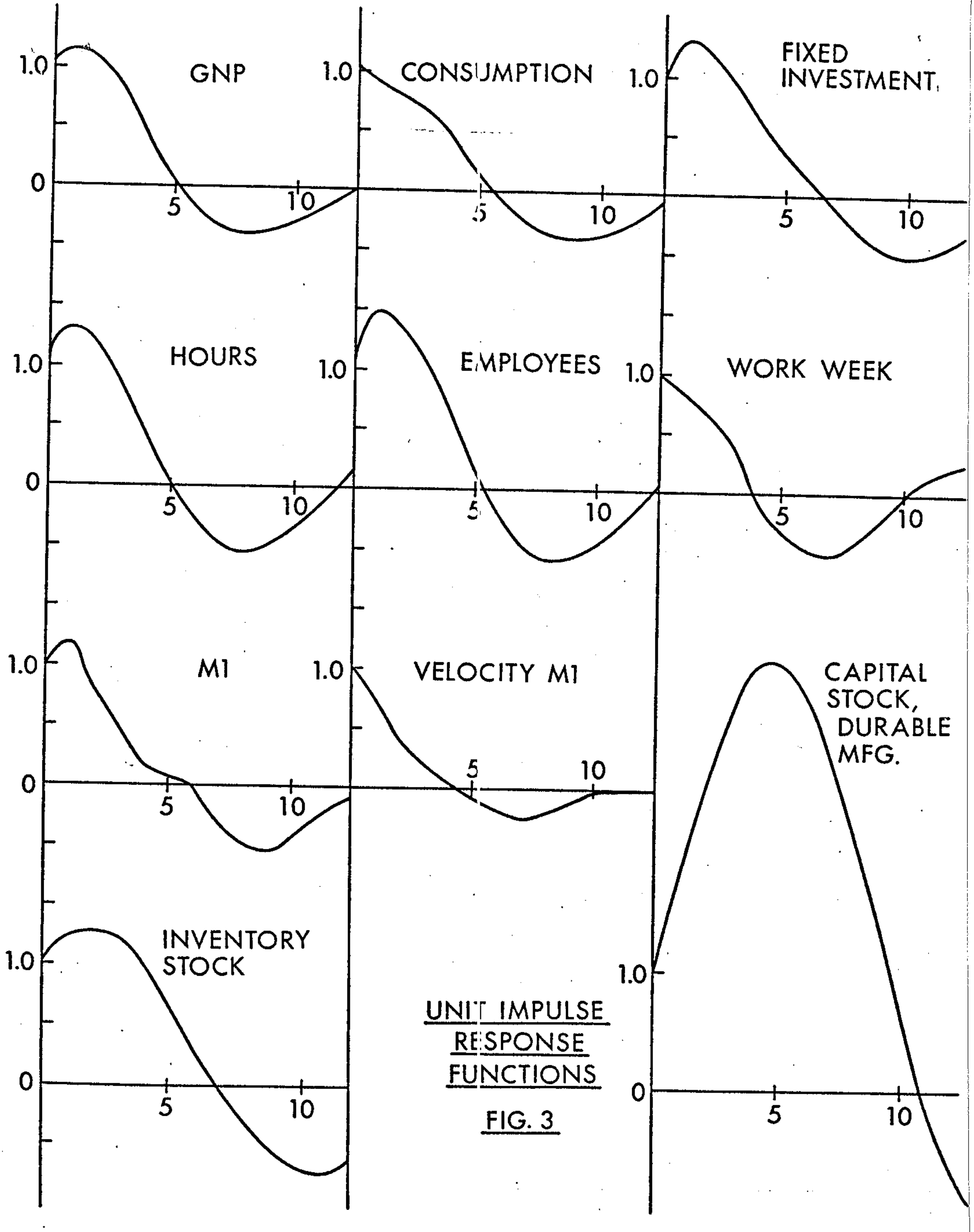
	Correlation with Real Output Squared	R^2 for Regressions $c_{jt} = \alpha_j + \sum_{i=-2}^{2} \beta_{ji} GNP_{t+1}$	Measure of Instability
M1			
Nominal Value	.437	.445	1.32
Velocity	.378	.408	4.37
Real Value	.319	.495	4.41
M2			
Nominal Value	.230	.371	.95
Velocity	.280	.376	7.41
Real Value	.187	.428	8.38
Interest Rate			
Short	.260	.506	7.37
Long	.037	.381	4.56
Price Index			
GNP Deflator	.057	.261	12.23
CPI	.010	.330	7.24

4. Serial Correlation Properties of Data Series

A sixth order autoregressive process was fit to a number of the series which displayed reasonable stable comovements with real output. In Figure 3 are plots of the unit impulse response functions for GNP and nine other series for the estimated autoregressive function. The function for GNP increases initially to a peak of 1.15 in period one and has a minimum of -.39 in period eight. The patterns for consumption and investment are similar except that for consumption the peak is in the initial period. The function for consumption and each of its three components (not pictured) are similar to the one for the aggregate.

The pattern for total hours and the number of employees, except for the greater amplitude, is very similar to the pattern for GNP. The average work week pattern, however, begins to decline immediately and the period of damped oscillation is shorter. The monetary variables have very different response patterns, indicating serial correlation properties very different than those of real output.

There is a dramatic difference in the response pattern for the capital stock in durable goods industries. The maximum amplitude of the response is much greater, being about 3.6, and occurs slightly over a year subsequent to the unit impulse. The pattern for the capital stock in the non-durable goods industries (not pictured) is similar though the maximum amplitude is smaller, being 2.8. For both capital stocks the peaks in the unit response function are in period five.



UNIT IMPULSE RESPONSE FUNCTIONS

FIG. 3

5. Concluding Comments

In this paper a method was developed for extracting the rapidly varying or cyclical component of economic time series and the method applied to aggregate post-war U.S. quarterly data. The motivation for reporting the variances, covariances and autocovariances of the cyclical components was recent developments in economic theory. It is now feasible to construct equilibrium models that place restrictions upon the statistics reported here. The hope is that these numbers will prove useful both in the search for an economic structure and the testing of theories of cyclical fluctuations. In this article no explanation of the cyclical regularities is offered. We think such an explanation can be provided only within the context of a well-specified eocnomic model. We do think it appropriate, however, to study the observations prior to theorizing.

APPENDIX

All the data were obtained from the Wharton Economic Forecasting Associates Quarterly Data Bank.

The short-term interest rate was the taxable three-month U.S. Treasury bill rate, and the long-term interest rate, the yield on U.S. Government long-term bonds.

FOOTNOTES

[1]Lucas [1980], for example, makes this point.

[2]Examples included Gordon [1980], Litterman and Sargent [1979], Neftci [1978], Nelson and Plosser [1980], Sargent and Sims [1977], Sims [1980, a,b] and Singleton [1980].

[3]We thank Paul Milgrom for bringing to our attention that the procedure we employed has been widely used by actuarians for a long time.

[4]This minimization has two elements, g_0 and $g_0 - g_{-1}$, which are treated as unknown parameters with diffuse priors. The Kalman smoothing technique (see Pagan [1980]) was used to efficiently compute the conditional expectations of the g_t, given the observed y_t. The posterior means of g_0 and $g_0 - g_{-1}$ are the generalized least squares estimates. The conditional expectation of the g_t for $t \geqq 1$ are linear functions of these parameters and the observations.

[5]Adding a white noise error, a_t, to (4) and determining the invertible moving average representations,

$$c_t = \sum_{i=0}^{\infty} \theta_i a_{t-i},$$

parameter θ_i equals the value of the unit response function in period i. One must take care in interpreting the response pattern. Two moving average processes can be observationally equivalent (same autocovariances function) yet have very different response patterns. We choose the invertible representation because it is unique. It is just one way to represent the serial correlation properties of a covariance stationary stochastic process. Others are the spectrum, the autoregressive representation and the autocovariance function. We thank Thomas J. Sargent who pointed out to us the fact that the response function is not identified unless one has prior knowledge to select among the moving average representations.

[6] See Miller [1946] for a derivation. There are certain implicit restrictions on the y_t sequence when the sample is infinite. Otherwise the g_{jt} may not exist. We require that the $\{y_t\}$ sequence belong to the space for which

$$\sum_{t=-\infty}^{\infty} .8941^{|t|} |y_{jt}| < \infty .$$

REFERENCES

Dension, E.F. 1974. Accounting for United States Growth 1929-1969. Washington, D.C.: Bookings Institute.

Litterman, R. and Sargent, T.J. 1979. "Detecting Neutral Price Level Changes and the Effects of Aggregate Demand with Index Models". University of Minnesota Working Paper.

Lucas, R.E., Jr. 1980. "Methods and Problems in Business Cycle Theory". "Journal of Money, Credit and Banking 12: 696-715

McCarthy, M.D. 1978. "The U.S. Productivity Growth Recession: History and Prospects for the Future." The Journal of Finance 33: 977-89.

Miller, M.D. 1946. Elements of Graduation. New York: Actuarial Society of America and American Institute of Actuaries.

Mitchell, W.C. 1913. Business Cycles. Berkeley, CA: University of California Press.

Neftci, S.N. 1978. "A Time-Series Analysis of the Real Wage-Employment Relationship." Journal of Political Economy 86: 281-91.

Nelson, C.L. and Plosser, C.I. 1980. "Trends and Random Walks in Macro-economic Time Series." University of Whashington Discussion Paper.

Pagan, A. 1980. "Some Identification and Estimation Results for Regression Models with Stochastically Varying Coefficients." Journal of Econometrics 13: 341-64.

Sargent, T.S. and Sims, C.A. 1977. "Business Cycle Modeling without Pretending to Have Too Much a Priori Economic Theory," in New Methods in Business Cycle Research: Proceedings from A Conference. Minneapolis: Federal Reserve Bank of Minneapolis, pp. 45-110.

Schumpeter, J.A. 1939. Business Cycles. Vol. 1. New York: McGraw-Hill.

Sims, C.A. 1980a. "Macroeconomics and Reality." Econometrica 48: 1-48.

Sims, C.A. 1980b. "Comparison of Interwar and Postwar Business Cycles: Monetarism Reconsidered." American Economic Review: Papers and Proceedings 70: 250-257

Singleton, K.J. 1980. "Real and Nominal Factors in the Cyclical Behavior of Interest Rates, Output, and Money". Carnegie-Mellon University Working Paper.

Stigler, S.M. 1978. "Mathematical Statistics in the Early States." Annals of Statistics 6: 239-65.

Whittaker, E.T. 1923. "On A New Method of Graduations". Proceedings of the Edinburgh Mathematical Society 41: 63-75.